The core and dual core inverses of morphisms with kernels

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Let $\mathscr{C}$ be an additive category with an involution $*$. Suppose that $\varphi : X \to X$ is a morphism with kernel $\kappa : K \to X$ in $\mathscr{C}$, then φ is core invertible if and only if φ has a cokernel $\lambda : X \to L$ and both $\kappa\lambda$ and $\varphi^*\varphi^3 + \kappa^*\kappa$ are invertible. In this case, we give the representation of the core inverse of φ. We also give the corresponding result about dual core inverse.

Keywords: Core inverse; morphism; kernel; cokernel; invertibility

AMS subject classifications: 15A09, 18A20, 18A99

1 Introduction

Throughout this paper, $\mathscr{C}$ is an additive category with an involution $*$, that is to say, there is a unary operation $*$ on the morphisms such that $\varphi : X \to Y$ implies $\varphi^* : Y \to X$ and that $(\varphi^*)^* = \varphi, (\varphi\psi)^* = \psi^*\varphi^*$ for any $\psi : Y \to Z$ and $(\varphi + \phi)^* = \varphi^* + \phi^*$ for any $\phi : X \to Y$. (See for example, [1, p. 131].) And R is a $*$-ring, which is an associative ring with 1 and an involution $*$.

Let $\varphi : X \to Y$ be a morphism of $\mathscr{C}$, we say that φ is regular (or $\{1\}$-invertible) if there is a morphism $\chi : Y \to X$ in $\mathscr{C}$ such that $\varphi\chi\varphi = \varphi$. In

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this case, χ is said to be an inner inverse of φ and is denoted by φ^-. If such a regular element φ also satisfies $\chi\varphi\chi = \chi$, then we call that χ is a reflexive inverse of φ. When $X = Y$, if χ is a reflexive inverse of φ and commutes with φ, then φ is group invertible and such a χ is called the group inverse of φ. The group inverse of φ is unique if it exists and is denoted by $\varphi^\#$.

Recall that φ is Moore-Penrose invertible if there is a morphism $\chi : Y \to X$ in $\mathscr{C}$ satisfying the following four equations:

$$(1)\ \varphi\chi\varphi = \varphi, \quad (2)\ \chi\varphi\chi = \chi, \quad (3)\ (\varphi\chi)^* = \varphi\chi, \quad (4)\ (\chi\varphi)^* = \chi\varphi.$$

If such a χ exists, then it is unique and denoted by $\varphi^\dagger$. Let $\varphi\{i, j, \cdots, l\}$ denote the set of morphisms χ which satisfy equations $(i), (j), \cdots, (l)$ from among equations (1)-(4), and in this case, χ is called the $\{i, j, \cdots, l\}$-inverse of φ. If $\chi \in \varphi\{1, 3\}$, then χ is called a $\{1, 3\}$-inverse of φ and is denoted by $\varphi^{(1,3)}$. A $\{1, 4\}$-inverse of φ can be similarly defined. Also, a regular element and a reflexive invertible element can be called a $\{1\}$-invertible element and a $\{1, 2\}$-invertible element, respectively.

Baksalary and Trenkler [2] introduced the core and dual core inverses for a complex matrix. Then, Rakić et al. [3] generalized this concept to an arbitrary $*$-ring, and they use five equations to characterize the core inverse. Later, Xu et al. [4] proved that these five equations can be dropped to three equations. In the following, we rewrite these three equations in the category case. Let $\varphi : X \to X$ be a morphism of $\mathscr{C}$, if there is a morphism $\chi : X \to X$ satisfying

$$(\varphi\chi)^* = \varphi\chi,\ \varphi\chi^2 = \chi,\ \chi\varphi^2 = \varphi,$$

then φ is core invertible and χ is called the core inverse of φ. If such χ exists, then it is unique and denoted by $\varphi^{\oplus}$. And the dual core inverse can be given dually and denoted by $\varphi_{\oplus}$.

Group inverses and Moore-Penrose inverses of morphisms were investigated some years ago. (See,[1] and [5]-[9].) In [5], Robinson and Puystjens give the characterizations about the Moore-Penrose inverse and the group inverse of a morphism with kernels. In [6], Miao and Robinson investigate the group and Moore-Penrose inverses of regular morphisms with kernel and cokernel. Inspired by them, we consider the core invertibility and dual core invertibility of a morphism with kernels and give their representations. In the process of proving the above results, we obtain some characterizations for core inverse and dual core inverse of an element in a $*$-ring by the properties of annihilators and units.

The following notations will be used in this paper: $aR = \{ax \mid x \in R\}$, $Ra = \{xa \mid x \in R\}$, ${}^{\circ}a = \{x \in R \mid xa = 0\}$, $a^{\circ} = \{x \in R \mid ax = 0\}$, $R^{\oplus} = \{a \in R \mid a \text{ is core invertible}\}$, $R_{\oplus} = \{a \in R \mid a \text{ is dual core invertible}\}$. Before beginning, there are some lemmas presenting for the further reference. It should be pointed out, the following Lemma 1.1 - 1.3 were put forward in a $*$- ring. It is easy to prove that they are valid in an additive category with an involution $*$. Thus, we rewrite them in the category case.

