

# The Sprague-Grundy Functions of Saturations of Misère Nim

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We consider misère Nim as a normal-play game obtained from Nim by removing the terminal position. While explicit formulas are known for the Sprague-Grundy functions of Nim and Welter’s game, no explicit formula is known for that of misère Nim. All three of these games can be considered as position restrictions of Nim. What are the differences between them? We point out that Nim and Welter’s game are saturated, but misère Nim is not. Moreover, we present explicit formulas for the Sprague-Grundy functions of saturations of misère Nim, which are obtained from misère Nim by adjoining some moves.

## 1 Introduction

The loser in Nim is the winner in misère Nim. Nim is a two-player game played with heaps of coins. Two players alternately choose a heap and take at least one coin from it. The player who takes the last coin wins in Nim and loses in misère Nim. In general, the player who moves last wins in the normal-play convention and loses in the misère-play convention, which has been extensively studied by using misère Sprague-Grundy functions, genera, and misère quotients (see, for example, [4, 5, 8, 12, 13]). In this paper, we will consider misère Nim as a normal-play game obtained from Nim by removing the terminal position (see Section 1.2).

Impartial games including Nim and misère Nim can be analyzed using their (normal) Sprague-Grundy functions [7, 15]. Sprague-Grundy functions are defined recursively, and computing them often leads to a combinatorial explosion. However, explicit formulas are known for the Sprague-Grundy functions of some games such as Nim [7, 15] and Welter’s game [16], which is a position restriction of Nim. Though misère Nim is also a position restriction of Nim, no explicit formula is currently available for its Sprague-Grundy function.<sup>1</sup>

What are the differences between Nim, Welter’s game, and misère Nim? One of the differences is that Nim and Welter’s game are *2-saturated*<sup>2</sup>, but misère Nim is not. The purpose of this paper is to present an explicit formula for the Sprague-Grundy functions of 2-saturations of misère Nim, which are obtained from misère Nim by adjoining some moves. More generally, for

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<sup>1</sup>By contrast, an explicit formula is known for the misère Sprague-Grundy function of Nim. See Remark 5.

<sup>2</sup>The concept of saturations was first introduced in [9] to connect Welter’s game with representations of symmetric groups.

a mixed-radix number system  $\beta$ , we give an explicit formula for the Sprague-Grundy functions of  $\beta$ -saturations of misère Nim.

## 1.1 Mixed-radix number systems

We introduce some notation for mixed-radix number systems.

Let  $\mathbb{N}$  be the set of nonnegative integers. Throughout this paper,  $\beta$  denotes a sequence  $(\beta_L)_{L \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$  with  $\beta_L \geq 2$  for every  $L \in \mathbb{N}$ . Define  $\beta^L = \beta_0 \cdot \beta_1 \cdots \beta_{L-1}$ . For example, if  $\beta = (2, 3, 2, \dots)$ , then  $\beta^0 = 1$ ,  $\beta^1 = 2$ , and  $\beta^2 = 6$ .

Let  $n \in \mathbb{N}$ . We denote by  $n_L^\beta$  the  $L$ th digit in the mixed base  $\beta$  expansion of  $n$ , that is, if  $n_{\geq L}^\beta$  is the integer quotient<sup>3</sup> of  $n$  divided by  $\beta^L$ , then  $n_L^\beta = n_{\geq L}^\beta \bmod \beta_L$ , where  $n_{\geq L}^\beta \bmod \beta_L$  is the remainder of  $n_{\geq L}^\beta$  divided by  $\beta_L$ . By definition,

$$n = \sum_{L \in \mathbb{N}} n_L^\beta \beta^L \quad \text{and} \quad n_L^\beta \in \{0, 1, \dots, \beta_L - 1\}.$$

For example, if  $\beta_L = b$  for every  $L \in \mathbb{N}$ , then  $n_L^\beta$  is the  $L$ th digit in the ordinary base  $b$  expansion of  $n$ , so it is convenient to write  $\beta = b$ . For a negative integer  $n$ , we define  $n_L^\beta$  similarly; then

$$n_L^\beta + (-n - 1)_L^\beta = \beta_L - 1.$$

For example,  $(-1)_L^\beta = \beta_L - 1$ . We drop the superscript  $\beta$  when no confusion can arise.

For  $n \in \mathbb{Z}$ , define

$$\text{ord}_\beta(n) = \begin{cases} \min \{ L \in \mathbb{N} : n_L \neq 0 \} \quad (= \max \{ L \in \mathbb{N} : \beta^L \text{ divides } n \}) & \text{if } n \neq 0, \\ \infty & \text{if } n = 0. \end{cases}$$

For example, if  $\beta = (3, 2, 5, 4, \dots)$ , then  $\text{ord}_\beta(54) = \text{ord}_\beta(4 \cdot \beta^2 + \beta^3) = 2$ .

## 1.2 Subtraction games

We define subtraction games, their misère versions, and Sprague-Grundy functions.

Fix a positive integer  $k$  and let  $\Omega$  denote  $\{0, 1, \dots, k-1\}$ . Let  $\mathcal{P} \subseteq \mathbb{N}^k$  and  $\mathcal{C} \subseteq \mathbb{N}^k \setminus \{(0, \dots, 0)\}$ . Define  $\Gamma(\mathcal{P}, \mathcal{C})$  to be the digraph with vertex set  $\mathcal{P}$  and edge set

$$\{(X, Y) \in \mathcal{P}^2 : X - Y \in \mathcal{C}\}.$$

We call  $\Gamma(\mathcal{P}, \mathcal{C})$  a *subtraction game* or a *take-away game*. The vertex set  $\mathcal{P}$  is called the *position set* of  $\Gamma(\mathcal{P}, \mathcal{C})$ .

*Remark 1.* We can consider  $\Gamma(\mathcal{P}, \mathcal{C})$  as a two-player game as follows. Before the game begins, we pick an initial position  $X_0 \in \mathcal{P}$ . The first player subtracts some  $C_0 \in \mathcal{C}$  from  $X_0$  so that  $X_0 - C_0 \in \mathcal{P}$ . Let  $X_1 = X_0 - C_0$ . Similarly, the second player subtracts some  $C_1 \in \mathcal{C}$  from  $X_1$  so that  $X_1 - C_1 \in \mathcal{P}$ . In this way, the two players alternately subtract some  $C \in \mathcal{C}$  from the current position. The player who subtracts last wins.

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<sup>3</sup> $n_{\geq L}^\beta$  is the unique integer satisfying  $n - n_{\geq L}^\beta \beta^L \in \{0, 1, \dots, \beta^L - 1\}$ .

**Example 2** (Nim). Let

$$\mathcal{C}_{[1]} = \{ C \in \mathbb{N}^k : \text{wt}(C) = 1 \},$$

where  $\text{wt}(C)$  is the Hamming weight of  $C$ , that is, the number of nonzero components of  $C$ . The subtraction game  $\Gamma(\mathbb{N}^k, \mathcal{C}_{[1]})$  is called *Nim*. For example, in Nim, the first player will win if we start from  $(1, 0)$ ; indeed, he can subtract  $(1, 0) \in \mathcal{C}_{[1]}$  from  $(1, 0)$ , but the second player cannot subtract any  $C \in \mathcal{C}_{[1]}$  from  $(0, 0)$ .

