

ESSENTIAL SPECTRUM FOR MAXWELL'S EQUATIONS

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ABSTRACT. We study the essential spectrum of operator pencils associated with anisotropic Maxwell equations, with permittivity ε , permeability μ and conductivity σ , on finitely connected unbounded domains. The main result is that the essential spectrum of the Maxwell pencil is the union of two sets: namely, the spectrum of the pencil $\operatorname{div}((\omega\varepsilon+i\sigma)\nabla\cdot)$, and the essential spectrum of the Maxwell pencil with constant coefficients. We expect the analysis to be of more general interest and to open avenues to investigation of other questions concerning Maxwell's and related systems.

1. INTRODUCTION

In this paper we consider the essential spectrum of linear operator pencils arising from the Maxwell system

$$(1) \quad \begin{cases} \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E & \text{in } \Omega, \\ \operatorname{curl} E = i\omega\mu H & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^3$ is a finitely connected domain, with boundary condition

$$\nu \times E = 0 \text{ on } \partial\Omega$$

if Ω has a boundary. In these equations ω is the pencil spectral parameter, ε the electric permittivity, μ the magnetic permeability and σ is the conductivity; ν is the unit normal to the boundary.

Lassas [16] already studied this problem on a bounded domain with $C^{1,1}$ boundary so in this article our primary concern is to treat unbounded domains which provides additional sources for essential spectrum. However, even for bounded domains, we are able to relax the required boundary regularity to Lipschitz continuity. Like Lassas we allow the permittivity, permeability and conductivity to be tensor valued (i.e. we allow anisotropy); however we make the physically realistic assumption that, at infinity, these coefficients approach isotropic constant values.

Maxwell systems in infinite domains are usually studied in the context of scattering, with a Silver-Müller radiation condition imposed at infinity, see, e.g. [18, p. 10] and [6, 5]. Scattering theory is sometimes regarded as the study of solutions when the spectral parameter lies in the essential spectrum, though the fact that the Maxwell system already has non-trivial essential spectrum in bounded domains

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indicates that such an interpretation involves local conditions as well as the study of radiation to infinity. The case of zero conductivity $\sigma \equiv 0$ is substantially simpler, both for bounded and unbounded domains. However it is also physically unrealistic in numerous applications, including imaging [7, 12, 14, 15].

The main technical difficulty in dealing with the essential spectrum of Maxwell systems in infinite domains is the fact that compactly supported perturbations to the coefficients do change the essential spectrum, as is clear even for bounded domains from [16]. This means that techniques such as Glazman decomposition, useful for Schrödinger operators, are no longer helpful. We use instead a Helmholtz decomposition inspired by [3, 1] together with further decompositions of the resulting 2×2 block operator matrices. As in [2], this approach allows us to substantially reduce Maxwell's system to an elliptic problem. The main result is stated in Theorem 5: the essential spectrum of the Maxwell pencil is the union of two sets: namely, the spectrum of the pencil $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ acting between suitable spaces, together with the essential spectrum of the Maxwell pencil with constant coefficients. The spectral geometric question of how the topology of Ω at infinity is reflected in the essential spectrum of a constant coefficient Maxwell operator is also interesting, and an avenue for future work.

Our original motivation for the investigations in this paper came from our study of inverse problems in a slab for the Maxwell system with conductivity. However a knowledge of the essential spectrum has much more fundamental importance. It is a first step towards determination of the absolutely continuous subspace of an operator and hence the behaviour of its semi-group, as required, e.g., for the study of Vlasov-Maxwell systems. It can also be a key component in the analysis of certain types of homogenisation problem.

2. MAIN RESULT

We shall study the Maxwell system on a finitely-connected domain $\Omega \subseteq \mathbb{R}^3$. Prototype examples include exterior domains $\Omega := \mathbb{R}^3 \setminus \overline{\Omega'}$ in which Ω' has finitely many simply connected components; the case of an infinite slab, $\Omega = \mathbb{R}^2 \times (0, 1)$, or a half-space $\Omega = \mathbb{R}^2 \times (0, \infty)$; domains with cylindrical ends, such as waveguides; and indeed the case $\Omega = \mathbb{R}^3$ (see Assumption 13 and Proposition 14 below for more details). The boundary $\partial\Omega$, if non-empty, will be of Lipschitz type, and the coefficients ε , σ and μ will be assumed to lie in $L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ and be such that for some $\Lambda > 0$ and every $\eta \in \mathbb{R}^3$

$$(2) \quad \Lambda^{-1}|\eta|^2 \leq \eta \cdot \varepsilon \eta \leq \Lambda|\eta|^2, \quad \Lambda^{-1}|\eta|^2 \leq \eta \cdot \mu \eta \leq \Lambda|\eta|^2, \quad 0 \leq \eta \cdot \sigma \eta \leq \Lambda|\eta|^2$$

almost everywhere in Ω .

As already mentioned, the case of bounded domains was treated by Lassas [16] under slightly stronger regularity assumptions; for infinite domains we assume that all the coefficients have a 'value at infinity' in the precise sense that

$$(3) \quad \lim_{x \rightarrow \infty} \mu(x) = \mu_0 I, \quad \lim_{x \rightarrow \infty} \varepsilon(x) = \varepsilon_0 I, \quad \lim_{x \rightarrow \infty} \sigma(x) = \sigma_0 I,$$

for some scalar values $\mu_0 > 0$, $\varepsilon_0 > 0$ and $\sigma_0 \geq 0$. To allow a unified treatment of unbounded and bounded domains, it is convenient to assign values to ε_0 , μ_0 and σ_0 when Ω is bounded, and we choose

$$(4) \quad \varepsilon_0 := 1, \quad \mu_0 := 1; \quad \sigma_0 := 0, \quad (\Omega \text{ bounded}).$$

Several function spaces arise commonly in the study of Maxwell systems; to fix notation, we denote

$$\begin{aligned}\mathcal{H}(\operatorname{curl}, \Omega) &:= \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{curl} u \in L^2(\Omega; \mathbb{C}^3)\}, \\ \mathcal{H}(\operatorname{div}, \Omega) &:= \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{div} u \in L^2(\Omega)\}.\end{aligned}$$

If $\partial\Omega$ is non-empty then we let ν denote the outward unit normal vector, and define

$$\mathcal{H}_0(\operatorname{curl}, \Omega) = \{u \in \mathcal{H}(\operatorname{curl}, \Omega) : \nu \times u|_{\partial\Omega} = 0\},$$

with the understanding that when $\Omega = \mathbb{R}^3$ then $\mathcal{H}_0(\operatorname{curl}, \Omega) = \mathcal{H}(\operatorname{curl}, \Omega)$.

We start by considering, in the Hilbert space

$$(5) \quad \mathcal{H}_1 := \mathcal{H}_0(\operatorname{curl}, \Omega) \oplus \mathcal{H}(\operatorname{curl}, \Omega),$$

the operator pencil $\omega \mapsto V_\omega$ defined from (1) in the space \mathcal{H}_1 by

$$(6) \quad \begin{aligned}V_\omega: \mathcal{H}_1 &\longrightarrow L^2(\Omega; \mathbb{C}^3)^2, \\ (E, H) &\longmapsto (\operatorname{curl} H + i(\omega\varepsilon + i\sigma)E, \operatorname{curl} E - i\omega\mu H).\end{aligned}$$

Our aim is to study the essential spectrum of the pencil V_ω .

Definition 1. Let H_1 and H_2 be two Hilbert spaces. For each $\omega \in \mathbb{C}$, let $L_\omega : H_1 \rightarrow H_2$ be a bounded linear operator. We say that ω lies in the essential spectrum of the pencil $\omega \mapsto L_\omega$ if 0 lies in the $\sigma_{e,2}$ essential spectrum of the operator L_ω as defined in [9, Ch. I, §4]; explicitly, if L_ω is not in the class \mathcal{F}_+ of semi-Fredholm operators with finite-dimensional kernel.

Remark 2.

- (a) By [9, Ch. I, Cor. 4.7], the statement ‘0 lies in $\sigma_{e,2}(L_\omega)$ ’ is equivalent to the statement that there exists a *Weyl singular sequence* (u_n) in H_1 with $\|u_n\|_{H_1} = 1$ and $u_n \rightarrow 0$ in H_2 such that $\|L_\omega u_n\|_{H_2} \rightarrow 0$.
- (b) We shall often abuse terminology and say ‘ ω lies in the essential spectrum of L_ω ’ or write ‘ $\omega \in \sigma_{ess}(L_\omega)$ ’.
- (c) In our situation we deal exclusively with densely defined operators having closed range. By [9, Ch. I, Thm. 3.7] such operators are semi-Fredholm with finite-dimensional kernel if and only if they are Fredholm; in the terminology of [9, Ch. I, §4], the essential spectra $\sigma_{e,2}$, $\sigma_{e,3}$ and $\sigma_{e,4}$ coincide and we have the following equivalent characterisations of the essential spectrum:

$$\begin{aligned}\omega \in \sigma_{ess}(L_\omega) &\iff L_\omega \text{ is not Fredholm} \\ &\iff \text{for every compact } K, L_\omega - K \text{ is not invertible.}\end{aligned}$$

The last equivalence is a classical characterisation of Fredholm operators, see, e.g., Kantorovich and Akilov [11, Chapter XIII, §5].

Finally, we introduce some homogeneous Sobolev spaces which are required for the Helmholtz decomposition for unbounded domains. For bounded domains these coincide with the usual Sobolev spaces.

Definition 3.

(1) (Ω unbounded) The homogeneous Sobolev spaces $\dot{H}_0^1(\Omega)$ and $\dot{H}^1(\Omega)$ are the completions of the Schwartz spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}(\bar{\Omega})$, respectively, with respect to the norm $\|u\| := \|\nabla u\|_{L^2(\Omega)}$.

(2) (Ω bounded) In this case we define the homogeneous Sobolev spaces to coincide with the usual Sobolev spaces: $\dot{H}_0^1(\Omega) = H_0^1(\Omega)$ and $\dot{H}^1(\Omega) = H^1(\Omega)$.

Remark 4.

- (a) Note that this definition does not coincide with Definition 1.31 in [4], which uses Fourier transforms to define $\dot{H}^s(\mathbb{R}^d)$ and results in spaces which are not complete if $s \geq d/2$. Our definition follows Dautray and Lions [8]. For clarity, we use our definition directly in Appendix A below.
- (b) If K is any compact subset of Ω with non-empty interior and Ω is bounded, then the usual H^1 -norm is equivalent to

$$(7) \quad \|u\|^2 := \|u\|_{L^2(K)}^2 + \|\nabla u\|_{L^2(\Omega)}^2,$$

see Maz'ya [17]. In the case when Ω is unbounded, the norms on \dot{H}^1 and \dot{H}_0^1 may be shown to be equivalent to the norm defined in (7), for any compact $K \subset \Omega$ with non-empty interior. Thus an equivalent definition of $\dot{H}^1(\Omega)$, valid for bounded and unbounded Ω , is the closure of $\mathcal{D}(\Omega)$ in the norm (7). However for unbounded Ω this is no longer equivalent to the H^1 -norm; e.g. the function given in polar coordinates by $u(r) = 1/(r+1)^{3/2}$ does not lie in $H^1(\mathbb{R}^3)$ but lies in $\dot{H}^1(\mathbb{R}^3)$.

