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Homogenization of a non–homogeneous heat conducting fluid

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Abstract

We consider a non–homogeneous incompressible and heat conducting fluid confined to a 3D domain perforated by tiny holes. The ratio of the diameter of the holes and their mutual distance is critical, the former being equal to ε^3, the latter proportional to ε, where ε is a small parameter. We identify the asymptotic limit for $\varepsilon \to 0$, in which the momentum equation contains a friction term of Brinkman type determined uniquely by the viscosity and geometric properties of the perforation. Besides the inhomogeneity of the fluid, we allow the viscosity and the heat conductivity coefficient to depend on the temperature, where the latter is determined via the Fourier law with homogenized (oscillatory) heat conductivity coefficient that is different for the fluid and the solid holes. To the best of our knowledge, this is the first result in the critical case for the inhomogenous heat–conducting fluid.

Keywords: Non-homogeneous Navier–Stokes system, homogenization, heat–conducting fluid, incompressible fluid, Brinkman law

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1 Introduction

We study the motion of a non–homogeneous, incompressible viscous and heat conducting fluid contained in a bounded spatial domain perforated by a system of tiny holes. The mass density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the temperature $\Theta = \Theta(t,x)$ satisfy a variant of the Navier–Stokes–Boussinesq system proposed by Chandrasekhar [3] (see also Lignières [12]) :

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\ \mathrm{div}_x \mathbf{u} = 0, \tag{1.1}$$

$$\begin{gathered} \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x P = \mathrm{div}_x \mathbb{S}(\Theta, \nabla_x \mathbf{u}) - \Theta \nabla_x F, \\ \mathbb{S}(\Theta, \nabla_x \mathbf{u}) = \mu(\Theta) \left(\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u} \right), \end{gathered} \tag{1.2}$$

$$-\,\mathrm{div}_x \left(\kappa \nabla_x \Theta \right) = \nabla_x F \cdot \mathbf{u}. \tag{1.3}$$

Here the last equation can be seen as a quasi–static (high Péclet number) approximation of the conventional heat equation

$$\partial_t(\varrho \Theta) + \mathrm{div}_x(\varrho \Theta \mathbf{u}) - \mathrm{div}_x \left(\kappa \nabla_x \Theta \right) = \nabla_x F \cdot \mathbf{u},$$

where F is the gravitational potential. We refer to [10, Chapter 4, Section 4.3] for a rigorous derivation of system (1.1–1.3) in the spatially homogeneous case $\varrho \equiv 1$.

The fluid is contained in a bounded domain $\Omega_\varepsilon \subset R^3$, on the boundary of which the velocity obeys the no-slip condition

$$\mathbf{u}|_{\partial \Omega_\varepsilon} = 0. \tag{1.4}$$

Extending $\mathbf{u}$ to be zero outside Ω_ε we may therefore assume that the equation of continuity (1.1) is satisfied in the whole space R^3. Similarly, we suppose $\Omega_\varepsilon \subset \Omega$ whereas (1.3) is satisfied in Ω, with

$$\kappa = \kappa_\varepsilon(x) = \begin{cases} \kappa_f > 0 \text{ if } x \in \Omega_\varepsilon, \\ \\ \kappa_s > 0 \text{ if } x \in \Omega \setminus \overline{\Omega}_\varepsilon \end{cases}$$

where, in general, we allow $\kappa_f \neq \kappa_s$. For definiteness, we prescribe the homogeneous Dirichlet boundary conditions for the temperature,

$$\Theta|_{\partial \Omega} = 0. \tag{1.5}$$

1.1 Perforated domain

We now introduce the perforated domain under consideration. Let $0 < \varepsilon < 1$ be a small parameter. We suppose

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{k=1}^{K(\varepsilon)} T_{k,\varepsilon},$$

where

$$T_{k,\varepsilon} = x_k + \varepsilon^3 \overline{U}_{k,\varepsilon}, \ k = 1, 2, \dots, K(\varepsilon), \tag{1.6}$$

with $\text{dist}[x_i, x_j] > c\,\varepsilon$ whenever $i \neq j$, $\text{dist}[x_i, \partial\Omega] > c\,\varepsilon$, for some constant $c > 0$ independent of ε. By a normalization process, we may assume $c = 1$.

Here $\{U_{k,\varepsilon}\}_{\varepsilon>0,k=1,\dots,K(\varepsilon)}$ are assumed to be uniformly $C^{2+\nu}$ simply connected domains satisfying

$$\left\{ |x| < \frac{1}{2} \right\} \subset U_{k,\varepsilon} \subset \overline{U}_{k,\varepsilon} \subset \left\{ |x| < \frac{3}{4} \right\} \text{ for any } \varepsilon, \ k. \tag{1.7}$$

Thus possible spatial configuration of the holes $T_{k,\varepsilon}$ includes the so–called critical case, the holes being of radius ε^3 with their mutual distance proportional to ε, cf. Allaire [1] among others. The assumptions (1.6)–(1.7) imposed on the distribution of holes guarantee the holes are pairwise disjoint. Note that no periodicity of the holes is a priori assumed.

Finally, for the sake of simplicity, we suppose that $\partial\Omega$ is smooth, of class $C^{2+\nu}$. We use C to denote a universal constant whose value is independent of ε.

1.2 Weak solutions

We consider weak solutions to problem (1.1–1.5) emanating from the initial data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon}, \ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}, \tag{1.8}$$

and belonging to the regularity class

$$\begin{aligned} &\varrho \in C([0,T]; L^1(\Omega)), \ 0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho} \text{ a.a. in } (0,T)\times\Omega, \\ &\Theta \in L^\infty(0,T; W_0^{1,2}(\Omega)), \ \Theta(t,\cdot) \in C^\nu(\overline{\Omega}), \ \|\Theta(t,\cdot)\|_{C^\nu(\overline{\Omega})} \leq C \text{ for a.a. } t \in (0,T), \ \nu > 0, \\ &\mathbf{u} \in L^\infty(0,T; L^2(\Omega; R^3)) \cap L^2(0,T; W_0^{1,2}(\Omega; R^3)). \end{aligned} \tag{1.9}$$

The equations (1.1), (1.2), (1.3) will be interpreted in the weak sense. More specifically,

$$\int_0^T \int_{R^3} [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x\,\mathrm{d}t = -\int_{R^3} \varrho_{0,\varepsilon} \varphi(0,\cdot) \ \mathrm{d}x \tag{1.10}$$

for any $\varphi \in C_c^1([0,T) \times R^3)$, where $\mathbf{u} \equiv 0$ outside Ω_ε;

$\text{div}_x \mathbf{u} = 0$ a.a. in $(0,T)\times\Omega$;

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} [\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi}] \ \mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Omega_\varepsilon} \mathbb{S}(\Theta, \nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x + \int_0^T \int_\Omega \Theta \nabla_x F \cdot \boldsymbol{\varphi} \ \mathrm{d}x\,\mathrm{d}t - \int_{\Omega_\varepsilon} \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x \end{aligned} \tag{1.11}$$

for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega_\varepsilon; R^3))$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$;

$$\int_\Omega \kappa_\varepsilon \nabla_x \Theta(\tau,\cdot) \cdot \nabla_x \phi \ \mathrm{d}x = \int_\Omega \mathbf{u}(\tau,\cdot) \cdot \nabla_x F \phi \ \mathrm{d}x \tag{1.12}$$

for a.a. $\tau \in (0,T)$ and $\phi \in C^1_c(\Omega)$.

In addition, we suppose that the energy inequality

$$\begin{aligned} \int_{\Omega_\varepsilon} \frac{1}{2} \varrho |\mathbf{u}|^2(\tau,\cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \kappa_\varepsilon |\nabla_x \Theta|^2 \ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_{\Omega_\varepsilon} \frac{\mu(\Theta)}{2} |\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}|^2 \ \mathrm{d}x\,\mathrm{d}t \\ \leq \int_{\Omega_\varepsilon} \frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 \ \mathrm{d}x \end{aligned} \tag{1.13}$$

holds for a.a. $\tau \in (0,T)$.

