The core and dual core inverse of a morphism with factorization

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Let $\mathscr{C}$ be a category with an involution $*$. Suppose that $\varphi : X \to X$ is a morphism and $(\varphi_1, Z, \varphi_2)$ is an (epic, monic) factorization of φ through Z, then φ is core invertible if and only if $(\varphi^*)^2\varphi_1$ and $\varphi_2\varphi_1$ are both left invertible if and only if $((\varphi^*)^2\varphi_1, Z, \varphi_2)$, $(\varphi_2^*, Z, \varphi_1^*\varphi^*\varphi)$ and $(\varphi^*\varphi_2^*, Z, \varphi_1^*\varphi)$ are all essentially unique (epic, monic) factorizations of $(\varphi^*)^2\varphi$ through Z. We also give the corresponding result about dual core inverse. In addition, we give some characterizations about the coexistence of core inverse and dual core inverse of an R-morphism in the category of R-modules of a given ring R.

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1 Introduction

Let $\mathscr{C}$ be a category. $\mathscr{C}$ is said to have an involution $*$ provided that there is a unary operation $*$ on the morphisms such that $\varphi : X \to Y$ implies $\varphi^* : Y \to X$ and that $(\varphi^*)^* = \varphi, (\varphi\psi)^* = \psi^*\varphi^*$ for all morphisms φ and ψ in $\mathscr{C}$. (See, for example, [1, p. 131].) Let $\varphi : X \to Y$ and $\chi : Y \to X$ be morphisms of $\mathscr{C}$. Consider the following four equations:

$$(1)\ \varphi\chi\varphi = \varphi, \quad (2)\ \chi\varphi\chi = \chi, \quad (3)\ (\varphi\chi)^* = \varphi\chi, \quad (4)\ (\chi\varphi)^* = \chi\varphi.$$

Let $\varphi\{i, j, \cdots, l\}$ denote the set of morphisms χ which satisfy equations $(i), (j), \cdots, (l)$ from among equations (1)-(4), and in this case, χ is called

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the $\{i, j, \cdots, l\}$-inverse of φ. If $\chi \in \varphi\{1,3\}$, then χ is called a $\{1,3\}$-inverse of φ and is denoted by $\varphi^{(1,3)}$. A $\{1,4\}$-inverse of φ can be similarly defined. If $\chi \in \varphi\{1,2,3,4\}$, then χ is called the Moore-Penrose inverse of φ. If such a χ exists, then it is unique and denoted by $\varphi^{\dagger}$. If $X = Y$, then a morphism φ is group invertible if there is a morphism $\chi \in \varphi\{1,2\}$ that commutes with φ. If the group inverse of φ exists, then it is unique and denoted by $\varphi^{\#}$. References to group inverses and Moore-Penrose inverses of morphisms can be seen in, for example, [2]-[6].

In 2010, O.M. Baksalary and G. Trenkler introduced the core and dual core inverse of a complex matrix in [7]. Rakić et al. [10] generalized core inverse of a complex matrix to the case of an element in a ring with an involution $*$, and they use five equations to characterize the core inverse. In the following, we rewrite these five equations in the category case. Let $\mathscr{C}$ be a category with an involution and $\varphi : X \to X$ a morphism of $\mathscr{C}$. If there is a morphism $\chi : X \to X$ satisfying

$$\varphi\chi\varphi = \varphi,\ \chi\varphi\chi = \chi,\ (\varphi\chi)^* = \varphi\chi,\ \varphi\chi^2 = \chi,\ \chi\varphi^2 = \varphi, \tag{1}$$

then φ is core invertible and χ is called the core inverse of φ. If such χ exists, then it is unique and denoted by $\varphi^{\oplus}$. In [8], Xu et al. proved that equations $\varphi\chi\varphi = \varphi$ and $\chi\varphi\chi = \chi$ in (1) can be dropped, that is to say, φ is core invertible with $\varphi^{\oplus} = \chi$ if and only if

$$(\varphi\chi)^* = \varphi\chi,\ \varphi\chi^2 = \chi,\ \chi\varphi^2 = \varphi.$$

And the dual core inverse can be given dually and denoted by $\varphi_{\oplus}$. References to core and dual core inverses of morphisms can be seen in, for example, [9].

In this paper, the convention is used of reading morphism composition from left to right, that is to say,

$$\varphi\psi : X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z.$$

A morphism φ is said to be epic if $\varphi\psi = \varphi\psi'$ implies $\psi = \psi'$, and monic if $\psi\varphi = \psi'\varphi$ implies $\psi = \psi'$. A morphism $\varphi : X \to Y$ is left invertible if there exists a morphism $\psi : Y \to X$ such that $\psi\varphi = 1_Y$, and right invertible if there exists a morphism $\psi : Y \to X$ such that $\varphi\psi = 1_X$.

Let $\varphi : X \to Y$ be a morphism in $\mathscr{C}$. If $\varphi_1 : X \to Z$ and $\varphi_2 : Z \to Y$ are morphisms and $\varphi = \varphi_1\varphi_2$, then $(\varphi_1, Z, \varphi_2)$ is called a factorization of φ through an object Z. A factorization $(\varphi_1, Z, \varphi_2)$ of φ through Z is called an (epic, monic) factorization of φ whenever φ_1 is epic and φ_2 is monic.

Furthermore, an (epic, monic) factorization $(\varphi_1, Z, \varphi_2)$ of φ through Z is said to be essentially unique (see, for example, [1]) if whenever $(\varphi_1', Z', \varphi_2')$ is also an (epic, monic) factorization of φ through an object Z', then there is an invertible morphism $\nu : Z \to Z'$ such that $\varphi_1 \nu = \varphi_1'$ and $\nu \varphi_2' = \varphi_2$. References to generalized inverses of a factorization can be seen in, for example, [11] and [12].