We are now ready to state our main result.

Theorem 5. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain satisfying Assumption 13 (given below) and $\varepsilon, \sigma, \mu \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ satisfy (2), (3) if Ω is unbounded, and (4) if Ω is bounded. We have*

$$\sigma_{ess}(V_\omega) = \sigma_{ess}(\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)) \cup \sigma_{ess}(V_\omega^0),$$

where $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ acts from $\dot{H}_0^1(\Omega; \mathbb{C})$ to its dual $\dot{H}^{-1}(\Omega; \mathbb{C})$ and V_ω^0 is the Maxwell pencil with constant coefficients ε_0, μ_0 and σ_0 .

Thanks to this result, the essential spectrum of the Maxwell pencil is decomposed into two parts.

- The essential spectrum of the operator $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$: this component depends on the coefficients ε and σ directly. In particular, in the case when the coefficients ε and σ are continuous, it consists of the closure of the set of $\omega = i\nu$, $\nu \in \mathbb{R}$, for which $\nu\varepsilon + \sigma$ is indefinite at some point in Ω : see Proposition 24.
- The essential spectrum of the constant coefficient Maxwell pencil: this component is related to the geometry of Ω , and depends on the coefficients only through their values at infinity. It can be computed explicitly in many cases of interest: we provide several examples below.

In the next examples we will calculate the essential spectrum of V_ω^0 , where

$$V_\omega^0(E, H) = (\operatorname{curl} H + i(\omega\varepsilon_0 + i\sigma_0)E, \operatorname{curl} E - i\omega\mu_0 H),$$

for different choices of domains Ω .

Example 6. The simplest case to consider in the calculation of $\sigma_{ess}(V_\omega^0)$ is when Ω is bounded. By (4) we have $\varepsilon_0 = \mu_0 = 1$ and $\sigma_0 = 0$. Thus, the pencil is self-adjoint, and we have

$$\sigma_{ess}(V_\omega^0) = \{0\},$$

see [16, 18, 13].

Example 7. We consider here the case of the full space $\Omega = \mathbb{R}^3$. We can make use of the Fourier transform to obtain a simple expression of this operator. Writing

$E(x) = \int_{\mathbb{R}^3} \hat{E}(\xi) e^{ix \cdot \xi} d\xi$, the expression of the operator $\text{curl } E$ in the Fourier domain is given by the multiplication operator $iC(\xi)\hat{E}(\xi)$, where

$$(8) \quad C(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Writing $\text{curl } H$ in a similar way, we immediately see that V_ω^0 is represented, in the Fourier domain, by the multiplication by the matrix

$$A_\omega(\xi) = \begin{pmatrix} i(\omega\varepsilon_0 + i\sigma_0)I & iC(\xi) \\ iC(\xi) & -i\omega\mu_0 I \end{pmatrix}.$$

A direct calculation gives

$$\det(A_\omega(\xi)) = k_\omega(|\xi|^2 - k_\omega)^2, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0).$$

By a standard argument, we obtain that $\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : \det(A_\omega(\xi)) = 0 \text{ for some } \xi \in \mathbb{R}^3\}$, so that

$$\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : k_\omega \geq 0\}.$$

In the particular case when the conductivity at infinity is zero, i.e. $\sigma_0 = 0$, we simply have $\sigma_{ess}(V_\omega^0) = \mathbb{R}$.

Example 8. Let us look at the case of the slab $\Omega = \{x = (x', x_3) \in \mathbb{R}^3 : 0 < x_3 < L\}$, for some $L > 0$. The derivation is very similar to the one presented above for the full space, the only difference being that the continuous Fourier transform in the third variable becomes a Fourier series. As a consequence, the continuous variable ξ_3 is replaced by a discrete variable $n = 0, 1, \dots$. More precisely,

$$E_j(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \hat{E}_j(\xi', n) e^{ix' \cdot \xi'} \sin\left(\frac{n\pi}{L} x_3\right) d\xi', \quad j = 1, 2,$$

$$E_3(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \hat{E}_3(\xi', n) e^{ix' \cdot \xi'} \cos\left(\frac{n\pi}{L} x_3\right) d\xi',$$

and, analogously,

$$H_j(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \hat{H}_j(\xi', n) e^{ix' \cdot \xi'} \cos\left(\frac{n\pi}{L} x_3\right) d\xi', \quad j = 1, 2,$$

$$H_3(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \hat{H}_3(\xi', n) e^{ix' \cdot \xi'} \sin\left(\frac{n\pi}{L} x_3\right) d\xi';$$

the range of n in each summation has been determined by the boundary conditions on $x_3 = 0$ and $x_3 = L$. Compared to the full space in Example 7, the continuous frequency variable $\xi \in \mathbb{R}^3$ has become $\xi := (\xi', \frac{n\pi}{L}) \in \mathbb{R}^2 \times (\frac{\pi}{L}\mathbb{N})$. By calculations similar to those for the full space, we see that the essential spectrum is the set of $\omega \in \mathbb{C}$ such that for some $\xi \in \mathbb{R}^2 \times (\frac{\pi}{L}\mathbb{N})$

$$k_\omega(|\xi|^2 - k_\omega)^2 = 0, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0);$$

and it is easy to see that this coincides with the essential spectrum for the full space problem.

Example 9. We now compute the essential spectrum of V_ω^0 in a cylinder $\Omega = \{x \in \mathbb{R}^3 : 0 < x_2 < L_1, 0 < x_3 < L_2\}$. As above, let us expand E and H in Fourier coordinates as

$$\begin{aligned} E_1(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_1(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi, \\ E_2(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_2(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi, \\ E_3(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_3(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi \end{aligned}$$

and

$$\begin{aligned} H_1(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_1(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi, \\ H_2(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_2(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi, \\ H_3(x_1, x_2, x_3) &= \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_3(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi. \end{aligned}$$

In order to guarantee uniqueness of the expansions, set

$$(9) \quad \begin{aligned} \hat{H}_2(0, n_2, \xi) &= 0, & \hat{H}_3(n_1, 0, \xi) &= 0, & \hat{E}_1(0, n_2, \xi) &= 0, \\ \hat{E}_1(n_1, 0, \xi) &= 0, & \hat{E}_2(n_1, 0, \xi) &= 0, & \hat{E}_3(0, n_2, \xi) &= 0, \end{aligned}$$

for every $n \in \mathbb{N}^2$ and $\xi \in \mathbb{R}$.

A direct calculation gives that the operators $E \mapsto \text{curl } E$ and $H \mapsto \text{curl } H$ may be written in Fourier coordinates as the multiplication operators by the matrices

$$C\left(i\xi, \frac{\pi}{L_1} n_1, \frac{\pi}{L_2} n_2\right) \quad \text{and} \quad C\left(i\xi, -\frac{\pi}{L_1} n_1, -\frac{\pi}{L_2} n_2\right),$$

respectively, where the matrix C is defined in (8). As a consequence, in the Fourier domain, V_ω^0 is a multiplication operator represented by the matrix

$$A_\omega(n, \xi) = \begin{pmatrix} i(\omega\varepsilon_0 + i\sigma_0)I & C\left(i\xi, -\frac{\pi}{L_1} n_1, -\frac{\pi}{L_2} n_2\right) \\ C\left(i\xi, \frac{\pi}{L_1} n_1, \frac{\pi}{L_2} n_2\right) & -i\omega\mu_0 I \end{pmatrix}.$$

A further calculation yields

$$\det(A_\omega(n, \xi)) = k_\omega \left(\xi^2 + \frac{\pi^2}{L_1^2} n_1^2 + \frac{\pi^2}{L_2^2} n_2^2 - k_\omega \right)^2, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0).$$

If ω is such that $\det(A_\omega(n, \xi)) \neq 0$ for every $n \in \mathbb{N}^2$ and $\xi \in \mathbb{R}$, then ω does not belong to the essential spectrum of V_ω^0 . On the other hand, suppose that ω is such that $\det(A_\omega(n, \xi)) = 0$ for some $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$. If $n_1 = n_2 = 0$, it is easy to see that there are no nonzero elements of $\text{Ker } A_\omega(n, \xi)$ satisfying (9). On the other hand, the vector $(0, \omega\mu_0 L_2, 0, \pi i, 0, \xi L_2)$ belongs to $\text{Ker } A_\omega(0, 1, \xi)$ and satisfies (9) (and similarly if $n_1 = 1$ and $n_2 = 0$). As a consequence, we have that

$$\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : k_\omega = 0 \text{ or } k_\omega \geq \frac{\pi^2}{L^2}\}, \quad L = \max(L_1, L_2).$$

In the particular case when $\sigma_0 = 0$, this set takes the simpler form

$$\sigma_{ess}(V_\omega^0) = \left(-\infty, -\frac{\pi}{L\sqrt{\varepsilon_0\mu_0}}\right] \cup \{0\} \cup \left[\frac{\pi}{L\sqrt{\varepsilon_0\mu_0}}, +\infty\right).$$

Note that this set approaches the essential spectrum for the slab as $L \rightarrow +\infty$. This is expected: as L increases the cylinder becomes larger and larger in one direction.

3. HELMHOLTZ DECOMPOSITION AND RELATED OPERATORS

We shall treat both bounded and unbounded Lipschitz domains $\Omega \subseteq \mathbb{R}^3$. The latter are our primary interest, as the bounded case has already been studied by Lassas [16], albeit under slightly stronger assumptions on the boundary regularity. However, in the definitions which follow, we deal with both cases.

The first decomposition result which we require is true without restrictions on the topology of Ω . Although it is standard, we present a proof since it shows how the homogeneous Sobolev spaces arise in a natural way.

