SHEAF QUANTIZATION OF LEGENDRIAN ISOTOPY

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ABSTRACT. Let $\{\Lambda_t^\infty\}$ be an isotopy of Legendrians (possibly singular) in a unit cosphere bundle S^*M. Let $\mathcal{C}_t = Sh(M, \Lambda_t^\infty)$ be the differential graded (dg) derived category of constructible sheaves on M with singular support at infinity contained in Λ_t^∞. We prove that if the isotopy of Legendrians embeds into an isotopy of Weinstein hypersurfaces, then the categories $\mathcal{C}_t$ are invariant.

Let M be a smooth compact manifold of real dimension m, T^*M the cotangent bundle, and

$$\dot{T}^*M = T^*M - T^*_M M, \quad T^\infty M := \dot{T}^*M/\mathbb{R}_{>0}$$

be the punctured cotangent bundle and contact cosphere bundle at infinity. Let $\Lambda^\infty \subset T^\infty M$ be a (singular) Legendrian, by which we mean a Whitney stratifiable subspace whose top dimensional strata are smooth Legendrian, and

$$\dot{\Lambda} = \mathbb{R}_{>0} \cdot \Lambda^\infty, \quad \Lambda = \dot{\Lambda} \cup T^*_M M$$

be two associated conical Lagrangians. We denote by $Sh(M, \Lambda^\infty)$ the the dg derived category of constructible sheaves on M where objects are sheaves F with singular support at infinity $SS^\infty(F) \subset \Lambda^\infty$, i.e. $SS(F) \subset \Lambda$. [1]

Definition 0.1. *Let $I \subset \mathbb{R}$ be an open interval, (C, ξ) be a contact manifold. An* **isotopy of Legendrian** *in C over I is a Whitney stratifiable closed subset $\mathcal{L}_I \subset C \times I$, such that $\mathcal{L}_t := \mathcal{L}_I \cap C \times \{t\}$ is a (singular) Legendrian for all $t \in I$. We also denote an isotopy as $\{\mathcal{L}_t\}_{t \in I}$ or simply $\{\mathcal{L}_t\}$.*

Remark 0.2. *If we choose a contact form α on (C, ξ), we may form a new contact manifold $(C \times T^*I, \alpha + \tau dt)$, and lift $\mathcal{L}_I$ to a Legendrian in $C \times T^*I$. The two description of isotopy are equivalent.*

We are interested in the following question:

Main Question: *Given an isotopy of Legendrians $\{\Lambda_t^\infty\}$ in $T^\infty M$, when is it 'non-characteristic' [N3], that is, the sheaf category $Sh(M, \Lambda_t^\infty)$ remains invariant? Or more concretely, if we deform Λ^∞, can we deform the sheaf F such that $SS^\infty(F)$ remains in Λ^∞?*

Before we state our main result, we first review two important results in this direction. The first one is due to Guillermou-Kashiwara-Schapira, which quantizes isotopy of the entire contact manifold $T^\infty M$.

Theorem 0.3 ([GKS] Theorem 3.7, Proposition 3.12)**.** *Let I be an open interval containing 0, and $\varphi : I \times T^\infty M \to T^\infty M$ be a smooth map with $\varphi_t = \varphi(t, -)$.*

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[1] One can work with either 'large', or 'traditional', or 'wrapped' constructible sheaves [N4]. Here for simplicity, we work with traditional constructible sheaf.

Assume φ satisfies (1) $\varphi_0 = id$, and (2) φ_t are contactomorphisms for all $t \in I$. Then for each $t \in I$, we have equivalences of category

$$\hat{\varphi}_t : Sh(M) \xrightarrow{\sim} Sh(M), \quad \text{such that } SS^\infty(\hat{\varphi}_t F) = \varphi_t(SS^\infty(F)).$$

One immediately get the following corollary.

Corollary 0.4. *If the isotopy of Legendrian $\{\Lambda_t^\infty\}_{t\in I}$ can be embedded into an isotopy $\{\varphi_t\}_{t\in I} : S^*M \to S^*M$ of the contact manifold, that is, $\Lambda_t^\infty = \varphi_t(\Lambda_0^\infty)$. Then we have equivalence of categories*

$$\hat{\varphi}_t : Sh(M, \Lambda_0^\infty) \xrightarrow{\sim} Sh(M, \Lambda_t^\infty).$$

Remark 0.5. *Any isotopy of smooth Legendrian can be extended to a contact isotopy of the ambient manifold. If the Legendrian is singular, and if the homeomorphism type of the Legendrian changes during the isotopy, then it cannot be extended to a contact isotopy.*

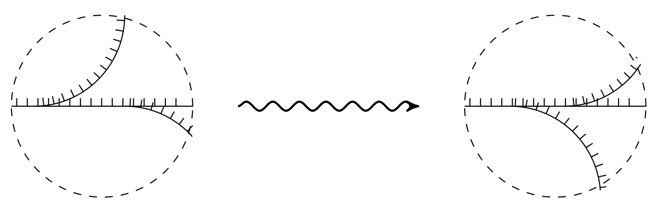


FIGURE 1. An example of Legendrian isotopy (shown as front projection from $S^*\mathbb{R}^2 \to \mathbb{R}^2$, with 'hairs' indicating co-direction) which cannot be embedded in a contact isotopy.

The second result is due to Nadler [N3], where he proves that any Legendrian singularity admits a non-characteristic deformation to an arboreal singularity (introduced in [N2]). In [N3], Nadler proposed the following geometric condition on Legendrian isotopies.

Definition 0.6 (Displaceable Legendrian)**.** *Let $(T^\infty M, \xi = \ker\alpha, R_\alpha)$ be the cosphere bundle with a choice of Reeb vector field R_α, and let $R_\alpha^t : T^\infty M \to T^\infty M$ be the Reeb flow for time t.*[2] *Let $\epsilon > 0$. A Legendrian $\Lambda^\infty \subset T^\infty M$ is* **displaceable** *for R_α if there exists a constant $\epsilon > 0$, such that*

$$\Lambda^\infty \cap R_\alpha^s(\Lambda^\infty) = \emptyset, \quad \forall 0 < |s| < \epsilon. \tag{1}$$

We say a family of Legendrian $\{\Lambda_t^\infty\}$ is **uniformly displaceable** *for R_α, if each Λ_t^∞ is displaceable for the same constant ϵ.*

[2]Throughout the paper, we will use the notation X^t for the flow generated by a vector field X for time t.

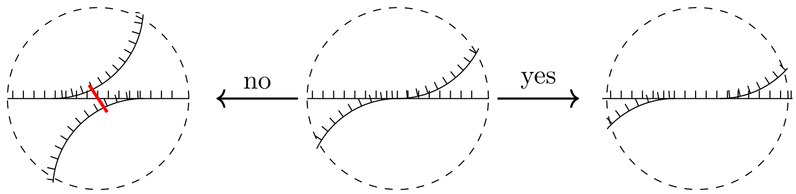


FIGURE 2. The deformation to the right is uniformly displaceable, and the one to the left is not, due to the appearance of new short Reeb chord (marked in red). (c.f. [N3], Example 1.5)

It turns out just having the uniform displaceablity for Legendrian is not enough, one need to impose some control on the topology of the Legendrians as well.

Example 0.7. *Let $C = J^1\mathbb{R}, \alpha = dz - ydx, R_\alpha = \partial_z$. For $t \in [0, 1)$, define a family of Legendrians*

$$\mathcal{L}_t = \{(x, 0, 0) : x \in \mathbb{R}\} \cup \{(x, x^2 + t, x^3/3 + tx) : x \in \mathbb{R}\}$$

there are no Reeb chords ending on $\mathcal{L}_t$ even at $t = 0$. However the sheaf category $\mathcal{C}_t$ associated to $\mathcal{L}_t$ (after identifying $J^1\mathbb{R}$ with $S^*\mathbb{R}^2$ with matching contact forms) jumps as $t \to 0^+$, since the topology of the Legendrian changed.

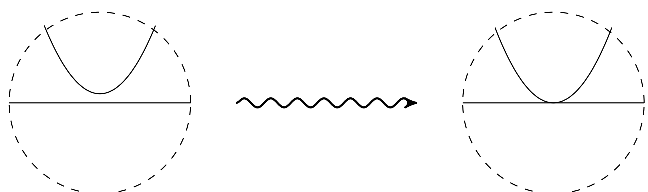


FIGURE 3. The deformation of Legendrian $\mathcal{L}$ in $J^1\mathbb{R} \simeq T^*\mathbb{R} \times \mathbb{R}$, draw as the Lagrangian projection to $T^*\mathbb{R}$. Note there is no Reeb chord ending on $\mathcal{L}$ throughout the deformation.