In [1], D.W. Robinson and R. Puystjens showed us the characterizations about the Moore-Penrose inverse of a morphism with a factorization. And in [13], R. Puystjens and D.W. Robinson gave the characterizations about the group inverse of a morphism with a factorization, and they also gave the characterizations about the Moore-Penrose inverse which is different from the results in [1]. Inspired by them, the second part in this paper will give some characterizations about the core inverse and the dual core inverse of a morphism with a factorization, respectively.

In [14] and [15], authors investigated the coexistence of core inverse and dual core inverse of an element in a $*$- ring which is a ring with an involution $*$ provided that there is an anti-isomorphism $*$ such that $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^* a^*$ for all $a, b \in R$. It makes sense to investigate the coexistence of core inverse and dual core inverse of an R-morphism in the category of R-modules of a given ring R. We give some characterizations about the coexistence of core inverse and dual core inverse of an R-morphism in the third part.

The following notations will be used in this paper: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^\circ a = \{x \in R \mid xa = 0\}$, $a^\circ = \{x \in R \mid ax = 0\}$. In addition, some auxiliary lemmas and results are presented for the further reference.

Lemma 1.1. *[1, Lemma 1] Let $\varphi : X \to Y$ be a morphism of $\mathscr{C}$ with a factorization $(\varphi_1, Z, \varphi_2)$ through an object Z. If $\varphi_1 : X \to Z$ is left invertible and $\varphi_2 : Z \to Y$ is right invertible, then $(\varphi_1, Z, \varphi_2)$ is an essentially unique (epic, monic) factorization of φ through Z.*

It should be pointed out, [8, Theorem 2.6 and 2.8], [15, Theorem 2.10], [15, Theorem 2.11] and [16, p. 201] were put forward in a $*$- ring. Actually, one can easily prove that these results are also valid in a category with an involution $*$. Thus, we can rewrite them in the category case as the following Lemma 1.2 - 1.5, respectively.

Lemma 1.2. *[8, Theorem 2.6 and 2.8] Let $\varphi : X \to X$ be a morphism of $\mathscr{C}$, we have the following results:*
(i) φ is core invertible if and only if φ is group invertible and $\{1,3\}$-invertible.

In this case, $\varphi^{\oplus} = \varphi^{\#}\varphi\varphi^{(1,3)}$.
(ii) φ *is dual core invertible if and only if* φ *is group invertible and* $\{1,4\}$*-invertible. In this case,* $\varphi_{\oplus} = \varphi^{(1,4)}\varphi\varphi^{\#}$.

Lemma 1.3. *[15, Theorem 2.10] Let* $\varphi : X \to X$ *be a morphism of* $\mathscr{C}$ *and* $n \geqslant 2$ *a positive integer, we have the following results:*
(i) φ *is core invertible if and only if there exist morphisms* $\varepsilon : X \to X$ *and* $\tau : X \to X$ *such that* $\varphi = \varepsilon(\varphi^*)^n\varphi = \tau\varphi^n$*. In this case,* $\varphi^{\oplus} = \varphi^{n-1}\varepsilon^*$.
(ii) φ *is dual core invertible if and only if there exist morphisms* $\theta : X \to X$ *and* $\rho : X \to X$ *such that* $\varphi = \varphi(\varphi^*)^n\theta = \varphi^n\rho$*. In this case,* $\varphi_{\oplus} = \theta^*\varphi^{n-1}$.

Lemma 1.4. *[15, Theorem 2.11] Let* $\varphi : X \to X$ *be a morphism of* $\mathscr{C}$ *and* $n \geqslant 2$ *a positive integer, the following statements are equivalent:*
(i) φ *is both Moore-Penrose invertible and group invertible.*
(ii) φ *is both core invertible and dual core invertible.*
(iii) There exist $\alpha : X \to X$ *and* $\beta : X \to X$ *such that* $\varphi = \alpha(\varphi^*)^n\varphi = \varphi(\varphi^*)^n\beta$.
In this case,

$$\begin{aligned}
\varphi^{\oplus} &= \varphi^{n-1}\alpha^*,\\
\varphi_{\oplus} &= \beta^*\varphi^{n-1},\\
\varphi^{\dagger} &= \beta\varphi^{2n-1}\alpha^*,\\
\varphi^{\#} &= (\varphi^{n-1}\alpha^*)^2\varphi = \varphi(\beta^*\varphi^{n-1})^2.
\end{aligned}$$

Lemma 1.5. *[16, p. 201] Let* $\varphi : X \to Y$ *be a morphism of* $\mathscr{C}$*, we have the following results:*
(i) φ *is* $\{1,3\}$*-invertible with* $\{1,3\}$*-inverse* $\chi : Y \to X$ *if and only if* $\chi^*\varphi^*\varphi = \varphi$;
(ii) φ *is* $\{1,4\}$*-invertible with* $\{1,4\}$*-inverse* $\zeta : Y \to X$ *if and only if* $\varphi\varphi^*\zeta^* = \varphi$.

2 Main Results

In [1], D.W. Robinson and R. Puystjens gave some results about the Moore-Penrose inverse of a morphism with a factorization. And in [13], R. Puystjens and D.W. Robinson gave the characterizations about the group inverse of a morphism with a factorization, as follows.

Lemma 2.1. *[13, Theorem 1] Let* $(\varphi_1, Z, \varphi_2)$ *be an (epic, monic) factorization of a morphism* $\varphi : X \to X$ *through an object* Z *of a category. Then the*

following statements are equivalent:
(i) φ *has a group inverse* $\varphi^{\#} : X \to X$,
(ii) $\varphi_2\varphi_1 : Z \to Z$ *is invertible,*
(iii) $(\varphi_1, Z, \varphi_2\varphi)$ *and* $(\varphi\varphi_1, Z, \varphi_2)$ *are both essentially unique (epic, monic) factorizations of* φ^2 *through* Z.