Lemma 1.1. *[4, Theorem 2.6 and 2.8] Let $\varphi : X \to X$ be a morphism of $\mathscr{C}$, we have the following results:*
(1) φ is core invertible if and only if φ is group invertible and $\{1,3\}$-invertible. In this case, $\varphi^{\oplus} = \varphi^{\#}\varphi\varphi^{(1,3)}$.
(2) φ is dual core invertible if and only if φ is group invertible and $\{1,4\}$-invertible. In this case, $\varphi_{\oplus} = \varphi^{(1,4)}\varphi\varphi^{\#}$.

Lemma 1.2. *[10, p. 201] Let $\varphi : X \to Y$ be a morphism of $\mathscr{C}$, we have the following results:*
(1) *φ is $\{1,3\}$-invertible with $\{1,3\}$-inverse $\chi : Y \to X$ if and only if $\chi^*\varphi^*\varphi = \varphi$;*
(2) *φ is $\{1,4\}$-invertible with $\{1,4\}$-inverse $\zeta : Y \to X$ if and only if $\varphi\varphi^*\zeta^* = \varphi$.*

Lemma 1.3. *[11, Theorem 2.10] Let $\varphi : X \to X$ be a morphism of $\mathscr{C}$ and $n \geqslant 2$ a positive integer, we have the following results:*
(i) φ is core invertible if and only if there exist morphisms $\varepsilon : X \to X$ and $\tau : X \to X$ such that $\varphi = \varepsilon(\varphi^*)^n\varphi = \tau\varphi^n$. In this case, $\varphi^{\oplus} = \varphi^{n-1}\varepsilon^*$.
(ii) φ is dual core invertible if and only if there exist morphisms $\theta : X \to X$ and $\rho : X \to X$ such that $\varphi = \varphi(\varphi^*)^n\theta = \varphi^n\rho$. In this case, $\varphi_{\oplus} = \theta^*\varphi^{n-1}$.

Lemma 1.4. *[10, Proposition 7] Let $a \in R$. $a \in R^{\#}$ if and only if $a = a^2x = ya^2$ for some $x, y \in R$. In this case, $a^{\#} = yax = y^2a = ax^2$.*

2 The Core and Dual Core Inverse of a Morphism with Kernel

In [5], Robinson and Puystjens gave the characterizations about the Moore-Penrose inverse and the group inverse of a morphism with kernels, see the following two lemmas.

Lemma 2.1. *[5, Theorem 1] Let $\varphi : X \to Y$ be a morphism in $\mathscr{C}$. If $\kappa : K \to X$ is a kernel of φ, then φ has a Moore-Penrose inverse $\varphi^{\dagger}$ with* *respect to* $*$ *if and only if*

$$\varphi\varphi^* + \kappa^*\kappa : X \to X$$

is invertible. In this case, κ *also has a Moore-Penrose inverse* $\kappa^\dagger$, $\kappa\kappa^*$ *is invertible,*

$$\kappa^\dagger = \kappa^*(\kappa\kappa^*)^{-1} = (\varphi\varphi^* + \kappa^*\kappa)^{-1}\kappa^*,$$

and

$$\varphi^\dagger = \varphi^*(\varphi\varphi^* + \kappa^*\kappa)^{-1}.$$

Dually, if $\lambda : Y \to L$ *is a cokernel of* φ, *then* φ *has a Moore-Penrose inverse* $\varphi^\dagger$ *with respect to* $*$ *if and only if*

$$\varphi^*\varphi + \lambda\lambda^* : Y \to Y$$

is invertible. In this case, λ *also has a Moore-Penrose inverse* $\lambda^\dagger$, $\lambda^*\lambda$ *is invertible,*

$$\lambda^\dagger = (\lambda^*\lambda)^{-1}\lambda^* = \lambda^*(\varphi^*\varphi + \lambda\lambda^*)^{-1},$$

and

$$\varphi^\dagger = (\varphi^*\varphi + \lambda\lambda^*)^{-1}\varphi^*.$$

Lemma 2.2. *[5, Corollary 2] Let* $\varphi : X \to X$ *be a morphism in* $\mathscr{C}$. *If* $\kappa : K \to X$ *is a kernel of* φ, *then* φ *has a group inverse if and only if* φ *has a cokernel* $\lambda : X \to L$ *and both* $\kappa\lambda : K \to L$ *and* $\varphi^2 + \lambda(\kappa\lambda)^{-1}\kappa : X \to X$ *are invertible. In this case,* $\gamma = \lambda(\kappa\lambda)^{-1} : X \to K$ *is a cokernel of* φ, $\varphi\varphi^\# + \gamma\kappa = 1_X$, *and*

$$\varphi^\# = \varphi(\varphi^2 + \gamma\kappa)^{-1} = (\varphi^2 + \gamma\kappa)^{-1}\varphi.$$

There are some papers characterizing the core and dual core inverse by units. (See for example, [12] and [11].) Inspired by them and the above two lemmas, we get characterizations of the core invertibility of a morphism with kernel.