In view of the DiPerna–Lions theory [8], and the anticipated regularity of ϱ, $\mathbf{u}$ stated in (1.9), the weak formulation (1.10) implies its renormalized variant

$$\int_0^T \int_{R^3} \left[b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x\,\mathrm{d}t = - \int_{R^3} b(\varrho_{0,\varepsilon}) \varphi(0,\cdot) \ \mathrm{d}x \tag{1.14}$$

for any $\varphi \in C^1_c([0,T) \times R^3)$ and any $b \in C((0,\infty))$ (actually any Borel b), due to the lower and upper bound restriction of ϱ.

We remark that, for any fixed $\varepsilon > 0$, the existence of renormalized weak solutions can be derived following the nowadays well understood argument in Lions's book [13].

1.3 Main result

Let $K \subset \{|x| < 1\}$ be a compact set. We define the matrix

$$\mathbb{C}_{i,j}(K) = \int_{\{|x|<1\} \setminus K} \nabla_x \mathbf{v}^i : \nabla_x \mathbf{v}^l \ \mathrm{d}x, \ i,j = 1, \dots 3,$$

where $\mathbf{v}^i$ is the unique solution of the *model problem*

$$- \Delta \mathbf{v}^i + \nabla_x q^i = 0, \ \mathrm{div}_x \mathbf{v}_i = 0 \text{ in } \{|x| < 1\} \setminus K, \ \mathbf{v}^i|_{\partial K} = \mathbf{e}^i, \ \mathbf{v}^i|_{\{|x|=1\}} = 0. \tag{1.15}$$

Here $\{\mathbf{e}^i\}_{i=1}^3$ denotes the standard orthogonal basis of the vector space R^3. Note that $\mathbf{v}^i$ is the unique minimizer of the Dirichlet integral

$$\int_{R^3} |\nabla_x \mathbf{v}|^2 \ \mathrm{d}x \text{ over the set } \{\mathbf{v} \in W^{1,2}(R), \ \mathrm{div}_x \mathbf{v} = 0, \ \mathbf{v}|_K = \mathbf{e}^i, \ \mathbf{v}|_{\{|x| \geq 1\}} = 0\}.$$

Finally, we suppose that there is a positive definite symmetric matrix field $\mathbb{D} \in L^\infty(\Omega; R^{3\times 3}_{\mathrm{sym}})$ such that

$$\lim_{\varepsilon \to 0} \sum_{T_{k,\varepsilon} \subset B} \mathbb{C}_{i,j}(\varepsilon^3 \overline{U}_{k,\varepsilon}) = \int_B \mathbb{D}_{i,j}(x) \ \mathrm{d}x \text{ for any Borel set } B \subset \Omega, \tag{1.16}$$

where $U_{k,\varepsilon}$ are related to $T_{k,\varepsilon}$ via (1.6). Note that the limit (1.16) exists in the spatially periodic case with holes of uniform shape studied in the nowadays classical papers by Allaire [1], [2]. Other relevant examples can be found in Desvillettes, Golse, and Ricci [6], or Marchenko, Khruslov [14].

We are ready to formulate our main result:

Theorem 1.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of perforated domains specified in Section 1.1, where the asymptotic distribution of holes satisfies (1.5). Let the initial data be given such that*

$$\begin{aligned} &\varrho_{0,\varepsilon} \in L^\infty(R^3),\ \varrho_{0,\varepsilon}(x) = \varrho_s > 0 - \textit{a positive constant for } x \in R^3 \setminus \overline{\Omega}_\varepsilon \\ &0 < \underline{\varrho} \leq \varrho_{0,\varepsilon}(x) \leq \overline{\varrho},\ \varrho_{0,\varepsilon} \to \varrho_0 \ \textit{in } L^1(\Omega); \end{aligned} \tag{1.17}$$

$$\mathrm{div}_x \mathbf{u}_{0,\varepsilon} = 0,\ \mathbf{u}_{0,\varepsilon} = 0 \ \textit{in } R^3 \setminus \overline{\Omega}_\varepsilon,\ \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \ \textit{in } L^2(\Omega; R^3); \tag{1.18}$$

Finally, suppose that $\nabla_x F \in L^\infty(\Omega; R^3)$ and that $\mu = \mu(\Theta)$ is a positive continuous function of Θ.

Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \Theta_\varepsilon)$ be a weak solution of the problem (1.1–1.3), (1.4), (1.5), (1.8). Then, up to a subsequence, we have

$$\begin{aligned} &\varrho_\varepsilon \to \varrho \in C([0,T]; L^1(\Omega)),\ 0 < \underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq \overline{\varrho}, \\ &\Theta_\varepsilon \to \Theta \ \textit{in } L^q((0,T)\times\Omega) \textit{ for any } 1 \leq q < \infty, \textit{ and weakly-(*) in } L^\infty(0,T; W_0^{1,2}(\Omega)), \\ &\mathbf{u}_\varepsilon \to \mathbf{u} \ \textit{in } L^2((0,T)\times\Omega; R^3) \textit{ and weakly in } L^2(0,T; W_0^{1,2}(\Omega; R^3)), \end{aligned}$$

where $(\varrho, \mathbf{u}, \Theta)$ is a weak solution of the problem

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,\ \mathrm{div}_x \mathbf{u} = 0,$$

$$\begin{aligned} \partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \mu(\Theta)\mathbb{D}\mathbf{u} + \nabla_x P &= \mathrm{div}_x \mathbb{S}(\Theta, \nabla_x \mathbf{u}) + \Theta\nabla_x F, \\ \mathbb{S}(\Theta, \nabla_x\mathbf{u}) &= \mu(\Theta)\left(\nabla_x\mathbf{u} + \nabla_x\mathbf{u}^t\right), \end{aligned}$$

$$-\mathrm{div}_x\left(\kappa_f \nabla_x \Theta\right) = \nabla_x F \cdot \mathbf{u},$$

in $(0,T)\times\Omega$, satisfying the boundary conditions (1.4), (1.5), and the initial conditions $(\varrho_0, \mathbf{u}_0)$.

The rest of the paper is devoted to the proof of Theorem 1.1. It is worth noting that the limit process includes in fact *two* homogenization procedures: the first one in the momentum equation due to the domain perforation, the second one in the heat equation due to the spatial oscillations of the heat conductivity coefficient. The two processes interact via the temperature dependent viscosity coefficient μ. Besided the nowadays standard homogenization technique developed in the pioneering paper by Allaire [1], our method leans essentially on the uniform estimates of the Hölder norm of the temperature Θ_ε. To the best of our knowledge, this is the first result concerning the critical case for the inhomogeneous heat conducting fluid. It is worth noting that the Brikman type term in the asymptotic limit is independent of the density of the fluid, cf. the nowadays classical paper of Cioranescu and Murat [4], [5] concerning the background of this extra term.

The paper is organized as follows. In Section 2, we derive some preliminary estimates that follow directly from the renormalized formulation and the available energy bounds, in particular we derive the uniform bounds on the Hölder norm of the temperature. The homogenization process in the momentum equation is performed in Section 3. Finally, the limit passage is completed in Section 4. To conclude, let us remark that, in contrast with the bulk of the available homogenization literature almost exclusively focused on stationary problems, the evolutionary setting requires essential modifications of the limit process.

2 Preliminaries - uniform estimates

We start with uniform bounds for ϱ and Θ. Using hypothesis (1.17) we can take

$$b(\varrho) = [\varrho - \overline{\varrho}]^+, \ b(\varrho) = -[\varrho - \underline{\varrho}]^-$$

as test functions in the renormalized equation (1.14) to deduce

$$0 < \underline{\varrho} \leq \varrho_\varepsilon(t,x) \leq \overline{\varrho} \text{ for a.a. } (t,x) \tag{2.1}$$

uniformly in $\varepsilon \to 0$. Next, using the lower bound for (2.1) for ϱ, we we deduce from the energy inequality (1.13), combined with (1.18), that

$$\text{ess} \sup_{t\in[0,T]} \|\mathbf{u}_\varepsilon(t,\cdot)\|_{L^2(\Omega;R^3)} + \int_0^T \|\nabla_x \Theta_\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.2}$$

Note that $\mathbf{u}_\varepsilon \equiv 0$ outside Ω_ε.