Lemma 2.2. *[1, Theorem 2 and Theorem 3] Let* $\varphi : X \to Y$ *be a morphism of a category with involution* $*$. *If* $(\varphi_1, Z, \varphi_2)$ *is an (epic, monic) factorization of* φ *through* Z, *then the following statements are equivalent:*
(i) φ *has a Moore-Penrose inverse with respect to* $*$,
(ii) $\varphi^*\varphi_1$ *is left invertible and* $\varphi_2^*\varphi$ *is right invertible,*
(iii) $(\varphi^*\varphi_1, Z, \varphi_2)$ *and* $(\varphi_1, Z, \varphi_2\varphi^*)$ *are, respectively, essentially unique (epic, monic) factorizations of* $\varphi^*\varphi$ *and* $\varphi\varphi^*$ *through* Z,
(iv) $\varphi_1^*\varphi_1$ *and* $\varphi_2\varphi_2^*$ *are both invertible.*
In this case,

$$\varphi^{\dagger} = \varphi_2^*(\varphi_2\varphi_2^*)^{-1}(\varphi_1^*\varphi_1)^{-1}\varphi_1^*.$$

From Lemma 2.2, we know that φ has a Moore-Penrose inverse if and only if $\varphi^*\varphi_1$ is left invertible and $\varphi_2\varphi^*$ is right invertible if and only if both $\varphi_1^*\varphi_1$ and $\varphi_2\varphi_2^*$ are invertible. Thus we can easily prove the following two lemmas in a similar way.

Lemma 2.3. *Let* $\varphi : X \to Y$ *be a morphism of a category with involution* $*$. *If* $(\varphi_1, Z, \varphi_2)$ *is an (epic, monic) factorization of* φ *through* Z, *then* φ *has a* $\{1,3\}$*-inverse with respect to* $*$ *if and only if* $\varphi^*\varphi_1 : Y \to Z$ *is left invertible.*

Proof. Since

$$\begin{aligned}\varphi_1 \cdot 1_Z \cdot \varphi_2 &= \varphi = \varphi\varphi^{(1,3)}\varphi\varphi^{(1,3)}\varphi = \varphi\varphi^{(1,3)}(\varphi\varphi^{(1,3)})^*\varphi \\ &= \varphi\varphi^{(1,3)}(\varphi^{(1,3)})^*\varphi^*\varphi = \varphi_1\varphi_2\varphi^{(1,3)}(\varphi^{(1,3)})^*\varphi^*\varphi_1\varphi_2,\end{aligned}$$

φ_1 is epic and φ_2 monic, then

$$\varphi_2\varphi^{(1,3)}(\varphi^{(1,3)})^*\varphi^*\varphi_1 = 1_Z.$$

Therefore, $\varphi^*\varphi_1$ is left invertible.

Conversely, there is a morphism $\mu : Z \to Y$ such that $\mu\varphi^*\varphi_1 = 1_Z$, then

$$\varphi = \varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi_1(\mu\varphi^*\varphi_1)\varphi_2 = \varphi_1\mu\varphi^*\varphi,$$

thus φ is $\{1,3\}$-invertible by Lemma 1.5. □

Similarly, we have a dual result for $\{1,4\}$-inverse.

Lemma 2.4. *Let $\varphi : X \to Y$ be a morphism of a category with involution $*$. If $(\varphi_1, Z, \varphi_2)$ is an (epic, monic) factorization of φ through Z, then φ has a $\{1,4\}$-inverse with respect to $*$ if and only if $\varphi_2\varphi^* : Z \to X$ is right invertible.*

Inspired by D.W. Robinson and R. Puystjens [13], we get some characterizations of the core invertibility of a morphism with an (epic, monic) factorization in a category $\mathscr{C}$.

Theorem 2.5. *Let $\varphi : X \to X$ be a morphism of a category with involution $*$. If $(\varphi_1, Z, \varphi_2)$ is an (epic, monic) factorization of φ through Z, then the following statements are equivalent:*
(i) φ has a core inverse with respect to $*$;
(ii) $\varphi^*\varphi_1 : X \to Z$ is left invertible and $\varphi_2\varphi_1 : Z \to Z$ is invertible;
(iii) $(\varphi^*)^n\varphi_1 : X \to Z$ and $\varphi_2\varphi_1 : Z \to Z$ are both left invertible for any positive integer $n \geqslant 2$;
(iv) $((\varphi^*)^2\varphi_1, Z, \varphi_2)$, $(\varphi_2^*, Z, \varphi_1^*\varphi^*\varphi)$ and $(\varphi^*\varphi_2^*, Z, \varphi_1^*\varphi)$ are all essentially unique (epic, monic) factorizations of $(\varphi^*)^2\varphi$ through Z.

Proof. $(i) \Leftrightarrow (ii)$. By Lemma 1.2, φ is core invertible if and only if φ is group invertible and $\{1,3\}$-invertible. Moreover, φ is group invertible if and only $\varphi_2\varphi_1 : Z \to Z$ is invertible by Lemma 2.1, and φ is $\{1,3\}$-invertible if and only if $\varphi^*\varphi_1 : X \to Z$ is left invertible by Lemma 2.3. In conclusion, φ is core invertible if and only if $\varphi^*\varphi_1 : X \to Z$ is left invertible and $\varphi_2\varphi_1 : Z \to Z$ is invertible.