Theorem 2.3. *Let* $\varphi : X \to X$ *be a morphism in* $\mathscr{C}$. *If* $\kappa : K \to X$ *is a kernel of* φ, *then* φ *has a core inverse in* $\mathscr{C}$ *if and only if* φ *has a cokernel* $\lambda : X \to L$ *and both* $\kappa\lambda : K \to L$ *and* $\varphi^*\varphi^3 + \kappa^*\kappa : X \to X$ *are invertible. In this case,* $\gamma = \lambda(\kappa\lambda)^{-1} : X \to K$ *is a cokernel of* φ, $\varphi^{\#\!\!\!\bigcirc}\varphi + \gamma\kappa = 1_X$, *and*

$$\varphi^{\#\!\!\!\bigcirc} = \varphi^2(\varphi^*\varphi^3 + \kappa^*\kappa)^{-1}\varphi^*.$$

Proof. Let $\lambda : X \to L$ be a cokernel of φ with both $\kappa\lambda$ and $\varphi^*\varphi^3 + \kappa^*\kappa$ invertible, and set $\gamma = \lambda(\kappa\lambda)^{-1}$. Since $\kappa\varphi = 0 = \varphi\lambda$ and $\varphi^*\kappa^* = 0 = \lambda^*\varphi^*$, then

$$\varphi^*\varphi^3 + \kappa^*\kappa = (\varphi^*\varphi + \kappa^*\kappa)(\varphi^2 + \gamma\kappa),$$

because $\varphi^*\varphi + \kappa^*\kappa$ is symmetric, both $\varphi^*\varphi + \kappa^*\kappa : X \to X$ and $\varphi^2 + \gamma\kappa : X \to X$ are invertible. In addition,

$$(\varphi^*\varphi + \kappa^*\kappa)\varphi = \varphi^*\varphi^2,$$

and for $s \geq 0$ an integer,

$$\varphi^{1+s}(\varphi^2 + \gamma\kappa) = \varphi^{3+s} = (\varphi^2 + \gamma\kappa)\varphi^{1+s}.$$

Consequently,

$$\varphi = (\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*\varphi^2, \tag{1}$$

$$\varphi^{1+s} = \varphi^{3+s}(\varphi^2 + \gamma\kappa)^{-1} = (\varphi^2 + \gamma\kappa)^{-1}\varphi^{3+s}, \tag{2}$$

$$\varphi^{1+s}(\varphi^2 + \gamma\kappa)^{-1} = (\varphi^2 + \gamma\kappa)^{-1}\varphi^{1+s}. \tag{3}$$

Let $\chi = \varphi^2(\varphi^*\varphi^3 + \kappa^*\kappa)^{-1}\varphi^* = \varphi^2(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*$, we now show that χ is the core inverse of φ. Since

$$\begin{aligned}\varphi\chi &= \varphi\varphi^2(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^* \\ &= [\varphi^3(\varphi^2 + \gamma\kappa)^{-1}](\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^* \\ &\overset{(2)}{=} \varphi(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*,\end{aligned}$$

thus $(\varphi\chi)^* = \varphi\chi$. In addition,

$$\begin{aligned}\chi\varphi^2 &= \varphi^2(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*\varphi^2 \overset{(1)}{=} \varphi^2(\varphi^2 + \gamma\kappa)^{-1}\varphi \\ &\overset{(3)}{=} \varphi^2\varphi(\varphi^2 + \gamma\kappa)^{-1} \overset{(2)}{=} \varphi,\end{aligned}$$

and

$$\begin{aligned}\varphi\chi^2 &= \varphi\varphi^2(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*\varphi^2(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^* \\ &= [\varphi^3(\varphi^2 + \gamma\kappa)^{-1}][(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^*\varphi^2](\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^* \\ &\overset{(1)(2)}{=} \varphi\varphi(\varphi^2 + \gamma\kappa)^{-1}(\varphi^*\varphi + \kappa^*\kappa)^{-1}\varphi^* = \chi.\end{aligned}$$

Therefore, φ is core invertible with core inverse $\varphi^{\#} = \varphi^2(\varphi^*\varphi^3 + \kappa^*\kappa)^{-1}\varphi^*$.