We next define the misère version of a subtraction game. Let  $X$  be a position in a subtraction game  $\Gamma(\mathcal{P}, \mathcal{C})$ . If  $X - C \in \mathcal{P}$  for some  $C \in \mathcal{C}$ , then  $X - C$  is called an *option* of  $X$  (in  $\Gamma(\mathcal{P}, \mathcal{C})$ ). If  $X$  has no options, then  $X$  is called a *terminal position*. Let  $\mathcal{P}'$  be the set of non-terminal positions in  $\Gamma(\mathcal{P}, \mathcal{C})$ . The subtraction game  $\Gamma(\mathcal{P}', \mathcal{C})$  is called the *misère version* of  $\Gamma(\mathcal{P}, \mathcal{C})$  [11].

**Example 3** (misère Nim). Let  $\mathcal{P}_{\text{Mis}} = \mathcal{P}_{\text{Mis}}^k = \mathbb{N}^k \setminus \{ (0, \dots, 0) \}$ . Then  $\mathcal{P}_{\text{Mis}}$  is the set of non-terminal positions of Nim, so the misère version of Nim is  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$ . We call  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$  *misère Nim*.<sup>4</sup> In misère Nim, the first player will lose if we start from  $(1, 0)$  because this position is terminal.

We now define Sprague-Grundy functions. See, for example, [1, 2, 5, 14] for details. Let  $\Gamma = \Gamma(\mathcal{P}, \mathcal{C})$  and  $X \in \mathcal{P}$ . The *Sprague-Grundy value*  $\text{sg}(X)$  of  $X$  is defined to be the minimum nonnegative integer  $n$  such that  $n$  is not equal to the Sprague-Grundy value of any option of  $X$ , that is,

$$\text{sg}(X) = \text{sg}_{\Gamma}(X) = \text{mex} \{ \text{sg}_{\Gamma}(Y) : Y \text{ is an option of } X \},$$

where  $\text{mex } S = \min \{ n \in \mathbb{N} : n \notin S \}$ . Note that if  $X$  is a terminal position, then  $\text{sg}(X) = \text{mex } \emptyset = 0$ . The nonnegative integer-valued function  $\text{sg} : \mathcal{P} \ni X \mapsto \text{sg}(X) \in \mathbb{N}$  is called the *Sprague-Grundy function* of the subtraction game  $\Gamma$ . For a position  $X$ , the following two statements are equivalent:

- (1)  $\text{sg}(X) = 0$ .
- (2) The second player can win when the initial position is  $X$ .

**Example 4.** Let  $k = 2$ . We calculate the Sprague-Grundy values of some positions in misère Nim (see Table 1 in Example 7). Since  $(0, 1)$  and  $(1, 0)$  are terminal positions, their Sprague-Grundy values are 0. Hence  $\text{sg}((0, 2)) = \text{sg}((1, 1)) = \text{sg}((2, 0)) = \text{mex} \{ 0 \} = 1$ . This implies that  $\text{sg}((1, 2)) = \text{sg}((2, 1)) = \text{mex} \{ 0, 1 \} = 2$ , so  $\text{sg}((2, 2)) = \text{mex} \{ 1, 2 \} = 0$ . We can verify that the second player can win when  $(2, 2)$  is the initial position.

*Remark 5.* The Sprague-Grundy function of misère Nim is different from the misère Sprague-Grundy function  $\mathcal{G}^-$  of Nim defined in [5]. The domain of the former function is  $\mathbb{N}^k \setminus \{ (0, \dots, 0) \}$  and that of the latter one is  $\mathbb{N}^k$ . Here, for  $X \in \mathbb{N}^k$ , we can compute  $\mathcal{G}^-(X)$  as follows:

$$\mathcal{G}^-(X) = \begin{cases} 1 & \text{if } X = (0, \dots, 0), \\ \text{mex} \{ \mathcal{G}^-(Y) : Y \text{ is an option of } X \} & \text{otherwise.} \end{cases}$$

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<sup>4</sup>Misère Nim is usually defined to be Nim in misère play, so the definition of misère Nim used in this paper is slightly different from the standard one. However, for  $X \in \mathbb{N}^k \setminus \{ (0, \dots, 0) \}$ , the outcomes of  $X$  in the two of these misère Nim are the same. In other words, when the initial position is  $X$ , the first player can win in one of the two misère Nim if and only if he can win in the other misère Nim.

The value  $\mathcal{G}^-(X)$  is generally not equal to  $\text{sg}_\Gamma(X)$ , where  $\Gamma$  is misère Nim  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$ . For example,  $\mathcal{G}^-((0, 2)) = 2 \neq 1 = \text{sg}_\Gamma((0, 2))$ . However,  $\mathcal{G}^-(X) = 0$  if and only if  $\text{sg}_\Gamma(X) = 0$ .<sup>5</sup>

### 1.3 $\beta$ -Saturation

We define  $\beta$ -saturation of subtraction games.

Elements in  $\mathbb{N}^k$  will be denoted by upper-case letters, and components of them by lower-case letters with superscripts. For example,  $C = (c^0, \dots, c^{k-1}) \in \mathbb{N}^k$ . Define

$$\mathcal{C}^\beta = \mathcal{C}^{\beta, k} = \left\{ C \in \mathbb{N}^k \setminus \{(0, \dots, 0)\} : \text{ord}_\beta \left( \sum_{i \in \Omega} c^i \right) = \text{mord}_\beta(C) \right\},$$

where

$$\text{mord}_\beta(C) = \min \{ \text{ord}_\beta(c^i) : i \in \Omega \}.$$

For example,  $(2, 2, 6) \in \mathcal{C}^2$  and  $(2, 2, 4) \notin \mathcal{C}^2$  because

$$\text{ord}_2(2 + 2 + 6) = 1 = \text{mord}_2((2, 2, 6)) \quad \text{and} \quad \text{ord}_2(2 + 2 + 4) = 3 > 1 = \text{mord}_2((2, 2, 4)).$$

A subtraction game  $\Gamma(\mathcal{P}, \mathcal{C})$  is said to be  $\beta$ -saturated if its Sprague-Grundy function is equal to that of  $\Gamma(\mathcal{P}, \mathcal{C}^\beta)$ . If  $\Gamma(\mathcal{P}, \mathcal{C})$  is  $\beta$ -saturated, then we also say that it is a  $\beta$ -saturation of  $\Gamma(\mathcal{P}, \mathcal{C}_{[1]})$ .

**Example 6** (Nim and Welter's game). Let

$$\mathcal{P}_{\text{Wel}} = \{ X \in \mathbb{N}^k : x^i \neq x^j \text{ whenever } i \neq j \}.$$

The subtraction game  $\Gamma(\mathcal{P}_{\text{Wel}}, \mathcal{C}_{[1]})$  is called *Welter's game*. It is known that Nim and Welter's game are 2-saturated [3, 6, 9, 16], that is, for  $\mathcal{P} \in \{\mathbb{N}^k, \mathcal{P}_{\text{Wel}}\}$ , the Sprague-Grundy function of  $\Gamma(\mathcal{P}, \mathcal{C}_{[1]})$  is equal to that of  $\Gamma(\mathcal{P}, \mathcal{C}^2)$ . Moreover,  $\Gamma(\mathbb{N}^k, \mathcal{C})$  is 2-saturated if and only if  $\mathcal{C}_{[1]} \subseteq \mathcal{C} \subseteq \mathcal{C}^2$  [3].