$(i) \Rightarrow (iii)$. Since

$$\begin{aligned}
\varphi_1 \cdot 1_Z \cdot \varphi_2 &= \varphi = \varphi\varphi^{\oplus}\varphi = \varphi(\varphi^{\oplus}\varphi\varphi^{\oplus})\varphi = \varphi\varphi^{\oplus}(\varphi\varphi^{\oplus})^*\varphi \\
&= \varphi\varphi^{\oplus}(\varphi^{\oplus})^*\varphi^*\varphi = \varphi\varphi^{\oplus}(\varphi\varphi^{\oplus}\varphi^{\oplus})^*\varphi^*\varphi \\
&= \varphi_1\varphi_2\varphi^{\oplus}(\varphi^{\oplus})^*(\varphi^{\oplus})^*(\varphi^*)^2\varphi_1\varphi_2 \\
&= \varphi_1\varphi_2\varphi^{\oplus}(\varphi^{\oplus})^*(\varphi\varphi^{\oplus}\varphi^{\oplus})^*(\varphi^*)^2\varphi_1\varphi_2 \\
&= \varphi_1\varphi_2\varphi^{\oplus}(\varphi^{\oplus})^*((\varphi^{\oplus})^2)^*(\varphi^*)^3\varphi_1\varphi_2 \\
&= \cdots \\
&= \varphi_1\varphi_2\varphi^{\oplus}(\varphi^{\oplus})^*((\varphi^{\oplus})^{n-1})^*(\varphi^*)^n\varphi_1\varphi_2,
\end{aligned}$$

φ_1 is epic and φ_2 is monic, then

$$\varphi_2\varphi^{\oplus}(\varphi^{\oplus})^*((\varphi^{\oplus})^{n-1})^*(\varphi^*)^n\varphi_1 = 1_Z,$$

thus $(\varphi^*)^n\varphi_1 : X \to Z$ is left invertible for any $n \geqslant 2$.

Similarly, from

$$\varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi = \varphi\varphi^{\oplus}\varphi = \varphi\varphi^{\oplus}(\varphi^{\oplus}\varphi^2) = \varphi_1\varphi_2\varphi^{\oplus}\varphi^{\oplus}\varphi_1\varphi_2\varphi_1\varphi_2,$$

φ_1 is epic and φ_2 is monic, we obtain

$$\varphi_2\varphi^{\oplus}\varphi^{\oplus}\varphi_1\varphi_2\varphi_1 = 1_Z, \tag{2}$$

thus $\varphi_2\varphi_1 : Z \to Z$ is left invertible.

$(iii) \Rightarrow (i)$. Suppose that $(\varphi^*)^n\varphi_1 : X \to Z$ and $\varphi_2\varphi_1 : Z \to Z$ are both left invertible for any $n \geqslant 2$, then there exist $\mu : Z \to X$ and $\nu : Z \to Z$ such that

$$\mu(\varphi^*)^n\varphi_1 = 1_Z = \nu\varphi_2\varphi_1.$$

Therefore,

$$\begin{aligned}\varphi &= \varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi_1(\nu\varphi_2\varphi_1)\varphi_2 = \varphi_1\nu(\nu\varphi_2\varphi_1)\varphi_2\varphi_1\varphi_2 = \varphi_1\nu^2\varphi_2\varphi^2 \\ &= \varphi_1\nu^2(\nu\varphi_2\varphi_1)\varphi_2\varphi^2 = \varphi_1\nu^3\varphi_2\varphi^3 = \cdots = \varphi_1\nu^n\varphi_2\varphi^n,\end{aligned}$$

$$\varphi = \varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi_1(\mu(\varphi^*)^n\varphi_1)\varphi_2 = \varphi_1\mu(\varphi^*)^n\varphi.$$

Thus φ has a core inverse with $\varphi^{\oplus} = \varphi^{n-1}(\varphi_1\mu)^*$ by Lemma 1.3.

$((i) \Leftrightarrow (iii)) \Rightarrow (iv)$. When taking $n = 2$, φ has a core inverse if and only if $(\varphi^*)^2\varphi_1 : X \to Z$ and $\varphi_2\varphi_1 : Z \to Z$ are both left invertible. To begin with, since $(\varphi^*)^2\varphi_1$ is left invertible, and equality (2) shows that φ_2 is right invertible, thus $((\varphi^*)^2\varphi_1, Z, \varphi_2)$ is an essentially unique (epic, monic) factorization of $(\varphi^*)^2\varphi$ through Z by Lemma 1.1.

Next, φ_2 is right invertible, which gives that $(\varphi_2)^*$ is left invertible. In addition, equality (2) gives

$$1_Z = \varphi_2\varphi^{\oplus}\varphi^{\oplus}\varphi_1\varphi_2\varphi_1 = \varphi_2\varphi^{\oplus}(\varphi^{\oplus}\varphi\varphi^{\oplus})\varphi\varphi_1 = \varphi_2\varphi^{\oplus}\varphi^{\oplus}(\varphi^{\oplus})^*\varphi^*\varphi\varphi_1,$$

so $\varphi^*\varphi\varphi_1$ is left invertible, that is to say, $\varphi_1^*\varphi^*\varphi$ is right invertible. Therefore, $(\varphi_2^*, Z, \varphi_1^*\varphi^*\varphi)$ is an essentially unique (epic, monic) factorization of $(\varphi^*)^2\varphi$ through Z by Lemma 1.1.

Finally, since $(\varphi^*)^2\varphi_1$ is left invertible, then $\varphi^*\varphi_1$ is also left invertible, which implies that $\varphi_1^*\varphi$ is right invertible. In addition, equality (2) shows

$$1_Z = \varphi_2\varphi^{\oplus}\varphi^{\oplus}\varphi_1\varphi_2\varphi_1 = \varphi_2(\varphi\varphi^{\oplus}\varphi^{\oplus})\varphi^{\oplus}\varphi\varphi_1,$$

thus $\varphi_2\varphi$ is right invertible, that is to say, $\varphi^*\varphi_2^*$ is left invertible. Hence $(\varphi^*\varphi_2^*, Z, \varphi_1^*\varphi)$ is an essentially unique (epic, monic) factorization of $(\varphi^*)^2\varphi$ through Z by Lemma 1.1.