Lemma 10. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

(1) *The space $L^2(\Omega; \mathbb{C}^3)$ admits the following orthogonal decompositions:*

$$(10a) \quad L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}_0^1(\Omega) \oplus \mathcal{H}(\operatorname{div} 0, \Omega),$$

$$(10b) \quad L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}^1(\Omega) \oplus \mathcal{H}_0(\operatorname{div} 0, \Omega),$$

in which

$$\mathcal{H}(\operatorname{div} 0, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u = 0\},$$

$$\mathcal{H}_0(\operatorname{div} 0, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) \mid \operatorname{div} u = 0, \nu \cdot u|_{\partial\Omega} = 0\}.$$

(2) *The spaces $\mathcal{H}_0(\operatorname{curl}, \Omega)$ and $\mathcal{H}(\operatorname{curl}, \Omega)$ admit the orthogonal decompositions*

$$(11a) \quad \mathcal{H}_0(\operatorname{curl}, \Omega) = \nabla \dot{H}_0^1(\Omega; \mathbb{C}) \oplus (\mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}(\operatorname{div} 0, \Omega)),$$

$$(11b) \quad \mathcal{H}(\operatorname{curl}, \Omega) = \nabla \dot{H}^1(\Omega; \mathbb{C}) \oplus (\mathcal{H}(\operatorname{curl}, \Omega) \cap \mathcal{H}_0(\operatorname{div} 0, \Omega)).$$

Proof. (1) The operator $\nabla: \dot{H}_0^1(\Omega) \rightarrow L^2(\Omega; \mathbb{C}^3)$ is an isometry, and so $\nabla \dot{H}_0^1(\Omega)$ is closed in $L^2(\Omega; \mathbb{C}^3)$. It remains to prove that $(\nabla \dot{H}_0^1(\Omega))^\perp = \mathcal{H}(\operatorname{div} 0, \Omega)$. Suppose that $\phi \in (\nabla \dot{H}_0^1(\Omega))^\perp$; then $\langle \phi, \nabla v \rangle = 0$ for all $v \in \mathcal{D}(\Omega)$, which means that $\langle \operatorname{div} \phi, v \rangle = 0$ for all $v \in \mathcal{D}(\Omega)$. This proves that $\phi \in \mathcal{H}(\operatorname{div} 0, \Omega)$. Conversely, if $\phi \in \mathcal{H}(\operatorname{div} 0, \Omega)$ then for any $v \in \mathcal{D}(\Omega)$ we have $0 = \langle \operatorname{div} \phi, v \rangle = \langle \phi, \nabla v \rangle$. Taking the closure in the $\dot{H}_0^1(\Omega)$ -topology shows that $\langle \phi, \nabla v \rangle = 0$ for all $v \in \dot{H}_0^1(\Omega)$, which proves (10a).

Analogously, $\nabla \dot{H}^1(\Omega)$ is closed in $L^2(\Omega; \mathbb{C}^3)$. To prove (10b) suppose that $\phi \in (\nabla \dot{H}^1(\Omega))^\perp$; then certainly $\operatorname{div} \phi = 0$ since $(\nabla \dot{H}^1(\Omega))^\perp \subseteq (\nabla \dot{H}_0^1(\Omega))^\perp$. Thus for all $v \in \mathcal{D}(\overline{\Omega})$, we have $0 = \langle \phi, \nabla v \rangle = \int_{\partial\Omega} (\nu \cdot \phi) \bar{v} ds$. This means that $\nu \cdot \phi = 0$ on $\partial\Omega$ and so $\phi \in \mathcal{H}_0(\operatorname{div} 0, \Omega)$. The proof that any $\phi \in \mathcal{H}_0(\operatorname{div} 0, \Omega)$ lies in $(\nabla \dot{H}^1(\Omega))^\perp$ is straightforward.

(2) The decompositions (11) follow immediately from (10) by taking the appropriate subspaces. \square

To decompose the Maxwell pencil we need to decompose the spaces $\mathcal{H}(\operatorname{div} 0, \Omega)$ and $\mathcal{H}_0(\operatorname{div} 0, \Omega)$ further, by using vector potentials in some suitable spaces, which we now introduce.

Definition 11. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.

- The space $\dot{X}_T(\Omega)$ is the closure of $\mathcal{H}(\operatorname{curl}, \Omega) \cap \mathcal{H}_0(\operatorname{div} 0, \Omega)$ with respect to the seminorm $\|u\| := \|\operatorname{curl} u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} + \|u \cdot \nu\|_{H^{-1/2}(\partial\Omega)}$.
- The space $\dot{X}_N(\Omega)$ is the closure of $\mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}(\operatorname{div} 0, \Omega)$ with respect to the seminorm $\|u\| := \|\operatorname{curl} u\|_{L^2(\Omega)} + \|\operatorname{div} u\|_{L^2(\Omega)} + \|u \times \nu\|_{H^{-1/2}(\partial\Omega)}$.
- The space $K_T(\Omega)$ is the kernel of the curl operator restricted to $\dot{X}_T(\Omega)$, namely

$$K_T(\Omega) = \{u \in \dot{X}_T(\Omega) : \operatorname{curl} u = 0\}.$$

- The space $K_N(\Omega)$ is the kernel of the curl operator restricted to $\dot{X}_N(\Omega)$, namely

$$K_N(\Omega) = \{u \in \dot{X}_N(\Omega) : \text{curl } u = 0\}.$$

The spaces $K_T(\Omega)$ and $K_N(\Omega)$ are closed in $\dot{X}_T(\Omega)$ and in $\dot{X}_N(\Omega)$, respectively, and so we can consider the quotient spaces

$$\dot{X}_T(\Omega)/K_T(\Omega), \quad \dot{X}_N(\Omega)/K_N(\Omega).$$

The curl operator is well-defined and injective on these spaces. To avoid cumbersome notation, we will in the following identify $\text{curl } \psi$ for $\psi \in \dot{X}_T(\Omega)/K_T(\Omega)$ or $\psi \in \dot{X}_N(\Omega)/K_N(\Omega)$ with the vector in $L^2(\Omega; \mathbb{C}^3)$ given by curl acting on any representative of the equivalence class ψ . The curl operator maps these quotient spaces into the space of divergence free fields, with appropriate boundary conditions.

Lemma 12. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

- (1) *The space $\text{curl}(\dot{X}_T(\Omega)/K_T(\Omega))$ is contained in $\mathcal{H}(\text{div } 0, \Omega)$.*
- (2) *The space $\text{curl}(\dot{X}_N(\Omega)/K_N(\Omega))$ is contained in $\mathcal{H}_0(\text{div } 0, \Omega)$.*

Proof. Part (1) follows immediately from $\text{div} \circ \text{curl} = 0$. Part (2) follows from the identities $\text{div} \circ \text{curl} = 0$ and $(\text{curl } u) \cdot \nu = \text{div}_{\partial\Omega}(u \times \nu)$ on $\partial\Omega$ [18, (3.52)]. \square

We make the following assumption.

Assumption 13. The spaces $K_T(\Omega)$ and $K_N(\Omega)$ are finite-dimensional and

$$(12a) \quad \mathcal{H}(\text{div } 0, \Omega) = \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \oplus K_N(\Omega),$$

$$(12b) \quad \mathcal{H}_0(\text{div } 0, \Omega) = \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \oplus K_T(\Omega).$$

This assumption is verified in many cases of theoretical and practical interest.

Proposition 14. *Assumption 13 is verified in any of the following cases:*

- (1) $\Omega = \mathbb{R}^3$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
- (2) Ω is a bounded Lipschitz domain, satisfying Hypothesis 3.3 of [4];
- (3) Ω is a C^2 exterior domain, satisfying assumptions (1.45) of [8, Chapter IXA];
- (4) Ω is the half space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
- (5) Ω is the slab $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < L\}$ for some $L > 0$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
- (6) Ω is a cylinder $\mathbb{R} \times \Omega'$, where $\Omega' \subseteq \mathbb{R}^2$ is a simply connected bounded domain of class $C^{1,1}$ or piecewise smooth with no re-entrant corners (with $K_T(\Omega) = K_N(\Omega) = \{0\}$).

Remark 15. We have decided not to provide the details of the assumptions of parts (2) and (3), since they are rather lengthy and are not needed for the rest of the paper. In simple words, these assumptions require $\partial\Omega$ to be a finite union of connected surfaces and that there exist a finite number of *cuts* within Ω which divide it into multiple simply connected domains. The number of cuts is given by $\dim K_T(\Omega)$, and the number of connected components of $\partial\Omega$ by $\dim K_N(\Omega) + 1$. Thus, for simply-connected domains with connected boundaries the decomposition is even simpler: $K_T(\Omega)$ and $K_N(\Omega)$ are trivial and can be omitted.

Proof. (1) The decompositions (12a) and (12b) coincide, and simply follow from the identity $\hat{u}(\xi) = -\xi \times (\frac{\xi \times \hat{u}}{|\xi|^2})$, valid for every divergence-free field u (which implies $\xi \cdot \hat{u} = 0$), where \hat{u} denotes the Fourier transform of u . Alternatively, this is also a consequence of Proposition 26 and Lemma 27.

(2) This part is proved in [4] (see also [8, Chapter IXA] and [10, Chapter I, §3] for the smooth case). The construction of the spaces $K_T(\Omega)$ and $K_N(\Omega)$ is described explicitly.

(3) The decompositions in this case are proved in [8, Chapter IXA].

(4)-(5)-(6) The arguments are standard and explicit, but it is not easy to find precise statements in the literature. We detail the derivation in Appendix A, which contains a general construction for a larger class of cylinders. \square

Combining (10) and (12), we obtain that the space $L^2(\Omega; \mathbb{C}^3)$ admits the following orthogonal decompositions:

$$(13a) \quad L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}_0^1(\Omega; \mathbb{C}) \oplus \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \oplus K_N(\Omega),$$

$$(13b) \quad L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}^1(\Omega; \mathbb{C}) \oplus \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \oplus K_T(\Omega).$$

In view of these decompositions, to every vector field in $L^2(\Omega; \mathbb{C}^3)$ we can associate the unique vector potentials in $\dot{X}_T(\Omega)/K_T(\Omega)$ and in $\dot{X}_N(\Omega)/K_N(\Omega)$.

Lemma 16. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain satisfying Assumption 13. There exist bounded operators $T_N : L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{X}_N(\Omega)/K_N(\Omega)$ and $T_T : L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{X}_T(\Omega)/K_T(\Omega)$ such that*

$$(14) \quad \begin{aligned} T_N \text{curl} \Phi &= \Phi, \quad \Phi \in \dot{X}_N(\Omega)/K_N(\Omega), \\ T_N \nabla q &= 0, \quad q \in \dot{H}^1(\Omega; \mathbb{C}); \quad T_N f = 0, \quad f \in K_T(\Omega); \\ T_T \text{curl} \Phi &= \Phi, \quad \Phi \in \dot{X}_T(\Omega)/K_T(\Omega), \\ T_T \nabla q &= 0, \quad q \in \dot{H}_0^1(\Omega; \mathbb{C}); \quad T_T f = 0, \quad f \in K_N(\Omega). \end{aligned}$$

Proof. In view of (13b), every $F \in L^2(\Omega; \mathbb{C}^3)$ admits a unique decomposition into three orthogonal vectors,

$$F = \nabla q + \text{curl} \Phi + f,$$

with $q \in \dot{H}^1(\Omega; \mathbb{C})$, $\Phi \in \dot{X}_N(\Omega)/K_N(\Omega)$ and $f \in K_T(\Omega)$. We define T_N by $T_N F = \Phi$, so that $T_N \text{curl} \Phi = \Phi$ for all $\Phi \in \dot{X}_N(\Omega)/K_N(\Omega)$. By the closed graph theorem, T_N is bounded. The definition of T_T follows similarly by using the other Helmholtz decomposition (13a). \square

4. PROOF OF THE MAIN RESULT

In a first part, we introduce a series of equivalent reformulations of our problem to obtain a form where the two contributions to the essential spectrum in our main result can easily be separated.