Now, seeing that Θ_ε solves for a.a. fixed time the elliptic equation (1.12), with the diffusion coefficient

$$0 < \min\{\kappa_s, \kappa_f\} \leq \kappa \leq \max\{\kappa_s, \kappa_f\},$$

we may use the standard elliptic theory, see e.g. Ladyzhenskaya, Uralceva [11, Chapter 3, Theorem 12.1], to obtain the estimate

$$\text{ess} \sup_{t\in(0,T)} \|\Theta_\varepsilon(t,\cdot)\|_{C^\nu(\overline{\Omega})} \leq C \tag{2.3}$$

for a certain $\nu > 0$. It is important that the bound in (2.3) depends solely on κ_s, κ_f, and the constant in (2.2), specifically on the norm of the initial data. In particular,

$$-\overline{\Theta} \leq \Theta_\varepsilon(t,x) \leq \overline{\Theta} \text{ for a.a. } (t,x). \tag{2.4}$$

Going back to the energy balance (1.13) and using the positivity of μ on the range $[-\overline{\Theta}, \overline{\Theta}]$ we may infer that

$$\int_0^T \|\nabla_x \mathbf{u}_\varepsilon\|^2_{L^2(\Omega;R^3)} \leq C. \tag{2.5}$$

Now, using (2.1), (2.5), the renormalized equation (1.14) and hypothesis (1.17), we get

$$\begin{aligned} &\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T]; L^q(\Omega)) \text{ for any } 1 < q < \infty, \\ &\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T; L^2(\Omega;R^3)) \text{ and weakly in } L^2(0,T; W_0^{1,2}(\Omega;R^3)), \end{aligned} \tag{2.6}$$

passing to suitable subsequences as the case may be. In addition, the standard Aubin–Lions argument yields immediately that ϱ, $\mathbf{u}$ satisfy (1.10). Finally, by DiPerna–Lions theory [8], the same equation holds in the renormalized sense (1.14). In particular, it can be shown that

$$\|\varrho_\varepsilon\|^2_{L^2(\Omega)} \to \|\varrho\|^2_{L^2(\Omega)} \text{ in } C[0,T];$$

whence

$$\varrho_\varepsilon \to \varrho \text{ in } C([0,T];L^q(\Omega)) \text{ for any } 1 \le q < \infty. \tag{2.7}$$

3 Homogenization

We start with the elliptic problem associated to the momentum equation (1.2):

$$-\operatorname{div}_x\left[\mu(\Theta)\left(\nabla_x\mathbf{U}_\varepsilon + \nabla_x^t\mathbf{U}_\varepsilon\right)\right] + \nabla_x P_\varepsilon = \mathbf{f}_\varepsilon,\ \operatorname{div}_x\mathbf{U}_\varepsilon = 0 \text{ in } \Omega_\varepsilon,\ \mathbf{U}_\varepsilon|_{\partial\Omega_\varepsilon} = 0. \tag{3.1}$$

For a given $\Theta \in C^\nu(\overline{\Omega})$ and $\mathbf{f}_\varepsilon \in W^{-1,2}(\Omega_\varepsilon;R^3)$, problem (3.1) admits a weak solution $\mathbf{U}_\varepsilon$, P_ε, unique in the class

$$\mathbf{U}_\varepsilon \in W_0^{1,2}(\Omega_\varepsilon;R^3),\ P_\varepsilon \in L^2(\Omega_\varepsilon),\ \int_{\Omega_\varepsilon} P_\varepsilon \ \mathrm{d}x = 0, \tag{3.2}$$

such that the equations in (3.1) are satisfied in the weak sense: for any $\phi \in C_c^\infty(\Omega_\varepsilon)$ and any $\boldsymbol{\varphi} \in C_c^\infty(\Omega_\varepsilon;R^3)$, there holds

$$\int_{\Omega_\varepsilon} \mathbf{U}_\varepsilon \cdot \nabla_x\phi \ \mathrm{d}x = 0, \tag{3.3}$$

and

$$\int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x\mathbf{U}_\varepsilon + \nabla_x^t\mathbf{U}_\varepsilon\right) : \nabla_x\boldsymbol{\varphi} - P_\varepsilon \operatorname{div}_x\boldsymbol{\varphi} \ \mathrm{d}x = \langle \mathbf{f}_\varepsilon, \boldsymbol{\varphi}\rangle_{W_0^{1,2}(\Omega_\varepsilon)}. \tag{3.4}$$

We remark that the solution can be obtained as the minimizer of the functional

$$\mathbf{U} \mapsto \int_{R^3} \frac{1}{2}\mu(\Theta)|\nabla_x\mathbf{U} + \nabla_x^t\mathbf{U}|^2 \ \mathrm{d}x - \langle \mathbf{f}_\varepsilon \cdot \mathbf{U}\rangle$$

over the space of functions

$$\left\{\mathbf{U} \in W^{1,2}(R^3;R^3) \ \middle|\ \operatorname{div}_x\mathbf{U} = 0,\ \mathbf{U}|_{R^3\setminus\Omega_\varepsilon = 0}\right\}.$$

Our goal in this section is to show the following result.

Proposition 3.1. *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains satisfying the same hypotheses as Theorem 1.1. Suppose that*

$$\Theta \in C^\nu(\overline{\Omega}),\ \limsup_{\varepsilon\to 0} \|\mathbf{f}_\varepsilon - \mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon;R^3)} \le M \tag{3.5}$$

for some $\mathbf{f} \in W^{-1,2}(\Omega;R^3)$ independent of ε and some $M \ge 0$.

Let $\mathbf{U}_\varepsilon$, P_ε be the unique (weak) solution of problem (3.1). Then, up to the zero extension and a substraction of subsequence, there holds

$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } W_0^{1,2}(\Omega;R^3),\ P_\varepsilon \to P \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0, \tag{3.6}$$

where $\mathbf{U}$, P is the solution of the problem

$$-\mathrm{div}_x \left[\mu(\Theta)\left(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U}\right)\right] + \mu(\Theta)\mathbb{D}\mathbf{U} + \nabla_x P = \mathbf{f} + \mathbf{r},\ \mathrm{div}_x \mathbf{U} = 0 \text{ in } \Omega,\ \mathbf{U}|_{\partial\Omega} = 0 \tag{3.7}$$

for some $\mathbf{r} \in W_0^{-1,2}(\Omega;R^3)$ satisfying

$$\|\mathbf{r}\|_{W_0^{-1,2}(\Omega;R^3)} \le CM.$$

The rest of this section is devoted to proving Proposition 3.1. This is done in the following subsections step by step by employing similar arguments as in [9], where the main idea goes back to [6].

We recall the the following pointwise and integral estimates of the solution $(\mathbf{v}^i, q^i)$ to the model problem (1.15). The proof follows from the proof of Lemma 4.1 in [9].

Lemma 3.2. *Let $(\mathbf{v}^i, q^i)$ be a solution to (1.15) with $K \subset B(0,r) \subset B(0,d) \subset B(0,1)$. Then there holes the estimates*

$$\left|\partial^\alpha \mathbf{v}^i\right| \le C\frac{r}{|x|^{1+|\alpha|}}, \quad \left|q^i\right| \le C\frac{r}{|x|^2}, \quad \forall x \in B(0,1)\setminus B(0,r), \tag{3.8}$$

where $\alpha \in \mathbb{N}^3$, $|\alpha| \le 2$, and

$$\begin{aligned}&\int_{B(0,d)} |\mathbf{v}^i|^2 \,\mathrm{d}x \le Cr^2 d, \quad \int_{B(0,d)} |\nabla_x \mathbf{v}^i|^2 \,\mathrm{d}x \le Cr,\\ &\int_{B(0,d)} |q^i|^2 \,\mathrm{d}x \le Cr,\end{aligned} \tag{3.9}$$

3.1 Uniform estimates

Since $\Theta \in C^\nu(\overline{\Omega})$, it admit a lower and upper bound. By the assumption that $\mu(\Theta)$ is positive and continuous function in Θ, we have that for some positive constants $\underline{\mu}$ and $\bar{\mu}$,

$$0 < \underline{\mu} \le \mu(\Theta) \le \bar{\mu} < \infty. \tag{3.10}$$

By a density argument, the weak formulation (3.4) is satisfied for any $\boldsymbol{\varphi} \in W_0^{1,2}(\Omega_\varepsilon;R^3)$. By (3.2), we can take the solution $\mathbf{U}_\varepsilon$ itself to be a test function in (3.4) and obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} \mu(\Theta)\left|\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon\right|^2 \,\mathrm{d}x &= 2\langle \mathbf{f}_\varepsilon, \mathbf{U}_\varepsilon\rangle_{W_0^{1,2}(\Omega_\varepsilon)}\\ &\le 2\|\mathbf{f}_\varepsilon\|_{W^{-1,2}(\Omega_\varepsilon)}\|\mathbf{U}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)} \le 2(\|\mathbf{f}\| + M)\|\mathbf{U}_\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon)}.\end{aligned} \tag{3.11}$$