Conversely, suppose that φ has a core inverse $\varphi^{\#}$, then φ is group invertible and $\varphi^{\#}\varphi = \varphi^{\#}\varphi$ by Lemma 1.1. Therefore, by applying Lemma 2.2, φ has a cokernel $\lambda : X \to L$, both $\kappa\lambda : K \to L$ and $\varphi^2 + \lambda(\kappa\lambda)^{-1}\kappa : X \to X$ are invertible and $1_X = \varphi\varphi^{\#} + \gamma\kappa = \varphi^{\#}\varphi + \gamma\kappa = \varphi^{\#}\varphi + \gamma\kappa$, where $\gamma = \lambda(\kappa\lambda)^{-1} : X \to K$ is a cokernel of φ. In addition, since $\kappa\varphi = 0 = \varphi\lambda$, thus $\kappa\varphi^{\#} = \kappa\varphi\varphi^{\#}\varphi^{\#} = 0$, $\varphi\gamma = 0$ and $\kappa\gamma = 1_K$, furthermore,

$$
\begin{aligned}
&(\varphi^{\#}(\varphi^{\#})^* + \gamma\gamma^*)(\varphi^*\varphi + \kappa^*\kappa) \\
&= \varphi^{\#}(\varphi^{\#})^*\varphi^*\varphi + \varphi^{\#}(\varphi^{\#})^*\kappa^*\kappa + \gamma\gamma^*\varphi^*\varphi + \gamma\gamma^*\kappa^*\kappa \\
&= \varphi^{\#}(\varphi\varphi^{\#})^*\varphi + \varphi^{\#}(\kappa\varphi^{\#})^*\kappa + \gamma(\varphi\gamma)^*\varphi + \gamma(\kappa\gamma)^*\kappa \\
&= \varphi^{\#}\varphi + \gamma\kappa = 1_X.
\end{aligned}
$$

Since $\varphi^*\varphi + \kappa^*\kappa$ is symmetric, it follows that $\varphi^*\varphi + \kappa^*\kappa$ is invertible with inverse $\varphi^{\#}(\varphi^{\#})^* + \gamma\gamma^*$. Consequently, $\varphi^*\varphi^3 + \kappa^*\kappa = (\varphi^*\varphi + \kappa^*\kappa)(\varphi^2 + \lambda(\kappa\lambda)^{-1}\kappa)$ is invertible. □

Dually, we obtain the following result.

Theorem 2.4. *Let $\varphi : X \to X$ be a morphism of an additive category $\mathscr{C}$. If $\lambda : X \to L$ is a cokernel of φ, then φ has a dual core inverse in $\mathscr{C}$ if and only if φ has a kernel $\kappa : K \to X$ and both $\kappa\lambda : K \to L$ and $\varphi^3\varphi^* + \lambda\lambda^* : X \to X$ are invertible. In this case, $\delta = (\kappa\lambda)^{-1}\kappa : L \to X$ is a kernel of φ, $\varphi_{\#}\varphi + \delta\kappa = 1_X$, and*

$$\varphi_{\#} = \varphi^*(\varphi^3\varphi^* + \lambda\lambda^*)^{-1}\varphi^2.$$

Remark 2.5. *In fact, one can easily find that Theorem 2.3 is true when we raise the 3 power to n power, that is to say, change $\varphi^*\varphi^3+\kappa^*\kappa$ to $\varphi^*\varphi^n+\kappa^*\kappa$, where $n \geqslant 3$. And in this case, $\varphi^{\#} = \varphi^{n-1}(\varphi^*\varphi^n + \kappa^*\kappa)^{-1}\varphi^*$. Similarly, it is valid for dual core inverse.*

Consider Theorem 2.3 in the ring case, we obtain the following result.

Theorem 2.6. *Let $a \in R$ and $n \geqslant 3$ a positive integer. Then $a \in R^{\#}$ if and only if there exists $b \in R$ such that ${}^{\circ}a = Rb$ and $u = a^*a^n + b^*b$ is invertible. In this case,*

$$a^{\#} = a^{n-1}u^{-1}a^*.$$

Proof. Suppose that a is core invertible with core inverse $a^{\#}$. Let $b = 1 - aa^{\#}$, then $b^* = b = b^2$ and $Rb = R(1 - aa^{\#}) = {}^{\circ}(aa^{\#})$. Obviously ${}^{\circ}a \subseteq$

${}^{\circ}(aa^{\oplus})$; if $xaa^{\oplus} = 0$, then $xa = xaa^{\oplus}a = 0$, hence ${}^{\circ}(aa^{\oplus}) \subseteq {}^{\circ}a$. Therefore, ${}^{\circ}a = {}^{\circ}(aa^{\oplus}) = Rb$. In addition,

$$u = a^*a^n + b^*b = a^*a^n + 1 - aa^{\oplus} = (a^* + 1 - aa^{\oplus})(a^n + 1 - aa^{\oplus}).$$

It is easy to verify that $(a^{\oplus} + 1 - a^{\oplus}a)^*$ and $(a^{\oplus})^n + 1 - a^{\oplus}a$ are inverses of $a^* + 1 - aa^{\oplus}$ and $a^n + 1 - aa^{\oplus}$, respectively. Thus $a^* + 1 - aa^{\oplus}$ and $a^n + 1 - aa^{\oplus}$ are both invertible, which implies that $u = a^*a^n + 1 - aa^{\oplus}$ is invertible.