Decomposing \mathcal{H}_1 using (11) and (12) allows us to transform the Maxwell operator V_ω . More precisely, consider the decompositions

$$(15) \quad E = \nabla q_E + \Psi_E + h_N, \quad H = \nabla q_H + \Psi_H + h_T,$$

where $q_E \in \dot{H}_0^1(\Omega; \mathbb{C})$, $q_H \in \dot{H}^1(\Omega; \mathbb{C})$, $\Psi_E \in \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega))$, $\Psi_H \in \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega))$, $h_T \in K_T(\Omega)$ and $h_N \in K_N(\Omega)$. We now wish to discard the contribution coming from $K_T(\Omega)$ and $K_N(\Omega)$. To this end, we introduce the space

$$\begin{aligned} \mathcal{H}_2 &= \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \\ &\quad \times \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \times \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \end{aligned}$$

equipped with the canonical product norm

$$(16) \quad \|(u_1, u_2, \Psi_1, \Psi_2)\|_{\mathcal{H}_2}^2 = \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 + \|\Psi_1\|_{\mathcal{H}(\text{curl}, \Omega)}^2 + \|\Psi_2\|_{\mathcal{H}(\text{curl}, \Omega)}^2.$$

Define the projection map

$$W: \mathcal{H}_1 \rightarrow \mathcal{H}_2, \quad W(E, H) = (\nabla q_E, \nabla q_H, \Psi_E, \Psi_H),$$

where E, H are given by (15), and its right inverse $W^{-1}: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ by

$$W^{-1}(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) = (\nabla q_E + \Psi_E, \nabla q_H + \Psi_H).$$

Since the decompositions in (11) and (12) are orthogonal, for any $(E, H) \in \mathcal{H}_1$ we have

$$(17) \quad \|(E, H)\|_{\mathcal{H}_1}^2 = \|W(E, H)\|_{\mathcal{H}_2}^2 + \|(h_N, h_T)\|_{L^2(\Omega)^2}^2.$$

Instead of the operator V_ω , we consider

$$\tilde{V}_\omega = V_\omega \circ W^{-1}: \mathcal{H}_2 \rightarrow L^2(\Omega; \mathbb{C}^3)^2.$$

This does not change the essential spectrum, as the following lemma shows.

Lemma 17. *The essential spectra of V_ω and of \tilde{V}_ω coincide.*

Proof. Using that W^{-1} is an isometry we immediately obtain that the essential spectrum of \tilde{V}_ω is contained in the essential spectrum of V_ω . It remains to show the reverse inclusion.

Let ω belong to the essential spectrum of V_ω . By Remark 2 (a), there exists a sequence of functions $u_n = (\nabla q_{E,n} + \Psi_{E,n} + h_{N,n}, \nabla q_{H,n} + \Psi_{H,n} + h_{T,n})$ in \mathcal{H}_1 , $\|u_n\|_{\mathcal{H}_1} = 1$, $u_n \rightarrow 0$ in \mathcal{H}_1 such that $\|V_\omega u_n\|_{L^2} \rightarrow 0$. Then there exists $c > 0$ such that $\|W u_n\|_{\mathcal{H}_2} \geq c$ for all sufficiently large n . This follows from the fact that otherwise by (17) we would have that $P_{NT} u_n := (h_{N,n}, h_{T,n})$ satisfies $\|P_{NT} u_n\|_{\mathcal{H}_1} \rightarrow 1$. However, the range of P_{NT} is the finite dimensional space $K_N(\Omega) \times K_T(\Omega)$. This contradicts that $u_n \rightarrow 0$ in \mathcal{H}_1 , which implies that $(h_{N,n}, h_{T,n}) \rightarrow 0$ in \mathcal{H}_1 .

Set $\tilde{u}_n = W u_n / \|W u_n\|_{\mathcal{H}_2}$. Then, $\|\tilde{u}_n\|_{\mathcal{H}_2} = 1$ and

$$\tilde{V}_\omega \tilde{u}_n = \frac{V_\omega(\nabla q_{E,n} + \Psi_{E,n}, \nabla q_{H,n} + \Psi_{H,n})}{\|W u_n\|_{\mathcal{H}_2}} = \frac{V_\omega u_n - V_\omega(h_{N,n}, h_{T,n})}{\|W u_n\|_{\mathcal{H}_2}} \rightarrow 0$$

in $L^2(\Omega; \mathbb{C}^3)^2$. Finally, for any $\varphi \in (\mathcal{H}_2)'$ we have $\varphi \circ W \in (\mathcal{H}_1)'$, so

$$\varphi(\tilde{u}_n) = \frac{(\varphi \circ W) u_n}{\|W u_n\|_{\mathcal{H}_2}} \rightarrow 0,$$

and hence ω is in the essential spectrum of \tilde{V}_ω . \square

By definition of \tilde{V}_ω and (6), we obtain

$$(18) \quad \tilde{V}_\omega(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) = \begin{pmatrix} \text{curl } \Psi_H + iM_\omega \nabla q_E + iM_\omega \Psi_E \\ \text{curl } \Psi_E - i\omega M_\mu \nabla q_H - i\omega M_\mu \Psi_H \end{pmatrix},$$

where $M_\omega F = (\omega\varepsilon + i\sigma)F$ and $M_\mu F = \mu F$.

In order to simplify this operator even further, we need the following elementary result.

Lemma 18. *Let P_H denote the orthogonal projection onto the space H .*

(1) The map $\zeta_1: L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\dot{X}_T(\Omega)/K_T(\Omega)) \times K_N(\Omega)$ defined by

$$F \mapsto (\operatorname{div} F, T_T F, P_{K_N(\Omega)} F)$$

is an isomorphism, where $\dot{H}^{-1}(\Omega; \mathbb{C})$ denotes the dual of $\dot{H}_0^1(\Omega; \mathbb{C})$.

(2) The map $\zeta_2: L^2(\Omega; \mathbb{C}^3) \rightarrow (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times K_T(\Omega)$ given by

$$F \mapsto (h(F), T_N F, P_{K_T(\Omega)} F),$$

where $h: L^2(\Omega; \mathbb{C}^3) \rightarrow (\nabla \dot{H}^1(\Omega; \mathbb{C}))'$ is defined by

$$\langle h(F), \nabla q \rangle := \int_{\Omega} F \cdot \nabla q \, dx,$$

is an isomorphism.

Proof. (1) Take $(\phi, \Phi, f) \in \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\dot{X}_T(\Omega)/K_T(\Omega)) \times K_N(\Omega)$. We need to show that there exists a unique $F \in L^2(\Omega; \mathbb{C}^3)$ such that $\zeta_1(F) = (\phi, \Phi, f)$. We use the Helmholtz decomposition (13a) and look for F of the form $F = \nabla q + \operatorname{curl} \tilde{\Phi} + f_N$, with $q \in \dot{H}_0^1(\Omega; \mathbb{C})$, $\tilde{\Phi} \in \dot{X}_T(\Omega)/K_T(\Omega)$ and $f_N \in K_N(\Omega)$. First, since $P_{K_N(\Omega)} F = f_N$, choose $f_N = f$. Now note that

$$\operatorname{div} F = \phi \iff \Delta q = \phi,$$

which is uniquely solvable for $q \in \dot{H}_0^1(\Omega; \mathbb{C})$ by the Lax-Milgram theorem.

Further,

$$T_T F = \Phi \iff \tilde{\Phi} = \Phi,$$

which is clearly uniquely solvable for $\tilde{\Phi} \in \dot{X}_T(\Omega)/K_T(\Omega)$. This shows that $\zeta_1(\nabla q + \operatorname{curl} \tilde{\Phi} + f_N) = (\phi, \Phi, f)$, as desired.

(2) The map ζ_2 is well-defined since $\nabla \dot{H}^1(\Omega) \subseteq L^2(\Omega; \mathbb{C}^3)$. We now show that ζ_2 is an isomorphism. Take

$$(\varphi, \Phi, f) \in (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times K_T(\Omega).$$

We use the Helmholtz decomposition (13b) and look for F of the form $F = \nabla p + \operatorname{curl} \tilde{\Phi} + f_T$, with $p \in \dot{H}^1(\Omega; \mathbb{C})$, $\tilde{\Phi} \in \dot{X}_N(\Omega)/K_N(\Omega)$ and $f_T \in K_T(\Omega)$. Then $T_N F = \tilde{\Phi}$ and $P_{K_T(\Omega)} F = f_T$, and so $\tilde{\Phi}$ and f_T are uniquely determined by $\tilde{\Phi} = \Phi$ and $f_T = f$.

It remains to show that p can be chosen so that $\nabla p + \operatorname{curl} \tilde{\Phi} + f = \varphi$ or $\nabla p = \varphi - \operatorname{curl} \tilde{\Phi} - f$ in $(\nabla \dot{H}^1(\Omega; \mathbb{C}))'$. Thus we need to find p such that

$$\int_{\Omega} \nabla p \cdot \nabla q \, dx = \int_{\Omega} (\varphi - \operatorname{curl} \tilde{\Phi} - f) \cdot \nabla q \, dx, \quad q \in \dot{H}^1(\Omega; \mathbb{C}).$$

Using that $L^2(\Omega; \mathbb{C}^3) \subseteq (\nabla \dot{H}^1(\Omega))'$, this is uniquely solvable for p using the Lax-Milgram theorem.

This shows that $\zeta_2(\nabla p + \operatorname{curl} \tilde{\Phi} + f) = (\varphi, \Phi, f)$, as desired. \square

Now, define $\zeta = (\zeta_1, \zeta_2)$ and $\tilde{\zeta}(F) = (\operatorname{div} F, h(F), T_N F, T_T F)$, i.e. $\tilde{\zeta}$ contains the parts of ζ not in $K_N(\Omega) \oplus K_T(\Omega)$. Let

$$\mathcal{H}_3 = \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega)).$$

Set

$$\tilde{V}_\omega = \tilde{\zeta} \circ \tilde{V}_\omega : \mathcal{H}_2 \rightarrow \mathcal{H}_3.$$

Lemma 19. *The essential spectra of \tilde{V}_ω and of $\tilde{\tilde{V}}_\omega$ coincide.*

Proof. This follows from the fact that by Lemma 18, ζ is a bijective continuous linear map, so both ζ and ζ^{-1} are continuous, and that $K_N(\Omega) \oplus K_T(\Omega)$ is finite dimensional. \square

Now, recalling that $\Psi_H \in \dot{X}_T(\Omega)$ and $\Psi_E \in \dot{X}_N(\Omega)$, by (14) and (18) we have that

$$(19) \quad \begin{aligned} \tilde{V}_\omega(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) &= \tilde{\zeta} \begin{pmatrix} \operatorname{curl} \Psi_H + iM_\omega \nabla q_E + iM_\omega \Psi_E \\ \operatorname{curl} \Psi_E - i\omega M_\mu \nabla q_H - i\omega M_\mu \Psi_H \end{pmatrix} \\ &= \begin{pmatrix} i \operatorname{div}(M_\omega \nabla q_E) + i \operatorname{div}(M_\omega \Psi_E) \\ -i\omega h(M_\mu \nabla q_H) - i\omega h(M_\mu \Psi_H) \\ [\Psi_E] - i\omega T_N M_\mu \nabla q_H - i\omega T_N M_\mu \Psi_H \\ [\Psi_H] + iT_T M_\omega \nabla q_E + iT_T M_\omega \Psi_E \end{pmatrix}, \end{aligned}$$

in which $[\cdot]$ denotes the equivalence class in the appropriate quotient space.