(iv) $\Rightarrow$ *(iii)*. Suppose that $((\varphi^*)^2\varphi_1, Z, \varphi_2)$, $(\varphi^*\varphi_2^*, Z, \varphi_1^*\varphi)$ and $(\varphi_2^*, Z, \varphi_1^*\varphi^*\varphi)$ are essentially unique (epic, monic) factorizations of $(\varphi^*)^2\varphi$ through Z. In particular, there exist invertible morphisms $\rho : Z \to Z, \sigma : Z \to Z$ such that

$$(\varphi^*)^2\varphi_1\rho = \varphi^*\varphi_2^*,$$

$$\rho\varphi_1^*\varphi = \varphi_2, \tag{3}$$

$$\varphi_2^*\sigma = \varphi^*\varphi_2^*, \tag{4}$$

and

$$\sigma\varphi_1^*\varphi = \varphi_1^*\varphi^*\varphi. \tag{5}$$

By calculation, we have

$$\varphi_2 \overset{(3)}{=} \rho\varphi_1^*\varphi = \rho\varphi_1^*\varphi_1\varphi_2,$$

$$\varphi_2\varphi_1\varphi_2 = \varphi_2\varphi = (\varphi^*\varphi_2^*)^* \overset{(4)}{=} (\varphi_2^*\sigma)^* = \sigma^*\varphi_2,$$

and

$$\sigma\varphi_1^*\varphi_1\varphi_2 = \sigma\varphi_1^*\varphi \overset{(5)}{=} \varphi_1^*\varphi^*\varphi = \varphi_1^*\varphi^*\varphi_1\varphi_2.$$

Since φ_2 is monic, then $1_Z = \rho\varphi_1^*\varphi_1$, $\varphi_2\varphi_1 = \sigma^*$ and $\sigma\varphi_1^*\varphi_1 = \varphi_1^*\varphi^*\varphi_1$. Thus $\varphi_2\varphi_1$ is invertible follows from $\varphi_2\varphi_1 = \sigma^*$. Moreover, $\sigma\varphi_1^*\varphi_1 = \varphi_1^*\varphi^*\varphi_1$ implies $\varphi_1^*\varphi_1 = \sigma^{-1}\varphi_1^*\varphi^*\varphi_1$, then

$$\begin{aligned}
1_Z &= \rho\varphi_1^*\varphi_1 = \rho\sigma^{-1}\varphi_1^*\varphi^*\varphi_1 = \rho\sigma^{-1}(\sigma^{-1}\sigma)\varphi_1^*\varphi^*\varphi_1 = \rho\sigma^{-2}(\sigma^*)^*\varphi_1^*\varphi^*\varphi_1 \\
&= \rho\sigma^{-2}(\varphi_2\varphi_1)^*\varphi_1^*\varphi^*\varphi_1 = \rho\sigma^{-2}\varphi_1^*(\varphi^*)^2\varphi_1 = \rho\sigma^{-2}(\sigma^{-1}\sigma)\varphi_1^*(\varphi^*)^2\varphi_1 \\
&= \rho\sigma^{-3}(\sigma^*)^*\varphi_1^*(\varphi^*)^2\varphi_1 = \rho\sigma^{-3}(\varphi_2\varphi_1)^*\varphi_1^*(\varphi^*)^2\varphi_1 \\
&= \rho\sigma^{-3}\varphi_1^*(\varphi^*)^3\varphi_1 = \cdots = \rho\sigma^{-n}\varphi_1^*(\varphi^*)^n\varphi_1,
\end{aligned}$$

hence $(\varphi^*)^n\varphi_1$ is left invertible. □

The following theorem is a corresponding result for dual core inverse.

Theorem 2.6. *Let $\varphi : X \to X$ be a morphism of a category with involution. If $(\varphi_1, Z, \varphi_2)$ is an (epic, monic) factorization of φ through Z, then the following statements are equivalent:*
(i) φ has a dual core inverse with respect to $*$;
(ii) $\varphi_2\varphi^* : Z \to X$ is right invertible and $\varphi_2\varphi_1 : Z \to Z$ is invertible;
(iii) $\varphi_2(\varphi^*)^n : Z \to X$ and $\varphi_2\varphi_1 : Z \to Z$ are both right invertible for any positive integer $n \geqslant 2$;
(iv) $(\varphi_1, Z, \varphi_2(\varphi^*)^2)$, $(\varphi\varphi^*\varphi_2^*, Z, \varphi_1^*)$ and $(\varphi\varphi_2^*, Z, \varphi_1^*\varphi^*)$ are all essentially unique (epic, monic) factorizations of $\varphi(\varphi^*)^2$ through Z.

In [14] and [15], authors characterized the coexistence of core inverse and dual core inverse of a regular element by units in a $*$-ring. And we give characterizations of the coexistence of core inverse and dual core inverse of a morphism with an (epic, monic) factorization in a category $\mathscr{C}$.

Theorem 2.7. *Let $\varphi : X \to X$ be a morphism of a category with involution and $n \geqslant 2$ a positive integer. If $(\varphi_1, Z, \varphi_2)$ is an (epic, monic) factorization of φ through Z, then the following statements are equivalent:*
(i) φ is both Moore-Penrose invertible and group invertible;
(ii) φ is both core invertible and dual core invertible;
(iii) $(\varphi^*)^n\varphi_1 : X \to Z$ is left invertible and $\varphi_2(\varphi^*)^n : Z \to X$ is right invertible;
(iv) $\varphi^n\varphi_2^* : X \to Z$ is left invertible and $\varphi_1^*\varphi^n : Z \to X$ is right invertible.
In this case,

$$\begin{aligned}
\varphi^{\oplus} &= \varphi^{n-1}\mu^*\varphi_1^*,\\
\varphi_{\oplus} &= \varphi_2^*\nu^*\varphi^{n-1},\\
\varphi^{\dagger} &= \nu\varphi_2\varphi^{2n-1}\mu^*\varphi_1^*,\\
\varphi^{\#} &= (\varphi^{n-1}\mu^*\varphi_1^*)^2\varphi = \varphi(\varphi_2^*\nu^*\varphi^{n-1})^2,
\end{aligned}$$

where $\mu(\varphi^*)^n\varphi_1 = 1_Z = \varphi_2(\varphi^*)^n\nu$ for some $\mu : Z \to X$ and $\nu : X \to Z$.