**Example 7.** Let  $k = 2$ . We compare the Sprague-Grundy function of misère Nim  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$  with that of  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}^2)$  (see Table 1). We first consider the position  $(2, 2)$ . The Sprague-Grundy value of  $(2, 2)$  equals 0 in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$ ; however, it equals 3 in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}^2)$  because  $(0, 1)$  is an option of  $(2, 2)$  in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}^2)$ . Thus misère Nim is not 2-saturated when  $k = 2$ . We next compute the Sprague-Grundy value of  $(2, 3)$  in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}^2)$ . Since  $(2, 3) - (0, 1) = (2, 2) \notin \mathcal{C}^2$  and  $(2, 3) - (1, 0) = (1, 3) \notin \mathcal{C}^2$ ,  $(2, 3)$  has no options with Sprague-Grundy value 0, and hence its Sprague-Grundy value is 0.

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<sup>5</sup>As we have mentioned,  $\mathcal{G}^-(X)$  can be written down explicitly. If  $\max X \geq 2$ , then  $\mathcal{G}^-(X) = \sigma^2(X)$ , where  $\sigma^2(X)$  is the Nim sum of the components of  $X$ . If  $\max X < 2$ , then  $\mathcal{G}^-(X) = 1 - \sigma^2(X)$ .

Table 1: Sprague-Grundy values in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}_{[1]})$  and  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C}^2)$ .

0	0	1	2	3	4	5	6	7	8	0	0	1	2	3	4	5	6	7	8
1	0	1	2	3	4	5	6	7	8	1	0	1	2	3	4	5	6	7	8
2	1	2	0	4	5	3	7	8	6	2	1	2	3	0	5	6	7	4	9
3	2	3	4	0	1	6	8	5	9	3	2	3	0	1	6	7	4	5	10
4	3	4	5	1	0	2	9	10	11	4	3	4	5	6	7	0	1	2	11
5	4	5	3	6	2	0	1	9	10	5	4	5	6	7	0	1	2	3	12
6	5	6	7	8	9	1	0	2	3	6	5	6	7	4	1	2	3	0	13
7	6	7	8	5	10	9	2	0	1	7	6	7	4	5	2	3	0	1	14
8	7	8	6	9	11	10	3	1	0	8	7	8	9	10	11	12	13	14	15

#### 1.4 A formula for $\beta$ -saturations of misère Nim

We present an explicit formula for the Sprague-Grundy functions of  $\beta$ -saturations of misère Nim.

Let  $\llbracket \beta_L \rrbracket$  denote  $\{0, 1, \dots, \beta_L - 1\}$  equipped with the following two operations: for  $a, b \in \llbracket \beta_L \rrbracket$ ,

$$a \oplus b = (a + b) \bmod \beta_L \quad \text{and} \quad a \ominus b = (a - b) \bmod \beta_L.$$

In other words,  $\llbracket \beta_L \rrbracket$  is the additive group of integers modulo  $\beta_L$ . For example, in  $\llbracket 5 \rrbracket$ ,  $2 \oplus 4 = 1$  and  $2 \ominus 4 = 3$ . For  $n \in \mathbb{N}$ , we will think of  $n_L^\beta$  as an element of  $\llbracket \beta_L \rrbracket$ . Let  $\mathbb{Z}_\beta$  denote  $\prod_{L \in \mathbb{N}} \llbracket \beta_L \rrbracket$  equipped with the following two operations: for  $n, m \in \mathbb{Z}_\beta$ ,

$$n \oplus m = [n_L \oplus m_L]_{L \in \mathbb{N}} \quad \text{and} \quad n \ominus m = [n_L \ominus m_L]_{L \in \mathbb{N}},$$

where  $n = [n_L]_{L \in \mathbb{N}}$  and  $m = [m_L]_{L \in \mathbb{N}}$ . For  $n \in \mathbb{Z}_\beta$ , define

$$\text{ord}_\beta(n) = \begin{cases} \min \{ L \in \mathbb{N} : n_L \neq 0 \} & \text{if } n \neq [0, 0, \dots], \\ \infty & \text{if } n = [0, 0, \dots]. \end{cases}$$

Consider the map  $\Phi : \mathbb{Z} \ni n \mapsto [n_L^\beta]_{L \in \mathbb{N}} \in \mathbb{Z}_\beta$ . For  $n \in \mathbb{Z}$ , we identify  $\Phi(n)$  with  $n$ . Let  $\mathcal{N}_\beta$  denote  $\Phi(\mathbb{N})$ . For  $n \in \mathcal{N}_\beta$ , it is convenient to write  $n = [n_0, n_1, \dots, n_{L-1}]$  when  $n_{\geq L} = 0$ . For example, if  $\beta = 10$ , then  $24 = [4, 2, 0, \dots] = [4, 2] \in \mathcal{N}_{10}$ .

For  $X \in \mathcal{N}_\beta^k$ , define

$$\sigma^\beta(X) = \sigma^{\beta, k}(X) = x^0 \oplus \dots \oplus x^{k-1} \quad (\in \mathcal{N}_\beta). \quad (1.1)$$

Let  $\sigma_L^\beta(X) = (\sigma^\beta(X))_L$  (= the  $L$ th digit of  $\sigma^\beta(X)$ ) for  $L \in \mathbb{N}$ .

**Example 8** ( $\beta$ -saturations of Nim [10]). Let  $X$  be a position in a  $\beta$ -saturation of Nim. Then

$$\text{sg}(X) = \sigma^\beta(X).$$

For example, if  $\beta = (3, 2, 5, \dots)$  and  $X = (16, 27) = ([1, 1, 2], [0, 1, 4]) \in \mathcal{N}_\beta^2$ , then

$$\text{sg}(X) = \sigma^\beta(X) = [1 \oplus 0, 1 \oplus 1, 2 \oplus 4] = [1, 0, 1] = 7.$$

**Example 9** (*b*-saturations of Welter's game [9]). Let  $X$  be a position in a  $b$ -saturation of Welter's game, where  $b$  is an integer greater than 1. Then

$$\text{sg}(X) = \sigma^b(X) \oplus \bigoplus_{i < j} \left( b^{\text{ord}_b(x^i - x^j) + 1} - 1 \right).$$

For example, if  $b = 3$  and  $X = (1, 4) \in \mathcal{N}_3^2$ , then  $\text{sg}(X) = 1 \oplus 4 \oplus (3^2 - 1) = 1$ .

We now give an explicit formula for the Sprague-Grundy functions of  $\beta$ -saturations of misère Nim. For  $X \in \mathcal{P}_{\text{Mis}} (\subseteq \mathcal{N}_\beta^k)$ , define

$$\phi^\beta(X) = \sigma^\beta(X) \oplus \left( \beta^{\text{mord}_\beta(X) + 1} - 1 \right). \quad (1.2)$$

**Theorem 10.** *The Sprague-Grundy function of a  $\beta$ -saturation of misère Nim is equal to  $\phi^\beta$ , that is,*

$$\text{sg}(X) = \phi^\beta(X)$$

for every position  $X$  in a  $\beta$ -saturation of misère Nim.