By (3.10)-(3.11), applying Korn's inequality and Poincaré's inequality gives

$$\begin{aligned}\|\mathbf{U}_\varepsilon\|^2_{W^{1,2}_0(\Omega_\varepsilon)} &\le C\|\nabla_x \mathbf{U}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le C\|\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \\ &\le C\int_{\Omega_\varepsilon} \mu(\Theta)\left|\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon\right|^2 \,\mathrm{d}x \le C(\|\mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon)} + M)\|\mathbf{U}_\varepsilon\|_{W^{1,2}_0(\Omega_\varepsilon)}.\end{aligned} \tag{3.12}$$

This implies the uniform estimate:

$$\sup_{0<\varepsilon<1} \|\mathbf{U}_\varepsilon\|_{W^{1,2}_0(\Omega_\varepsilon)} \le C(\|\mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon)} + M). \tag{3.13}$$

Theorem 2.3 in [7] applies to the setting of perforated domains in this paper. As a result, there exits a linear uniform bounded Bogovskii type operator

$$\mathcal{B}_\varepsilon : L^2_0(\Omega_\varepsilon) \to W^{1,2}_0(\Omega_\varepsilon; R^3),$$

such that for any $f \in L^2_0(\Omega_\varepsilon)$,

$$\mathrm{div}_x \mathcal{B}_\varepsilon(f) = f \text{ in } \Omega_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W^{1,2}_0(\Omega_\varepsilon;R^3)} \le C\,\|f\|_{L^2(\Omega_\varepsilon)}, \tag{3.14}$$

for some constant C independent of ε.

Since $P_\varepsilon \in L^2_0(\Omega_\varepsilon)$ which is the collection of $L^2(\Omega_\varepsilon)$ functions with zero mean value, we have $\mathcal{B}_\varepsilon(P_\varepsilon) \in W^{1,2}_0(\Omega_\varepsilon; R^3)$. Taking $\mathcal{B}_\varepsilon(P_\varepsilon)$ as a test function in the weak formulation (3.4) implies

$$\int_{\Omega_\varepsilon} |P_\varepsilon|^2 \,\mathrm{d}x = \int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon\right) : \nabla_x \mathcal{B}_\varepsilon(P_\varepsilon) \,\mathrm{d}x - \langle \mathbf{f}_\varepsilon, \mathcal{B}_\varepsilon(P_\varepsilon)\rangle_{W^{1,2}_0(\Omega_\varepsilon)}. \tag{3.15}$$

Together with (3.13) and (3.14), we obtain from (3.15) that

$$\|P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le C(\|\mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon)} + M)\|\mathcal{B}_\varepsilon(P_\varepsilon)\|_{W^{1,2}_0(\Omega_\varepsilon)} \le C(\|\mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon)} + M)\|P_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{3.16}$$

which implies

$$\sup_{0<\varepsilon<1} \|P_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C(\|\mathbf{f}\|_{W^{-1,2}(\Omega_\varepsilon)} + M). \tag{3.17}$$

Hence, by the uniform estimates in (3.13) and (3.17), up to the zero extensions and a substraction of subsequence, there holds

$$\mathbf{U}_\varepsilon \to \mathbf{U} \text{ weakly in } W^{1,2}_0(\Omega; R^3),\ P_\varepsilon \to P \text{ weakly in } L^2(\Omega) \text{ as } \varepsilon \to 0. \tag{3.18}$$

By the divergence free property of $\mathbf{U}_\varepsilon$, we have $\mathrm{div}_x \mathbf{U} = 0$. It is left to prove that the limit $(\mathbf{U}, P)$ solves the Brinkman type equations in (3.7). This is done in the next subsection.

3.2 Decompositions

Let χ be a function satisfying

$$\chi \in C_c^\infty(-1,1), \quad \chi = 1 \text{ on } \Big[-\frac{3}{4}, \frac{3}{4}\Big], \quad 0 \le \chi \le 1,\ 0 \le \chi' \le 4. \tag{3.19}$$

Define the cut-off function $\phi_{k,\varepsilon}$ near each hole $T_{k,\varepsilon}$ by

$$\phi_{k,\varepsilon}(x) := \chi\left(\frac{|x-x_k|}{\varepsilon^3}\right). \tag{3.20}$$

By the assumptions (1.6)–(1.7) of the distribution of holes, there holds

$$\phi_{k,\varepsilon}(x) = 1 \text{ on } T_{k,\varepsilon}, \quad \phi_{k,\varepsilon}\phi_{k',\varepsilon} = 0 \text{ whence } k \neq k'.$$

Let $\mathbf{v}_\varepsilon^i$ be the solution to the model problem (1.15) with $K = \overline{U}_{k,\varepsilon}$. For any $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^i)_{i=1,2,3} \in C_c^\infty(\Omega; R^3)$, we define $\boldsymbol{\varphi}^\varepsilon \in W_0^{1,2}(\Omega_\varepsilon; R^3)$ as

$$\boldsymbol{\varphi}^\varepsilon := \boldsymbol{\varphi} - \boldsymbol{\varphi}_1^\varepsilon - \boldsymbol{\varphi}_2^\varepsilon, \tag{3.21}$$

with

$$\begin{aligned} \boldsymbol{\varphi}_1^\varepsilon(x) &:= \sum_{k=1}^{K(\varepsilon)} \left(\boldsymbol{\varphi}(x) - \boldsymbol{\varphi}(x_k)\right)\phi_{k,\varepsilon}(x), \\ \boldsymbol{\varphi}_2^\varepsilon(x) &:= \sum_{k=1}^{K(\varepsilon)}\sum_{i=1}^{3} \boldsymbol{\varphi}^i(x_k)\mathbf{v}_\varepsilon^i(x-x_k)\phi_{k,\varepsilon}(x). \end{aligned} \tag{3.22}$$

Give the above definition, it is immediately to find

$$\boldsymbol{\varphi}^\varepsilon = 0, \quad \text{on } \bigcup_{k=1}^{K(\varepsilon)} T_{k,\varepsilon}. \tag{3.23}$$

So there does holds $\boldsymbol{\varphi}^\varepsilon \in W_0^{1,2}(\Omega_\varepsilon; R^3)$ with

$$\|\boldsymbol{\varphi}^\varepsilon\|_{W_0^{1,2}(\Omega_\varepsilon;R^3)} \le C\|\boldsymbol{\varphi}\|_{W_0^{1,2}(\Omega;R^3)}.$$

Moreover, similarly as Lemma 5.2 in [9], a direct calculation gives

$$\begin{aligned} &\boldsymbol{\varphi}_1^\varepsilon \to 0, \text{ strongly in } W_0^{1,2}(\Omega; R^3), \text{ as } \varepsilon \to 0, \\ &\boldsymbol{\varphi}_2^\varepsilon \to 0, \text{ weakly in } W_0^{1,2}(\Omega; R^3), \text{ as } \varepsilon \to 0. \end{aligned} \tag{3.24}$$

Let $(\mathbf{U}, P)$ be the limit we obtained in (3.18). For any given $\kappa > 0$, there exits $\mathbf{U}^0 \in C_c^\infty(\Omega; R^3)$ such that

$$\mathbf{U} = \mathbf{U}^0 + \mathbf{U}^\kappa, \quad \|\mathbf{U}^\kappa\|_{W^{1,2}(\Omega)} \le \kappa. \tag{3.25}$$

As in (3.21) and (3.22), we consider the decomposition

$$\mathbf{U}^0 = \mathbf{U}^0_\varepsilon + \mathbf{U}^0_{\varepsilon,1} + \mathbf{U}^0_{\varepsilon,2}, \tag{3.26}$$

where $\mathbf{U}^0_{\varepsilon,1}$ and $\mathbf{U}^0_{\varepsilon,2}$ are defined in the same manner as in (3.22) and satisfy the same convergence results as in (3.24).