0.1. **Definitions and Result.** To state our main result, we need some definitions. Recall that a hypersurface in a contact manifold is *convex* [Gi] if it admits a transverse contact vector field. We want to consider nested tubular neighborhoods of Legendrians with convex boundaries.

Definition 0.8. *Let $\mathcal{L}$ be a singular Legendrian in (C, ξ). A* **convex tubular neighborhood** *for $\mathcal{L}$ is the following data (U, ρ, X),*
(1) $U =: U(\mathcal{L})$ is an open tubular neighborhood of $\mathcal{L}$,
(2) $\rho : U \to [0, 1)$ is a C^1-function,
(3) X is a smooth contact vector field on U.
Such that if we define

$$U_r(\mathcal{L}) = \{x \in U : \rho(x) < r\}, \quad \forall 0 < r < 1$$

then, $\cap_r U_r(\mathcal{L}) = \mathcal{L}$; $\partial U_r(\mathcal{L})$ are C^1-smooth and C^1-diffeomorphic; X is transverse to all $\partial U_r(\mathcal{L})$; and $d\rho(X) > c\rho$ for some constant $c > 0$.

Let $\{\mathcal{L}_t\}$ be an isotopy of Legendrian in (C, ξ). An **isotopy of convex tubular neighborhoods** *$\{(U, \rho, X)_t\}$ of $\{\mathcal{L}_t\}$ is a one-parameter family of such data with uniform bound on constant.*

The notion of Weinstein hypersurface is introduced in [Av]. We recall its definition following [Eli, Section 2].

Definition 0.9. (1) *A codimension-*1 *submanifold $\mathcal{H}$ in a contact manifold (C, ξ) with boundary $\partial\mathcal{H}$ is called* **Weinstein hypersurface** *if there exists a contact form α such that $(\mathcal{H}, \lambda := \alpha|_{\mathcal{H}})$ is compatible with a Weinstein structure on $\mathcal{H}$, i.e. $d\lambda$ is symplectic and the Liouville vector field X on $\mathcal{H}$ dual to the Liouville form λ is outward transverse to $\partial\mathcal{H}$ and admits a Lyapunov function $\phi : \mathcal{H} \to \mathbb{R}$.*

We denote a Weinstein hypersurface (including a specification of compatible Weinstein structure) by $(\mathcal{H}, \lambda, X, \phi)$.

(2) *If $\mathcal{L}$ is the skeleton of $(\mathcal{H}, \lambda, X, \phi)$, we say $\mathcal{H}$ is a* **Weinstein hypersurface thickening** *of $\mathcal{L}$.*
(3) *An isotopy of Weinstein hypersurface is a smooth family of $\{(\mathcal{H}, \lambda, X, \phi)_t\}$ where the choice of contact 1-form α_t has smooth and bounded variation with t.*

We can show that if the Legendrian admits a Weinstein hypersurface thickening and is uniformly displaceable, then it admits a canonical tube thickening (Proposition 1.10).

Main Theorem (Theorem 3.1)**.** *Let $\{\Lambda^\infty_t\}_{t\in I}$ be an isotopy of Legendrian in $T^\infty M$, such that Λ^∞_t is constant for t outside a closed interval $[a, b] \subset I$. If $\{\Lambda^\infty_t\}$ is uniformly displaceable for some Reeb vector field R_α on $T^\infty M$, and*

(1) *there exists an isotopy of convex tubular neighborhoods $\{(U(\Lambda^\infty_t), \rho_t, X_t)\}$ of $\{\Lambda^\infty_t\}$*
(2) *or, there exists an isotopy of Weinstein hypersurface thickening $\{(H, \lambda, X, \phi)_t\}$ of $\{\Lambda^\infty_t\}$,*

then the sheaf categories $Sh(M, \Lambda^\infty_t)$ remain invariant.

Remark 0.10. *The notion of convex tubular neighborhood is related to the 'frozen boundary' for a Liouville sector* [GPS]*, and the Weinstein hypersurface is related to 'stop' in partially wrapped Fukaya category* [Syl]*. See* [Eli, Section 2,3] *for related work on Weinstein hypersurface and Weinstein pair.*

0.2. **Idea of the Proof.** We first give an heuristic derivation for why we might expect such a theorem, though we do not follow this approach literally. See the previous section for notations $X_s, U_r(\mathcal{L}_s), \cdots$. The main idea is to use the retracting contact flow $-X_s$ toward Λ^∞_s, properly cut-off outside $U_\delta(\mathcal{L}_s)$ for some $1/2 < \delta < 1$, to deform and squeeze a nearby Legendrian skeleton $\Lambda^\infty_t \subset U_\delta(\Lambda^\infty_s)$ into Λ^∞_s in the limit. We consider the sheaf quantization of the retracting flow for time $T > 0$,

$$X_s^{-T} : T^\infty M \xrightarrow{\sim} T^\infty M \quad \rightsquigarrow \quad \hat{X}_s^{-T} : Sh(M) \xrightarrow{\sim} Sh(M).$$

Then we define the projection functors as the limit of the flow

$$\Pi_s : Sh(M, U_\delta(\Lambda^\infty_s)) \to Sh(M, \Lambda^\infty_s), \quad \Pi_s(F) := \lim_{T\to\infty} \hat{X}_s^{-T}(F_t), \tag{2}$$

where $Sh(M, U_\delta(\Lambda^\infty_s))$ means constructible sheaves with $SS^\infty(F) \subset U_\delta(\Lambda^\infty_s)$. The limit is not inductive, or projective limit, and is defined (in the style of a nearby cycle functor) in Section 2.6 . Then, one only need to show that for any t, s closed enough (contained in each other's tubular neighborhoods U_δ), we have a pair of inverse functors

$$\Pi_s|_{Sh(M,\Lambda_t^\infty)} : Sh(M, \Lambda_t^\infty) \overset{\sim}{\leftrightarrow} Sh(M, \Lambda_s^\infty) : \Pi_t|_{Sh(M,\Lambda_s^\infty)}.$$

To see they are inverses, we consider a constructible sheaf $F_t \in Sh(M, \Lambda_t^\infty)$ as functors $\mathrm{Hom}(-, F_t)$, and test on 'probe' sheaves P such that

$$SS^\infty(P) \cap [U_\delta(\Lambda_t^\infty) \cup U_\delta(\Lambda_s^\infty)] = \emptyset \tag{3}$$

then $\mathrm{Hom}(P, \hat{X}_t^{-T_1} \hat{X}_s^{-T_2} F_t)$ is independent of T_1, T_2. One way to construct such probe uses *wrapped constructible sheaves* (see [N4] for definition), we have [N4, Theorem 1.6]

$$Sh(M, \Lambda_t^\infty) \xrightarrow{\sim} \mathrm{Fun}^{ex}(Sh^w(M, \Lambda_t^\infty)^{op}, \mathrm{Perf}_{\mathbb{C}}).$$

To achieve (3), we use small *negative* Reeb flow to displace P without changing the homs (Proposition 2.9). We get, for all $P \in Sh^w(M, \Lambda_t^\infty), F_t \in Sh(M, \Lambda_t^\infty)$

$$\begin{aligned}\mathrm{Hom}(P, F_t) &\simeq \mathrm{Hom}(\hat{R}^{-\epsilon} P, F_t) \simeq \mathrm{Hom}(\hat{R}^{-\epsilon} P, \Pi_s(F_t)) \\ &\simeq \mathrm{Hom}(\hat{R}^{-\epsilon} P, \Pi_t \Pi_s(F_t)) \simeq \mathrm{Hom}(P, \Pi_t \Pi_s(F_t)))\end{aligned}$$

Hence Π_t, Π_s are inverses.

Our actual approach is as following: let $\Lambda_I^\infty \subset T^\infty(M \times I)$ be an isotopy of Legendrians and let $F_t \in Sh(M, \Lambda_t^\infty)$. We will extend F_t to a sheaf $F_I \subset Sh(M \times I, \Lambda_I^\infty)$ such that $F_I|_t = F_t$.

One first show that such extension is unique (if exists), this is equivalent to show that restriction functor $F_I \mapsto F_t$ is fully-faithful, i.e. (Proposition 3.2)

$$\mathrm{Hom}(F_I, G_I) \xrightarrow{\sim} \mathrm{Hom}(F_t, G_t), \forall F_I, G_I \in Sh(M \times I, \Lambda_I^\infty).$$

One need to show that $\mathcal{H}om(F_I, G_I)(M \times (a, b))$ is independent of the size of the interval, hence one can interpolate from $(a, b) = I$ to infinitesimal small neighborhood around t. The key technical point is to use the uniform displaceability condition to perturb G_I slice-wise by positive Reeb flow for time s, $G_I \to K_s^! G_I$, to separate $SS^\infty(F_I)$ and $SS^\infty(K_s^! G_I)$.

