Rate of approximation by logarithmic derivatives of polynomials whose zeros lie on a circle[1]

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Abstract We obtain an estimate for uniform approximation rate of bounded analytic in the unit disk functions by logarithmic derivatives of C-polynomials, i.e., polynomials, all of whose zeros lie on the unit circle $C : |z| = 1$.

Keywords logarithmic derivatives of polynomials, simple partial fractions, C-polynomials, uniform approximation

Mathematics Subject Classification 41A25, 41A20, 41A29, 30E10

1. Let C denote the unit circle $|z| = 1$ and D denote the unit disk $|z| < 1$. It's proved in [12] for any bounded analytic in D function f and in [11] for any analytic in D function, that there is a sequence of rational functions $S_n(z) = S_n(f; z)$ of the form

$$S_n(z) = \sum_{k=1}^{m_n} (z - z_{n,k})^{-1}, \qquad |z_{n,k}| = 1,$$

which converges to $f(z)$ uniformly on every closed subset of D. Obviously, S_n is a logarithmic derivative of C-polynomial $(z - z_{n,1}) \dots (z - z_{n,m_n})$ (C-*polynomials* defined in abstract). Analogous problems with more general constraints on poles (for example, if $z_{n,k}$ belong to rectifiable Jordan curve) investigated in [6, 1, 2]. Approximation by sums $\sum_{k=1}^{n} (z - z_k)^{-1}$ with *free* poles studied in [3, 7, 5] (see also bibliography in [5]).

We study a rate of approximation of bounded analytic in D functions by logarithmic derivatives of C-polynomials on closed subsets of D.

Theorem. *For any bounded analytic in D function $f(z)$ there is a sequence of C-polynomials $P_N(z)$, $N \geq N_0$, such that* $\deg P_N(z) = N$ *and*

$$\left| \frac{P_N'(z)}{P_N(z)} - f(z) \right| < \frac{(a + \varepsilon)^{n+1}}{\varepsilon(1 - a - \varepsilon)} (1 + o(1)), \qquad n = [N/2], \qquad \text{as} \quad N \to \infty \tag{1}$$

in any disk $K_a = \{|z| \leq a\}$ $(a < 1)$ for every $\varepsilon \in (0, 1 - a)$.

Let $d_n(f, K_a)$ denote the error in best approximation to f on the disk K_a by logarithmic derivatives Q'/Q of polynomials Q of degree at most n with *free* zeros. It's interesting, that, generally speaking, the convergence $d_n(f, K_a)$ is also geometric (sf. (1)):

$$\limsup_{n\to\infty} \sqrt[n]{d_n(f, K_a)} \leq a.$$

This estimate follows from [7] (see also [3]), where the analog of polynomial Walsh's theorem was proved for the problem of approximation by such fractions Q'/Q.

2. To construct polynomials $P_N(z)$ we use the approach [10]. We need the next lemma, stated in [10] (for the case $m = 0$ see [8, p. 108]). Further $\overline{D} = D + C$.

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Lemma [10]. *Let* $Q(z) = a_0(z-z_1)\dots(z-z_q)$, $a_0 \neq 0$, *be a polynomial of degree* $q \geq 1$ *and* $Q^*(z) := z^q\overline{Q}(1/z) = \overline{a}_0(1-\overline{z}_1 z)\dots(1-\overline{z}_q z)$. *If* $Q(z)$ *zero free in* $\overline{D}$, *then* $Q(z)+z^mQ^*(z)$ *is C-polynomial for every* $m = 0, 1, 2, \dots,$ *and* $|Q^*(z)| \leq |Q(z)|$ *in* $\overline{D}$.

To prove this lemma it is sufficient to consider the equation

$$\Phi(z) = -a_0/\overline{a}_0, \qquad \Phi(z) := \frac{a_0}{\overline{a}_0}\frac{z^mQ^*(z)}{Q(z)} \equiv z^m \prod_{j=1}^{q} \frac{1-\overline{z}_j z}{z-z_j}, \qquad |z_j| > 1. \tag{2}$$

Absolute values of all factors in last product are less, equal or more than 1 iff $|z| < 1$, $|z| = 1$ or $|z| > 1$, respectively, therefore all roots of equation (2) (and so, all zeros of $Q(z) + z^mQ^*(z)$) lie on C, and in the disk $\overline{D}$ we have $|\Phi(z)| \leq 1$ and $|Q^*(z)| \leq |Q(z)|$.

Remark 1. If $Q(z) \equiv a_0 = \text{const} \neq 0$, then $Q^*(z) \equiv \overline{a}_0$, consequently, $Q(z) + z^mQ^*(z) \equiv a_0 + z^m\overline{a}_0$ is C-polynomial for $m > 0$ only.

Remark 2. Lemma is also true, if zeros of $Q(z)$ (but not all of them) lie on C, because $\overline{z}_j = 1/z_j$ and $|1-\overline{z}_j z|/|z-z_j| \equiv 1$ as $|z_j| = 1$. But if $Q(z)$ is C-polynomial, then $Q^*(z) \equiv tQ(z)$, $t = \overline{Q(0)}/a_0$, and we need again to assume $m > 0$, if $1 + t = 0$.