Conversely, assume that there exists $b \in R$ such that ${}^{\circ}a = Rb$ and $u = a^*a^n + b^*b$ is invertible, where $n \geqslant 3$ is a positive integer. Then $b \in {}^{\circ}a$, that is to say, $ba = 0 = a^*b^*$. Since $(a^*)^{n-1}u = (a^*)^n a^n$ is symmetric, namely, $(a^*)^{n-1}u = u^*a^{n-1}$, which implies the following equation

$$(u^*)^{-1}(a^*)^{n-1} = a^{n-1}u^{-1}. \tag{4}$$

Also, $a^*u = (a^*)^2a^n$ implies $a^* = (a^*)^2a^nu^{-1}$, hence we have

$$\begin{aligned} a &= (u^*)^{-1}(a^*)^n a^2 = [(u^*)^{-1}(a^*)^{n-1}]a^*a^2 \\ &\overset{(4)}{=} a^{n-1}u^{-1}a^*a^2 \in a^2R \cap Ra^2, \end{aligned}$$

so a is group invertible with group inverse $a^{\#}$ according to Lemma 1.4. Since $1 - a^{\#}a \in {}^{\circ}a = Rb$, thus $1 - a^{\#}a = yb$ for some $y \in R$. Pre-multiplication of $1 - a^{\#}a = yb$ by a and b yield $ayb = 0$ and $b = byb$, respectively. Therefore, $u = a^*a^n + b^*b$ can be decomposed as

$$u = a^*a^n + b^*b = (a^*a + b^*b)(a^{n-1} + yb).$$

Since u is invertible and $a^*a + b^*b$ is symmetric, thus both $a^*a + b^*b$ and $a^{n-1} + yb$ are invertible. Set $x = a^{n-1}u^{-1}a^* = a^{n-1}(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^*$, we show that x is the core inverse of a. Since

$$\begin{aligned} ax &= a^n(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^* \\ &= a(a^{n-1} + yb)(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^* \\ &= a(a^*a + b^*b)^{-1}a^* \end{aligned}$$

shows that $(ax)^* = ax$,

$$\begin{aligned} ax^2 &= (ax)x = a(a^*a + b^*b)^{-1}a^*a^{n-1}(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^* \\ &= a(a^*a + b^*b)^{-1}(a^*a + b^*b)a^{n-2}(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^* \\ &= aa^{n-2}(a^{n-1} + yb)^{-1}(a^*a + b^*b)^{-1}a^* = x, \end{aligned}$$

and

$$\begin{aligned} xa^2 &= a^{n-1}(a^{n-1}+yb)^{-1}(a^*a+b^*b)^{-1}a^*a^2 \\ &= a^{n-1}(a^{n-1}+yb)^{-1}(a^*a+b^*b)^{-1}(a^*a+b^*b)a \\ &= a^{n-1}(a^{n-1}+yb)^{-1}a = (a^{n-1}+yb)^{-1}a^{n-1}a \\ &= (a^{n-1}+yb)^{-1}(a^{n-1}+yb)a = a, \end{aligned}$$

thus $x = a^{\oplus}$. □

In the same way, there is a corresponding result for dual core inverse.

Theorem 2.7. *Let $a \in R$ and $n \geqslant 3$ a positive integer, then $a \in R_{\oplus}$ if and only if there exists $c \in R$ such that $a^{\circ} = cR$ and $v = a^n a^* + cc^*$ is invertible. In this case,*

$$a_{\oplus} = a^* v^{-1} a^{n-1}.$$

Let $\varphi : X \to Y$ be a morphism in $\mathscr{C}$. If $\eta\varphi = 0 : N \to Y$ is the zero morphism, then we shall call the morphism $\eta : N \to X$ an annihilator of the morphism φ. Dually, we call φ a coannihilator of η. (See for example, [9].)

Theorem 2.8. *Let $\varphi : X \to X$ be a morphism in $\mathscr{C}$ and $n \geqslant 2$ a positive integer. Then φ is core invertible if and only if there exists an annihilator $\eta : N \to X$ of φ such that $\mu = \varphi^n + \eta^*\eta : X \to X$ is invertible. In this case,*

$$\varphi^{\oplus} = \varphi^{n-1}\mu^{-1}.$$

Proof. Suppose that φ is core invertible with core inverse $\varphi^{\oplus}$. Since $(1_X - \varphi\varphi^{\oplus})\varphi = 0$, then $\eta = 1_X - \varphi\varphi^{\oplus}$ is a annihilator of φ such that $\eta^* = \eta = \eta^2$. In this case, $\mu = \varphi^n + \eta^*\eta = \varphi^n + 1_X - \varphi\varphi^{\oplus}$. Since

$$(\varphi^n + 1_X - \varphi\varphi^{\oplus})((\varphi^{\oplus})^n + 1_X - \varphi^{\oplus}\varphi) = 1_X = ((\varphi^{\oplus})^n + 1_X - \varphi^{\oplus}\varphi)(\varphi^n + 1_X - \varphi\varphi^{\oplus}),$$

$\mu = \varphi^n + \eta^*\eta$ is invertible.