Proof. $(i) \Leftrightarrow (ii)$. Obviously.

$(ii) \Rightarrow (iii)$. It is clear by Theorem 2.5 and Theorem 2.6.

$(iii) \Rightarrow (ii)$. Suppose that $\mu(\varphi^*)^n\varphi_1 = 1_Z = \varphi_2(\varphi^*)^n\nu$ for some $\mu : Z \to X$ and $\nu : X \to Z$, where $n \geqslant 2$ is a positive integer. Then we have

$$\varphi = \varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi_1(\mu(\varphi^*)^n\varphi_1)\varphi_2 = \varphi_1\mu(\varphi^*)^n\varphi$$

and

$$\varphi = \varphi_1 \cdot 1_Z \cdot \varphi_2 = \varphi_1(\varphi_2(\varphi^*)^n\nu)\varphi_2 = \varphi(\varphi^*)^n\nu\varphi_2.$$

Hence, the conclusion is now a consequence of Lemma 1.4.

$(ii) \Leftrightarrow (iv)$. Since φ^* exists and has an (epic, monic) factorization $(\varphi_2^*, Z, \varphi_1^*)$, and φ is both core invertible and dual core invertible if and only if φ^* is both core invertible and dual core invertible. Therefore, the conclusion is a consequence of the preceding argument.

The expressions can be deduced by Lemma 1.4. □

Let $\mathbb{C}_{m,n}$ be the set of all $m \times n$ complex matrices. In [17], H.X. Wang and X.L. Liu showed us that if $A \in \mathbb{C}_n^{\mathrm{CM}}$ has a full-rank decomposition $A = BC$, then $A^{\#\!\!\!\bigcirc} = B(CB)^{-1}(B^*B)^{-1}B^*$, where $\mathbb{C}_n^{\mathrm{CM}} = \{A \in \mathbb{C}_{n,n} : \mathrm{rank}(A^2) = \mathrm{rank}(A)\}$. We will show another derivation for this result as follow.

Corollary 2.8. *[17, Theorem 2.4] Let $A \in \mathbb{C}_n^{\mathrm{CM}}$ with $\mathrm{rank}(A) = r$. If A has a full-rank decomposition $A = BC$, then*

$$A^{\#\!\!\!\bigcirc} = B(CB)^{-1}(B^*B)^{-1}B^*.$$

Proof. Let $U = (B^*B)^{-1}((CB)^*)^{-1}(CC^*)^{-1}C$, then

$$\begin{aligned} U(A^*)^2B &= [(B^*B)^{-1}((CB)^*)^{-1}(CC^*)^{-1}C](A^*)^2B \\ &= [(B^*B)^{-1}((CB)^*)^{-1}(CC^*)^{-1}C](BC)^*(BC)^*B \\ &= [(B^*B)^{-1}((CB)^*)^{-1}(CC^*)^{-1}C]C^*(CB)^*B^*B \\ &= I_r, \end{aligned}$$

thus U is a left inverse of $(A^*)^2B$. According to the proof $(iii) \Rightarrow (i)$ of Theorem 2.5, we deduce that $A^{\#\!\!\!\bigcirc} = A(BU)^*$. Therefore,

$$\begin{aligned} A^{\#\!\!\!\bigcirc} &= A(BU)^* \\ &= A[B(B^*B)^{-1}((CB)^*)^{-1}(CC^*)^{-1}C]^* \\ &= (BC)C^*(CC^*)^{-1}(CB)^{-1}(B^*B)^{-1}B^* \\ &= B(CB)^{-1}(B^*B)^{-1}B^*. \end{aligned}$$

□

Likewise, we have the following result.

Corollary 2.9. *Let $A \in \mathbb{C}_n^{\mathrm{CM}}$ with $\mathrm{rank}(A) = r$. If A has a full-rank decomposition $A = BC$, then*

$$A_{\#\!\!\!\bigcirc} = C^*(CC^*)^{-1}(CB)^{-1}C.$$

3 Applications

Let R be a ring, and let ${}_R\mathrm{Mod}$ be the category of R-modules and R-morphisms. In [1], R. Puystjens and D.W. Robinson mentioned that associated with every morphism $\tau : M \to N$ of ${}_R\mathrm{Mod}$ are the R-modules $\mathrm{Im}\tau = M\tau = \{x\tau | x \in M\}$, $\mathrm{Ker}\tau = \{x | x\tau = 0\}$ and the R-morphisms $\tau_1 : M \to \mathrm{Im}\tau$, $x \mapsto x\tau$, and $\tau_2 : \mathrm{Im}\tau \to N$, $x \mapsto x$. In particular, $(\tau_1, \mathrm{Im}\tau, \tau_2)$ is an (epic, monic) factorization of τ through the object $\mathrm{Im}\tau$, which is herein called the standard factorization of τ in ${}_R\mathrm{Mod}$.

Now we consider the coexistence of the core inverse and dual core inverse of an R-morphism in the category of R-modules of a given ring R.

Lemma 3.1. *Let $\tau : M \to M$ be a morphism of ${}_R\mathrm{Mod}$ with standard factorization $(\tau_1, \mathrm{Im}\tau, \tau_2)$, and let n be a positive integer. If the full subcategory determined by M has an involution $*$, then*
(i) $(\tau^*)^n\tau_1$ is epic if and only if $\mathrm{Im}(\tau^*)^n\tau = \mathrm{Im}\tau$;
(ii) $\tau_2(\tau^*)^n$ is monic if and only if $\mathrm{Ker}\tau(\tau^*)^n = \mathrm{Ker}\tau$;
(iii) $\mathrm{Ker}(\tau^*)^n\tau_1 = \mathrm{Ker}(\tau^*)^n\tau$;
(iv) $\mathrm{Im}\tau_2(\tau^*)^n = \mathrm{Im}\tau(\tau^*)^n$.