In order to compute the essential spectrum of \tilde{V}_ω we now decompose the coefficients in the Maxwell system. As a consequence of our hypotheses (3,4), whether Ω be bounded or unbounded, for each $\delta > 0$ the Maxwell coefficients admit a decomposition

$$(20) \quad \mu = \mu_0 + \mu_c + \mu_\delta, \quad \varepsilon = \varepsilon_0 + \varepsilon_c + \varepsilon_\delta, \quad \sigma = \sigma_0 + \sigma_c + \sigma_\delta,$$

in which the terms μ_0 , ε_0 and σ_0 are constant and do not depend on δ , the terms μ_c , ε_c and σ_c are compactly supported, and the terms μ_δ , ε_δ , σ_δ are essentially bounded, with

$$(21) \quad m_\delta := \max(\|\mu_\delta\|_{L^\infty(\Omega)}, \|\varepsilon_\delta\|_{L^\infty(\Omega)}, \|\sigma_\delta\|_{L^\infty(\Omega)}) < \delta,$$

where the norms are defined by $\|a\|_{L^\infty(\Omega)} := \operatorname{ess\,sup}_{x \in \Omega} \|a(x)\|_2$ for $a \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$, where $\|A\|_2$ denotes the induced norm $\sup_{v \in \mathbb{R}^3 \setminus \{0\}} \frac{|Av|}{|v|}$ for $A \in \mathbb{R}^{3 \times 3}$.

In the expression for \tilde{V}_ω appearing in (19) the Maxwell coefficients appear linearly in the multiplication operators M_μ (multiplication by μ) and M_ω (multiplication by $\omega\varepsilon + i\sigma$). The decomposition (20) of the coefficients is partially reflected in the following decomposition of \tilde{V}_ω :

$$\tilde{V}_\omega = \tilde{V}_{\omega,0} + \tilde{V}_{\omega,c} + \tilde{V}_{\omega,\delta},$$

in which

$$(22) \quad \begin{aligned} \tilde{V}_{\omega,0} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} &= \begin{pmatrix} i \operatorname{div}((\omega\varepsilon + i\sigma)\nabla q_E) \\ -i\omega h(\mu\nabla q_H) \\ -i\omega T_N((\mu_0 + \mu_c)\nabla q_H) + [\Psi_E] - i\omega T_N(\mu_0\Psi_H) \\ iT_T((\omega(\varepsilon_0 + \varepsilon_c) + i(\sigma_0 + \sigma_c))\nabla q_E) + iT_T((\omega\varepsilon_0 + i\sigma_0)\Psi_E) + [\Psi_H] \end{pmatrix}, \\ \tilde{V}_{\omega,c} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} &= \begin{pmatrix} i \operatorname{div}((\omega(\varepsilon_0 + \varepsilon_c) + i(\sigma_0 + \sigma_c))\Psi_E) \\ -i\omega h((\mu_0 + \mu_c)\Psi_H) \\ -i\omega T_N(\mu_c\Psi_H) \\ iT_T((\omega\varepsilon_c + i\sigma_c)\Psi_E) \end{pmatrix} \end{aligned}$$

and

$$\tilde{V}_{\omega,\delta} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} = \begin{pmatrix} i \operatorname{div}((\omega\varepsilon_\delta + i\sigma_\delta)\Psi_E) \\ -i\omega h(\mu_\delta\Psi_H) \\ -i\omega T_N(\mu_\delta\nabla q_H) - i\omega T_N(\mu_\delta\Psi_H) \\ iT_T((\omega\varepsilon_\delta + i\sigma_\delta)\nabla q_E) + iT_T((\omega\varepsilon_\delta + i\sigma_\delta)\Psi_E) \end{pmatrix}.$$

The operator $\tilde{V}_{\omega,c}$ is compact and the operator $\tilde{V}_{\omega,\delta}$ is $O(\delta)$ -small in a suitable norm, as we show in the following two lemmata.

Lemma 20. *The operator $\tilde{V}_{\omega,c} : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ is compact.*

Proof. By a direct calculation it is easy to see that $\operatorname{div}((\omega\varepsilon_0 + i\sigma_0)\Psi_E) = 0$ and $h(\mu_0\Psi_H) = 0$, using that ε_0, σ_0 and μ_0 are scalar. Since the operators

$$(23) \quad \begin{aligned} \operatorname{div} : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}), & T_T : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{X}_T(\Omega)/K_T(\Omega), \\ h : L^2(\Omega; \mathbb{C}^3) &\rightarrow (\nabla\dot{H}^1(\Omega; \mathbb{C}))', & T_N : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{X}_N(\Omega)/K_N(\Omega), \end{aligned}$$

are bounded, it is enough to show that the operators

$$\begin{aligned} F_T : \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) &\rightarrow L^2(\Omega; \mathbb{C}^3), & \Psi_E &\mapsto (\omega\varepsilon_c + i\sigma_c)\Psi_E, \\ F_N : \mathcal{H}(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) &\rightarrow L^2(\Omega; \mathbb{C}^3), & \Psi_H &\mapsto \mu_c\Psi_H, \end{aligned}$$

are compact. We now prove that F_T is compact, the other proof is completely analogous. Let $R > 0$ be big enough so that $K := \operatorname{supp}(\omega\varepsilon_c + i\sigma_c) \subseteq B(0, R) \cap \overline{\Omega}$ and $\chi \in C^\infty(\Omega)$ be a cutoff function such that $\chi \equiv 1$ in K and $\operatorname{supp}\chi \subseteq B(0, R) \cap \overline{\Omega}$. Setting $\Omega_R = B(0, R) \cap \Omega$, the operator F_T may be expressed via the following compositions

$$\begin{array}{ccc} \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) & & \Psi_E \\ \downarrow & & \downarrow \\ \mathcal{H}_0(\operatorname{curl}, \Omega_R) \cap \mathcal{H}(\operatorname{div}, \Omega_R) & & (\chi\Psi_E)|_{\Omega_R} \\ \downarrow & & \downarrow \\ L^2(\Omega_R; \mathbb{C}^3) & & (\chi\Psi_E)|_{\Omega_R} \\ \downarrow & & \downarrow \\ L^2(\Omega_R; \mathbb{C}^3) & & ((\omega\varepsilon_c + i\sigma_c)\Psi_E)|_{\Omega_R} \\ \downarrow & & \downarrow \\ L^2(\Omega; \mathbb{C}^3) & & (\omega\varepsilon_c + i\sigma_c)\Psi_E \end{array},$$

where the third operator is the multiplication by $\omega\varepsilon_c + i\sigma_c$ and the fourth operator is simply the extension by zero. Therefore, since the embedding $\mathcal{H}_0(\operatorname{curl}, \Omega_R) \cap \mathcal{H}(\operatorname{div}, \Omega_R) \hookrightarrow L^2(\Omega_R; \mathbb{C}^3)$ is compact [20] (see also [4, Theorem 2.8]), the operator F_T is compact. \square

Lemma 21. *There exists a constant $C > 0$ depending only on Ω and on the coefficients μ, ε and σ , such that for each $\delta > 0$ we have*

$$\|\tilde{V}_{\omega,\delta}\|_{\mathcal{H}_2 \rightarrow \mathcal{H}_3} \leq C(1 + |\omega|)\delta.$$

Proof. Note that by (16) we have

$$\|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{L^2(\Omega; \mathbb{C}^3)^4} \leq \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_2}.$$

Thus, since the four operators in (23) are bounded, there exists a constant $C > 0$ depending only on Ω and on the coefficients μ, ε and σ , such that

$$\begin{aligned} \|\tilde{V}_{\omega,\delta}(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_3} &\leq C(1 + |\omega|)m_\delta \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{L^2(\Omega; \mathbb{C}^3)^4} \\ &\leq C(1 + |\omega|)\delta \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_2}, \end{aligned}$$

where the second inequality follows from (21). This concludes the proof. \square

It is helpful to recall that $\tilde{V}_\omega: \mathcal{H}_2 \rightarrow \mathcal{H}_3$, where

$$\begin{aligned} \mathcal{H}_2 &= \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \\ &\quad \times \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \times \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \end{aligned}$$

and

$$\mathcal{H}_3 = \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega)).$$

Proposition 22. *The essential spectrum of \tilde{V}_ω is the union of the essential spectra of the two block operator pencils*

$$\begin{aligned} \mathcal{A}_\omega: \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) &\rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \\ \mathcal{D}_\omega: \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) &\times \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \\ &\rightarrow (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega)) \end{aligned}$$

determined by the expressions

$$(24) \quad \mathcal{A}_\omega = \begin{pmatrix} i \operatorname{div}((\omega \varepsilon + i\sigma) \cdot) & 0 \\ 0 & -i\omega h(\mu \cdot) \end{pmatrix}, \quad \mathcal{D}_\omega = \begin{pmatrix} I_N & -i\omega \mu_0 T_N \\ i(\omega \varepsilon_0 + i\sigma_0) T_T & I_T \end{pmatrix}.$$

Here the operators I_N and I_T are the canonical mappings from $\dot{X}_N(\Omega)$ and $\dot{X}_T(\Omega)$ to the quotient spaces $\dot{X}_N(\Omega)/K_N(\Omega)$ and $\dot{X}_T(\Omega)/K_T(\Omega)$ respectively.