We thus consider the decomposition of $\mathbf{U}_\varepsilon$ as

$$\mathbf{U}_\varepsilon = \mathbf{U}^0_\varepsilon + \mathbf{U}^r_\varepsilon, \tag{3.27}$$

where $\mathbf{U}^0_\varepsilon$ comes from the decomposition (3.26). It is crucial to study the property of the remainder $\mathbf{U}^r_\varepsilon = \mathbf{U}_\varepsilon - \mathbf{U}^0_\varepsilon \in W^{1,2}_0(\Omega_\varepsilon; R^3)$. Due to the fact

$$\mathbf{U}_\varepsilon \to \mathbf{U} = \mathbf{U}^0 + \mathbf{U}^\kappa,\ \mathbf{U}^0_\varepsilon \to \mathbf{U}^0,\ \text{weakly in } W^{1,2}_0(\Omega; R^3), \quad \text{as } \varepsilon \to 0,$$

it is straightforward to obtain that

$$\mathbf{U}^r_\varepsilon \to \mathbf{U} - \mathbf{U}^0 = \mathbf{U}^\kappa,\ \text{weakly in } W^{1,2}_0(\Omega; R^3), \quad \text{as } \varepsilon \to 0. \tag{3.28}$$

Then the Rellich-Kondrachov compact embedding theorem implies, up to a substraction of subsequence, that

$$\mathbf{U}^r_\varepsilon \to \mathbf{U}^\kappa,\ \text{strongly in } L^q(\Omega; R^3), \quad \forall q \in [2,6), \quad \text{as } \varepsilon \to 0. \tag{3.29}$$

A consequence is that

$$\limsup_{\varepsilon\to 0} \|\mathbf{U}^r_\varepsilon\|_{L^q(\Omega)} \le C(q)\|\mathbf{U}^\kappa\|_{L^q(\Omega)} \le C(q)\|\mathbf{U}^\kappa\|_{W^{1,2}_0(\Omega)} \le C(q)\kappa, \quad \forall q \in [2,6). \tag{3.30}$$

Moreover, it can be shown that

$$\limsup_{\varepsilon\to 0} \|\nabla \mathbf{U}^r_\varepsilon\|_{L^2(\Omega)} \le h(\kappa) \to 0,\ \text{as } \kappa \to 0. \tag{3.31}$$

In order to estimate $\|\nabla \mathbf{U}^r_\varepsilon\|_{L^2(\Omega)}$, choosing $\mathbf{U}^r_\varepsilon$ as a test function in the weak formulation (3.4) implies

$$\begin{aligned}
\int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}^r_\varepsilon + \nabla^t_x \mathbf{U}^r_\varepsilon\right) : \nabla_x \mathbf{U}^r_\varepsilon &= -\int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}^0_\varepsilon + \nabla^t_x \mathbf{U}^0_\varepsilon\right) : \nabla_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x \\
&\quad + \int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x + \langle \mathbf{f}_\varepsilon, \mathbf{U}^r_\varepsilon\rangle_{W^{1,2}_0(\Omega_\varepsilon)} \\
&= -\int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}^0 + \nabla^t_x \mathbf{U}^0\right) : \nabla_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x + \int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x + \langle \mathbf{f}_\varepsilon, \mathbf{U}^r_\varepsilon\rangle_{W^{1,2}_0(\Omega_\varepsilon)} \\
&\quad + \int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}^0_{\varepsilon,1} + \nabla^t_x \mathbf{U}^0_{\varepsilon,1}\right) : \nabla_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x + \int_{\Omega_\varepsilon} \mu(\Theta)\left(\nabla_x \mathbf{U}^0_{\varepsilon,2} + \nabla^t_x \mathbf{U}^0_{\varepsilon,2}\right) : \nabla_x \mathbf{U}^r_\varepsilon \,\mathrm{d}x.
\end{aligned} \tag{3.32}$$

Then starting from (3.32), by (3.28)–(3.30), together with the strong convergence of $\mathbf{U}^0_{\varepsilon,1}$ and weak convergence of $\mathbf{U}^0_{\varepsilon,2}$, by using the property of $\mathbf{v}^i_\varepsilon$ as the solution to the model problem (1.15) with $K = \overline{U}_{k,\varepsilon}$ (see Lemma 3.2), a similar argument as the proof of Lemma 5.1 in [9] implies our desired result in (3.31).

3.3 Limit equations

Now we deduce the equations satisfied by the limit couple $(\mathbf{U}, P)$. Let $\boldsymbol{\varphi} = (\boldsymbol{\varphi}^i)_{i=1,2,3} \in C_c^\infty(\Omega; R^3)$ and let $\boldsymbol{\varphi}^\varepsilon \in W_0^{1,2}(\Omega_\varepsilon; R^3)$ be defined as in (3.21). Employing the decomposition (3.27) and taking $\boldsymbol{\varphi}^\varepsilon$ as a test function in (3.4) gives

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x\big(\mathbf{U}^0 - \mathbf{U}^0_{\varepsilon,1} - \mathbf{U}^0_{\varepsilon,2}\big) + \nabla_x^t\big(\mathbf{U}^0 - \mathbf{U}^0_{\varepsilon,1} - \mathbf{U}^0_{\varepsilon,2}\big)\right) : \nabla_x \left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1^\varepsilon - \boldsymbol{\varphi}_2^\varepsilon\right) \mathrm{d}x \\
&+ \int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}^r_\varepsilon + \nabla_x^t \mathbf{U}^r_\varepsilon\right) : \nabla_x \boldsymbol{\varphi}^\varepsilon \ \mathrm{d}x = \int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1^\varepsilon - \boldsymbol{\varphi}_2^\varepsilon\right) \mathrm{d}x + \langle \mathbf{f}_\varepsilon, \boldsymbol{\varphi}^\varepsilon \rangle_{W_0^{1,2}(\Omega_\varepsilon)}.
\end{aligned} \tag{3.33}$$

We first look at the right-hand side of (3.33). By the convergence in (3.24) and the assumption (3.5), we have

$$\lim_{\varepsilon\to 0} \langle \mathbf{f}, \boldsymbol{\varphi}^\varepsilon \rangle_{W_0^{1,2}(\Omega_\varepsilon)} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{W_0^{1,2}(\Omega)}$$

and

$$\limsup_{\varepsilon\to 0} |\langle \mathbf{f}_\varepsilon - \mathbf{f}, \boldsymbol{\varphi}^\varepsilon \rangle_{W_0^{1,2}(\Omega_\varepsilon)}| \le CM \|\boldsymbol{\varphi}\|_{W_0^{1,2}(\Omega)}.$$

Thus

$$\lim_{\varepsilon\to 0} \langle \mathbf{f}_\varepsilon, \boldsymbol{\varphi}^\varepsilon \rangle_{W_0^{1,2}(\Omega_\varepsilon)} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle_{W_0^{1,2}(\Omega)} + \langle \mathbf{r}, \boldsymbol{\varphi} \rangle_{W_0^{1,2}(\Omega)}, \tag{3.34}$$

for some $\mathbf{r} \in W_0^{-1,2}(\Omega; R^3)$ satisfying

$$\|\mathbf{r}\|_{W_0^{-1,2}(\Omega; R^3)} \le CM.$$

By the weak convergence of P_ε in (3.18) and the strong convergence of $\boldsymbol{\varphi}_1^\varepsilon$ in (3.24), we have

$$\int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x \to \int_\Omega P \mathrm{div}_x \boldsymbol{\varphi} \ \mathrm{d}x, \ \int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \boldsymbol{\varphi}_1^\varepsilon \ \mathrm{d}x \to 0, \ \text{as } \varepsilon \to 0. \tag{3.35}$$

By the definition of $\boldsymbol{\varphi}_2^\varepsilon$ in (3.22), and the divergence free property of $\mathbf{v}_\varepsilon^i$, we have

$$\mathrm{div}_x \boldsymbol{\varphi}_2^\varepsilon(x) = \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \boldsymbol{\varphi}^i(x_k) \mathbf{v}_\varepsilon^i(x - x_k) \cdot \nabla_x \phi_{k,\varepsilon}(x).$$