Before giving an example of Theorem 10, we introduce some notation. For  $n \in \mathcal{N}_\beta$  and  $L \in \mathbb{N}$ , let  $n_{<L} = [n_0, n_1, \dots, n_{L-1}] \in \mathcal{N}_\beta$ , that is,  $n_{<L} = n \bmod \beta^L$ . For  $X \in \mathcal{N}_\beta^k$ , let  $X_{<L} = (x_{<L}^0, x_{<L}^1, \dots, x_{<L}^{k-1}) \in \mathcal{N}_\beta^k$ . When  $X = X_{<L}$ , it is convenient to write  $X$  as follows:

$$X = \begin{bmatrix} x_0^0 & x_1^0 & \cdots & x_{L-1}^0 \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{k-1} & x_1^{k-1} & \cdots & x_{L-1}^{k-1} \end{bmatrix}.$$

Note that if  $\text{mord}_\beta(X) = N$ , then

$$X = \begin{bmatrix} & x_N^0 & \cdots & x_{L-1}^0 \\ 0 & \vdots & \cdots & \vdots \\ & x_N^{k-1} & \cdots & x_{L-1}^{k-1} \end{bmatrix}.$$

**Example 11.** Let us consider Example 7 again. Let

$$X = (2, 2) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in \mathcal{N}_2^2 \quad \text{and} \quad Y = (2, 3) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \in \mathcal{N}_2^2.$$

Then  $\text{mord}_2(X) = 1$  and  $\text{mord}_2(Y) = 0$ , so

$$\phi^2(X) = \sigma^2(X) \oplus (2^{1+1} - 1) = 3 \quad \text{and} \quad \phi^2(Y) = \sigma^2(Y) \oplus (2^{0+1} - 1) = 0.$$

### 1.5 The weight of $\phi^{\beta,k}$

We give the minimum of the *weight* of  $\mathcal{C}$  such that  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C})$  is a  $\beta$ -saturation of misère Nim.

Let  $\mathcal{P} \subseteq \mathbb{N}^k$ . For a nonnegative integer-valued function  $\psi : \mathcal{P} \rightarrow \mathbb{N}$ , let  $\Delta(\psi)$  be the set of  $\mathcal{C} \subseteq \mathbb{N}^k \setminus \{(0, \dots, 0)\}$  such that the Sprague-Grundy function of  $\Gamma(\mathcal{P}, \mathcal{C})$  equals  $\psi$ . Note that if  $\mathcal{C}, \mathcal{D} \in \Delta(\psi)$  and  $\mathcal{C} \subseteq \mathcal{E} \subseteq \mathcal{D}$ , then  $\mathcal{E} \in \Delta(\psi)$ . By definition,  $\Gamma(\mathcal{P}, \mathcal{C})$  is  $\beta$ -saturated if and

only if  $\mathcal{C} \in \Delta(\psi^\beta)$ , where  $\psi^\beta$  is the Sprague-Grundy function of  $\Gamma(\mathcal{P}, \mathcal{C}^\beta)$ . If  $\Delta(\psi) \neq \emptyset$ , then define

$$\text{wt}(\psi) = \min_{\mathcal{C} \in \Delta(\psi)} \text{wt}(\mathcal{C}),$$

where  $\text{wt}(\mathcal{C}) = \max \{ \text{wt}(C) : C \in \mathcal{C} \}$  and  $\max \emptyset = 0$ . For example, if  $\psi^2$  is the Sprague-Grundy function of a 2-saturation of Nim or that of a 2-saturation of Welter's game, then

$$\text{wt}(\psi^2) = 1$$

since  $\mathcal{C}_{[1]} \in \Delta(\psi^2)$ . In other words, Nim and Welter's game themselves are 2-saturated. However, as we have seen in Example 7, if  $k = 2$ , then  $\mathcal{C}_{[1]} \notin \Delta(\phi^{2,2})$ , so  $\text{wt}(\phi^{2,2}) = 2$ . In fact, if  $k \geq 2$ , then

$$\text{wt}(\phi^{2,k}) = 2.$$

Let  $B$  be the supremum of  $\{ \beta_L : L \geq 1 \}$  in  $\mathbb{N} \cup \{ \infty \}$ . In general, we will prove that

$$\begin{aligned} \text{wt}(\phi^{\beta,k}) &= \max \left\{ \min \left\{ \beta_L - \delta(L), k - \delta(L) [\beta_0 < 2k] \right\} : L \in \mathbb{N} \right\} \\ &= \begin{cases} k & \text{if } B \geq k \text{ or } \beta_0 \geq 2k, \\ k - 1 & \text{if } B < k \text{ and } k \leq \beta_0 < 2k, \\ \max \{ \beta_0 - 1, B \} & \text{if } B < k \text{ and } \beta_0 < k, \end{cases} \end{aligned} \quad (1.3)$$

where  $\delta(L) = [L = 0]$  and  $[ \ ]$  is the Iverson bracket notation, that is,  $[P] = 1$  if a statement  $P$  holds, and  $[P] = 0$  otherwise. In particular, if  $\beta = b$  for some  $b \in \mathbb{N}$ , then

$$\text{wt}(\phi^{b,k}) = \min \{ b, k \}.$$

## 2 Proofs

When no confusion can arise, we write  $\sigma$  and  $\phi$  instead of  $\sigma^\beta$  and  $\phi^\beta$ , respectively.

### 2.1 Preliminaries

Let  $\mathcal{C} \subseteq \mathcal{N}_\beta^k \setminus \{ (0, \dots, 0) \}$ . The Sprague-Grundy function of  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C})$  equals  $\phi$  if and only if  $\mathcal{C}$  satisfies the following two conditions:

- (SG1)** If  $X \in \mathcal{P}_{\text{Mis}}$ , then  $X$  has no option  $Y$  with  $\phi(Y) = \phi(X)$  in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C})$ , that is,  $\phi(X - C) \neq \phi(X)$  for every  $C \in \mathcal{C}$  with  $X - C \in \mathcal{P}_{\text{Mis}}$ .
- (SG2)** If  $X \in \mathcal{P}_{\text{Mis}}$  and  $0 \leq m < \phi(X)$ , then  $X$  has an option  $Y$  with  $\phi(Y) = m$  in  $\Gamma(\mathcal{P}_{\text{Mis}}, \mathcal{C})$ , that is,  $\phi(X - C) = m$  for some  $C \in \mathcal{C}$  with  $X - C \in \mathcal{P}_{\text{Mis}}$ .

To prove Theorem 10 and (1.3), it therefore suffices to show the following three assertions:

- (A1)**  $\mathcal{C}^\beta$  satisfies (SG1).
- (A2)**  $\{ C \in \mathcal{N}_\beta^k : \text{wt}(C) < w \}$  does not satisfy (SG2), where  $w$  is the right-hand side of (1.3).
- (A3)**  $\mathcal{C}_{[w]}^\beta$  satisfies (SG2), where  $\mathcal{C}_{[w]}^\beta = \{ C \in \mathcal{C}^\beta : \text{wt}(C) \leq w \}$ .