Conversely, if there exists an annihilator $\eta : N \to X$ of φ such that $\mu = \varphi^n + \eta^*\eta : X \to X$ is invertible, where $n \geqslant 2$ is a positive integer. On the one hand, $\varphi^*\mu = \varphi^*\varphi^n$, which implies $\varphi^* = \varphi^*\varphi^n\mu^{-1}$, so $\varphi = (\mu^{-1})^*(\varphi^*)^n\varphi$. On the other hand, $\mu\varphi = \varphi^{n+1}$ shows that $\varphi = \mu^{-1}\varphi^{n+1}$. Therefore, φ is core invertible with $\varphi^{\oplus} = \varphi^{n-1}\mu^{-1}$ by Lemma 1.3. □

Corollary 2.9. *Let $a \in R$ and $n \geqslant 2$ a positive integer, then $a \in R^{\oplus}$ if and only if there exists $b \in {}^{\circ}a$ such that $u = a^n + b^*b$ is invertible. In this case,*

$$a^{\oplus} = a^{n-1}u^{-1}.$$

Analogously, there are similar conclusions for dual core inverses, which are not to be repeated here.

3 Core and Dual Core Inverses of Regular Morphisms with Kernels and Cokernels

In [6], Miao and Robinson investigated the group and Moore-Penrose inverses of regular morphisms with kernels and cokernels, and they showed us two results as follows. Let $\varphi : X \to Y$ be a morphism with kernel $\kappa : K \to X$ and cokernel $\lambda : Y \to L$ in an additive category $\mathscr{C}$. (1) If $X = Y$, then φ has a group inverse $\varphi^{\#}$ if and only if φ is regular and $\kappa\lambda$ is invertible. (2) φ has a Moore-Penrose inverse $\varphi^{\dagger}$ if and only if φ is regular and both $\kappa\kappa^*$ and $\lambda^*\lambda$ are invertible.

Inspired by them, we investigate the core and dual core inverses of regular morphisms with kernels and cokernels.

Lemma 3.1. *Let $\varphi : X \to Y$ be a morphism with kernel $\kappa : K \to X$, then φ is $\{1,3\}$-invertible if and only if φ is regular and $\kappa\kappa^* : K \to K$ is invertible. In this case, if $\psi : Y \to X$ is such that $\varphi\psi\varphi = \varphi$, then*

$$\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \in \varphi\{1,3\}.$$

Proof. Suppose that φ is $\{1,3\}$-invertible with $\{1,3\}$-inverse $\varphi^{(1,3)}$. Since $\varphi\varphi^{(1,3)}\varphi = \varphi$, then φ is regular. Moreover, since $(1_X - \varphi\varphi^{(1,3)})\varphi = 0$, then by the definition of a kernel, $1_X - \varphi\varphi^{(1,3)} = \zeta\kappa$ for some $\zeta : X \to K$. In addition, since $\kappa\varphi = 0$, then $\kappa\zeta\kappa = \kappa(1_X - \varphi\varphi^{(1,3)}) = \kappa$, and since κ is monic, $\kappa\zeta = 1_K$. Therefore, $\zeta\kappa\zeta = \zeta$. Consequently, κ is Moore-Penrose invertible with $\kappa^{\dagger} = \zeta$. Since,

$$(\kappa\kappa^*)(\zeta^*\zeta) = \kappa(\zeta\kappa)^*\zeta = \kappa\zeta\kappa\zeta = 1_K$$

and $\kappa\kappa^*$ is symmetric, then $\kappa\kappa^*$ is invertible with $(\kappa\kappa^*)^{-1} = \zeta^*\zeta = (\kappa^*)^{\dagger}\kappa^{\dagger}$.

Conversely, suppose that $\varphi\psi\varphi = \varphi$ and that $\kappa\kappa^* : K \to K$ is invertible, where $\psi : Y \to X$ is morphism. We prove that $\chi = \psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa]$ is a $\{1,3\}$-inverse of φ. Indeed, since $(1_X - \varphi\psi)\varphi = 0$, then $1_X - \varphi\psi = \delta\kappa$ for some $\delta : X \to K$. Therefore,

$$\begin{aligned}\varphi\chi &= \varphi\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] = (1_X - \delta\kappa)[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \\ &= 1_X - \delta\kappa - \kappa^*(\kappa\kappa^*)^{-1}\kappa + \delta\kappa\kappa^*(\kappa\kappa^*)^{-1}\kappa \\ &= 1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa\end{aligned}$$

is symmetric. Furthermore, Since $\kappa\varphi = 0$, then

$$\varphi\chi\varphi = (1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa)\varphi = \varphi.$$

Thus, $\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \in \varphi\{1,3\}$. □

Similarly, we have the following result.