Thus, by the property of $\mathbf{v}_\varepsilon^i$ shown in Lemma 3.2 and the property of the cut-off function $\phi_{k,\varepsilon}(x)$

in (3.19)–(3.20), we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x \boldsymbol{\varphi}_2^\varepsilon \,\mathrm{d}x &= \int_{\Omega_\varepsilon} P_\varepsilon \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \boldsymbol{\varphi}^i(x_k) \mathbf{v}_\varepsilon^i(x-x_k) \cdot \nabla_x \phi_{k,\varepsilon}(x) \,\mathrm{d}x \\
&= \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \int_{\{\frac{3}{4}\varepsilon^3 \le |x-x_k| \le \varepsilon^3\}} P_\varepsilon \boldsymbol{\varphi}^i(x_k) \mathbf{v}_\varepsilon^i(x-x_k) \cdot \nabla_x \phi_{k,\varepsilon}(x) \,\mathrm{d}x \\
&\le C\varepsilon^{-3} \int_{\bigcup_{k=1}^{K(\varepsilon)} \{\frac{3}{4}\varepsilon^3 \le |x-x_k| \le \varepsilon^3\}} |P_\varepsilon| \,\mathrm{d}x \\
&\le C \int_{\bigcup_{k=1}^{K(\varepsilon)} \{\frac{3}{4}\varepsilon^3 \le |x-x_k| \le \varepsilon^3\}} |P_\varepsilon|^2 \,\mathrm{d}x \\
&\to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned} \tag{3.36}
$$

We turn to consider the left-hand side of (3.33). By (3.31), we have

$$
\int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}_\varepsilon^r + \nabla_x^t \mathbf{U}_\varepsilon^r \right) : \nabla_x \boldsymbol{\varphi}^\varepsilon \,\mathrm{d}x \le Ch(\kappa), \tag{3.37}
$$

which tends to 0 when $\kappa \to 0$.

By the strong convergence of $\boldsymbol{\varphi}_1^\varepsilon$ and weak convergence of $\boldsymbol{\varphi}_2^\varepsilon$ in (3.24) and similar convergence for $\mathbf{U}_{\varepsilon,1}^0$ and $\mathbf{U}_{\varepsilon,2}^0$, there holds

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \big(\mathbf{U}^0 - \mathbf{U}_{\varepsilon,1}^0 - \mathbf{U}_{\varepsilon,2}^0 \big) + \nabla_x^t \big(\mathbf{U}^0 - \mathbf{U}_{\varepsilon,1}^0 - \mathbf{U}_{\varepsilon,2}^0 \big) \right) : \nabla_x \left(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1^\varepsilon \right) \mathrm{d}x \\
&\to \int_{\Omega} \mu(\Theta) \left(\nabla_x \mathbf{U}^0 + \nabla_x^t \mathbf{U}^0 \right) : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}x
\end{aligned} \tag{3.38}
$$

and

$$
\int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \big(\mathbf{U}^0 - \mathbf{U}_{\varepsilon,1}^0 \big) + \nabla_x^t \big(\mathbf{U}^0 - \mathbf{U}_{\varepsilon,1}^0 \big) \right) : \nabla_x \boldsymbol{\varphi}_2^\varepsilon \,\mathrm{d}x \to 0 \tag{3.39}
$$

as $\varepsilon \to 0$.

It is left to study the limit

$$
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}_{\varepsilon,2}^0 + \nabla_x^t \mathbf{U}_{\varepsilon,2}^0 \right) : \nabla_x \boldsymbol{\varphi}_2^\varepsilon \,\mathrm{d}x. \tag{3.40}
$$

By the definition in (3.22), $\nabla_x \boldsymbol{\psi}^\varepsilon$, where $\boldsymbol{\psi}^\varepsilon \in \{\mathbf{U}_{\varepsilon,2}^0, \boldsymbol{\varphi}_2^\varepsilon\}$, has two parts:

$$
\nabla_x \boldsymbol{\psi}^\varepsilon = \boldsymbol{\psi}^{\varepsilon,1} + \boldsymbol{\psi}^{\varepsilon,2}, \tag{3.41}
$$

where

$$\boldsymbol{\psi}^{\varepsilon,1} := \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \boldsymbol{\psi}^i(x_k) \mathbf{v}_\varepsilon^i(x - x_k) \nabla_x \phi_{k,\varepsilon}(x), \quad \boldsymbol{\psi}^{\varepsilon,2} := \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \boldsymbol{\psi}^i(x_k) \nabla_x \mathbf{v}_\varepsilon^i(x - x_k) \phi_{k,\varepsilon}(x). \tag{3.42}$$

By using (3.41)–(3.42), we then can write (3.40) into four parts, and by a similarly argument as (3.36), any part involves $\boldsymbol{\psi}^{\varepsilon,2}$ convergence to 0 as $\varepsilon \to 0$. Thus

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}^0_{\varepsilon,2} + \nabla_x^t \mathbf{U}^0_{\varepsilon,2} \right) : \nabla_x \boldsymbol{\varphi}_2^\varepsilon \, \mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \sum_{k=1}^{K(\varepsilon)} \sum_{i=1}^{3} \mathbf{U}^{0,i}(x_k) \big(\nabla_x \mathbf{v}_\varepsilon^i + \nabla_x^t \mathbf{v}_\varepsilon^i \big)(x - x_k) \phi_{k,\varepsilon}(x) : \sum_{l=1}^{K(\varepsilon)} \sum_{j=1}^{3} \boldsymbol{\varphi}^j(x_l) \nabla_x \mathbf{v}_\varepsilon^j(x - x_l) \phi_{l,\varepsilon}(x) \, \mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \sum_{k=1}^{K(\varepsilon)} \sum_{i,j=1}^{3} \mathbf{U}^{0,i}(x_k) \big(\nabla_x \mathbf{v}_\varepsilon^i + \nabla_x^t \mathbf{v}_\varepsilon^i \big)(x - x_k) \phi_{k,\varepsilon}(x) : \boldsymbol{\varphi}^j(x_k) \nabla_x \mathbf{v}_\varepsilon^j(x - x_k) \phi_{k,\varepsilon}(x) \, \mathrm{d}x.
\end{aligned} \tag{3.43}$$

Again by a similarly argument as (3.36), and by the divergence free property of $\mathbf{v}_\varepsilon^j$, we deduce from (3.43) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \sum_{k=1}^{K(\varepsilon)} \sum_{i,j=1}^{3} \mathbf{U}^{0,i}(x_k) \nabla_x^t \mathbf{v}_\varepsilon^i(x - x_k) \phi_{k,\varepsilon}(x) : \boldsymbol{\varphi}^j(x_k) \nabla_x \mathbf{v}_\varepsilon^j(x - x_k) \phi_{k,\varepsilon}(x) \, \mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \sum_{k=1}^{K(\varepsilon)} \sum_{i,j=1}^{3} \mathbf{U}^{0,i}(x_k) \mathbf{v}_\varepsilon^i(x - x_k) \phi_{k,\varepsilon}(x) \cdot \boldsymbol{\varphi}^j(x_k) \nabla_x \mathbf{v}_\varepsilon^j(x - x_k) \nabla_x^t \phi_{k,\varepsilon}(x) \, \mathrm{d}x \\
&= 0.
\end{aligned} \tag{3.44}$$

By (3.43)–(3.44), by the definition of the cut-off function $\phi_{k,\varepsilon}$ in (3.19)–(3.20), there holds

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}^0_{\varepsilon,2} + \nabla_x^t \mathbf{U}^0_{\varepsilon,2} \right) : \nabla_x \boldsymbol{\varphi}_2^\varepsilon \, \mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \mu(\Theta) \sum_{k=1}^{K(\varepsilon)} \sum_{i,j=1}^{3} \mathbf{U}^{0,i}(x_k) \boldsymbol{\varphi}^j(x_k) \int_{|x-x_k|\le\varepsilon^3} \phi_{k,\varepsilon}^2(x) \nabla_x \mathbf{v}_\varepsilon^i : \nabla_x \mathbf{v}_\varepsilon^j(x - x_k) \, \mathrm{d}x.
\end{aligned} \tag{3.45}$$

Hence, by the assumption (1.16), together with Lemma 3.2, we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \mu(\Theta) \left(\nabla_x \mathbf{U}^0_{\varepsilon,2} + \nabla_x^t \mathbf{U}^0_{\varepsilon,2} \right) : \nabla_x \boldsymbol{\varphi}_2^\varepsilon \, \mathrm{d}x &= \sum_{i,j=1}^{3} \int_\Omega \mu(\Theta) \mathbb{D}_{i,j}(x) \mathbf{U}^{0,i}(x) \boldsymbol{\varphi}^j(x) \, \mathrm{d}x \\
&= \int_\Omega \mu(\Theta) \mathbb{D} \mathbf{U}^0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x.
\end{aligned} \tag{3.46}$$

The final step is to pass $\kappa \to 0$. Recall the fact $\|\mathbf{U} - \mathbf{U}^0\|_{W_0^{1,2}(\Omega)} \leq \kappa$. Then, by summarizing the limits in (3.34), (3.35), (3.36), (3.37), (3.38), (3.39) and (3.46) and by passing $\kappa \to 0$, we deduce that

$$\int_\Omega \mu(\Theta) \left(\nabla_x \mathbf{U} + \nabla_x^t \mathbf{U} \right) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x + \int_\Omega \mu(\Theta) \mathbb{D} \mathbf{U} \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_\Omega P \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x + \langle \mathbf{f} + \mathbf{r}, \boldsymbol{\varphi} \rangle_{W_0^{1,2}(\Omega)}.$$

The proof of Proposition 3.1 is completed.