Proof. (i). Assume that $(\tau^*)^n\tau_1 : M \to \mathrm{Im}\tau$ is epic. It is obvious that $\mathrm{Im}(\tau^*)^n\tau \subseteq \mathrm{Im}\tau$, so we only need to prove that $\mathrm{Im}\tau \subseteq \mathrm{Im}(\tau^*)^n\tau$. Since $(\tau^*)^n\tau_1$ is epic, if $z \in \mathrm{Im}\tau$, then there is a $y \in M$ such that $z = y(\tau^*)^2\tau_1$, and

$$z = z\tau_2 = (y(\tau^*)^n\tau_1)\tau_2 = y(\tau^*)^n\tau \in \mathrm{Im}(\tau^*)^n\tau.$$

Thus, $\mathrm{Im}(\tau^*)^n\tau = \mathrm{Im}\tau$. Conversely, Assume that $\mathrm{Im}(\tau^*)^n\tau = \mathrm{Im}\tau$ and let $z \in \mathrm{Im}\tau$, then there is a $y \in M$ such that

$$z = y(\tau^*)^n\tau = y(\tau^*)^n\tau_1\tau_2 = y(\tau^*)^n\tau_1.$$

That is to say, $(\tau^*)^n\tau_1$ is surjective as a function and hence is epic as an R-morphism.

(ii). Suppose that $\tau_2(\tau^*)^n : \mathrm{Im}\tau \to M$ is monic. It is easy to see that $\mathrm{Ker}\tau \subseteq \mathrm{Ker}\tau(\tau^*)^n$, hence, we only need to show that $\mathrm{Ker}\tau(\tau^*)^n \subseteq \mathrm{Ker}\tau$. Since $\tau_2(\tau^*)^n$ is monic, for any $z \in \mathrm{Ker}\tau(\tau^*)^n$, we have $0 = z\tau(\tau^*)^n = z\tau_1\tau_2(\tau^*)^n$, thus $0 = z\tau_1 = z\tau_1\tau_2 = z\tau$, that is $z \in \mathrm{Ker}\tau$. Conversely, suppose $\mathrm{Ker}\tau(\tau^*)^n = \mathrm{Ker}\tau$ and $z\tau_2(\tau^*)^n = 0$, where $z \in \mathrm{Im}\tau$. Since τ_1 is epic, there exists a $y \in M$ such that $z = y\tau_1$. Therefore,

$$0 = z\tau_2(\tau^*)^n = (y\tau_1)\tau_2(\tau^*)^n = y\tau(\tau^*)^n,$$

which follows that $0 = y\tau = y\tau_1\tau_2 = y\tau_1 = z$. Hence, $\tau_2(\tau^*)^n$ is monic.

Part (iii) follows from the fact that τ_2 is an insertion and part (iv) is a consequence of the fact that τ_1 is epic. $\square$

Theorem 3.2. *Let $\tau : M \to M$ be a morphism of the category ${}_R\mathrm{Mod}$, and let $n \geqslant 2$ be a positive integer. If the full subcategory determined by M has an involution $*$, then the following statements are equivalent:*
(i) τ is both core invertible and dual core invertible;
(ii) τ is both Moore-Penrose invertible and group invertible;
(iii) both $\mathrm{Ker}(\tau^*)^n\tau$ and $\mathrm{Im}\tau(\tau^*)^n$ are direct summands of M, and $\mathrm{Im}(\tau^*)^n\tau = \mathrm{Im}\tau$, $\mathrm{Ker}\tau(\tau^*)^n = \mathrm{Ker}\tau$;
(iv) both $\mathrm{Ker}\tau^n\tau^*$ and $\mathrm{Im}\tau^*\tau^n$ are direct summands of M, and $\mathrm{Im}\tau^n\tau^* = \mathrm{Im}\tau^*$, $\mathrm{Ker}\tau^*\tau^n = \mathrm{Ker}\tau^*$;
(v) $M = \mathrm{Ker}\tau \oplus \mathrm{Im}(\tau^*)^n$, $M = \mathrm{Ker}(\tau^*)^n \oplus \mathrm{Im}\tau$;
(vi) $M = \mathrm{Ker}\tau^* \oplus \mathrm{Im}\tau^n$, $M = \mathrm{Ker}\tau^n \oplus \mathrm{Im}\tau^*$.

Proof. $(i) \Leftrightarrow (ii)$. Clearly.

$(i) \Leftrightarrow (iii)$. As is known that an epic morphism in ${}_R\mathrm{Mod}$ is left invertible if and only if its kernel is a direct summand of its domain. (See for example [18, p. 12].) In particular, $(\tau^*)^n\tau_1$ is left invertible if and only if $(\tau^*)^n\tau_1$ is epic and $\mathrm{Ker}(\tau^*)^n\tau_1$ is a direct summand of M. Thus, by (i) and (iii) in Lemma 3.1, $(\tau^*)^n\tau_1$ is left invertible if and only if $\mathrm{Im}(\tau^*)^n\tau = \mathrm{Im}\tau$ and $\mathrm{Ker}(\tau^*)^n\tau$ is a direct summand of M. In a similar way, since a monic morphism in Mod_R is right invertible if and only if its image is a direct summand of its codomain. then from (ii) and (iv) in Lemma 3.1, $\tau_2(\tau^*)^n$ is right invertible if and only if $\mathrm{Ker}\tau(\tau^*)^n = \mathrm{Ker}\tau$ and $\mathrm{Im}\tau(\tau^*)^n$ is a direct summand of M. Consequently, we get the conclusion by Theorem 2.7.

$(i) \Leftrightarrow (iv)$. Since τ is both core invertible and dual core invertible if and only if τ^* is both core invertible and dual core invertible. Then, we can get this conclusion by replacing τ with τ^* in the preceding argument.