Lemma 3.2. *Let $\varphi : X \to Y$ be a morphism with cokernel $\lambda : Y \to L$, then φ is $\{1,4\}$-invertible if and only if φ is regular and $\lambda^*\lambda : L \to L$ is invertible. In this case, if $\psi : Y \to X$ is such that $\varphi\psi\varphi = \varphi$, then*

$$[1_X - \lambda(\lambda^*\lambda)^{-1}\lambda^*]\psi \in \varphi\{1,4\}.$$

Theorem 3.3. *Let $\varphi : X \to X$ be a morphism with kernel $\kappa : K \to X$ and cokernel $\lambda : X \to L$ in an additive category $\mathscr{C}$, then φ has a core inverse in $\mathscr{C}$ if and only if φ is regular and both $\kappa\lambda : K \to L$ and $\kappa\kappa^* : K \to K$ are invertible. In this case, if $\psi : X \to X$ is such that $\varphi\psi\varphi = \varphi$, then*

$$\varphi^{\#} = [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa].$$

Proof. By Lemma 1.1, Lemma 3.1 and [6, Theorem], it is clear that φ is core invertible if and only if φ is regular and both $\kappa\lambda : K \to L$ and $\kappa\kappa^* : K \to K$ are invertible.

Suppose that $\psi : X \to X$ is such that $\varphi\psi\varphi = \varphi$, we show that $\chi = [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa]$ is the core inverse of φ. Indeed, since $(1_X - \varphi\psi)\varphi = 0 = \varphi(1_X - \psi\varphi)$, then $1_X - \varphi\psi = \delta\kappa$ and $1_X - \psi\varphi = \lambda\zeta$ for some $\delta : X \to K$ and $\zeta : L \to X$, respectively. Since $\kappa\varphi = 0 = \varphi\lambda$, then

$$\begin{aligned}
\varphi\chi &= \varphi[1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \\
&= \varphi\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \\
&= (1_X - \delta\kappa)[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \\
&= 1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa
\end{aligned}$$

is symmetric. In addition,

$$\begin{aligned}
\chi\varphi^2 &= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa]\varphi^2 \\
&= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi\varphi^2 = [1_X - \lambda(\kappa\lambda)^{-1}\kappa](1_X - \lambda\zeta)\varphi \\
&= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\varphi = \varphi
\end{aligned}$$

and

$$\begin{aligned}
\varphi\chi^2 &= [1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa][1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa] \\
&= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa = \chi.
\end{aligned}$$

Therefore, φ is core invertible with core inverse $\varphi^{\#} = [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa]$. □

Similarly, we can get a dually result about dual core inverse.

Theorem 3.4. *Let $\varphi : X \to X$ be a morphism with kernel $\kappa : K \to X$ and cokernel $\lambda : X \to L$ in an additive category $\mathscr{C}$, then φ has a dual core inverse in $\mathscr{C}$ if and only if φ is regular and both $\kappa\lambda : K \to L$ and $\lambda^*\lambda : L \to L$ are invertible. In this case, if $\psi : X \to X$ is such that $\varphi\psi\varphi = \varphi$, then*

$$\varphi_{\oplus} = [1_X - \lambda(\lambda^*\lambda)^{-1}\lambda^*]\psi[1_X - \lambda(\kappa\lambda)^{-1}\kappa].$$

Corollary 3.5. *Let $\varphi : X \to X$ be a morphism with kernel $\kappa : K \to X$ and cokernel $\lambda : X \to L$ in an additive category $\mathscr{C}$, then the following statements are equivalent:*
(1) φ is both core invertible and dual core invertible in $\mathscr{C}$;
(2) φ is both Moore-Penrose invertible and group invertible in $\mathscr{C}$;
(3) φ is regular and $\kappa\lambda : K \to L$, $\kappa\kappa^* : K \to K$ and $\lambda^*\lambda : L \to L$ are all invertible.
In this case, if $\psi : X \to X$ is such that $\varphi\psi\varphi = \varphi$, then

$$\begin{aligned}
\varphi^{\dagger} &= [1_X - \lambda(\lambda^*\lambda)^{-1}\lambda^*]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa],\\
\varphi^{\#} &= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \lambda(\kappa\lambda)^{-1}\kappa],\\
\varphi^{\oplus} &= [1_X - \lambda(\kappa\lambda)^{-1}\kappa]\psi[1_X - \kappa^*(\kappa\kappa^*)^{-1}\kappa],\\
\varphi_{\oplus} &= [1_X - \lambda(\lambda^*\lambda)^{-1}\lambda^*]\psi[1_X - \lambda(\kappa\lambda)^{-1}\kappa].
\end{aligned}$$

4 Bordered Inverses

Recall that a morphism $\varphi : X \to Y$ is $*$-left invertible if there is a morphism $\psi : Y \to X$ such that $\psi\varphi = 1_Y$ and $(\varphi\psi)^* = \varphi\psi$. Similarly, $\varphi : X \to Y$ is $*$-right invertible if there is a morphism $\psi : Y \to X$ such that $\varphi\psi = 1_X$ and $(\psi\varphi)^* = \psi\varphi$. (See, [5, p. 76].)

Lemma 4.1. *[5, Lemma] If $\varphi : X \to Y$ is a morphism in a category with an involution. Then*
(1) φ is $*$-left invertible if and only if $\varphi^*\varphi$ is invertible, and in this case, $\varphi^{\dagger} = (\varphi^*\varphi)^{-1}\varphi^*$;
(2) φ is $*$-right invertible if and only if $\varphi\varphi^*$ is invertible, and in this case, $\varphi^{\dagger} = \varphi^*(\varphi\varphi^*)^{-1}$.