4 Asymptotic limit

Our ultimate goal is to perform the asymptotic limit in the *evolutionary* system (1.1–1.3).

4.1 Compactness in time of the velocities

We start by showing compactness in time of the family $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of the velocity fields. Let $\boldsymbol{\varphi} \in C_c^\infty(\Omega; R^3)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$. Set

$$\mathbf{f} = -\Delta \boldsymbol{\varphi} + \mathbb{D} \cdot \boldsymbol{\varphi} \in L^\infty(\Omega; R^3).$$

Let $\boldsymbol{\varphi}_\varepsilon$ be the unique solution of the Stokes problem

$$-\Delta \boldsymbol{\varphi}_\varepsilon + \nabla_x P_\varepsilon = \mathbf{f} \ , \ \mathrm{div}_x \boldsymbol{\varphi} = 0 \text{ in } \Omega_\varepsilon, \ \boldsymbol{\varphi} \in W_0^{1,2}(\Omega_\varepsilon; R^3).$$

In accordance with Proposition 3.1,

$$\boldsymbol{\varphi}_\varepsilon \to \boldsymbol{\varphi} \text{ weakly in } W_0^{1,2}(\Omega; R^3); \text{ whence } \boldsymbol{\varphi}_\varepsilon \to \boldsymbol{\varphi} \text{ in } L^2(\Omega; R^3).$$

Now, we have

$$\int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi} \ \mathrm{d}x = \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon) \ \mathrm{d}x = \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x,$$

where, by virtue of the bounds (2.1), (2.2),

$$\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_\varepsilon) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0. \tag{4.1}$$

In addition, using $\psi(t)\boldsymbol{\varphi}_\varepsilon$, $\psi \in C_c^\infty(0,T)$ as a test function in the variational formulation of the momentum balance (1.11), we deduce that the family

$$\left\{ t \mapsto \int_{\Omega_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi}_\varepsilon \ \mathrm{d}x \right\}_{\varepsilon>0} \text{ is precompact in } C([0,T]). \tag{4.2}$$

Combining (4.1), (4.2), we conclude that

$$\left[t \mapsto \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right] \to \left[t \mapsto \int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right] \text{ in } L^\infty(0,T) \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^3), \ \mathrm{div}_x \boldsymbol{\varphi} = 0. \tag{4.3}$$

Using the density of smooth compactly supported functions in $W^{1,2}_{0,{\rm div}}(\Omega;R^3)$ - the Sobolev space $W^{1,2}_0(\Omega;R^3)$ of solenoidal vector fields - we deduce from (4.3) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \text{ in } L^q([0,T];W^{-1,2}_{\rm div}(\Omega;R^3)). \tag{4.4}$$

Thus, finally, relation (4.4), together with (2.6), (2.7), imply that

$$\int_0^T \int_\Omega \varrho |\mathbf{u}_\varepsilon|^2 \ {\rm d}x \to \int_0^T \int_\Omega \varrho |\mathbf{u}|^2 \ {\rm d}x,$$

yielding

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2((0,T)\times\Omega;R^3). \tag{4.5}$$

4.2 Strong convergence of the temperature

In view of (2.2), we may assume

$$\Theta_\varepsilon \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}_0(\Omega)) \text{ and weakly-(*) in } L^\infty((0,T)\times\Omega),$$

passing to a subsequence as the case may be. Moreover, by virtue of (4.5), the limit Θ solves (1.3) in the sense of (1.12), with $\kappa_\varepsilon = \kappa_f$. Note that

$$\kappa_\varepsilon \to \kappa_f \text{ weakly-(*) in } L^\infty(\Omega) \text{ and in } L^1(\Omega). \tag{4.6}$$

As a matter of fact, Θ being a solution of the limit problem with *constant* heat conductivity coefficient enjoys more regularity than Θ_ε, specifically,

$$\Theta \in L^\infty(0,T;W^{2,2}\cap W^{1,2}_0(\Omega)). \tag{4.7}$$

Finally, writing

$$\begin{aligned}\|\Theta_\varepsilon - \Theta\|^2_{W^{1,2}_0(\Omega)} \leq C\int_\Omega \kappa_\varepsilon |\nabla_x\Theta_\varepsilon - \nabla_x\Theta|^2 \ {\rm d}x &\leq C\int_\Omega (\kappa_\varepsilon\nabla_x\Theta_\varepsilon - \kappa_f\nabla_x\Theta)\cdot(\nabla_x\Theta_\varepsilon - \nabla_x\Theta) \ {\rm d}x \\ &+ C\int_\Omega (\kappa_f - \kappa_\varepsilon)\nabla_x\Theta\cdot(\nabla_x\Theta_\varepsilon - \nabla_x\Theta) \ {\rm d}x \\ &= C\int_\Omega (\mathbf{u}_\varepsilon - \mathbf{u})\cdot\nabla_x F(\Theta_\varepsilon - \Theta) \ {\rm d}x + C\int_\Omega (\kappa_f - \kappa_\varepsilon)\nabla_x\Theta\cdot(\nabla_x\Theta_\varepsilon - \nabla_x\Theta) \ {\rm d}x\end{aligned}$$

we deduce from (4.5–4.7) that

$$\Theta_\varepsilon \to \Theta \text{ in } L^2(0,T;W^{1,2}_0(\Omega)). \tag{4.8}$$

Relation (4.8), together with (2.3), yields the final conclusion

$$\Theta_\varepsilon \to \Theta \text{ in } L^q(0,T;C^\nu(\overline{\Omega})) \text{ for all } 1\leq q<\infty \text{ and some } \nu>0. \tag{4.9}$$

Note that ν in (4.9) is strictly smaller than its companion in (2.3).

4.3 Asymptotic limit in the momentum equation

Our ultimate goal is to perform the asymptotic limit in the momentum equation (1.11). To this end, we use the time regularization by means of a convolution with a family of regularization kernels $\chi_\delta = \chi_\delta(t)$,

$$\chi_\delta(t) = \frac{1}{\delta}\chi\left(\frac{t}{\delta}\right),\ \chi \in C^\infty_c(-1,1),\ \chi \geq 0,\ \chi(-z) = \chi(z),\ \chi'(z) \leq 0 \text{ for } z \geq 0,\ \int_{-1}^{1} \chi(z)\ \mathrm{d}z = 1.$$

As we are interested only in the behavior of $\mathbf{u}_\varepsilon$ on compact subsets of $(0,T) \times \Omega_\varepsilon$, this step can be performed rigorously by considering $\chi_\delta(\tau - t)\phi(x)$, $\phi \in C^\infty_c(\Omega; R^3)$, $\mathrm{div}_x \phi = 0$ as a test function in the weak formulation (1.11). Denoting $[v]_\delta = \chi_\delta * v$ we get

$$\int_{\Omega_\varepsilon} \left[\mu(\Theta_\varepsilon)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta : \nabla_x \phi\ \mathrm{d}x = \int_{\Omega_\varepsilon} \left[(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)\right]_\delta : \nabla_x \phi\ \mathrm{d}x - \int_{\Omega_\varepsilon} \partial_t [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta \cdot \phi\ \mathrm{d}x. \tag{4.10}$$

at any fixed $\tau \in (\delta, T-\delta)$.

