HIGGS BUNDLES OVER NON-COMPACT GAUDUCHON MANIFOLDS

CHUANJING ZHANG, PAN ZHANG AND XI ZHANG

Abstract. In this paper, we prove a generalized Donaldson-Uhlenbeck-Yau theorem on Higgs bundles over a class of non-compact Gauduchon manifolds.

1. Introduction

Let X be a complex manifold of dimension n and g a Hermitian metric with associated Kähler form ω. The metirc g is called Gauduchon if ω satisfies $\partial\bar{\partial}\omega^{n-1}=0$. A Higgs bundle $(E,\overline{\partial}_E,\theta)$ over X is a holomorphic bundle $(E,\overline{\partial}_E)$ coupled with a Higgs field $\theta\in\Omega_X^{1,0}(\mathrm{End}(E))$ such that $\overline{\partial}_E\theta=0$ and $\theta\wedge\theta=0$. Higgs bundles were introduced by Hitchin ([12]) in his study of the self duality equations. They have rich structures and play an important role in many areas including gauge theory, Kähler and hyperkähler geometry, group representations and nonabelian Hodge theory. Let H be a Hermitian metric on the bundle E, we consider the Hitchin-Simpson connection

$$\overline{\partial}_\theta:=\overline{\partial}_E+\theta,\quad D_{H,\theta}^{1,0}:=D_H^{1,0}+\theta^{*H},\quad D_{H,\theta}=\overline{\partial}_\theta+D_{H,\theta}^{1,0},$$

where D_H is the Chern connection of $(E,\overline{\partial}_E,H)$ and θ^{*H} is the adjoint of θ with respect to the metric H. The curvature of this connection is

$$F_{H,\theta}=F_H+[\theta,\theta^{*H}]+\partial_H\theta+\bar{\partial}_E\theta^{*H},$$

where F_H is the curvature of D_H and ∂_H is the $(1,0)$-part of D_H. H is said to be a Hermitian-Einstein metric on Higgs bundle $(E,\overline{\partial}_E,\theta)$ if the curvature of the Hitchin-Simpson connection satisfies the Einstein condition, i.e.

$$\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])=\lambda\cdot\mathrm{Id}_E,$$

where Λ_ω denotes the contraction with ω, and λ is a constant.

When the base space (X,ω) is a compact Kähler manifold, the stability of Higgs bundles, in the sense of Mumford-Takemoto, was a well established concept. Hitchin ([12]) and Simpson ([29], [30]) obtained a Higgs bundle version of the Donaldson-Uhlenbeck-Yau theorem ([28], [9], [32]), i.e. they proved that a Higgs bundle admits the Hermitian-Einstein metric if and only if it's Higgs poly-stable. Simpson ([29]) also considered some non-compact Kähler manifolds case, he introduced the concept of analytic stability for Higgs bundle, and proved that the analytic stability implies the existence of

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Hermitian-Einstein metric. There are many other interesting and important works related ([1, 2, 3, 4, 8, 14, 16, 17, 18, 19, 23, 24, 25, 27, 33], etc.). The non-Kähler case is also very interesting. The Donaldson-Uhlenbeck-Yau theorem is valid for compact Gauduchon manifolds (see [6, 20, 21, 22]).

In this paper, we want to study the non-compact and non-Kähler case. In the following, we always suppose that (X, g) is a Gauduchon manifold unless otherwise stated. By [29], we will make the following three assumptions:

Assumption 1. (X, g) has finite volume.

Assumption 2. There exists a non-negative exhaustion function ϕ with $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}\phi$ bounded.

Assumption 3. There is an increasing function $a : [0, +\infty) \to [0, +\infty)$ with $a(0) = 0$ and $a(x) = x$ for $x > 1$, such that if f is a bounded positive function on X with $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f \geq -B$ then

$$\sup_X |f| \leq C(B)a(\int_X |f|\frac{\omega^n}{n!}).$$

Furthermore, if $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f \geq 0$, then $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}f = 0$.

We fix a background metric K in the bundle E, and suppose that

$$\sup_X |\Lambda_\omega F_{K,\theta}|_K < +\infty.$$

Define the analytic degree of E to be the real number

$$\deg_\omega(E, K) = \sqrt{-1}\int_X \mathrm{tr}(\Lambda_\omega F_{K,\theta})\frac{\omega^n}{n!}.$$

According to the Chern-Weil formula with respect to the metric K (Lemma 3.2 in [29]), we can define the analytic degree of any saturated sub-Higgs sheaf V of $(E, \overline{\partial}_E, \theta)$ by

$$\deg_\omega(V, K) = \int_X \sqrt{-1}\mathrm{tr}(\pi\Lambda_\omega F_{K,\theta}) - |\overline{\partial}_\theta\pi|_K^2\frac{\omega^n}{n!}, \tag{1.1}$$

where π denotes the projection onto V with respect to the metric K. Following [29], we say that the Higgs bundle $(E, \overline{\partial}_E, \theta)$ is K-analytic stable (semi-stable) if for every proper saturated sub-Higgs sheaf $V \subset E$,

$$\frac{\deg_\omega(V, K)}{\mathrm{rank}(V)} < (\leq)\frac{\deg_\omega(E, K)}{\mathrm{rank}(E)}.$$

In this paper, we will show that, under some assumptions on the base space (X, g), the analytic stability implies the existence of Hermitian-Einstein metric on $(E, \overline{\partial}_E, \theta)$, i.e. we obtain the following Donaldson-Uhlenbeck-Yau type theorem.