One then show that such extension exists locally, i.e., given F_t, we may find a small neighborhood $(t - \delta, t + \delta)$, where δ is uniform, to extend F_t on $M \times \{t\}$ to $M \times (t - \delta, t + \delta)$. This is done using limit of the retracting flow, as done in defining Π_s in (2). For general contact flow, there is no way to take limit. Here we can take limit since the singular support of the sheaf $SS^\infty(F_t)$ converges under $-X_s$ to the sink of the flow Λ_s^∞. We thus get the limiting sheaf with desired bound on singular support.

Finally, we use uniqueness of extension to patch together local extensions, and get the global extension result. (c.f. Lemma 1.13 in [GKS]).

Remark 0.11. *We thank V. Shende for informing us the up-coming work of Nadler-Shende about quantization of exact symplectic category, which include a result on invariance of microlocal sheaf category $Sh(W)$ for Weinstein manifold (W, λ) [Sh] under Weinstein homotopy.*

0.3. **Acknowledgements.** I would like to thank my advisor Eric Zaslow for suggesting the idea of 'invariance of hom under Reeb perturbation'. I also thank P. Schapira for many warm discussions, and for suggesting finding a sheaf-theoretic proof for Proposition 2.9. I thank D. Nadler for encouragements and comment on an early draft of this paper using almost retraction. I also thank S. Guillermou for explaining many points in the [GKS] paper, and V. Shende for many useful discussions.

1. Convex Tubular Neighborhoods and Weinstein Hypersurfaces

We give basic definition and construction for Weinstein hypersurface and convex tubular neighborhood. We will work with general contact manifold (C, ξ) instead of S^*M so that the results may generalize to other Weinstein domain.

1.1. **Basic of Contact Geometry.** We recall the definition of co-oriented contact manifold as follow. Let C be a $2n+1$ dimensional manifold, $\xi \subset TC$ be a rank $2n$ sub-bundle, such that there exists a one-form (contact one-form) α (up to multiplication of non-negative function) satisfying $\xi = \ker \alpha$ and $\alpha \wedge (d\alpha)^n \neq 0$. If we fix such a α, we have a Reeb vector field R_α given by

$$\iota_{R_\alpha} \alpha = 1, \quad \iota_{R_\alpha} d\alpha = 0.$$

We note that different choices of α will lead to different choices of R_α.

A contact vector field X is one that perserves ξ.

Definition 1.1. *Given a smooth function $H : C \to \mathbb{R}$, the* **contact Hamiltonian vector field** *X_H is uniquely determined by*

$$\begin{cases} \langle X_H, \alpha \rangle = H \\ \iota_{X_H} d\alpha = \langle H, R \rangle \alpha - dH \end{cases} \tag{4}$$

Reeb vector field is a speical case of X_H for $H = 1$.

Proposition 1.2 ([Ge] Theorem 2.3.1)**.** *With a fixed choice of contact form α there is a one-to-one correspondence between contact vector field X and smooth functions $H : C \to \mathbb{R}$. The correspondence is given by*

$$X \mapsto H = \langle \alpha, X \rangle, \quad H \mapsto X_H.$$

Unlike symplectic Hamiltonian vector field, X_H does not preserve level set of H.

Lemma 1.3.

$$\langle X_H, dH \rangle = H \langle R, dH \rangle$$

In particular, X_H preserves the zero set of H.

Proof. Since $X_H = HR + X_H^{\parallel}$, where $X_H^{\parallel} \in \ker(\alpha)$, we have

$$\begin{aligned} \langle X_H - HR, dH \rangle &= \langle X_H - HR, dH - \alpha \rangle = \langle X_H - HR, -\iota_{X_H}(d\alpha) \rangle \\ &= d\alpha(X_H - HR, X_H) = 0 \end{aligned}$$

where we have used $R \in \ker(d\alpha)$. ∎

Example 1.4. *Let M be a smooth manifold, and T^*M the cotangent bundle with canonical Liouville one-form λ and symplectic two-form $\omega = d\lambda$. If we put local Darboux coordinate $(q, p) = (q_1, \cdots, q_m; p_1, \cdots, p_m)$ on T^*M where $m = \dim_{\mathbb{R}} M$, then $\lambda = \sum_{i=1}^m p_i dq_i$ and $\omega = \sum_i dp_i \wedge dq_i$, and we will suppress the indices and* *summation to write* $\lambda = pdq, \omega = dpdq$*. Also define* $\dot{T}^*M = T^*M \backslash T^*_M M$*,* $T^\infty M = \dot{T}^*M/\mathbb{R}_{>0}$*. The Liouville vector field for* λ *is defined by defined by* $\iota_{V_\lambda}\omega = \lambda$*, and here it is given by* $V_\lambda = p\partial_p$*. On* $T(\dot{T}^*M)$*, the symplectic orthogonal to the Liouville vector field defines a distribution*

$$\widetilde{\xi} = \{(q, p; v_q, v_p) \in T(\dot{T}^*M) : \omega((v_q, v_p), V_\lambda) = 0\},$$

which project to a canonical contact distribution ξ *on* $T^\infty M$*. Let* g *be any Riemaninan metric on* M*, then* T^*M *has induced norm. Let* $S^*M = \{(q, p) \in T^*M \mid |p| = 1\}$ *be the unit cosphere bundle with contact form* $\alpha = \lambda|_{S^*M}$*, then the contact distribution can also be written as* $\xi = \ker(\alpha)$*.*

Define the symplectization of $(C, \xi = \ker\alpha)$ by

$$S := C \times \mathbb{R}_u, \quad \lambda = e^u\alpha, \quad \omega_S = d(e^u\alpha).$$

We have projection along $\mathbb{R}_u$, and inclusion of zero section as:

$$\pi_S : S \to C, \quad \iota_C : C \simeq C \times \{0\} \hookrightarrow S.$$

Different choice of α gives the same S up to fiber preserving symplectomorphism, that identifies the 'zero-section' $\mathrm{Im}(\iota_C)$.

A Hamiltonian function $H : C \to \mathbb{R}$ can be extended to a homoegeneous degree one function $\widetilde{H} : S \to \mathbb{R}$ by $\widetilde{H} = e^u H$. Then the symplectic Hamiltonian vector field $\xi_{\widetilde{H}}$, given by $\omega_S(-, \xi_{\widetilde{H}}) = d\widetilde{H}(-)$, preserves the fiber of π_S and descend to X_H.

1.2. **Weinstein Hypersurface.** Let $(\mathcal{H}, \lambda, X, \phi)$ be a Weinstein hypersurface in $(C, \xi = \ker(\alpha))$, in particular $\lambda = \alpha|_{\mathcal{H}}$ (Definition 0.9). [3] For small enough $\epsilon > 0$, we may thicken $\mathcal{H}$ to $U_\epsilon(\mathcal{H})$ by Reeb flow for time $|t| < \epsilon$. We will take contact Hamiltonian function $H = t$ on $U_\epsilon(\mathcal{H})$, where t is the time coordinate (not to be confused with the isotopy parameter later). Then

$$\langle R, dH\rangle = \langle \partial t, dt\rangle = 1, \quad \mathcal{H} = \{H = 0\}.$$

Proposition 1.5. *Under the identification* $U_\epsilon(\mathcal{H}) \simeq \mathcal{H} \times (-\epsilon, \epsilon)$*, the contact form* α *and Reeb vector field can be written as*

$$\alpha = \lambda + dt, \quad R = \partial_t$$

The contact vector field X_H *can be written as*

$$X_H = X + t\partial_t,$$

where λ *is the Liouville form on* $\mathcal{H}$*,* X *the Liouville vector field along* $\mathcal{H}$*.*

Proof. We call $\mathcal{H}$ the horizontal direction and $(-\epsilon, +\epsilon)$ the vertical direction. Since the Reeb flow is translation the t coordinate, and the Reeb flow preserves α, we have $\alpha = \lambda$ on the horizontal direction. Since $\iota_R\alpha = 1$, we have $\alpha = dt$ along the vertical direction. Thus $\alpha = \lambda + dt$. We note that $\iota_X\lambda = \iota_X(\iota_X(d\lambda)) = 0$, and $\iota_X(dt) = 0$, hence $X \in \ker\alpha$. Thus we may easily check the given formula X_H satisfy the definition. □

[3] If we only fix $\mathcal{H}$ but allowing λ and α to vary, then we may change α to $e^f\alpha$ for some smooth function f as long as $k := 1 + \langle df, X\rangle > 0$ on $\mathcal{H}$, in this case the Liouville field changes to $\frac{1}{k}X$, and is gradient-like for the same ϕ. Moreover, the skeleton for $\mathcal{H}$ remains the same. See [Eli, Section 2] and [CE, Lemma 12.1].