$(i) \Rightarrow (v)$. Given $\tau^{\#}$ and $\tau_{\#}$, then $M = M(1_M - \tau\tau^{\#}) \oplus M\tau\tau^{\#}$. Clearly $M(1_M - \tau\tau^{\#}) = \mathrm{Ker}\tau$. Since

$$\begin{aligned} \tau\tau^{\#} &= (\tau\tau^{\#})^* = (\tau^{\#})^*\tau^* = (\tau\tau^{\#}\tau^{\#})^*\tau^* = ((\tau^{\#})^2)^*(\tau^*)^2 \\ &= (\tau^{\#})^*(\tau\tau^{\#}\tau^{\#})^*(\tau^*)^2 = ((\tau^{\#})^3)^*(\tau^*)^3 \\ &= \cdots = ((\tau^{\#})^n)^*(\tau^*)^n \end{aligned}$$

and

$$(\tau^*)^n = (\tau^*)^{n-1}\tau^* = (\tau^*)^{n-1}(\tau\tau^{\#}\tau)^* = (\tau^*)^{n-1}\tau^*\tau\tau^{\#},$$

then $M\tau\tau^{\#} = \mathrm{Im}(\tau^*)^n$. Thus $M = \mathrm{Ker}\tau \oplus \mathrm{Im}(\tau^*)^n$.

In addition, for any $z \in M$, $z = (z - z\tau_{\#}\tau) + z\tau_{\#}\tau$, where $z\tau_{\#}\tau \in \mathrm{Im}\tau$. Now we show that $z - z\tau_{\#}\tau \in \mathrm{Ker}(\tau^*)^n$. Since

$$(\tau^*)^n = (\tau\tau_{\#}\tau)^*(\tau^*)^{n-1} = \tau_{\#}\tau\tau^*(\tau^*)^{n-1} = \tau_{\#}\tau(\tau^*)^n,$$

then $(z - z\tau_{\#}\tau)(\tau^*)^n = z(\tau^*)^n - z\tau_{\#}\tau(\tau^*)^n = 0$. Let $y \in \mathrm{Ker}(\tau^*)^n \cap \mathrm{Im}\tau$, then $y(\tau^*)^n = 0$ and there exists an $x \in M$ such that $y = x\tau$. Hence,

$$\begin{aligned} y &= x\tau = x(\tau\tau_{\#}\tau) = x\tau\tau^*(\tau_{\#})^* = x\tau\tau^*(\tau_{\#}\tau_{\#}\tau)^* \\ &= y(\tau^*)^2(\tau_{\#}^2)^* = y(\tau^*)^2(\tau_{\#}\tau_{\#}\tau)^*\tau_{\#}^* \\ &= y(\tau^*)^3(\tau_{\#}^3)^* = \cdots = y(\tau^*)^n(\tau_{\#}^n)^* = 0. \end{aligned}$$

Therefore, we have $M = \mathrm{Ker}(\tau^*)^n \oplus \mathrm{Im}\tau$.

$(v) \Rightarrow (iii)$. Let $M = \mathrm{Ker}\tau \oplus \mathrm{Im}(\tau^*)^n$. Then, for any $z \in M$, $z = k + y(\tau^*)^n$, where $k \in \mathrm{Ker}\tau$. Therefore,

$$z\tau = y(\tau^*)^n\tau \in \mathrm{Im}(\tau^*)^n\tau,$$

which implies $\mathrm{Im}\tau \subseteq \mathrm{Im}(\tau^*)^n\tau$. Hence, $\mathrm{Im}\tau = \mathrm{Im}(\tau^*)^n\tau$. In addition, if $y(\tau^*)^n\tau = 0$, then

$$y(\tau^*)^n \in \mathrm{Ker}\tau \cap \mathrm{Im}(\tau^*)^n = \{0\},$$

thus $\mathrm{Ker}(\tau^*)^n\tau \subseteq \mathrm{Ker}(\tau^*)^n$. Moreover, $\mathrm{Ker}(\tau^*)^n\tau = \mathrm{Ker}(\tau^*)^n$

Likewise, let $M = \mathrm{Ker}(\tau^*)^n \oplus \mathrm{Im}\tau$, then $\mathrm{Ker}\tau(\tau^*)^n = \mathrm{Ker}\tau$ and $\mathrm{Im}(\tau^*)^n = \mathrm{Im}\tau(\tau^*)^n$.

$(vi) \Leftrightarrow (v)$. We can get this conclusion immediately by replacing τ with τ^* in the statement (v). □

Remark 3.3. *It should be noted that when taking $n = 1$, Lemma 3.1 is consistent with [1, Lemma 4], and the statements (iii), (iv), (v)and (vi) in Theorem 3.2 are all equivalent to that τ is Moore-Penrose invertible. (See [1, Theorem 4].)*

Corollary 3.4. *Let R be a $*$-ring and $a \in R$ and $n \geqslant 2$ a positive integer, then the following statements are equivalent:*
(i) a is both Moore-Penrose invertible and group invertible;
(ii) a is both core invertible and dual core invertible;
(iii) $R = {}^\circ a \oplus R(a^*)^n$, $R = {}^\circ((a^*)^n) \oplus Ra$;
(iv) $R = (a^*)^\circ \oplus a^nR$, $R = (a^n)^\circ \oplus a^*R$;
(v) $R = {}^\circ(a^*) \oplus Ra^n$, $R = {}^\circ(a^n) \oplus Ra^*$;
(vi) $R = a^\circ \oplus (a^*)^nR$, $R = ((a^*)^n)^\circ \oplus aR$.

Proof. As is known that $(i) \Leftrightarrow (ii)$. And $(ii) \Leftrightarrow (iii) \Leftrightarrow (v)$ follows from Theorem 3.2. When taking involution on statements (iii) and (v), we obtain statements (iv) and (vi), respectively. □

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