Proof. By inspection of (22), the operator pencil $\tilde{V}_{\omega,0}$ may be written as the block lower triangular operator matrix pencil

$$(25) \quad \tilde{V}_{\omega,0} = \begin{pmatrix} \mathcal{A}_\omega & 0 \\ \mathcal{C}_\omega & \mathcal{D}_\omega \end{pmatrix},$$

in which \mathcal{A}_ω and \mathcal{D}_ω are as in equation (24) and the off-diagonal component $\mathcal{C}_\omega: \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \rightarrow (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega))$ is given by

$$\mathcal{C}_\omega = \begin{pmatrix} 0 & -i\omega T_N((\mu_0 + \mu_c) \cdot) \\ iT_T((\omega(\varepsilon_0 + \varepsilon_c) + i(\sigma_0 + \sigma_c)) \cdot) & 0 \end{pmatrix}.$$

Take $\omega \notin \sigma_{ess}(\mathcal{A}_\omega) \cup \sigma_{ess}(\mathcal{D}_\omega)$. By Remark 2(c), there exist compact operators K_1 and K_2 such that $\mathcal{A}_\omega + K_1$ and $\mathcal{D}_\omega + K_2$ are invertible. Thus, $\tilde{V}_{\omega,0} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ is invertible, and its inverse is given by

$$\left(\tilde{V}_{\omega,0} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \right)^{-1} = \begin{pmatrix} (\mathcal{A}_\omega + K_1)^{-1} & 0 \\ -(\mathcal{D}_\omega + K_2)^{-1} \mathcal{C}_\omega (\mathcal{A}_\omega + K_1)^{-1} & (\mathcal{D}_\omega + K_2)^{-1} \end{pmatrix}.$$

Since \mathcal{A}_ω and \mathcal{D}_ω are independent of δ and the operator norm of \mathcal{C}_ω may be bounded independently of δ , this inverse is bounded in norm by a constant independent of δ .

As a consequence, by Lemma 21 there exists $\delta > 0$ such that $\tilde{V}_{\omega,0} + \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} + \tilde{V}_{\omega,\delta}$ is invertible. Using again Remark 2(c), we obtain that $\omega \notin \sigma_{ess}(\tilde{V}_{\omega,0} + \tilde{V}_{\omega,\delta})$. Finally, Lemma 20 yields $\omega \notin \sigma_{ess}(\tilde{V}_{\omega,0} + \tilde{V}_{\omega,\delta} + \tilde{V}_{\omega,c}) = \sigma_{ess}(\tilde{V}_\omega)$, since the essential spectrum is invariant under compact perturbations.

Now take $\omega \in \sigma_{ess}(\tilde{V}_\omega)$. By Remark 2(c) there exists a compact operator K such that $\tilde{V}_\omega + K$ is invertible. By Lemma 21 there exists $\delta > 0$ such that $\tilde{V}_{\omega,0} + \tilde{V}_{\omega,c} + K = \tilde{V}_\omega + K - \tilde{V}_{\omega,\delta}$ is invertible, whence $\omega \notin \sigma_{ess}(\tilde{V}_{\omega,0} + \tilde{V}_{\omega,c}) = \sigma_{ess}(\tilde{V}_{\omega,0})$. Assume by contradiction that $\omega \in \sigma_{ess}(\mathcal{A}_\omega) \cup \sigma_{ess}(\mathcal{D}_\omega)$. If $\omega \in \sigma_{ess}(\mathcal{D}_\omega)$, by Remark 2(a) there

exists a singular sequence $(v_n)_n$ for \mathcal{D}_ω , and so $(0, v_n)_n$ is a singular sequence for $\tilde{V}_{\omega,0}$ by (25), which implies that $\omega \in \sigma_{ess}(\tilde{V}_{\omega,0})$, a contradiction. Otherwise, if $\omega \in \sigma_{ess}(\mathcal{A}_\omega) \setminus \sigma_{ess}(\mathcal{D}_\omega)$, there exists a compact operator K' such that $\mathcal{D}_\omega + K'$ is invertible and a singular sequence $(u_n)_n$ for \mathcal{A}_ω . A direct calculation then shows that $(u_n, -(\mathcal{D}_\omega + K')^{-1}\mathcal{C}_\omega u_n)_n$ is a singular sequence for $\tilde{V}_{\omega,0}$, which again implies that $\omega \in \sigma_{ess}(\tilde{V}_{\omega,0})$. \square

Remark 23. The text [19] contains many interesting results on essential spectra of block-operator matrices and pencils; Theorem 2.4.1 is very close to what we would need, but our pencil $\tilde{V}_{\omega,0}$ is lower triangular rather than diagonally dominant.

We are now ready to prove our main result.

Proof of Theorem 5. We commence the proof by observing the following identity:

$$(26) \quad \sigma_{ess}(V_\omega) = \sigma_{ess}(\mathcal{A}_\omega) \cup \sigma_{ess}(\mathcal{D}_\omega).$$

This is an immediate consequence of Lemmas 17 and 19 and of Proposition 22. We now consider $\sigma_{ess}(\mathcal{A}_\omega)$ and $\sigma_{ess}(\mathcal{D}_\omega)$ in more detail.

The essential spectrum of \mathcal{A}_ω consists of the point $\{0\}$, arising from the $(2, 2)$ diagonal entry of \mathcal{A}_ω , which has $\omega = 0$ as an eigenvalue of infinite multiplicity and is otherwise invertible; and of the essential spectrum of the pencil in the $(1, 1)$ entry, which is as stated in the theorem, namely

$$(27) \quad \sigma_{ess}(\mathcal{A}_\omega) = \{0\} \cup \sigma_{ess}(\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)).$$

In order to deal with the essential spectrum of \mathcal{D}_ω we observe that if we replace V_ω by a new pencil V_ω^0 in which the coefficients have the constant values ε_0 , μ_0 and σ_0 , then \mathcal{D}_ω will be unchanged while \mathcal{A}_ω will be replaced by a pencil $\mathcal{A}_{\omega,0}$ in which all the coefficients are constant. For the constant coefficient pencil $\mathcal{A}_{\omega,0}$ we see that 0 lies in the essential spectrum as we reasoned before, while the $(1, 1)$ term is invertible and Fredholm precisely when $\omega\varepsilon_0 + i\sigma_0 \neq 0$, by the Babuška-Lax-Milgram theorem; hence $\sigma_{ess}(\mathcal{A}_{\omega,0}) = \{0, -i\sigma_0/\varepsilon_0\}$. Using (26) for the constant coefficient pencil, we now have

$$(28) \quad \sigma_{ess}(V_\omega^0) = \{0, -i\sigma_0/\varepsilon_0\} \cup \sigma_{ess}(\mathcal{D}_\omega).$$

We now prove that the essential spectrum of \mathcal{A}_ω already contains $\{0, -i\sigma_0/\varepsilon_0\}$. The $(2, 2)$ component has 0 as an eigenvalue of infinite multiplicity. If Ω is bounded, we have $\sigma_0 = 0$ and so the claim is proven. Otherwise, for the point $-i\sigma_0/\varepsilon_0$ we observe that by the hypothesis (3), given $n > 0$ there exists $R_n > 0$ such that if $\omega_0 := -i\sigma_0/\varepsilon_0$ then

$$\sup_{|x| \geq R_n} \|\omega_0\varepsilon(x) + i\sigma(x)\|_2 < \frac{1}{n}.$$

Choosing any function $\phi_n \in C_0^\infty(\Omega)$ with support in $\{x \in \Omega : |x| > R_n\}$, with $\|\nabla\phi_n\|_{L^2(\Omega)} = 1$, we see that

$$\|\operatorname{div}((\omega_0\varepsilon + i\sigma)\nabla\phi_n)\|_{\dot{H}^{-1}(\Omega)} \leq \frac{1}{n}.$$

Since the supports of the sequence $(\nabla\phi_n)_{n \in \mathbb{N}}$ move off to infinity, the sequence converges weakly to zero; it is therefore a singular sequence in $\nabla\dot{H}_0^1(\Omega)$ for the $(1, 1)$ element of \mathcal{A}_{ω_0} . Thus ω_0 lies in the essential spectrum of \mathcal{A}_ω . Combining

the observations (26), (27) and (28) with the fact that $\sigma_{ess}(\mathcal{A}_\omega) \ni \{0, -i\sigma_0/\varepsilon_0\}$ completes the proof. \square

We conclude this section with a more explicit description of the essential spectrum of the divergence form operator $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ in the case of continuous coefficients.

Proposition 24. *When the coefficients ε and σ are continuous in $\overline{\Omega}$, the essential spectrum of $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$, acting from $\dot{H}_0^1(\Omega; \mathbb{C})$ to $\dot{H}^{-1}(\Omega; \mathbb{C})$, consists of the closure of the set of all $\omega = i\nu$, $\nu \in \mathbb{R}$, such that $\nu\varepsilon + \sigma$ is indefinite at some point in Ω . Equivalently, when Ω is bounded, it is the set of $\omega = i\nu$, $\nu \in \mathbb{R}$, such that $\nu\varepsilon + \sigma$ is indefinite at some point in $\overline{\Omega}$.*

Proof. If $\Re(\omega) \neq 0$ then the real part of $\omega\varepsilon + i\sigma$ is definite, and the result follows by the Lax-Milgram theorem. If $\omega = i\nu$ is purely imaginary, this reasoning still works if $\nu\varepsilon + \sigma$ is uniformly definite in Ω . It remains only to show that if $\nu\varepsilon + \sigma$ is indefinite at some point $x_0 \in \Omega$, then 0 lies in the essential spectrum of $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$.

We prove the result by constructing a Weyl singular sequence. Define $a := \nu\varepsilon + \sigma$ and $a_0 := a(x_0)$. Let $\chi : [0, \infty) \mapsto [0, 1]$ be a smooth cutoff function such that $\chi(t) = 1$ for $0 \leq t \leq 1$ and $\chi(t) = 0$ for all $t \geq 2$. Let $\theta \in \mathbb{R}^3$ be a unit vector chosen such that $\theta^T a(x_0)\theta = 0$. For each sufficiently small $\delta > 0$ and large $r > 0$ let

$$(29) \quad \chi_\delta(x) := \frac{1}{\delta^{3/2}} \chi\left(\frac{|x - x_0|}{\delta}\right), \quad u_{r,\delta}(x) := \chi_\delta(x) r^{-1} \exp(ir\theta \cdot x).$$

A direct calculation shows that $\nabla u_{r,\delta}$ in sup-norm is $O(r^{-1}\delta^{-5/2}) + O(\delta^{-3/2})$. We suppose that $r\delta^{5/2} \gg 1$, so that the $\delta^{-3/2}$ term dominates; we have $\|\nabla u_{r,\delta}\|_{L^\infty(B_{2\delta}(x_0))} = O(\delta^{-3/2})$ and $\|u_{r,\delta}\|_{H_0^1(\Omega)} \geq c$ for some $c > 0$ independent of r and δ . If v is any smooth test function then

$$|\langle \nabla u_{r,\delta}, \nabla v \rangle| \leq C\delta^3 \|\nabla u_{r,\delta}\|_{L^\infty(B_{2\delta}(x_0))} \|\nabla v\|_{L^\infty(B_{2\delta}(x_0))} \leq C\|\nabla v\|_\infty \delta^3 \delta^{-3/2} = O(\delta^{3/2}),$$

so that the $u_{r,\delta}$ tend to zero weakly in $H_0^1(\Omega)$ as $r \nearrow +\infty$ and $\delta \searrow 0$, with $r \geq \delta^{-5/2}$.