Now, in view of (2.6), (2.7), and (4.5), it is easy to show that

$$[\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon(\tau,\cdot)]_\delta \to [\varrho \mathbf{u} \otimes \mathbf{u}(\tau,\cdot)]_\delta \text{ in } L^2(\Omega; R^{3\times 3}) \text{ as } \varepsilon \to 0, \text{ for any } \tau \in (\delta, T-\delta), \tag{4.11}$$

and, similarly,

$$\partial_t[\varrho_\varepsilon \mathbf{u}_\varepsilon(\tau,\cdot)]_\delta \to \partial_t[\varrho \mathbf{u}(\tau,\cdot)]_\delta \text{ in } L^2(\Omega; R^3) \text{ as } \varepsilon \to 0 \text{ for any } \tau \in (\delta, T-\delta). \tag{4.12}$$

Using (4.11), (4.12), we obtain the desired conclusion from (4.9) by application of Proposition 3.1 as soon as we show a suitable estimate for the "commutator"

$$\left[\mu(\Theta_\varepsilon)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta - \mu([\Theta]_\omega)\left(\nabla_x [\mathbf{u}_\varepsilon]_\delta + \nabla_x^t [\mathbf{u}_\varepsilon]_\delta\right),$$

where $\delta > 0$, $\omega > 0$ are small parameters.

To begin, by virtue of (2.2), (4.9), observe that

$$\left[\mu(\Theta_\varepsilon)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta - \left[\mu(\Theta)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta \to 0 \text{ in } L^2(\Omega; R^{3\times 3}) \text{ as } \varepsilon \to 0$$
$$\text{uniformly for } \tau \in (\delta, T-\delta);$$

whence it is enough to control

$$\left[\mu(\Theta)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta - \mu([\Theta]_\omega)\left(\nabla_x [\mathbf{u}_\varepsilon]_\delta + \nabla_x^t [\mathbf{u}_\varepsilon]_\delta\right). \tag{4.13}$$

To handle (4.13), we write

$$\begin{aligned}
&\left[\mu(\Theta)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta - \mu([\Theta]_\omega)\left(\nabla_x [\mathbf{u}_\varepsilon]_\delta + \nabla_x^t [\mathbf{u}_\varepsilon]_\delta\right) \\
&= \left[(\mu(\Theta) - \mu([\Theta]_\omega)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta + \left[\mu([\Theta]_\omega)\left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\right)\right]_\delta \\
&\quad - \mu([\Theta]_\omega)\left(\nabla_x [\mathbf{u}_\varepsilon]_\delta + \nabla_x^t [\mathbf{u}_\varepsilon]_\delta\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\| \left[(\mu(\Theta) - \mu([\Theta]_\omega) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right) \right]_\delta \right\|_{L^2(\Omega;R^{3\times3})} \\
&\leq \left\| (\mu(\Theta) - \mu([\Theta]_\omega) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right) \right\|_{L^2(\Omega;R^{3\times3})} \\
&\leq \| \mu(\Theta) - \mu([\Theta]_\omega) \|_{L^\infty(\Omega)} \left\| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right\|_{L^2(\Omega;R^{3\times3})} .
\end{aligned}$$

Thus, in view of (2.2), (4.9),

$$\left[(\mu(\Theta) - \mu([\Theta]_\omega) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right) \right]_\delta \to 0 \text{ in } L^q(0,T;L^2(\Omega;R^{3\times3})) \to 0 \text{ as } \omega \to 0 \tag{4.14}$$

for any $1 \leq q < 2$, uniformly in ε and δ.

On the other hand, if $\omega > 0$ is fixed, the function $[\Theta]_\omega$ is continuously differentiable with respect to the spatial variable. Thus we deduce that

$$\begin{aligned}
&\left[\mu([\Theta]_\omega) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right) \right]_\delta - \mu([\Theta]_\omega) \left(\nabla_x \left[\mathbf{u}_\varepsilon \right]_\delta + \nabla_x^t \left[\mathbf{u}_\varepsilon \right]_\delta \right) \\
&= \left[\mu([\Theta]_\omega) \left(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right) \right]_\delta - \mu([\Theta]_\omega) \left[\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \right]_\delta \\
&\to 0 \text{ in } L^2((0,T) \times \Omega; R^{3\times3}) \text{ as } \delta \to 0
\end{aligned} \tag{4.15}$$

uniformly in ε for any fixed $\omega > 0$.

Summing up relations (4.14), (4.15), we may rewrite (4.10) in the form

$$\begin{aligned}
\int_{\Omega_\varepsilon} \mu([\Theta]_\omega) \left(\nabla_x [\mathbf{u}_\varepsilon]_\delta + \nabla_x^t [\mathbf{u}_\varepsilon]_\delta \right) : \nabla_x \phi \, \mathrm{d}x = \int_{\Omega_\varepsilon} [(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)]_\delta : \nabla_x \phi \, \mathrm{d}x - \int_{\Omega_\varepsilon} \partial_t [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta \cdot \phi \, \mathrm{d}x \\
+ \int_{\Omega_\varepsilon} \left(\mathbb{R}^1_{\omega,\delta,\varepsilon} + \mathbb{R}^2_{\omega,\delta,\varepsilon} \right) : \nabla_x \phi \, \mathrm{d}x
\end{aligned}$$

at any fixed $\tau \in (\delta, T - \delta)$, where

$$\begin{aligned}
&\mathbb{R}^1_{\omega,\delta,\varepsilon} \to 0 \text{ in } L^q(0,T;L^2(\Omega;R^{3\times3})) \text{ as } \omega \to 0,\ 1 \leq q < 2, \text{ uniformly in } \varepsilon, \delta, \\
&\mathbb{R}^2_{\omega,\delta,\varepsilon} \to 0 \text{ in } L^2(0,T;L^2(\Omega;R^{3\times3})) \text{ as } \delta \to 0 \text{ for any fixed } \omega > 0 \text{ and uniformly in } \varepsilon.
\end{aligned}$$

Thus performing successively the limits $\varepsilon \to 0$, $\delta \to 0$, and, finally, $\omega \to 0$, we deduce the desired conclusion. Theorem 1.1 has been proved.

References

[1] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes. *Arch. Rational Mech. Anal.*, 113(3), 209–259, 1990.

[2] G. Allaire. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. II. Noncritical sizes of the holes for a volume distribution and a surface distribution of holes. *Arch. Rational Mech. Anal.*, 113(3), 261–298, 1990.

[3] S. Chandrasekhar. *Hydrodynamic and hydrodynamic stability*. Clarendon Press, Oxford, 1961.

[4] D. Cioranescu and F. Murat. Un terme étrange venu d'ailleurs. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. II (Paris, 1979/1980)*, volume 60 of *Res. Notes in Math.*, pages 98–138, 389–390. Pitman, Boston, Mass., 1982.

[5] Doïna Cioranescu and François Murat. Un terme étrange venu d'ailleurs. II. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981)*, volume 70 of *Res. Notes in Math.*, pages 154–178, 425–426. Pitman, Boston, Mass., 1982.

[6] L. Desvillettes, F. Golse, and V. Ricci. The meanfield limit for solid particles in a Navier–Stokes flow. *J. Stat. Physics*, **131**, 941–967, 2008.

[7] L. Diening, E. Feireisl, Y. Lu. The inverse of the divergence operator on perforated domains with applications to homogenization problems for the compressible Navier–Stokes system. *ESAIM: Control Optim. Calc. Var.,* **23** (3) (2017), 851-868.

[8] R.J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, **98**:511–547, 1989.

[9] E. Feireisl, Y. Namlyeyeva, Šárka Nečasová. Homogenization of the evolutionary Navier–Stokes system. Manuscripta Math. 149, 251–27 (2016)

[10] E. Feireisl and A. Novotný. *Singular limits in thermodynamics of viscous fluids*. Birkhäuser-Verlag, Basel, 2009.

[11] O. A. Ladyzhenskaya and N. N. Uralceva. *Linear and quasilinear elliptic equations*. Academic Press, New York and London, 1968.

[12] F. Lignières. The small Péclet number approximation in stellar radiative zones. *Astronomy and astrophysics*, pages 1–10, 1999.

[13] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.1, Incompressible models*. Oxford Science Publication, Oxford, 1998.

[14] V. A. Marchenko and E. Ya. Khruslov. *Homogenization of partial differential equations*, volume 46 of *Progress in Mathematical Physics*. Birkhäuser Boston, Inc., Boston, MA, 2006. Translated from the 2005 Russian original by M. Goncharenko and D. Shepelsky.