## 2.2 Proof of (A1)

Let  $X \in \mathcal{P}_{\text{Mis}}$ ,  $C \in \mathcal{C}^\beta$  with  $X - C \in \mathcal{P}_{\text{Mis}}$ , and  $Y = X - C$ . Let  $N = \text{mord}_\beta(X)$ ,  $M = \text{mord}_\beta(Y)$ , and  $H = \text{mord}_\beta(C)$ . We show that  $\phi_H(Y) \neq \phi_H(X)$ , where  $\phi_H(X) = (\phi(X))_H$ . Since  $(\beta^{N+1} - 1)_L = \ominus[L \leq N]$  for  $L \in \mathbb{N}$ , it follows that

$$\phi_L(X) = \sigma_L(X) \ominus [L \leq N] \quad \text{and} \quad \phi_L(Y) = \sigma_L(Y) \ominus [L \leq M]. \quad (2.1)$$

We first show that

$$[H \leq N] = [H \leq M]. \quad (2.2)$$

Since  $C_{<H} = (0, \dots, 0)$ , we see that  $Y_{<H} = (X - C)_{<H} = X_{<H}$ . Suppose that  $H > N$ . Then  $Y_{\leq N} = X_{\leq N} \neq (0, \dots, 0)$ , where  $Y_{\leq N} = Y_{<N+1}$ . Hence  $M = N$ , so (2.2) holds. If  $H \leq N$ , then  $Y_{<H} = X_{<H} = (0, \dots, 0)$ , so  $H \leq M$ . Therefore (2.2) holds.

We next show that

$$\sigma_H(X) \neq \sigma_H(Y). \quad (2.3)$$

Since  $C_{<H} = (0, \dots, 0)$ , it follows that  $y_H^i = x_H^i \ominus c_H^i$  for  $i \in \Omega$ . Hence

$$\sigma_H(Y) = \sigma_H(X) \ominus \sigma_H(C).$$

Recall that  $H = \text{mord}_\beta(C) = \text{ord}_\beta(\sum_i c^i)$  since  $C \in \mathcal{C}^\beta$ . This implies that  $(\sum_i c^i)_H = \sigma_H(C) \neq 0$ . Thus (2.3) holds.

Combining (2.1)–(2.3), we see that  $\phi_H(X) \neq \phi_H(Y)$ . Therefore  $\mathcal{C}^\beta$  satisfies (SG1).  $\square$

## 2.3 Proof of (A2)

A position  $Y \in \mathcal{P}_{\text{Mis}}$  is called a *descendant* of a position  $X \in \mathcal{P}_{\text{Mis}}$  if  $X - Y \in \mathcal{N}_\beta^k$ .

If  $k = 1$ , then  $w = 1$ , so (A2) is obvious. Suppose that  $k \geq 2$ . It suffices to show that there exist  $X \in \mathcal{P}_{\text{Mis}}$  and  $m$  with  $0 \leq m < \phi(X)$  satisfying the following condition: if  $Y$  is a descendant of  $X$  with  $\phi(Y) = m$ , then  $\text{wt}(X - Y) \geq w$ . By (1.3),

$$w = \min \left\{ \beta_N - \delta(N), k - \delta(N)[\beta_0 < 2k] \right\} \quad \text{for some } N \in \mathbb{N}.$$

Note that  $w \geq 1$ . We divide the proof into two cases.

**Case 1** ( $N > 0$  or  $\beta_0 < 2k$ ). We see that  $w = \min \{ \beta_N, k \} - \delta(N)$ . Let

$$X = \underbrace{(\beta^N, \dots, \beta^N)}_{w+\delta(N)}, 0, \dots, 0 \in \mathcal{P}_{\text{Mis}}^k.$$

Since  $w + \delta(N) \leq \beta_N$  and  $\text{mord}_\beta(X) = N$ , it follows that

$$\begin{aligned} \phi(X) &= \sigma(X) \oplus (\beta^{N+1} - 1) = \underbrace{[0, \dots, 0]}_N \oplus \underbrace{[1 \oplus \dots \oplus 1]}_{w+\delta(N)} \ominus \underbrace{[1, \dots, 1]}_N \\ &= \begin{cases} w\beta^N - 1 & \text{if } N > 0, \\ w & \text{if } N = 0. \end{cases} \end{aligned}$$



In particular,  $\phi(X) > 0$ . Let  $Y$  be a descendant of  $X$  with  $\phi(Y) = 0$ . To prove that  $\text{wt}(X - Y) = w$ , we show that  $\sum_i y_N^i = \delta(N)$ . Since  $\phi_0(Y) = 0$ , we see that  $\text{mord}_\beta(Y) = 0$ . Hence  $\phi_N(Y) = \sigma_N(Y) \ominus [N \leq 0] = \sigma_N(Y) \ominus \delta(N)$ . Since  $\phi_N(Y) = 0$ ,

$$\sigma_N(Y) = \delta(N). \quad (2.4)$$

We also see that  $\sum_i y_N^i < \sum_i x_N^i = w + \delta(N) \leq \beta_N$  because  $Y$  is a descendant of  $X$  with  $Y \neq X$ . Hence  $\sum_i y_N^i = \sigma_N(Y) = \delta(N)$ . Therefore  $\text{wt}(X - Y) = w$ .

**Case 2** ( $N = 0$  and  $\beta_0 \geq 2k$ ). Since  $k \geq 2$ , we see that  $\beta_0 \geq 4$  and  $w = \min\{\beta_0 - 1, k\} = k$ . Let

$$X = (2, \dots, 2) \in \mathcal{P}_{\text{Mis}}^k.$$

Then  $\phi(X) = \phi_0(X) = 2k - 1 > 0$ . Let  $Y$  be a descendant of  $X$  with  $\phi(Y) = 0$ . Then  $\phi_0(Y) = \sigma_0(Y) \ominus 1 = 0$ . Since  $\sum_i y_0^i < \sum_i x_0^i = 2k \leq \beta_0$ , it follows that  $\sum_i y_0^i = \sigma_0(Y) = 1$ . This implies that  $y^i \in \{0, 1\}$  for  $i \in \Omega$ . Hence  $\text{wt}(X - Y) = k = w$ .  $\square$

**Example 12.** Let  $\beta = (6, 2, 2, \dots)$ ,  $k = 3$ , and  $X = (2, 2, 2)$ . Then  $\phi(X) = 5$ . If  $Y$  is a descendant of  $X$  with  $\phi(Y) = 0$ , then  $Y \in \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}$ , and hence  $\text{wt}(X - Y) = 3$ . Note that if  $\beta = (5, 2, 2, \dots)$ , then  $\phi((2, 2, 2)) = 0$ .

## 2.4 Proof of (A3)

To prove (A3), we present two lemmas.

For  $X \in \mathcal{P}_{\text{Mis}}$ , the next lemma allows us to express  $x^0$  with  $x^1, \dots, x^{k-1}$ , and  $\phi(X)$ .