Theorem 1.1. *Let (X, g) be a non-compact Gauduchon manifold satisfying the Assumptions 1,2,3, and $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$, $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over X with a Hermitian metric K satisfying $\sup_X |\Lambda_\omega F_{K,\theta}|_K < +\infty$. If $(E, \overline{\partial}_E, \theta)$ is K-analytic stable, then there exists a Hermitian metric H with $\overline{\partial}_\theta(\log K^{-1}H) \in L^2$, H and K are mutually bounded, such that*

$$\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) = \lambda_{K,\omega} \cdot \mathrm{Id}_E,$$

where the constant $\lambda_{K,\omega} = \frac{\deg_\omega(E,K)}{\mathrm{rank}(E)\mathrm{Vol}(X,g)}$.

From the Chern-Weil formula (1.1), it is easy to see that the existence of Hermitian-Einstein metric H implies $(E, \bar{\partial}_E, \theta)$ is H-analytic poly-stable. Our result is slightly better than that in [29], where Simpson only obtained a Hermitian metric with vanishing trace-free curvature. The reason is that, in Section 4, we can solve the following Poisson equation

$$-2\sqrt{-1}\Lambda_\omega \bar{\partial}\partial f = \psi \tag{1.2}$$

on the non-Kähler and non-compact manifold (X, g) when $\int_X \psi \frac{\omega^n}{n!} = 0$. In [29], Simpson used Donaldson's heat flow method to attack the existence problem of the Hermitian-Einstein metrics on Higgs bundles, and his proof relies on the properties of the Donaldson functional. However, the Donaldson functional is not well-defined when g is only Gauduchon. So Simpson's argument is not applicable in our situation directly. In this paper, we follow the argument of Uhlenbeck-Yau in [32], where they used the continuity method and their argument is more natural. We first solve the following perturbed equation on (X, g):

$$\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(K^{-1}H) = 0. \tag{1.3}$$

The above perturbed equation can be solved by using the fact that the elliptic operators are Fredholm if the base manifold is compact. Generally speaking, this fact is not true in the non-compact case, which means we can not directly apply this method to solve the perturbed equation on the non-compact manifold. To fix this, we combine the method of heat flow and the method of exhaustion to solve the perturbed equation on (X, g) for any $0 < \varepsilon \leq 1$, see Section 5 for details. For simplicity, we set

$$\Phi(H, \theta) = \sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda_{K,\omega} \cdot \mathrm{Id}_E. \tag{1.4}$$

Under the assumptions as that in Theorem 1.1, we can prove the following identity:

$$\int_X \mathrm{tr}(\Phi(K, \theta)s) + \langle \Psi(s)(\overline{\partial}_\theta s), \overline{\partial}_\theta s\rangle_K \frac{\omega^n}{n!} = \int_X \mathrm{tr}(\Phi(H, \theta)s), \tag{1.5}$$

where $s = \log(K^{-1}H)$ and

$$\Psi(x, y) = \begin{cases} \frac{e^{y-x}-1}{y-x}, & x \neq y; \\ 1, & x = y. \end{cases} \tag{1.6}$$

By the above identity (1.5) and Uhlenbeck-Yau's result ([32]), that L_1^2 weakly holomorphic sub-bundles define coherent sub-sheaves, we can obtain the existence result of Hermitian-Einstein metric by using the continuity method. It should be pointed out that application of the identity (1.5) plays a key role in our argument (see Section 6), which is slightly different with that in [32] (or [6, 20, 21]).

In the end of this paper, we also study the semi-stable case. A Higgs bundle is said to be admitting an approximate Hermitian-Einstein structure, if for every $\delta > 0$, there exists a Hermitian metric H such that

$$\sup_X |\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda_{K,\omega} \cdot \mathrm{Id}_E|_H < \delta.$$

This notion was firstly introduced by Kobayashi([15]) in holomorphic vector bundles (i.e. $\theta = 0$). He proved that over projective manifolds, a semi-stable holomorphic vector bundle must admit an approximate Hermitian-Einstein structure. In [17], Li and the third author proved this result is valid for Higgs bundles over compact Kähler manifolds. There are also some other interesting works related, see references [5, 7, 13, 26] for details. In this paper, we obtain an existence result of approximate Hermitian-Einstein structures on analytic semi-stable Higgs bundles over a class of non-compact Gauduchon manifolds. In fact, we prove that:

Theorem 1.2. *Under the same assumptions as that in Theorem 1.1, if the Higgs bundle $(E, \bar{\partial}_E, \theta)$ is K-analytic semi-stable, then there must exist an approximate Hermitian-Einstein structure, i.e. for every $\delta > 0$, there exists a Hermitian metric H with H and K mutually bounded, such that*

$$\sup_X |\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda_{K,\omega} \cdot \mathrm{Id}_E|_H < \delta.$$

This paper is organized as follows. In Section 2, we give some estimates and preliminaries which will be used in the proof of Theorems 1.1 and 1.2. At the end of Section 2, we prove the identity (1.5). In Section 3, we get the long-time existence result of the related heat flow. In Section 4, we consider the Poisson equation (1.2) on some non-compact Gauduchon manifolds. In Section 5, we solve the perturbed equation (1.3). In Section 6, we complete the proof of Theorems 1.1 and 1.2.