3. *Proof of the theorem.* Set $g(z) = \exp\left(\int_0^z f(\zeta)d\zeta\right)$,

$$g(z) = s_n(z) + R_n(z), \qquad s_n(z) = 1 + \sum_{k=1}^{n} g_k z^k, \qquad R_n(z) = \sum_{k=n+1}^{\infty} g_k z^k.$$

Derivative $g' \equiv gf$ is bounded and analytic in D. In particular, g' belongs to the Hardy space $H^1(D)$, hence the series $\sum |g_k|$ converges [4, Theorem 15].

Function $\varphi(x,y) = \operatorname{Re}\int_0^z f(\zeta)d\zeta$ is bounded in D, so $\varphi > -\infty$ and $\inf_D |g(z)| = \inf_D \exp\varphi(x,y) = M_0 > 0$. Choose $n_0 \in \mathbb{N}$, such that $\sum_{n+1}^{\infty} |g_k| \leq M_0 - M_0/2$ as $n \geq n_0$. Hence polynomials $s_n(z)$ are zero free in $\overline{D}$ as $n \geq n_0$.

Further $N \geq 2n_0$ and $n := [N/2] \geq n_0$. Set $p(z) = s_n^*(z) = z^q\overline{s}_n(1/z)$, where $q = \deg s_n(z)$, $0 \leq q \leq n$. By lemma and remark 1 we have $|p(z)| \leq |s_n(z)|$ in $\overline{D}$, and sums

$$P(z) = P_{q+m}(z) := s_n(z) + z^mp(z) \qquad \text{as} \quad m = 1, 2, \dots$$

are C-polynomials. Rewrite P as $P(z) \equiv g(z) + z^mp(z) - R_n(z)$. We now have

$$P'(z) = g(z)f(z) + mz^{m-1}p(z) + z^mp'(z) - R_n'(z),$$

$$\frac{P'(z)}{P(z)} - f(z) = \frac{z^{m-1}(m - zf(z))p(z) + z^mp'(z) - R_n'(z) + f(z)R_n(z)}{P(z)}. \tag{3}$$

Denote $M_1 = \max\{1, |g_1|, \dots, |g_n|\}$. Since $|g_k| < M_0$ (as $k \geq n_0 + 1$), we have

$$|p(z)| \leq |s_n(z)| < M_1/(1-a), \qquad |R_n(z)| < M_0a^{n+1}/(1-a) \qquad \text{as} \quad |z| \leq a, \tag{4}$$

$$|P(z)| \geq |g(z)| - |z^mp(z) - R_n(z)| > M_0 - (M_1a^m + M_0a^{n+1})/(1-a) \qquad \text{as} \quad |z| \leq a. \tag{5}$$

If $|z| < r < 1$ and function F is analytic in D, then

$$2\pi|F'(z)| = \left|\int_{|\zeta|=r} \frac{F(\zeta)d\zeta}{(\zeta-z)^2}\right| \leq \max_{|\zeta|=r}|F(\zeta)| \int_{|\zeta|=r} \frac{|d\zeta|}{|\zeta-z|^2} = \max_{|\zeta|=r}|F(\zeta)| \frac{2\pi r}{r^2-|z|^2}$$

(we apply Poisson's integral), and hence if $r = a + \varepsilon < 1$, then

$$|F'(z)| < \varepsilon^{-1} \max_{|\zeta|=a+\varepsilon} |F(\zeta)| \qquad \text{as} \quad |z| \le a.$$

Thus, by this and (4) we have

$$|p'(z)| < \frac{M_1}{\varepsilon(1-a-\varepsilon)}, \qquad |R_n'(z)| < \frac{M_0(a+\varepsilon)^{n+1}}{\varepsilon(1-a-\varepsilon)} \qquad \text{as} \quad |z| \le a. \tag{6}$$

We put $m = N - q \ge n$ and obtain (1) from (3)—(6), and theorem follows.

Remark 3. $P_N'(z)/P_N(z) - f(z) = O(z^l)$ as $l \ge n-1$ (see (3)).

Remark 4. It follows from $g' \in H^1(D)$, that $g(z)$ continuous in $\overline{D}$ and absolutely continuous on C [9, Ch.II, §5(5.7)]. In particular, $g(e^{i\theta})$ has bounded variation and $|g_k| = O(1/k)$, so we can to write $O(1/n)$ instead of $1 + o(1)$ in (1). If g is a zero free in $\overline{D}$ polynomial of degree $q \ge 0$ and $f = g'/g$, then $R_n(z) \equiv 0$ as $n \ge q$, and approximation rate is higher. For example, if $f(z) \equiv 0$, then $P_N(z) = 1 + z^N$ and $\sup_{K_a} |P_N'/P_N - f| = Na^{N-1}/(1-a^N)$.

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