**Lemma 13.** *Let  $X \in \mathcal{P}_{\text{Mis}}$  and  $m \in \mathcal{N}_\beta$ . For  $i \in \Omega$ , let*

$$M^{(i)} = \text{mord}_\beta \left( (m \ominus (-1), x^0, \dots, x^{i-1}, x^{i+1}, \dots, x^{k-1}) \right),$$

$$y^{(i)} = m \ominus (\beta^{M^{(i)}+1} - 1) \ominus x^0 \ominus \dots \ominus x^{i-1} \ominus x^{i+1} \ominus \dots \ominus x^{k-1},$$

and

$$Y^{(i)} = (x^0, \dots, x^{i-1}, y^{(i)}, x^{i+1}, \dots, x^{k-1}).$$

Then  $\text{mord}_\beta(Y^{(i)}) = M^{(i)}$ . In particular,  $\phi(Y^{(i)}) = m$ .

*Proof.* It suffices to prove the lemma when  $i = 0$ . Let  $M = M^{(0)}$  and  $Y = Y^{(0)}$ . We show that  $\text{mord}_\beta(Y) = M$ . For  $L < M$ ,

$$\begin{aligned} y_L^0 &= m_L \ominus (\beta^{M+1} - 1)_L \ominus x_L^1 \ominus \dots \ominus x_L^{k-1} \\ &= m_L \ominus (\ominus 1) = (m \ominus (-1))_L = 0. \end{aligned}$$

This implies that  $\text{mord}_\beta(Y) \geq M$ . By the definition of  $M$ , we see that  $(m \ominus (-1))_M \neq 0$  or  $x_M^j \neq 0$  for some  $j \geq 1$ . If the latter holds, then  $\text{mord}_\beta(Y) = M$ . Suppose that  $(m \ominus (-1))_M \neq 0$  and  $x_M^j = 0$  for every  $j \geq 1$ . Then  $y_M^0 = m_M \ominus (\beta^{M+1} - 1)_M \ominus x_M^1 \ominus \dots \ominus x_M^{k-1} = (m \ominus (-1))_M \neq 0$ , so  $\text{mord}_\beta(Y) = M$ . Therefore

$$m = y^0 \oplus x^1 \oplus \dots \oplus x^{k-1} \oplus (\beta^{M+1} - 1) = \sigma^\beta(Y) \oplus (\beta^{\text{mord}_\beta(Y)+1} - 1) = \phi(Y).$$

$\square$

**Example 14.** Let  $\beta = 3$ ,  $X = (3, 4)$ , and  $m = 2$ . Note that  $m \ominus (-1) = [0, 1, 1, \dots]$  and  $\text{ord}_3(m \ominus (-1)) = 1$ . Since

$$M^{(0)} = \text{mord}_3((m \ominus (-1), 4)) = 0 \quad \text{and} \quad M^{(1)} = \text{mord}_3((m \ominus (-1), 3)) = 1,$$

it follows that

$$y^{(0)} = m \ominus (3^1 - 1) \ominus 4 = 8 \quad \text{and} \quad y^{(1)} = m \ominus (3^2 - 1) \ominus 3 = 0.$$

By Lemma 13,  $\phi((y^{(0)}, x^1)) = \phi((x^0, y^{(1)})) = m = 2$ . Indeed,

$$\phi((8, 4)) = 8 \oplus 4 \oplus (3^1 - 1) = 2 \quad \text{and} \quad \phi((3, 0)) = 3 \oplus 0 \oplus (3^2 - 1) = 2.$$

The following trivial lemma will be used to construct appropriate options. For  $X \in \mathcal{N}_\beta^k$  and  $L \in \mathbb{N}$ , let  $x_L = (x_L^0, x_L^1, \dots, x_L^{k-1}) \in \llbracket \beta_L \rrbracket^k$ . For example, if  $\beta = 3$  and  $X = ([1, 0, 2], [2, 1])$ , then  $x_0 = (1, 2)$  and  $x_1 = (0, 1)$ .

**Lemma 15.** Let  $X \in \mathcal{N}_\beta^k$ ,  $m \in \mathcal{N}_\beta$ , and  $R \in \mathbb{N}$ . Choose  $j \in \Omega$  so that  $x_R^j \geq x_R^i$  for every  $i \in \Omega$ . If  $m_R \leq \sum_i x_R^i$ , then there exists  $u \in \llbracket \beta_R \rrbracket^k$  satisfying the following three conditions:

- (1)  $\bigoplus_{i \in \Omega} u^i = m_R$ , where  $u = (u^0, \dots, u^{k-1})$ .
- (2)  $0 \leq x_R^i - u^i \leq x_R^j - u^j$  for every  $i \in \Omega$ .
- (3)  $\sum_{i \in \Omega} (x_R^i - u^i) \leq \beta_R - 1$ .

Before proving Lemma 15, let us give an example. Let  $\beta = (7, 2, \dots)$ ,  $X = (4, 4, 3)$ ,  $R = j = 0$ , and  $m = 6$ . Then  $m_0 = 6 < 11 = \sum_i x_0^i$ . Since  $(\bigoplus_i x_0^i) \ominus m_0 = 4 \ominus 6 = 5 < 11$ , we can obtain  $u$  satisfying (1)–(3) by subtracting a vector  $(t^0, t^1, t^2)$  with  $\sum t^i = 5$  from  $x_0$ . For example, let  $t = (4, 1, 0)$ ; then  $x_0 - t = (0, 3, 3)$ , and it satisfies (1)–(3).

*Proof.* By rearranging  $x^0, \dots, x^{k-1}$  if necessary, we may assume that  $j = 0$ . Let  $d = (\bigoplus_i x_R^i) \ominus m_R$ . Since  $m_R \leq \sum_i x_R^i$ , it follows that  $d \leq \sum_i x_R^i - m_R \leq \sum_i x_R^i$ . For  $i \in \Omega$ , let

$$t^i = \min \left\{ x_R^i, d - \sum_{h=0}^{i-1} t^h \right\}.$$

Then  $\sum_i t^i = d$ . It follows that  $x_R - (t^0, \dots, t^{k-1})$  satisfies (1)–(3).  $\square$

We now prove (A3). Let  $n = \phi(X)$  and  $m \in \mathcal{N}_\beta$  with  $0 \leq m < n$ . We show that  $\phi(X - C) = m$  for some  $C \in \mathcal{C}_{[w]}^\beta$  with  $X - C \in \mathcal{P}_{\text{Mis}}$ . Let  $N = \text{mord}_\beta(X)$  and

$$R = \max \{ L \in \mathbb{N} : m_L \neq n_L \}.$$

We divide the proof into two cases.