Lemma 4.2. *[5, Corollary 3] Let $\varphi : X \to X$ be a morphism of an additive category $\mathscr{C}$. If $\kappa : K \to X$ is a kernel and $\lambda : X \to L$ is a cokernel of φ, then φ has a group inverse in $\mathscr{C}$ if and only if*

$$\mathscr{G} = \begin{pmatrix} \varphi & \lambda \\ \kappa & 0 \end{pmatrix} : (X, K) \to (X, L)$$

is invertible in $\mathscr{M}_{\mathscr{C}}$. *In this case,* $\kappa\lambda : K \to L$ *is invertible and*

$$\mathscr{G}^{-1} = \begin{pmatrix} \varphi^{\#} & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & 0 \end{pmatrix} : (X, L) \to (X, K).$$

Theorem 4.3. *Let* $\varphi : X \to X$ *be a morphism of an additive category* $\mathscr{C}$ *with an involution* $*$. *If* $\kappa : K \to X$ *is a kernel of* φ *and* $\lambda : X \to L$ *is a cokernel of* φ, *then* φ *has a core inverse* $\varphi^{\oplus}$ *in* $\mathscr{C}$ *if and only if*

$$\begin{pmatrix} \varphi \\ \kappa \end{pmatrix} : (X, K) \to (X)$$

is $*$*-left invertible in* $\mathscr{M}_{\mathscr{C}}$ *and*

$$\mathscr{G} = \begin{pmatrix} \varphi & \lambda \\ \kappa & 0 \end{pmatrix} : (X, K) \to (X, L)$$

is invertible in $\mathscr{M}_{\mathscr{C}}$. *In this case,* $\kappa\lambda : K \to L$ *is invertible and*

$$\mathscr{G}^{-1} = \begin{pmatrix} (\varphi^{\oplus})^2\varphi & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & 0 \end{pmatrix} : (X, L) \to (X, K).$$

Proof. By Theorem 2.3, φ has a core inverse in $\mathscr{C}$ if and only if both $\kappa\lambda$ and $\varphi^*\varphi^3 + \kappa^*\kappa$ are invertible. Since $\varphi^*\varphi^3 + \kappa^*\kappa = (\varphi^*\varphi + \kappa^*\kappa)(\varphi^2 + \lambda(\kappa\lambda)^{-1}\kappa)$ and $\varphi^*\varphi + \kappa^*\kappa$ is symmetric, then $\varphi^*\varphi^3 + \kappa^*\kappa$ is invertible if and only if $\varphi^*\varphi + \kappa^*\kappa$ and $\varphi^2 + \lambda(\kappa\lambda)^{-1}\kappa$ are both invertible. By Lemma 4.1, $\varphi^*\varphi + \kappa^*\kappa$ is invertible if and only if $\begin{pmatrix} \varphi \\ \kappa \end{pmatrix}^* \begin{pmatrix} \varphi \\ \kappa \end{pmatrix}$ is invertible in $\mathscr{C}$ if and only if $\begin{pmatrix} \varphi \\ \kappa \end{pmatrix}$ is $*$-left invertible in $\mathscr{M}_{\mathscr{C}}$. Therefore, the conclusion is obtained by the previous proof, Lemma 2.2 and Lemma 4.2. And it is easy to verify that $\begin{pmatrix} (\varphi^{\oplus})^2\varphi & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & 0 \end{pmatrix} : (X, L) \to (X, K)$ is the inverse of $\mathscr{G}$. $\square$

We have a dually theorem about dual core inverse.

Theorem 4.4. *Let* $\varphi : X \to X$ *be a morphism of an additive category* $\mathscr{C}$ *with an involution* $*$. *If* $\kappa : K \to X$ *is a kernel of* φ *and* $\lambda : X \to L$ *is a cokernel of* φ, *then* φ *has a dual core inverse* $\varphi_{\oplus}$ *in* $\mathscr{C}$ *if and only if*

$$(\varphi, \lambda) : (X) \to (X, L)$$

is $*$*-right invertible and*

$$\mathscr{G} = \begin{pmatrix} \varphi & \lambda \\ \kappa & 0 \end{pmatrix} : (X, K) \to (X, L)$$

is invertible in $\mathscr{M}_{\mathscr{C}}$. In this case, $\kappa\lambda : K \to L$ is invertible and

$$\mathscr{G}^{-1} = \begin{pmatrix} \varphi\varphi_{\oplus}^{2} & \lambda(\kappa\lambda)^{-1} \\ (\kappa\lambda)^{-1}\kappa & 0 \end{pmatrix} : (X, L) \to (X, K).$$

Acknowledgements

This research is supported by the National Natural Science Foundation of China (No.11771076 and No.11471186); the Fundamental Research Funds For the Central Universities (No.KYCX17_0037); Postgraduate Research & Practice Innovation Program of Jiangsu Province (No.KYCX17_0037).

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