Corollary 1.6. *Let* $(\mathcal{H}, \lambda, X, \phi) \hookrightarrow (C, \xi = \ker \alpha)$ *be a Weinstein hypersurface. If* $\{(\mathcal{H}, \lambda_\delta, X_\delta, \phi_\delta)\}_{|\delta|<c}$ *is an isotopy of Weinstein domain, where*

$$\lambda_\delta = \lambda + \delta df$$

for some smooth and uniformly bounded $f : \mathcal{H} \to \mathbb{R}$*. Then there exists an isotopy of Weinstein hypersurface realizing the given isotopy of Weinstein domain for* $|\delta| < c'$ *where* $c' < c$*.*

Proof. We work in the neighborhood $U_\epsilon(\mathcal{H}) \simeq \mathcal{H} \times (-\epsilon, \epsilon)$ with coordinate (x, t). Since $\alpha = \lambda + dt$ in $U_\epsilon(\mathcal{H})$, we may define a family of hypersurface as graph of δf

$$\mathcal{H}_\delta = \{(x, t) \in \mathcal{H} \times (-\epsilon, \epsilon) \mid t = \delta f(x)\}, \forall |\delta| < c'$$

where

$$c' := \max\{\delta \mid \delta < c, \sup_{x \in \mathcal{H}} \delta f(x) < \epsilon\}.$$

We have canonical identification $\pi_\delta : \mathcal{H}_\delta \to \mathcal{H}$ by project away the t coordinate, and $\lambda_{\mathcal{H}_\delta} := \alpha|_{\mathcal{H}} = \lambda + \delta df$. □

1.3. **Construction of Convex Tubular Neighborhood.**

Proposition 1.7. *Let* $(\mathcal{H}, \lambda, X, \phi)$ *be a Weinstein domain, such that* $\partial\mathcal{H} = \phi^{-1}(c)$*. There exists a unique* C^1*-function* $\psi : \mathcal{H} \to [0, 1]$*, such that* ψ *is smooth on* $\mathcal{H} \backslash (\partial\mathcal{H} \cup \mathcal{L})$*,* $\psi|_{\partial\mathcal{H}} = 1$*,* $\psi|_{\mathcal{L}} = 0$*, and* $\langle X, d\psi \rangle = 2\psi$ *on* $\mathcal{H} \backslash \mathcal{L}$*.*

Proof. We have a diffeomorphism generated by the downward Liouville flow $-X$

$$\Psi : \partial\mathcal{H} \times \mathbb{R}_{\geq 0} \xrightarrow{\sim} \mathcal{H} \backslash \mathcal{L}, \quad (x, t) \mapsto X^{-t}(x).$$

Thus we may define ψ on $\mathcal{H} \backslash \mathcal{L}$ by

$$\psi|_{\mathcal{H} \backslash \mathcal{L}}(\Psi^{-1}(x, t)) = e^{-2t}.$$

Thus $-\partial_t e^{-2t} = \langle d\psi, X \rangle = 2\psi$. ψ has a C^1 extension by zero to $\mathcal{L}$, since $d\psi|_{\mathcal{L}} = 0$. □

Let $U = U_\epsilon(\mathcal{H}) \simeq \mathcal{H} \times (-\epsilon, \epsilon)$ with x coordinate on $\mathcal{H}$ and t coordinate on $(-\epsilon, \epsilon)$. We define

$$\rho(x, u) = \psi(x) + t^2 : U \to \mathbb{R}. \tag{5}$$

Then

Proposition 1.8. ρ *is a* C^1*-function on* U*, vanishing only on* $\mathcal{L}$*; and* $\langle d\rho, X \rangle = 2\rho > 0$ *on* $U \backslash \mathcal{L}$*.*

Proof. The regularity and vanishing statement is clear. Using Proposition 1.5 and 1.7, we have

$$\langle d(\psi + t^2), X_H \rangle = \langle d\psi + 2tdt, X + t\partial_t \rangle = 2\psi + 2t^2 = 2\rho$$

away from $\mathcal{L}$. □

Proposition 1.9. *If* $(\mathcal{H}, \lambda, V, \phi)$ *is a Weinstein hypersurface thickening of Legendrian* $\mathcal{L}$*, then there exist a convex tubular neighborhood thickening* (U, ρ, X) *of* $\mathcal{L}$*.*

Proof. Let α be the family of contact 1-form, such that $\lambda = \alpha|_{\mathcal{H}}$. Let $1 \gg \epsilon > 0$ be small enough, such that

$$R_\alpha^\tau(\mathcal{H}) \cap \mathcal{H} = \emptyset, \quad \forall 0 < |\tau| < 2\epsilon.$$

Then we define $U^0 = U_\epsilon(\mathcal{H})$ and H using 1-form α and the associated Reeb flow. Let ρ^0 be the C^1-function ρ as constructed in (5). We see U^0 and ρ^0 depends on $(\mathcal{H}, \lambda, X, \phi)$ and α canonically, hence U^0 has piecewise smooth boundary

$$\partial U^0 = \partial\mathcal{H} \times (-\epsilon, \epsilon) \cup \mathcal{H} \times \{-\epsilon, \epsilon\}$$

and ρ^0 is a globally C^1-function defined on U^0 and it is smooth away from $\mathcal{L} \times (-\epsilon, \epsilon)$. The vector field X_H is smooth in U and is smoothly varying in t.

Since $\{\rho^0(x) < \epsilon^2\}$ is contained in U^0, we can trim U^0 and rescale ρ^0 by

$$U := \{\rho^0(x) < \epsilon^2\}, \quad \rho := \rho^0/\epsilon^2.$$

Let $X := X_H$, we still have

$$\langle X, d\rho\rangle = \epsilon^{-2}\langle X, d\rho^0\rangle = 2\epsilon^{-2}\rho^0 = 2\rho.$$

Then the data (U, ρ, X) forms a convex tubular neighborhood thickening $\mathcal{L}$. □

Proposition 1.10. *Assume $\{\mathcal{L}_t\}_{t\in I}$ is an isotopy of Legendrian, uniformly displaceable for some Reeb vector field, and can be thickened to a Weinstein hypersurface isotopy $\{(\mathcal{H}, \lambda, V, \phi)_t\}$. Then there exist a convex tubular neighborhood thickening $\{(U, \rho, X)_t\}$.*

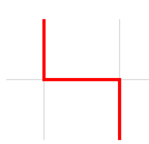
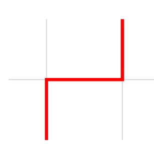
Proof. Since all the parameters have smooth and bounded dependence on t (needed if I is not compact), hence the proof of Proposition 1.10 goes through verbatim. □

2. Non-Characteristic Isotopy of Sheaves

2.1. **Constructible Sheaves.** We give a quick working definition for constructible sheaf used here, and point to [KS, S] for proper treatment. A constructible sheaf F on M is a sheaf valued in chain complex of $\mathbb{C}$-vector spaces, such that its cohomology is locally constant with finite rank with respect to some Whitney stratification $\mathcal{S} = \{\mathcal{S}_\alpha\}_{\alpha\in A}$ on M, where $\mathcal{S}_\alpha$ are disjoint locally closed smooth submanifolds with nice adjacency condition and $M = \sqcup_{\alpha\in A}\mathcal{S}_\alpha$. The singular support $SS(F)$ of F is a closed conical Lagrangian in T^*M, contained in $\cup_{\alpha\in A} T^*_{\mathcal{S}_\alpha}M$, such that $SS(F) \cap T^*_M M$ equals the support of F, and $(p, q) \in SS(F)\backslash T^*_M M$ if there exists a locally defined function f with $f(q) = 0, df(q) = p$, such that the restriction map $F(B_\epsilon(q) \cap \{f < \delta\}) \to F(B_\epsilon(q) \cap \{f < -\delta\})$ fails to be a quasi-isomorphism for $0 < \delta \ll \epsilon \ll 1$. We denote by $SS^\infty(F) = SS(F) \cap S^*M$ the singular support of F at infinity.

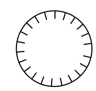
If $\Lambda \subset T^*M$ is a conical Lagrangian containing zero section (as always), we write $Sh(M, \Lambda^\infty)$ for the dg derived category of constructible sheaves [N1] with object F satisfying $SS^\infty(F) \subset \Lambda^\infty$.

Example 2.1. *For example, on $\mathbb{R}$, if $\mathbb{C}_{[0,1]}$ (resp. $\mathbb{C}_{(0,1)}$) denote constant sheaf with stalk $\mathbb{C}$ on $[0, 1]$ (resp. on $(0, 1)$) and zero stalk elsewhere, then their singular supports in $T^*\mathbb{R}$ are*

$SS(\mathbb{C}_{[0,1]}) =$  , $SS(\mathbb{C}_{(0,1)}) =$  .