**Case 1 ( $N > R$ ).** By rearranging  $x^0, \dots, x^{k-1}$  if necessary, we may assume that  $x_N^0 \neq 0$ . For  $i \geq 1$ , let  $y^i = x^i$ . Let

$$\begin{aligned} M &= \text{mord}_\beta((m \ominus (-1), y^1, \dots, y^{k-1})), \\ y^0 &= m \ominus (\beta^{M+1} - 1) \ominus y^1 \ominus \dots \ominus y^{k-1}, \end{aligned}$$

and  $Y = (y^0, y^1, \dots, y^{k-1})$ . By Lemma 13,  $\phi(Y) = m$  and  $\text{mord}_\beta(Y) = M$ . It remains to prove that  $y^0 < x^0$ . We show that  $y_N^0 = x_N^0 - 1$  and  $y_{\geq N+1}^0 = x_{\geq N+1}^0$ . Since  $m_R < n_R \leq \beta_R - 1$ , we see that  $(m \ominus (-1))_R = m_R \oplus 1 \neq 0$ . Hence  $M \leq R < N$ . Since  $\text{mord}_\beta(Y) = M$ ,

$$m_N = y_N^0 \oplus y_N^1 \oplus \dots \oplus y_N^{k-1} \ominus [N \leq M] = y_N^0 \oplus x_N^1 \oplus \dots \oplus x_N^{k-1}. \quad (2.5)$$

Moreover, since  $N > R$ ,

$$m_N = n_N = x_N^0 \oplus x_N^1 \oplus \dots \oplus x_N^{k-1} \ominus [N \leq N] = x_N^0 \oplus x_N^1 \oplus \dots \oplus x_N^{k-1} \ominus 1. \quad (2.6)$$

By (2.5) and (2.6),  $y_N^0 = x_N^0 \ominus 1$ . Since  $x_N^0 \neq 0$ , it follows that  $y_N^0 = x_N^0 \ominus 1 = x_N^0 - 1$ . Similarly, for  $L \geq N + 1$ ,

$$y_L^0 \oplus x_L^1 \oplus \dots \oplus x_L^{k-1} \ominus [L \leq M] = m_L = n_L = x_L^0 \oplus x_L^1 \oplus \dots \oplus x_L^{k-1} \ominus [L \leq N].$$

Since  $[L \leq M] = [L \leq N] = 0$ , we see that  $y_L^0 = x_L^0$ . Therefore  $y^0 < x^0$  and  $X - Y = (x^0 - y^0, 0, \dots, 0) \in \mathcal{C}_{[w]}^\beta$ .

**Case 2 ( $N \leq R$ ).** By rearranging  $x^0, \dots, x^{k-1}$  if necessary, we may assume that  $x_R^0 \geq x_R^1 \geq \dots \geq x_R^{k-1}$ . Let  $M = \text{mord}_\beta((m \ominus (-1), x^1, \dots, x^{k-1}))$ . Since  $m_R < n_R \leq \beta_R - 1$ , it follows that  $M \leq R$ .

**Claim.** There exists  $u \in [[\beta_R]^k$  satisfying the following four conditions:

**(C1)**  $\bigoplus_{i \in \Omega} u^i = m_R \oplus [R \leq M]$ , where  $u = (u^0, \dots, u^{k-1})$ .

**(C2)**  $0 \leq x_R^i - u^i \leq x_R^0 - u^0$  for every  $i \in \Omega$ .

**(C3)**  $u^0 < x_R^0$  unless  $N < R = M$ .

**(C4)**  $\text{wt}(x_R - u) \leq w$ .

Assuming the claim for the moment, we construct  $Y$  with  $\phi(Y) = m$  and  $X - Y \in \mathcal{C}_{[w]}^\beta$ . For  $i \geq 1$ , let

$$y^i = [x_0^i, \dots, x_{R-1}^i, u^i, x_{R+1}^i, x_{R+2}^i, \dots] = x^i - (x_R^i - u^i)\beta^R (\leq x^i). \quad (2.7)$$

Then

$$\text{mord}_\beta((m \ominus (-1), y^1, \dots, y^{k-1})) = M. \quad (2.8)$$

Indeed, if  $M < R$ , then (2.8) is obvious. If  $M = R$ , then  $(m \ominus (-1))_M = (m \ominus (-1))_R \neq 0$ , so (2.8) holds. Let

$$y^0 = m \ominus (\beta^{M+1} - 1) \ominus y^1 \ominus \dots \ominus y^{k-1} \quad (2.9)$$

and  $Y = (y^0, y^1, \dots, y^{k-1})$ . It follows from (2.8) and Lemma 13 that  $\phi(Y) = m$  and  $\text{mord}_\beta(Y) = M$ .

Let  $C = X - Y$ . We next show that  $C \in \mathcal{C}_{[w]}^\beta$ , that is,

(a)  $C \in \mathcal{N}_\beta^k \setminus \{(0, \dots, 0)\}$ ,

(b)  $\text{wt}(C) \leq w$ , and

(c)  $\text{ord}_\beta(\sum c^i) = \text{mord}_\beta(C)$ .

(a) Since  $\phi(Y) = m \neq n = \phi(X)$ , it follows that  $C \neq (0, \dots, 0)$ . By (2.7), we see that  $y^i \leq x^i$  for  $i \geq 1$ . To prove that  $y^0 \leq x^0$ , we show that  $y_{\geq R+1}^0 = x_{\geq R+1}^0$  and  $y_R^0 = u^0$ . By (2.9), for  $L > R (\geq M, N)$ ,

$$y_L^0 = m_L \oplus [L \leq M] \ominus y_L^1 \ominus \dots \ominus y_L^{k-1} = n_L \ominus x_L^1 \ominus \dots \ominus x_L^{k-1} = x_L^0 \ominus [L \leq N] = x_L^0.$$

We also see that

$$y_R^0 = m_R \oplus [R \leq M] \ominus y_R^1 \ominus \dots \ominus y_R^{k-1} = m_R \oplus [R \leq M] \ominus u^1 \ominus \dots \ominus u^{k-1} = u^0$$

by (C1). Hence if  $u^0 < x_R^0$ , then  $y^0 < x^0$ . Suppose that  $u^0 = x_R^0$ . Then  $M = R$  by (C3). Since  $M = \text{mord}_\beta(Y)$ ,

$$y^0 = \underbrace{[0, \dots, 0]}_M, u^0, x_{M+1}^0, x_{M+2}^0, \dots] = \underbrace{[0, \dots, 0]}_M, x_M^0, x_{M+1}^0, x_{M+2}^0, \dots] \leq x^0.$$

(b) If  $u^0 \neq x_R^0$ , then  $\text{wt}(C) = \text{wt}(x_R - u)$ , and hence  $\text{wt}(C) \leq w$  by (C4). Suppose that  $u^0 = x_R^0$ . By (C2),  $u = x_R$ , so  $\text{wt}(C) = 1 \leq w$ .