To complete the proof that 0 lies in the essential spectrum of our operator we show that $\|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)}$ can be made arbitrarily small. It is easy to see that

$$(30) \quad \|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)} \leq \|a - a_0\|_{L^\infty(B_{2\delta}(x_0))} + \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle|}{\|v\|_{H_0^1(\Omega)}}.$$

We compute $\nabla u_{r,\delta}$ by direct differentiation of eqn. (29) and deduce that for each $v \in H_0^1(\Omega)$,

$$\begin{aligned} |\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| &\leq \left| \left\langle a_0 \frac{1}{r\delta^{5/2}} \frac{x - x_0}{|x - x_0|} \chi' \left(\frac{|x - x_0|}{\delta} \right) \exp(ir\theta \cdot x), \nabla v \right\rangle \right| \\ &\quad + |\langle \chi_\delta a_0 \nabla (r^{-1} \exp(ir\theta \cdot x)), \nabla v \rangle| \\ &\leq C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + |\langle \operatorname{div}(\chi_\delta a_0 \nabla (r^{-1} \exp(ir\theta \cdot x))), v \rangle| \\ &= C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + |\langle (\nabla \chi_\delta) \cdot a_0 \nabla (r^{-1} \exp(ir\theta \cdot x)), v \rangle|; \end{aligned}$$

in the last step we have used the fact that $\operatorname{div}(a_0 \nabla(r^{-1} \exp(ir\theta \cdot x))) = 0$, which follows immediately from $\theta^T a_0 \theta = 0$. Integration by parts yields

$$|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| \leq C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + \left| \langle r^{-1} \exp(ir\theta \cdot x), \operatorname{div}(v a_0^T \nabla \chi_\delta) \rangle \right|.$$

We estimate the final inner product by observing that χ_δ is $O(\delta^{-3/2})$, its gradient is $O(\delta^{-5/2})$ and its second derivatives $O(\delta^{-7/2})$, while its support is a ball whose volume is $O(\delta^3)$: thus

$$|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| \leq C \frac{|a_0|}{r} \left\{ \frac{1}{\delta} + \frac{1}{\delta^2} \right\} \|v\|_{H_0^1(\Omega)},$$

for some constant $C > 0$. Substituting this back into (30) we obtain

$$\|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)} \leq \|a - a_0\|_{L^\infty(B_{2\delta}(x_0))} + C \frac{|a_0|}{r} \left\{ \frac{1}{\delta} + \frac{1}{\delta^2} \right\}.$$

Letting $r \nearrow \infty$ and then letting $\delta \searrow 0$ we obtain the required result. \square

APPENDIX A. THE HELMHOLTZ DECOMPOSITION FOR CYLINDERS

This appendix is devoted to the study of the decompositions (12a) and (12b) for a large class of cylinders of the form $\Omega = \mathbb{R} \times \Omega'$, with $\Omega' \subseteq \mathbb{R}^2$. We will then show that this class includes the full space, the half-space, the slab, and the cylinders with bounded sections as in Proposition 14 part(6), thereby providing a proof to the corresponding parts of Proposition 14.

We denote coordinates in Ω by (x_1, x') where $x' = (x_2, x_3) \in \Omega'$, with similar conventions for components of vectors and operators, such as gradient and Laplacian. For simplicity of notation, we shall write $a \lesssim b$ to mean $a \leq Cb$ for some positive constant C depending only on Ω' . We assume that the cross-section Ω' satisfies the following additional hypothesis.

Assumption 25. Let $g, h \in L^2(\Omega')$. If $\psi' \in \mathcal{D}'(\Omega')$ satisfies

$$(31) \quad \begin{cases} \operatorname{curl}' \psi' = g & \text{in } \Omega', \\ \operatorname{div}' \psi' = h & \text{in } \Omega', \\ \psi' \cdot \nu' = 0 \text{ or } \psi' \cdot \tau' = 0 & \text{on } \partial\Omega', \end{cases}$$

where $\nu' = (\nu_2, \nu_3)$ and $\tau' = (-\nu_3, \nu_2)$ denote the unit normal and tangent vectors to $\partial\Omega'$, respectively, then

$$\|\nabla' \psi'\|_{L^2(\Omega')} \lesssim \|g\|_{L^2(\Omega')} + \|h\|_{L^2(\Omega')}.$$

This assumption guarantees the existence of the decompositions (12a) and (12b) with the spaces $K_T(\Omega)$ and $K_N(\Omega)$ (Definition 11) both trivial.

Proposition 26. Let $\Omega = \mathbb{R} \times \Omega'$, where $\Omega' \subseteq \mathbb{R}^2$ is a Lipschitz domain satisfying Assumption 25. Then $K_N(\Omega) = K_T(\Omega) = \{0\}$ and

- (a) $\mathcal{H}(\operatorname{div} 0, \Omega) = \operatorname{curl}\{\psi \in \dot{H}^1(\Omega) : \operatorname{div} \psi = 0 \text{ in } \Omega, \psi \cdot \nu = 0 \text{ on } \partial\Omega\}$,
- (b) $\mathcal{H}_0(\operatorname{div} 0, \Omega) = \operatorname{curl}\{\psi \in \dot{H}^1(\Omega) : \operatorname{div} \psi = 0 \text{ in } \Omega, \psi \times \nu = 0 \text{ on } \partial\Omega\}$.

Proof. We divide the proof into three steps.

- (1) First, we prove that every function f in $\mathcal{H}_0(\operatorname{div} 0, \Omega)$ may be written as the curl of a unique divergence-free function ψ such that $\psi \times \nu = 0$ on $\partial\Omega$. In particular, this implies that the space $K_N(\Omega)$ is trivial.

- (2) Second, we prove that every function f in $\mathcal{H}(\operatorname{div} 0, \Omega)$ may be written as the curl of a unique divergence-free function ψ such that $\psi \cdot \nu = 0$ on $\partial\Omega$. In particular, this implies that the space $K_T(\Omega)$ is trivial.
- (3) Third, we prove that the potentials ψ constructed in steps (1) and (2) belong to $\dot{H}^1(\Omega)$.

Step (1). Given $f \in \mathcal{H}_0(\operatorname{div} 0, \Omega)$, we look for ψ such that

$$(32) \quad \operatorname{curl} \psi = f \quad \text{in } \Omega,$$

$$(33) \quad \operatorname{div} \psi = 0 \quad \text{in } \Omega,$$

$$(34) \quad \psi \times \nu = 0 \quad \text{on } \partial\Omega.$$

Since $\nu_1 = 0$, the second and third components of (34) yield $\psi_1 \nu_3 = 0$ and $\psi_1 \nu_2 = 0$, giving $\psi_1 = 0$ on $\partial\Omega$. Taking the curl of equation (32) we obtain

$$-\Delta \psi_1 = \partial_2 f_3 - \partial_3 f_2 \quad \text{in } \Omega;$$

upon taking the Fourier transform with respect to the first coordinate x_1 we obtain the boundary value problem

$$(35) \quad \begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \partial_2 \hat{f}_3 - \partial_3 \hat{f}_2 & \text{in } \Omega', \\ \hat{\psi}_1 = 0 & \text{on } \partial\Omega', \end{cases}$$

in which Δ' denotes the Laplacian with respect to $x' \in \Omega'$ and $\xi \in \mathbb{R}$ is the dual variable of x_1 under Fourier transformation. For almost every $\xi \in \mathbb{R}$, this Dirichlet boundary value problem admits a unique solution $\hat{\psi}_1(\xi) \in \dot{H}_0^1(\Omega')$ by the Lax Milgram theorem, and so ψ_1 is uniquely determined. To obtain the remaining components of ψ we rewrite (32) and (33) as

$$\partial_2 \psi_3 - \partial_3 \psi_2 = f_1, \quad \partial_3 \psi_1 - \partial_1 \psi_3 = f_2, \quad \partial_1 \psi_2 - \partial_2 \psi_1 = f_3, \quad \partial_1 \psi_1 + \partial_2 \psi_2 + \partial_3 \psi_3 = 0.$$

Again take the Fourier transform with respect to x_1 and obtain

$$(36) \quad \partial_2 \hat{\psi}_3 - \partial_3 \hat{\psi}_2 = \hat{f}_1, \quad \partial_3 \hat{\psi}_1 - i\xi \hat{\psi}_3 = \hat{f}_2, \quad i\xi \hat{\psi}_2 - \partial_2 \hat{\psi}_1 = \hat{f}_3, \quad i\xi \hat{\psi}_1 + \partial_2 \hat{\psi}_2 + \partial_3 \hat{\psi}_3 = 0.$$

Using the second and third identities in (36) yields

$$\hat{\psi}_2 = -i \frac{\partial_2 \hat{\psi}_1 + \hat{f}_3}{\xi}, \quad \hat{\psi}_3 = i \frac{\hat{f}_2 - \partial_3 \hat{\psi}_1}{\xi}, \quad \text{for a.e. } \xi \in \mathbb{R}.$$

It remains to check the first and fourth identities in (36) and the first component of (34). For the first identity in (36) we observe that

$$\partial_2 \hat{\psi}_3 - \partial_3 \hat{\psi}_2 = \frac{i}{\xi} (\partial_2 \hat{f}_2 - \partial_2 \partial_3 \hat{\psi}_1 + \partial_3 \partial_2 \hat{\psi}_1 + \partial_3 \hat{f}_3) = \frac{i}{\xi} (-i\xi \hat{f}_1) = \hat{f}_1.$$

Here we have used, for the second equality, the fact that $\operatorname{div} f = 0$. For the fourth identity in (36), by (35) we have

$$\begin{aligned} i\xi \hat{\psi}_1 + \partial_2 \hat{\psi}_2 + \partial_3 \hat{\psi}_3 &= i\xi \hat{\psi}_1 + \frac{i}{\xi} (-\partial_2^2 \hat{\psi}_1 - \partial_2 \hat{f}_3 + \partial_3 \hat{f}_2 - \partial_3^2 \hat{\psi}_1) \\ &= \frac{i}{\xi} ((\xi^2 \hat{\psi}_1 - \partial_2^2 \hat{\psi}_1 - \partial_3^2 \hat{\psi}_1) - (\partial_2 \hat{f}_3 - \partial_3 \hat{f}_2)) \\ &= 0. \end{aligned}$$

Finally, for the first component of (34), using \mathcal{F} to denote the Fourier transform,

$$\nu_3 \hat{\psi}_2 - \nu_2 \hat{\psi}_3 = \frac{i}{\xi} (-\partial_2 \hat{\psi}_1 \nu_3 - \hat{f}_3 \nu_3 - \hat{f}_2 \nu_2 + \partial_3 \hat{\psi}_1 \nu_2) = \frac{i}{\xi} \mathcal{F}(\nabla' \psi_1 \cdot \tau - f \cdot \nu) = 0,$$

where we have used the fact that $\psi_1 = 0$ and $f \cdot \nu = 0$ on $\partial\Omega$ in the last step.