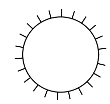
Example 2.2. *Let $j : U = B(0,1) \hookrightarrow \mathbb{R}^2$ be the inclusion of an open unit ball in $\mathbb{R}^2$. Then $j_*\mathbb{C}_U$ is supported on the closed set $\overline{U}$, with singular support at infinity as*

$$SS^\infty(j_*\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = -d|x|\} =$$



And $j_!\mathbb{C}_U$ is supported on the open set U, with singular support at infinity as

$$SS^\infty(j_!\mathbb{C}_U) = \{(x,\eta) \in S^*\mathbb{R}^2 \mid x \in \partial U, \eta = d|x|\} =$$



Here the Legendrians are represented by co-oriented hypersurfaces in $\mathbb{R}^2$ with hairs indicating the co-orientation.

2.2. **Operation on Constructible Sheaves.** Let X, Y be manifolds. We use $f^*, f_*, f^!, f_!, \mathcal{H}om, \otimes$ to mean the corresponding dg derived functors:

$$\begin{aligned} - \otimes F : \quad & Sh(X) \leftrightarrow Sh(X) \quad : \mathcal{H}om(F,-) \\ f^* : \quad & Sh(X) \leftrightarrow Sh(Y) \quad : f_* \\ f_! : \quad & Sh(Y) \leftrightarrow Sh(X) \quad : f^! \end{aligned}$$

where $f : Y \to X$ is a map of real analytic manifolds.

The Verdier duality $\mathbb{D} : Sh(X) \to Sh(X)$ is an anti-involution. It interchanges shriek with star

$$\mathbb{D}\mathbb{D} = id, \quad f_! = \mathbb{D}f_*\mathbb{D}, \quad f^! = \mathbb{D}f^*\mathbb{D}.$$

The shrieks and stars are directly related in two cases: when f is proper $f_! = f_*$; when f is a smooth morphism of relative dimension d_f, $f^!(-) \simeq f^*(-) \otimes \omega_{Y/X} \simeq f^*(-) \otimes \mathfrak{or}_{Y/X}[d_f]$, where $\mathfrak{or}_{Y/X}$ is the orientation sheaf of the fiber.

Given an open subset U of X and its closed complement Z,

$$\text{open inclusion:} \quad U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z, \quad \text{closed inclusion,}$$

we have $j^* = j^!$ and $i_* = i_!$. Furthermore, there are exact triangles

$$i_!i^! \to id \to j_*j^* \xrightarrow{[1]}, \quad j_!j^! \to id \to i_*i^* \xrightarrow{[1]} .$$

These are sheaf-theoretic incarnations of excisions: applied to the constant sheaf on X and taking global sections, we get

$$H^*(Z, i^!\mathbb{C}) \to H^*(X,\mathbb{C}) \to H^*(U,\mathbb{C}) \xrightarrow{[1]}, \quad H^*_c(U,\mathbb{C}) \to H^*_c(X,\mathbb{C}) \to H^*_c(Z,\mathbb{C}) \xrightarrow{[1]} .$$

If Y is a locally closed $\mathcal{C}$-submanifold of X, we use $j_Y : Y \hookrightarrow X$ to denote the inclusion. Let $\mathbb{C}_Y \in Sh(Y)$ denote the constant sheaf on Y, and $\omega_Y = \mathbb{D}\mathbb{C}_Y$ be the Verdier dualizing complex of Y, then ω_Y is the canonically isomorphic to the shifted orientation sheaf $\mathfrak{or}_Y[\dim Y]$ on Y. The standard sheaf on Y is $j_{Y*}\mathbb{C}_Y$, and the costandard sheaf on Y is $j_{Y!}\omega_Y$.

Let X_i, $i = 1, 2$, be spaces, and $K \in Sh(X_1 \times X_2)$. We define the following pair of adjoint functors

$$K_! : Sh(X_1) \leftrightarrow Sh(X_2) : K^! \tag{6}$$

$$K_! : F \mapsto \pi_{2!}(K \otimes \pi_1^* F), \qquad K^! : G \mapsto \pi_{1*}(\mathcal{H}om(K, \pi_2^! G)) \tag{7}$$

In [KS],$K_! = \Phi_K$ and $K^! = \Psi_K$ and with X_1, X_2 switched. The notation here is suggestive for them to be adjoint functors.

2.3. **Isotopy of Legendrian and Sheaves.** Let $I = (0,1) \subset \mathbb{R}$. For any $t \in I$, let

$$j_t : M_t := M \times \{t\} \hookrightarrow M_I := M \times I$$

be the inclusion of t-slice M_t into the total space M_I, and let $\pi_I : M_I \to I$ be the projection. Let $\mathbb{C}_{M_t}$ be the constant sheaf on M_t with stalk $\mathbb{C}$. We have then

$$SS(\mathbb{C}_{M_t}) = \{(x,t;0,\tau) \in T^*M_I\}, \quad SS^\infty(\mathbb{C}_{M_t}) = \{(x,t;0,\pm 1) \in S^*M_I \simeq T^\infty M\}.$$

We give another definition of isotopy of Legendrian and sheaves, equivalent to the one given in the introduction for the case $C = S^*M$.

Definition 2.3. *Let M be a smooth manifold, I an open interval of $\mathbb{R}$.*

(1) *An* **isotopy of Legendrians** *over I is a Legendrian $\Lambda_I^\infty \subset T^\infty(M \times I)$ such that*

$$\Lambda^\infty \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset, \text{ for all } t \in I.$$

For any $t \in I$, we define the **restriction of Λ_I^∞ at** *t as the Legendrian Λ_t^∞ for the conical Lagrangian Λ_t,*

$$\Lambda_t = \{(x,\xi) \in T^*M \mid \exists (x,t;\xi,\tau) \in \Lambda_I\}.$$

(2) *An* **isotopy of sheaves** *is a sheaf $F_I \in Sh(M \times I)$, such that*

$$SS^\infty(F_I) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset, \text{ for all } t \in I.$$

For any $t \in I$, we define **restriction of F_I at** *t as*

$$F_t := F_I|_{M_t} \in Sh(M).$$

(3) *Two isotopies of sheaves $F_I, G_I \in Sh(M \times I)$ are* **non-characteristic** *if*

$$SS^\infty(F_t) \cap SS^\infty(G_t) = \emptyset, \text{ for all } t \in I.$$

Some easy to check properties are in order.

Proposition 2.4. *(1) If F_I is an isotopy of sheaf, $\Lambda_I^\infty = SS^\infty(F_I)$, then*

$$\Lambda_t^\infty = SS^\infty(F_t).$$

(2) If F_I is an isotopy of sheaf, $\pi_I : M_I \to I$, then $(\pi_I)_*F_I$ is a local system on I.

2.4. **Invariance of morphism under non-characteristic isotopy.** We use the same notations for $M_I = M \times I, M_t, \mathbb{C}_{M_t}, \cdots$ as in the previous subsection.

Lemma 2.5. *Let $F \in Sh(M)$. Let $\varphi : M \to \mathbb{R}$ be a C^1 function, such that $d\varphi(x) \neq 0$ for $x \in \varphi^{-1}([0,1])$.*
(1) For $s \in (0,1)$, let $U_s = \{x : \varphi(x) < s\}$, and let $U_1 = \cup_s U_s$. If

$$SS^\infty(\mathbb{C}_{U_s}) \cap SS^\infty(F) = \emptyset, \ \forall\, 0 < s < 1,$$

then

$$\mathrm{Hom}(\mathbb{C}_{U_1}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{U_s}, F), \ \forall\, 0 < s < 1.$$

(2) For $s \in (0,1)$, let $Z_s = \{x : \varphi(x) \leq s\}$, and let $Z_0 = \cap_s Z_s$. If

$$SS^\infty(\mathbb{C}_{Z_s}) \cap SS^\infty(F) = \emptyset, \ \forall\, 0 < s < 1,$$

then

$$\mathrm{Hom}(\mathbb{C}_{Z_s}, F) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{Z_0}, F), \ \forall\, 0 < s < 1.$$

Proof. (1) is a special case in [GKS, Prop 1.8]. (2) follows from (1) and

$$0 \to \mathbb{C}_{M\setminus\mathbb{Z}_s} \to \mathbb{C}_M \to \mathbb{C}_{Z_s} \to 0.$$

□

The following lemma is also often used.

Lemma 2.6 (Petrowsky theorem for sheaves, Corollary 4.6 [S])**.** *Let $F, G \in Sh(M)$. If $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then the natural morphism*

$$\mathcal{H}om(F, \mathbb{C}_M) \otimes G \to \mathcal{H}om(F, G)$$

is an isomorphism.