(c) For  $i \geq 1$ , we know that  $c^i = x^i - y^i = (x_R^i - u^i)\beta^R$ . Hence if  $\text{ord}_\beta(c^0) < R$ , then

$$\text{mord}_\beta(C) = \text{ord}_\beta(c^0) = \text{ord}_\beta\left(\sum_{i \in \Omega} c^i\right).$$

Suppose that  $\text{ord}_\beta(c^0) \geq R$ . Then  $R \leq \text{mord}_\beta(C) \leq \text{ord}_\beta(\sum c^i)$ , so we need only show that  $\text{ord}_\beta(\sum c^i) = R$ , that is,

$$\bigoplus_{i \in \Omega} c_R^i \neq 0. \quad (2.10)$$

We first show that  $M = N$ . Since  $Y_{<R} = X_{<R}$  and  $M, N \leq R$ , it follows that if  $M < R$  or  $N < R$ , then  $M = N$ . If  $M \geq R$  and  $N \geq R$ , then  $M = R = N$ . We now show (2.10). Since  $y_R^i = x_R^i \ominus c_R^i$ , we see that

$$\begin{aligned} m_R &= \bigoplus_{i \in \Omega} y_R^i \ominus [R \leq M] = \bigoplus_{i \in \Omega} x_R^i \ominus \left(\bigoplus_{i \in \Omega} c_R^i\right) \ominus [R \leq N] \\ &= n_R \ominus \left(\bigoplus_{i \in \Omega} c_R^i\right). \end{aligned}$$

Since  $m_R \neq n_R$ , it follows that (2.10) holds, so  $\text{ord}_\beta(\sum c^i) = R = \text{mord}_\beta(C)$ . Therefore  $C \in \mathcal{C}_{[w]}^\beta$ .

It remains to prove the claim. We first show that  $x_R \neq (0, \dots, 0)$ . If  $R = N$ , then the assertion is obvious. If  $R > N$ , then  $[R \leq N] = 0$ , so  $m_R < n_R = \bigoplus_i x_R^i$ , which implies that  $x_R \neq (0, \dots, 0)$ .

We next show that

$$n_R + [R \leq N] \leq \sum_{i \in \Omega} x_R^i. \quad (2.11)$$

Since  $n_R \oplus [R \leq N] = \bigoplus_i x_R^i \leq \sum_i x_R^i$ , if  $n_R + [R \leq N] = n_R \oplus [R \leq N]$ , then (2.11) holds. Suppose that  $n_R + [R \leq N] \neq n_R \oplus [R \leq N]$ . Then  $[R \leq N] = 1$  and  $n_R \oplus [R \leq N] = 0$ . Hence  $\bigoplus_i x_R^i = 0$ . Since  $x_R \neq (0, \dots, 0)$ , it follows that  $\sum x_R^i \geq \beta_R = n_R + [R \leq N]$ .

We now construct  $u$  satisfying (C1)–(C4). Since  $[R \leq M], [R \leq N] \in \{0, 1\}$  and  $m_R < n_R$ , we see that

$$m_R \oplus [R \leq M] = m_R + [R \leq M] \leq n_R + [R \leq N] \leq \sum_{i \in \Omega} x_R^i.$$

By Lemma 15, there exists  $u \in \llbracket \beta_R \rrbracket^k$  satisfying (C1), (C2), and

$$\text{wt}(x_R - u) \leq \min \{ \beta_R - 1, k \}.$$

We divide the proof into two cases.

**Case i ( $R > 0$ ).** Since  $\min \{ \beta_R - 1, k \} \leq w$ , we see that  $u$  satisfies (C4). If  $u \neq x_R$ , then  $u^0 < x_R^0$ , so  $u$  also satisfies (C3). Suppose that  $u = x_R$ . Then

$$m_R \oplus [R \leq M] = \bigoplus_{i \in \Omega} u^i = \bigoplus_{i \in \Omega} x_R^i = n_R \oplus [R \leq N]. \quad (2.12)$$

It follows that  $[R \leq M] \neq [R \leq N]$  since  $m_R < n_R$ . Suppose that  $[R \leq M] = 1$  and  $[R \leq N] = 0$ . Then  $R \leq M$  and  $R > N$ , so  $N < R = M$  since  $M \leq R$ . Hence  $u$  satisfies (C3). Suppose that  $[R \leq M] = 0$  and  $[R \leq N] = 1$ . By (2.12),

$$m_R = \bigoplus_{i \in \Omega} x_R^i = n_R \oplus 1 < n_R.$$

This implies that  $\bigoplus_i x_R^i = 0$ , and hence that  $\sum_i x_R^i \geq \beta_R$  since  $x_R \neq (0, \dots, 0)$ . We can now find  $(\tilde{u}^0, \dots, \tilde{u}^{k-1})$  satisfying  $\sum_i \tilde{u}^i = \sum_i x_R^i - \beta_R$  and (C1)–(C3) in the same way as in the proof of Lemma 15. Let  $\tilde{u} = (\tilde{u}^0, \dots, \tilde{u}^{k-1})$ . Since  $R > 0$ ,

$$\text{wt}(x_R - \tilde{u}) \leq \min \{ \beta_R, k \} \leq w,$$

so  $\tilde{u}$  also satisfies (C4).

**Case ii ( $R = 0$ ).** We see that  $x_0 \neq u$  because

$$\bigoplus_{i \in \Omega} x_0^i = n_0 \oplus [0 \leq N] = n_0 \oplus 1$$

and

$$\bigoplus_{i \in \Omega} u^i = m_0 \oplus [0 \leq M] = m_0 \oplus 1.$$

Thus  $u$  satisfies (C3). Since  $\text{wt}(x_0 - u) \leq \min \{ \beta_0 - 1, k \}$ , we see that (C4) holds when  $\min \{ \beta_0 - 1, k - [\beta_0 < 2k] \} = \beta_0 - 1$  or  $k$ . Suppose that

$$\min \{ \beta_0 - 1, k - [\beta_0 < 2k] \} = k - 1 < \beta_0 - 1.$$

Then  $k < \beta_0 < 2k$  and  $k-1 \leq w$ . We show that there exists  $\tilde{u}$  satisfying (C1)–(C4). If  $x_0^{k-1} = 0$ , then  $u^{k-1} = 0$ , so  $u$  itself satisfies (C1)–(C4). Suppose that  $x_0^{k-1} \geq 1$ . We show that

$$x_0^0 + \cdots + x_0^{k-2} \geq m_0 \oplus 1 \ominus x_0^{k-1}. \quad (2.13)$$

If  $x_0^{k-1} = 1$ , then  $n_0 = (\bigoplus_i x_0^i) \ominus 1 = x_0^0 \oplus \cdots \oplus x_0^{k-2}$ , and hence

$$x_0^0 + \cdots + x_0^{k-2} \geq x_0^0 \oplus \cdots \oplus x_0^{k-2} = n_0 > m_0 = m_0 \oplus 1 \ominus x_0^{k-1}.$$

If  $x_0^{k-1} \geq 2$ , then, since  $x_0^0 \geq x_0^1 \geq \cdots \geq x_0^{k-1} \geq 2$  and  $2k > \beta_0$ ,

$$x_0^0 + \cdots + x_0^{k-2} \geq 2(k-1) \geq \beta_0 - 1 \geq m_0 \oplus 1 \ominus x_0^{k-1}.$$

Thus (2.13) holds. By Lemma 15, there exists  $(\tilde{u}^0, \dots, \tilde{u}^{k-2})$  such that  $\tilde{u}^0 \oplus \cdots \oplus \tilde{u}^{k-2} = m_0 \oplus 1 \ominus x_0^{k-1}$  and  $0 \leq x_0^i - \tilde{u}^i \leq x_0^0 - \tilde{u}^0$ . Thus  $(\tilde{u}^0, \dots, \tilde{u}^{k-2}, x_0^{k-1})$  satisfies (C1)–(C4). This completes the proof.  $\square$

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