Step (2). The only difference between this case and the one above lies in the boundary conditions. We no longer have $f \cdot \nu = 0$ on the boundary. This time $f \in \mathcal{H}(\text{div } 0, \Omega)$ and we seek ψ such that

$$(37) \quad \text{curl } \psi = f \text{ in } \Omega,$$

$$(38) \quad \text{div } \psi = 0 \text{ in } \Omega,$$

$$(39) \quad \psi \cdot \nu = 0 \text{ on } \partial\Omega.$$

The calculations follow as above except that problem (35) is replaced by

$$(40) \quad \begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \text{div}'(\hat{f}_3, -\hat{f}_2) & \text{in } \Omega', \\ -\nabla' \hat{\psi}_1 \cdot \nu' = (\hat{f}_3, -\hat{f}_2) \cdot \nu' & \text{on } \partial\Omega', \end{cases}$$

in which the reason for the slightly curious Neumann boundary condition will become clear shortly. As above, for almost every $\xi \in \mathbb{R}$, this problem admits a unique solution $\hat{\psi}_1(\xi) \in \dot{H}^1(\Omega')$ by the Lax Milgram theorem. Having found ψ_1 , we construct $\hat{\psi}_2$ and $\hat{\psi}_3$ as before, and the verification of (37) and (38) is similar to the calculations for (32) and (33). This leaves the boundary condition (39): since $\hat{\psi} \cdot \nu = \hat{\psi}_2 \nu_2 + \hat{\psi}_3 \nu_3$, we have

$$\hat{\psi} \cdot \nu = \frac{-i}{\xi} \{(\partial_2 \hat{\psi}_1 + \hat{f}_3) \nu_2 - (\hat{f}_2 - \partial_3 \hat{\psi}_1) \nu_3\} = \frac{-i}{\xi} \{\nabla' \hat{\psi}_1 \cdot \nu' + (\hat{f}_3, -\hat{f}_2) \cdot \nu'\} = 0,$$

the equality at the last step coming from the boundary equation in (40).

Step (3). We now verify that ψ lies in $\dot{H}^1(\Omega)$ in both cases. From (35,40) we have

$$(41) \quad \begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \text{div}'(\hat{f}_3, -\hat{f}_2) & \text{in } \Omega', \\ \hat{\psi}_1 = 0 \text{ or } -\nabla' \hat{\psi}_1 \cdot \nu' = (\hat{f}_3, -\hat{f}_2) \cdot \nu' & \text{on } \partial\Omega'. \end{cases}$$

An integration against $\hat{\psi}_1$ gives

$$\|\nabla' \hat{\psi}_1\|_{L^2(\Omega')}^2 + \xi^2 \|\hat{\psi}_1\|_{L^2(\Omega')}^2 = ((-\hat{f}_3, \hat{f}_2), \nabla' \hat{\psi}_1)_{L^2(\Omega')}$$

whence

$$(42) \quad \|\nabla' \hat{\psi}_1(\xi)\|_{L^2(\Omega')} \leq \|\hat{f}'(\xi)\|_{L^2(\Omega')}, \quad \|\hat{\psi}_1(\xi)\|_{L^2(\Omega')} \leq \frac{\|\hat{f}'(\xi)\|_{L^2(\Omega')}}{|\xi|}.$$

From the first and fourth identities of (36) we get, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned} \text{curl}' \hat{\psi}' &= \hat{f}_1 & \text{in } \Omega', \\ \text{div}' \hat{\psi}' &= -i\xi \hat{\psi}_1 & \text{in } \Omega'. \end{aligned}$$

We also have the desired boundary conditions:

$$\hat{\psi}' \cdot \tau' = 0 \text{ on } \partial\Omega' \text{ for (b), or } \hat{\psi}' \cdot \nu' = 0 \text{ on } \partial\Omega' \text{ for (a).}$$

By Assumption 25 we have

$$(43) \quad \|\nabla' \hat{\psi}'(\xi)\|_{L^2(\Omega')} \lesssim \|\hat{f}_1(\xi)\|_{L^2(\Omega')} + \|\xi \hat{\psi}_1(\xi)\|_{L^2(\Omega')} \lesssim \|\hat{f}(\xi)\|_{L^2(\Omega')},$$

the last inequality following from the second inequality in (42).

We now regularise $\hat{\psi}_i$ for $\xi \rightarrow 0$. Define, for $\varepsilon > 0$,

$$\hat{\psi}_{i,\varepsilon}(\xi, x') = \frac{|\xi|}{|\xi| + \varepsilon} \hat{\psi}_i(\xi, x'), \quad i = 1, 2, 3.$$

By a direct calculation we have $\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon} = \frac{\varepsilon}{|\xi| + \varepsilon} \nabla' \hat{\psi}_i$, and so we get

$$\|\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} = \frac{\varepsilon}{|\xi| + \varepsilon} \|\nabla' \hat{\psi}_i\|_{L^2(\Omega')} \leq \|\nabla' \hat{\psi}_i\|_{L^2(\Omega')} \lesssim \|\hat{f}(\xi)\|_{L^2(\Omega')},$$

where the last inequality follows from (42) for $i = 1$ and (43) for $i = 2, 3$. By the Dominated Convergence Theorem, therefore,

$$\lim_{\varepsilon \rightarrow 0} \|\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega)} = 0.$$

By direct calculation,

$$(44) \quad \|\xi \hat{\psi}_i - \xi \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} = \frac{|\xi| \varepsilon}{|\xi| + \varepsilon} \|\hat{\psi}_i\|_{L^2(\Omega')}.$$

Now, using (42) for $i = 1$ and the two identities

$$\xi \hat{\psi}_2 = -i(\partial_2 \hat{\psi}_1 + \hat{f}_3), \quad \xi \hat{\psi}_3 = i(\hat{f}_2 - \partial_2 \hat{\psi}_1),$$

together with (42) for $i = 2, 3$, we have $|\xi| \cdot \|\hat{\psi}_i\|_{L^2(\Omega')} \lesssim \|\hat{f}\|_{L^2(\Omega')}$, whence $\|\xi \hat{\psi}_i - \xi \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} \lesssim \|\hat{f}\|_{L^2(\Omega')}$. Taking inverse Fourier transforms in (44), dominated convergence yields

$$\lim_{\varepsilon \rightarrow 0} \|\partial_1 \psi_{i,\varepsilon} - \partial_1 \psi_i\|_{L^2(\Omega)} = 0.$$

Altogether we have that $\|\nabla \psi_{i,\varepsilon} - \nabla \psi_i\|_{L^2(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, namely

$$(45) \quad \lim_{\varepsilon \rightarrow 0} \|\psi_{i,\varepsilon} - \psi_i\|_{\dot{H}^1(\Omega)} = 0.$$

Since $|\hat{\psi}_{i,\varepsilon}(\xi, x')| \leq \varepsilon^{-1} |\xi \hat{\psi}_i(\xi, x')|$ we have that $\|\psi_{i,\varepsilon}\|_{L^2(\Omega)} \leq \varepsilon^{-1} \|\partial_1 \psi_i\|_{L^2(\Omega)} < +\infty$, and so $\psi_{i,\varepsilon} \in H^1(\Omega)$. Since $H^1(\Omega)$ is dense in $\dot{H}^1(\Omega)$, by (45) we conclude that $\psi \in \dot{H}^1(\Omega)$. \square

We now observe that Assumption 25 is verified in many situations of interest.

Lemma 27. *Assumption 25 is verified in each of the following cases:*

- (1) Ω' is the full space \mathbb{R}^2 ;
- (2) Ω' is the half space $\{(x_2, x_3) \in \mathbb{R}^2 : x_3 > 0\}$;
- (3) Ω' is a strip $\{(x_2, x_3) \in \mathbb{R}^2 : 0 < x_3 < L\}$ for some $L > 0$;
- (4) Ω' is a simply connected bounded domain of class $C^{1,1}$ or piecewise smooth with no re-entrant corners.

Proof. (1) Taking Fourier transforms in (31) we obtain

$$i\xi_2 \hat{\psi}'_3 - i\xi_3 \hat{\psi}'_2 = \hat{g}, \quad i\xi_2 \hat{\psi}'_2 + i\xi_3 \hat{\psi}'_3 = \hat{h},$$

with unique solution

$$\hat{\psi}'_2 = \frac{-i\xi_2 \hat{h} + i\xi_3 \hat{g}}{|\xi'|^2}, \quad \hat{\psi}'_3 = \frac{-i\xi_3 \hat{h} - i\xi_2 \hat{g}}{|\xi'|^2}.$$

Hence $|\xi_2 \hat{\psi}'_2| = \frac{|\xi_2^2 \hat{h}|}{|\xi'|^2} + \frac{|\xi_2 \xi_3 \hat{g}|}{|\xi'|^2} \leq |\hat{h}| + |\hat{g}|$, so that

$$\|\partial_2 \psi'_2\|_{L^2(\mathbb{R}^2)} \leq \|\hat{g}\|_{L^2(\mathbb{R}^2)} + \|\hat{h}\|_{L^2(\mathbb{R}^2)},$$

and similarly for the other conditions.

(2) In this case the boundary condition is either $\psi'_2 = 0$ or $\psi'_3 = 0$ on $\{x_3 = 0\}$. We study the case $\psi'_2 = 0$; the other is similar.

Taking Fourier transforms we get

$$\begin{aligned}\psi'_2(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \hat{\psi}'_2(\xi_2, \xi_3) e^{i\xi_2 x_2} \sin(\xi_3 x_3) d\xi_3 d\xi_2, \\ \psi'_3(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \hat{\psi}'_3(\xi_2, \xi_3) e^{i\xi_2 x_2} \cos(\xi_3 x_3) d\xi_3 d\xi_2, \\ g(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} g(\xi_2, \xi_3) e^{i\xi_2 x_2} \cos(\xi_3 x_3) d\xi_3 d\xi_2, \\ h(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} h(\xi_2, \xi_3) e^{i\xi_2 x_2} \sin(\xi_3 x_3) d\xi_3 d\xi_2.\end{aligned}$$

The equations in (31) become

$$i\xi_2 \hat{\psi}'_3 - \xi_3 \hat{\psi}'_2 = \hat{g}, \quad i\xi_2 \hat{\psi}'_2 - \xi_3 \hat{\psi}'_3 = \hat{h},$$

and then everything proceeds as for the case $\Omega' = \mathbb{R}^2$.

(3) This follows by using the Fourier transform with respect to the variable x_2 and the Fourier series in the variable x_3 , as in cases (1) and (2) above; the calculations are completely analogous.

(4) This part was proven in [10, Chapter 1, Remark 3.5]). \square

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