Corollary 2.7. *If F_I be an isotopy of sheaves, then*

$$\mathcal{H}om(\mathbb{C}_{M_t}, F_I) \simeq \mathbb{C}_{M_t}[-1] \otimes F_I$$

Proposition 2.8. *Let G_I and F_I be non-characteristic isotopy of sheaves, then $\mathcal{H}om(F_I, G_I)$ is an isotopy of sheaves. In particular,*

$$\mathrm{Hom}(F_t, G_t) \simeq \mathrm{Hom}(F_s, G_s) \quad \textit{for all } t, s \in I$$

Proof. G_I and F_I being non-characteristic implies $SS^\infty(G_I) \cap SS^\infty(F_I) = \emptyset$, hence we can bound singular support of the hom sheaf as [KS]

$$SS(\mathcal{H}om(F_I, G_I)) \subset SS(G_I) + SS(F_I)^a.$$

Again, using G_I and F_I being non-characteristic, we have

$$SS^\infty(\mathcal{H}om(F_I, G_I)) \cap SS^\infty(\mathbb{C}_{M_t}) = \emptyset \quad \text{for all } t, s \in I.$$

Hence $\mathcal{H}om(F_I, G_I))$ is an isotopy of sheaves. For the second statement, we have

$$\begin{aligned} & \mathrm{Hom}(F_t, G_t) \\ = \ & \mathrm{Hom}({j_t}^* F_I, {j_t}^* G_I) \simeq \mathrm{Hom}(F_I, j_{t*} j_t^* G_I) \simeq \mathrm{Hom}(F_I, \mathbb{C}_{M_t} \otimes G_I) \\ \simeq \ & \mathrm{Hom}(F_I, \mathcal{H}om(\mathbb{C}_{M_t}, G_I)[1]) \simeq \mathrm{Hom}(\mathbb{C}_{M_t}, \mathcal{H}om(F_I, G_I))[1] \\ \simeq \ & \mathrm{Hom}(\mathbb{C}_t, \pi_{I*}\mathcal{H}om(F_I, G_I))[1] \simeq [\pi_{I*}\mathcal{H}om(F_I, G_I)]_t \end{aligned} \tag{8}$$

then the result follows since $\pi_{I*}(\mathcal{H}om(F_I, G_I))$ is a local system. □

2.5. **Invariance of Morphism under Reeb Perturbation.** Sometimes we want to vary G, F while preserving $\mathrm{Hom}(F, G)$, but $SS^\infty(G) \cap SS^\infty(F) \neq \emptyset$, e.g. $F = G$. Here we borrow an idea from infinitesimally wrapped Fukaya-category [NZ], that to compute $\mathrm{Hom}_{Fuk}(L_1, L_2)$ one need to do perturbation to separate L_1, L_2 at infinity, one can perturb $L_2 \rightsquigarrow R^t L_2$ or $L_1 \rightsquigarrow R^{-t} L_1$ where R^t is Reeb flow [4] for positive small time t, small enough so that no new intersections are created between L_1, L_2 at infinity.[5]

Fix a Riemannian metric g on M, and identify S^*M with $T^\infty M$, so that Reeb flow R^t is the unit speed geodesic flow. Let $r_{inj}(M, g)$ be the injective radius of (M, g). Let $\hat{R}^t$ be the GKS quantization of R^t. The remaining part of this subsection will be devoted to prove the following Proposition.

[4]Note that in (partially) wrapped Fukaya category, one wraps L_1 positively (or L_2 negatively) in Reeb direction. This difference in sign is due to an opposite sign convention for ω. Hence Reeb flow here should be termed 'geodesic flow' to be precise.

[5]We thank P. Schapira and S. Guillermou for discussion about positive Reeb perturbation on sheaves.

Proposition 2.9. *Let $\Lambda^\infty \subset T^\infty M$ be a Legendrian, and $0 < \epsilon < r_{inj}(M, g)$ be small enough such that*

$$\Lambda^\infty \cap R^t \Lambda^\infty = \emptyset, \quad \forall\, 0 < |t| < \epsilon.$$

(1) For any $F \in Sh(M, \Lambda), 0 \leq t < \epsilon$, we have canonical morphism

$$F \to \hat{R}^t F.$$

(2) For any $F, G \in Sh(M, \Lambda), 0 \leq t < \epsilon$, we have canonical quasi-isomorphisms

$$\mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(F, \hat{R}^t G), \quad \mathrm{Hom}(F, G) \xrightarrow{\sim} \mathrm{Hom}(\hat{R}^{-t} F, G)$$

Proof. For any $0 \leq t < \epsilon$, define

$$K_t = \mathbb{C}_{\{(x,y) | d_g(x,y) \leq t\}} \in Sh(M \times M).$$

Then from [GKS], we have

$$\hat{R}^t F = \pi_{1*} \mathcal{H}om(K_t, \pi_2^! F),$$

and

$$\hat{R}^{-t} F = \pi_{2!} \mathcal{H}om(K_t \otimes \pi_1^* F),$$

where π_1 and π_2 are the projection from $M \times M$ to the first and second factor, and $\mathcal{H}om$ is the (dg derived) sheaf-hom. From the canonical restriction morphism $K_t \to K_0 = \mathbb{C}_\Delta$, where $\Delta \subset M \times M$ is the diagonal subset, we have

$$F = \pi_{1*} \mathcal{H}om(K_0, \pi_2^! F) \to \pi_{1*} \mathcal{H}om(K_t, \pi_2^! F) = \hat{R}^t F.$$

For the second statement, we first prove the following lemma.

Lemma 2.10.

$$SS^\infty(K_t) \cap SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G)) = \emptyset, \quad \forall 0 < t < \epsilon. \tag{9}$$

Proof. Assuming the intersection is non-empty and contains $(x_1, x_2; p_1, p_2)$ in its cone. Since $(x_1, x_2; p_1, p_2) \in \mathbb{R}_{>0} \cdot SS^\infty(K_t)$, we have

$$d_g(x_1, x_2) = t.$$

Using the boundary defining inequality $d(x_1, x_2) \leq t$ for K_t, we found its inward conormal at point (x_1, x_2) is given by

$$(p_1, p_2) \in \mathbb{R}_{>0} \cdot (-\partial_{x_1} d_g(x_1, x_2), -\partial_{x_2} d_g(x_1, x_2)),$$

In particular, since $0 < d_g(x_1, x_2) = t < \epsilon < r_{inj}(M, g)$, x_1, x_2 are conjugate pairs, hence from the geometry of geodesic flow, we have

$$R^t(x_1, p_1) = (x_2, -p_2), \quad R^t(x_2, p_2) = (x_1, -p_1), \quad p_1, p_2 \neq 0 \tag{10}$$

On the other hand, since $(x_1, x_2; p_1, p_2) \in \mathbb{R}_{>0} \cdot SS^\infty(\mathcal{H}om(\pi_1^* F, \pi_2^! G))$, and since $p_1, p_2 \neq 0$, we have

$$(x_1, -p_1) \in \mathbb{R}_{>0} \cdot SS^\infty(F), \quad (x_2, p_2) \in \mathbb{R}_{>0} \cdot SS^\infty(G). \tag{11}$$

Hence, combining (10) and (11), we have

$$[(x_1, -p_1)] \in R^t(SS^\infty(G)) \cap SS^\infty(F) \subset R^t \Lambda^\infty \cap \Lambda^\infty$$

This contradicts with the condition on ϵ, hence finishes the proof of the Lemma. $\square$

Now we come back to the proof of the main proposition. We have

$$\begin{aligned}\mathrm{Hom}(F,G) &\simeq \Gamma(M,\mathcal{H}om(F,G))\\ &\simeq \Gamma(M\times M,\mathcal{H}om(\mathbb{C}_\Delta,\mathcal{H}om(\pi_1^*F,\pi_2^!G))\\ &\xrightarrow{\sim} \Gamma(M\times M,\mathcal{H}om(K_t,\mathcal{H}om(\pi_1^*F,\pi_2^!G))\\ &\simeq \Gamma(M\times M,\mathcal{H}om(\pi_1^*F,\mathcal{H}om(K_t,\pi_2^!G))\\ &\simeq \Gamma(M,\mathcal{H}om(F,\pi_{1*}\mathcal{H}om(K_t,\pi_2^!G))\\ &\simeq \mathrm{Hom}(F,\hat{R}^tG).\end{aligned}$$

where in the third step when we replace $\mathbb{C}_\Delta$ by K_t, we used the canonical morphism $K_t\to\mathbb{C}_\Delta$, and used Lemma 2.10 and Lemma 2.5(2) to show it is an quasi-isomorphism. □

We will use the following purely sheaf-theoretical statement later to study family of GKS quantization.

Proposition 2.11. *Let $I=(0,1)$, and $K_I\in Sh(M\times M\times I)$ be an isotopy of sheaves, such that $K_t=\mathbb{C}_{\Delta_t}$ for some closed subsets $\{\Delta_t\}_{0<t<1}$ satisfying*

$$\Delta_t\subset\Delta_s,\quad \forall 0<t<s<1,\ \text{and}\ \bigcap_{t\in I}\Delta_t=\Delta_M=\{(x,x):x\in M\}$$

Let $F,G\in Sh(M,\Lambda)$, and $\mathcal{H}om(\pi_1^*F,\pi_2^!G)\in Sh(M\times M)$ be the hom-sheaf. Assume

$$SS^\infty(K_t)\cap SS^\infty(\mathcal{H}om(\pi_1^*F,\pi_2^!G))=\emptyset,\quad \forall t\in I$$

then

$$\mathrm{Hom}(F,G)\simeq\mathrm{Hom}(F,K_t^!G)\simeq\mathrm{Hom}(K_{t!}F,G),\quad \forall t\in I$$

where $K_t^!,K_{t!}$ are defined in (7).

Its proof is exactly as in Proposition 2.9 (2), where the condition provided in Lemma 2.10 is put into the hypothesis, hence we do not repeat here.

2.6. **Limit of Contact Isotopy.** Here we consider the compactification of $I=(0,1)$ at 0 to $[0,1)$. Let $\Lambda_I^\infty,\Lambda_t^\infty$ be as before.

Denote the inclusions as

$$(0,1)\xhookrightarrow{j_I}[0,1)\xhookleftarrow{j_0}\{0\}.$$

Proposition 2.12. *Let F_I be an isotopy of sheaves, and $\Lambda_I^\infty=SS^\infty(F_I)$. Suppose the family $(\Lambda_t^\infty,t)\subset T^\infty M\times(0,1)$ has a closure in $T^\infty M\times[0,1)$ whose intersection with $T^\infty M\times\{0\}$ is a Legendrian Λ_0^∞. Then the sheaf*

$$F_0:=(j_0)^*(j_I)_*F_I. \tag{12}$$

is a constructible sheaf with $SS^\infty(F_0)\subset\Lambda_0^\infty$.

Proof. Suppose $(x,\xi)\notin\Lambda_0$, with $\xi\neq 0$. We build test function f in a small coordinate ball B around x, that $f(x)=0,df(x)=\xi$. We then want to show the following

$$F_I((B\cap\{f<\epsilon\})\times(0,\delta))\xrightarrow{\sim}F_I((B\cap\{f<-\epsilon\})\times(0,\delta)) \tag{13}$$

for small enough ϵ,δ and B. Since the limit of Λ_t^∞ does not contain $[(x,\xi)]$, hence for $0<t<t_0\ll 1$, we have an open conic neighborhood $\Omega\subset\dot{T}^*M$ of $(x,\xi)\in\dot{\Lambda}_0$, such that $\Lambda_t\cap\Omega=\emptyset$. In particular, we have

$$(\Omega\times T^*(0,t_0))\cap\Lambda_I=\emptyset$$

Thus, we may choose ϵ, δ and B small enough, that the retraction $(B \cap \{f < \epsilon\}) \times (0, \delta)$ to $(B \cap \{f < -\epsilon\}) \times (0, \delta)$ is non-characteristic, hence (13) is an quasi-isomorphism. □

Remark 2.13. *We thank E. Zaslow for suggesting this condition on the family. For general behavior of how singular support of sheaves behave under pushforward or pullback of constructible sheaf, we refer the reader to* [KS, Chapter 5,6].

3. Existence and Uniqueness of Extension

Theorem 3.1. *Let $\{\Lambda_t^\infty\}_{t\in I}$ be an isotopy of Legendrian in $T^\infty M$, such that Λ_t^∞ is constant for t outside a closed interval $[a,b] \subset I$. Assume $\{\Lambda_t^\infty\}$ is uniformly displaceable for some Reeb vector field R_α on $T^\infty M$, and there exists an isotopy of convex tubular neighborhoods $\{(U(\Lambda_t^\infty), \rho_t, X_t)\}$ of $\{\Lambda_t^\infty\}$. Denote the inclusion of slice by*

$$\iota_t : M_t := M \times \{t\} \hookrightarrow M_I := M \times I.$$

Then the restriction functor

$$\iota_t^* : Sh(M_I, \Lambda_I^\infty) \to Sh(M_t, \Lambda_t^\infty)$$

is an equivalence of category for all $t \in I$.

This theorem together with Proposition 1.10 implies our main theorem in the introduction.

In the remaining part of this section, we will sometimes identify $\Lambda_t^\infty \subset T^\infty M$ with $\mathcal{L}_t \subset S^*M$, and identify Reeb flow with geodesic flow.

3.1. Uniqueness of Extension.

Proposition 3.2. *Let Λ_t^∞ be a family of Legendrian in $T^\infty M$ that are uniformly displaceable with parameter ϵ. Then, the restriction functor ι_t^* is fully-faithful for all t.*

Proof. For $0 \leq s < \epsilon$, we define a family of kernels in $Sh((M_1 \times I_1) \times (M_2 \times I_2))$.

$$K_s := \mathbb{C}_{d(x_1,x_2)\leq s} \boxtimes \mathbb{C}_{t_1=t_2}.$$

One can check that K_s generate slice-wise geodesic flow, i.e., if $F_I \in Sh(M_I)$, and

$$K_s^! F_I := \pi_{1*}\mathcal{H}om(K_s, \pi_2^! F_I)$$

then we have

$$SS^\infty((K_s^! F_I)|_{M_t}) = R^s SS^\infty(F_I|_{M_t})$$

where π_i is the projection from $(M_1 \times I_1) \times (M_2 \times I_2)$ to $M_i \times I_i$, and R^s is the Reeb (geodesic) flow for time s.

We first prove the following claim: for any $F_I, G_I \in Sh(M_I, \Lambda_I^\infty)$, we have

$$\mathrm{Hom}(\mathbb{C}_{M\times(a,b)}, \mathcal{H}om(F_I, G_I)) \text{ is independent of } a < b.$$

Suffice to prove the case for the right end-point b. To use the estimate of the singular support of the hom-sheaf, we would like to perturb G_I by the fiberwise Reeb flow.

Lemma 3.3. *For any $0 < s < \epsilon$, we have*

$$\mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, G_I)) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I)).$$

The same is true if we replace $\{t\}$ by any sub-interval, eg. $[a,b]$, (a,b) of I. Furthermore, $\mathrm{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I))$ is independent of t.

Proof. Unwind the definition of $K_s^!$, we have

$$\begin{aligned} & \operatorname{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, K_s^! G_I)) \\ = \; & \operatorname{Hom}(\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(F_I, \pi_{1*}\mathcal{H}om(K_s, \pi_2^! G_I))) \\ = \; & \operatorname{Hom}(\mathbb{C}_{M\times\{t\}}, \pi_{1*}\mathcal{H}om(\pi_1^* F_I, \mathcal{H}om(K_s, \pi_2^! G_I))) \\ = \; & \operatorname{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \end{aligned}$$

We claim that

$$SS^\infty(\pi_1^*\mathbb{C}_{M\times\{t\}}) \cap SS^\infty \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)) = \emptyset, \; \forall\, 0 < s < \epsilon. \tag{14}$$

By the same argument as Proposition 2.9 and Lemma 2.10, we have

$$SS^\infty(K_s) \cap SS^\infty(\mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)) = \emptyset, \; \forall\, 0 < s < \epsilon.$$

Hence

$$SS(\mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \subset (-\Lambda_I, \Lambda_I) - SS(K_s). \tag{15}$$

On the other hand

$$\begin{aligned} SS^\infty(\pi_1^*\mathbb{C}_{M\times\{t\}}) = \{&[(x_1, t_1; \xi_1, \tau_1), (x_2, t_2; \xi_2, \tau_2)] : \\ &\xi_1 = \xi_2 = 0, \tau_2 = 0, (t_1, \tau_1) = (t, \pm 1)\} \end{aligned} \tag{16}$$

If (14) is false, and contains a non-empty intersection point, then at the intersection we have

$$\tau_1 + \tau_2 = \pm 1 \neq 0, \xi_1 = \xi_2 = 0$$

from (16). From (15), suppose that we

$$(x_1', t_1'; \xi_1', \tau_1'), (x_2', t_2'; \xi_2', \tau_2') \in (-\Lambda_I, \Lambda_I)$$

where $\xi_i' = 0$ imply $\tau_i' = 0$ for $i = 1, 2$ respectively, and

$$(x_1', t_1'; \xi_1'', \tau_1''), (x_2', t_2'; \xi_2'', \tau_2'') \in -SS(K_s)$$

where $t_1' = t_2'$, $\tau_1'' + \tau_2'' = 0$, and if $\xi_1'' \neq 0$ iff $\xi_2' \neq 0$ and if true implies $d(x_1', x_2') = s$. Since $K_s|_t \circ \Lambda_t$ is disjoint from Λ_t away from zero-section, hence there is no non-trivial solution to

$$\xi_1' + \xi_1'' = 0, \; \xi_2' + \xi_2'' = 0$$

ie. each summand in each equation vanishes. That implies $\tau_i' = 0$. Then $\tau_1'' + \tau_2'' = 0$ contracdicts with $\tau_1 + \tau_2 \neq 0$. Hence we proved the Lemma.

From this claim, and

$$\begin{aligned} &\operatorname{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}}, \mathcal{H}om(K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \\ &\simeq \operatorname{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_s, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \end{aligned}$$

we may apply Lemma 2.5 (2) on shrinking closed set, to get

$$\operatorname{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_0, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I))) \simeq \operatorname{Hom}(\pi_1^*\mathbb{C}_{M\times\{t\}} \otimes K_0, \mathcal{H}om(\pi_1^* F_I, \pi_2^! G_I)))$$

for all $0 < s < \epsilon$. This proves the first statement of the Lemma.

The final statement of the Lemma follows from (14), then we may apply Proposition 2.8 where the first slot is $\mathbb{C}_{M\times\{t\}}$ and the second slot in hom $\mathcal{H}om(F_I, K_s^! G_I)$ is taken as a constant isotopy of sheaf with any fixed $0 < s < \epsilon$. The case for sub-interval can be proved similarly, and we omit the details. ∎

Now, we finish to prove the proposition. By Lemma 3.3,

$$\mathrm{Hom}(\mathbb{C}_{M\times(a,b)}, \mathcal{H}om(F_I, G_I))$$

is independent of (a,b), hence we may shrink from $(0,1)$ to an arbitrary small neighborhood of t. Then we have

$$\begin{aligned}\mathrm{Hom}(F_I, G_I)) &\simeq [\pi_{I*}(\mathcal{H}om(F_I, G_I)]_t \simeq [\pi_{I*}(\mathcal{H}om(F_I, K_s^! G_I)]_t \\ &\simeq \mathrm{Hom}(\iota_t^* F_I, \iota_t^* K_s^! G_I) \simeq \mathrm{Hom}(F_t, R^s G_t) \simeq \mathrm{Hom}(F_t, G_t)\end{aligned}$$

where $0 < s < \epsilon$, and we used small Reeb perturbation to make $F_I, K_s^! G_I$ non-charactersitic isotopy of sheaves, then apply (8) in Proposition 2.8. □

Proposition 3.4. *Let $\{\Lambda_t^\infty\}$ be a family of Legendrian in $T^\infty M$ that are uniformly displaceable with parameter ϵ. For a given t, let $F_t \in Sh(M, \Lambda_t^\infty)$. Suppose we have F_I' and F_I'' in $Sh(M_I, \Lambda_I^\infty)$ and isomorhpism*

$$f : F_I'|_t \xrightarrow{\sim} F_t, \quad g : F_I''|_t \xrightarrow{\sim} F_t,$$

then there exist canonical isomorphism

$$\Phi : F_I' \to F_I''$$

such that $\Phi|_t = g^{-1} \circ f : F_I'|_t \to F_I''|_t$.

Proof. By Proposition 3.2, we have $\mathrm{Hom}(F_I'|_t, F_I''|_t) \simeq \mathrm{Hom}(F_I', F_I'')$. Thus,

$$g^{-1} \circ f \in \mathrm{Hom}(F_I'|_t, F_I''|_t) \mapsto \Phi \in \mathrm{Hom}(F_I', F_I'').$$

Similarly

$$f \circ g^{-1} \in \mathrm{Hom}(F_I''|_t, F_I'|_t) \mapsto \Psi \in \mathrm{Hom}(F_I'', F_I').$$

Hence we have

$$\Phi_t \circ \Psi_t \simeq id_{F_t''} \in \mathrm{Hom}(F_I''|_t, F_I''|_t) \mapsto \Phi \circ \Psi \simeq id_{F_I''} \in \mathrm{Hom}(F_I'', F_I'').$$

and similarly $\Psi \circ \Phi \simeq id_{F_I'}$. □

3.2. **Existence of Local Extension.**

Proposition 3.5. *Let $\{\mathcal{L}_t\}$ be a family of Legendrian in S^*M that admits a family of convex tubular neighborhood thickening $\{(U, \rho, X)_t\}$. Then for any compact subset $K \subset I$, there exists $\delta > 0$ such that for any $t \in K$ and $F_t \in Sh(M, \mathcal{L}_t)$, there exists $F_J \in Sh(M \times J, \mathcal{L}_J)$ where $J = I \cap (t-\delta, t+\delta) \subset I$, such that $F_J|_t \xrightarrow{\sim} F_t$ canonically.*

Proof. Define Q as neighborhood of diagonal in $I \times I$

$$Q = \{(s,t) \in I \times I \mid \mathcal{L}_s \in U_{1/2}(\mathcal{L}_t),\ \mathcal{L}_t \in U_{1/2}(\mathcal{L}_s), \}$$

Then, we may find $\delta = \delta(K, \{U_t, \rho_t\})$ small enough such that

$$\Delta_{K,\delta} = \bigcup_{t\in K} [t-\delta, t+\delta] \times [t-\delta, t+\delta]$$

is contained in Q.

For any $t \in K$, let $J = I \cap (t-\delta, t+\delta) \subset I$. For any $s \in J$, we consider the trajectory of $\mathcal{L}_t$ under the retracting flow $-X_s$, and get an isotopy of Legendrians over $[0,\infty)$ as $X_s^{-T}(\mathcal{L}_t)$. We claim that the Gromov-Hausdorff limit of $X_s^{-T}(\mathcal{L}_t) \subset S^*M$ is $\mathcal{L}_s$, since

$$X_s^{-T}(\mathcal{L}_t) \subset X_s^{-T}(U_{1/2}(\mathcal{L}_s)) \subset U_{e^{-c_s T}/2}(\mathcal{L}_s) \to \mathcal{L}_s, \text{ as } T \to \infty,$$

where c_s is the shrinking rate $\langle d\rho_s, X_s \rangle > c_s \rho_s$ in the definition of (U, ρ, X). Thus we may define the limit of the corresponding isotopy of sheaves

$$\Pi_s(F_t) := (j_\infty)^*(j_{[0,\infty)})_*(\hat{X}_s^{-[0,\infty)} F_t)$$

where

$$\hat{X}_s^{-[0,\infty)} : Sh(M) \to Sh(M \times [0,\infty))$$

is the sheaf quantization of the flow X_s^{-T} and

$$j_{[0,\infty)} : \ [0,\infty) \hookrightarrow \ [0,\infty] \ \hookleftarrow \{\infty\} : \ j_\infty$$

are inclusion into the compactification, and we also abuse notation to denote $id_M \times j$ as j. By Proposition 2.12, we have

$$SS^\infty(\Pi_s(F_t)) \subset \mathcal{L}_s.$$

We claim that the collection of sheaves $\{\Pi_s(F_t)\}_{s\in J}$ assemble into an isotopy of sheaf, $\Pi_J(F_t) \in Sh(M_J, \mathcal{L}_J)$. Indeed since the contact flow X_s varies smoothly with parameter s, we have a (tensor) kernel for the family

$$K_J \in Sh(M_J \times (M_J \times [0,\infty))),$$

such that we have

$$\Pi_J(F_t) := (id_{M_J} \times j_\infty)^*(id_{M_J} \times j_{[0,\infty)})_*((K_J)_!(F_t \boxtimes \mathbb{C}_J)).$$

Thus we get the extension sheaf $\Pi_J(F_t)$, and one can check $\Pi_J(F_t)|_t \simeq F_t$ since the retraction flow X_t^{-T} preserves the Legendrian $\mathcal{L}_t$. □

3.3. **Proof of Theorem 3.1.** Let $K = [a,b] \subset (0,1)$, and we apply Proposition 3.5 to get the positive constant $\delta > 0$, such that for any $t \in K$, we may extend a sheaf $F_t \in Sh(M, \mathcal{L}_t)$ to a neighborhood $B_\delta(t) = (t-\delta, t+\delta)$, compatible with the Legendrian condition $\mathcal{L}_I$ restricted on the interval. We may take a finite set of points

$$A = \{t_n \in I \mid t_n = t + (n/2)\delta/2, n \in \mathbb{Z}, t_n \in [a,b]\}$$

and extend the sheaf F_t from $M \times \{t\}$ inductively to $M \times B_{(1+n/2)\delta}(t)$ for $n = 0, 1, 2, \cdots$, using existence of local extension and uniqueness of extension. Finally, since the isotopy is constant outside $[a,b]$, we may trivially extend from $[a,b]$ to $(0,1)$. This finishes the proof of Theorem 3.1.

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