2. Preliminary results

Let (M, g) be an n-dimensional Hermitian manifold. Let $(E, \bar{\partial}_E, \theta)$ be a rank r Higgs bundle over M and H_0 be a Hermitian metric on E. We consider the following heat flow.

$$H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H)), \tag{2.1}$$

where $H(t)$ is a family of Hermitian metrics on E and ε is a nonnegative constant. Choosing local complex coordinates $\{z^i\}_{i=1}^n$ on M, then $\omega = \sqrt{-1}g_{i\bar{j}}dz^i \wedge d\bar{z}^j$. We define the complex Laplace operator for functions

$$\widetilde{\Delta} f = -2\sqrt{-1}\Lambda_\omega \bar{\partial}\partial f = 2g^{i\bar{j}}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j},$$

where $(g^{i\bar{j}})$ is the inverse matrix of the metric matrix $(g_{i\bar{j}})$. As usual, we denote the Beltrami-Laplcaian operator by Δ. It is well known that the difference of the two Laplacians is given by a first order differential operator as follows

$$(\widetilde{\Delta} - \Delta)f = \langle V, \nabla f\rangle_g,$$

where V is a well-defined vector field on M.

Proposition 2.1. *Let $H(t)$ be a solution of the flow* (2.1)*, then*

$$(\frac{\partial}{\partial t} - \widetilde{\Delta})\{e^{2\varepsilon t} \cdot \mathrm{tr}(\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log h)\} = 0 \tag{2.2}$$

and

$$(\frac{\partial}{\partial t}-\widetilde{\Delta})|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*_H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log h|_H^2\leq 0. \tag{2.3}$$

Proof. For simplicity, we denote $\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*_H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log h=\Phi_\varepsilon$. By calculating directly, we have

$$\frac{\partial}{\partial t}\Phi_\varepsilon=\sqrt{-1}\Lambda_\omega\{\bar{\partial}_E(\partial_H(h^{-1}\frac{\partial h}{\partial t}))+[\theta,[\theta^{*_H},h^{-1}\frac{\partial h}{\partial t}]]\}+\varepsilon\frac{\partial}{\partial t}(\log h), \tag{2.4}$$

and

$$\begin{aligned}\widetilde{\Delta}|\Phi_\varepsilon|_H^2&=-2\sqrt{-1}\Lambda_\omega\bar{\partial}\partial\mathrm{tr}\{\Phi_\varepsilon H^{-1}\bar{\Phi}_\varepsilon^t H\}\\&=-2\sqrt{-1}\Lambda_\omega\bar{\partial}\mathrm{tr}\{\partial\Phi_\varepsilon H^{-1}\bar{\Phi}_\varepsilon^t H-\Phi_\varepsilon H^{-1}\partial H H^{-1}\bar{\Phi}_\varepsilon^t H\\&\quad+\Phi H^{-1}\overline{\bar{\partial}\Phi_\varepsilon}^t H+\Phi_\varepsilon H^{-1}\bar{\Phi}_\varepsilon^t H H^{-1}\partial H\}\\&=2\mathrm{Re}\langle-2\sqrt{-1}\Lambda_\omega\bar{\partial}_E\partial_H\Phi_\varepsilon,\Phi_\varepsilon\rangle_H+2|\partial_H\Phi_\varepsilon|_H^2+2|\bar{\partial}_E\Phi_\varepsilon|_H^2.\end{aligned}$$

From (2.4), it is easy to conclude that

$$(\frac{\partial}{\partial t}-\widetilde{\Delta})\mathrm{tr}\Phi_\varepsilon=-2\varepsilon\mathrm{tr}\Phi_\varepsilon. \tag{2.5}$$

Then, (2.5) implies (2.2).

From [22, p. 237], we can choose an open dense subset $W\subset M\times[0,T_0]$ satisfying at each $(x_0,t_0)\in W$ there exist an open neighborhood U of (x_0,t_0), a local unitary basis $\{e_i\}_{i=1}^r$ with respect to H and functions $\{\lambda_i\in C^\infty(U,\mathbb{R})\}_{i=1}^r$ such that

$$h(y,t)=\sum_{i=1}^r e^{\lambda_i(y,t)}e_i(y,t)\otimes e^i(y,t)$$

for all $(y,t)\in U$, where $\{e^i\}_{i=1}^r$ is the corresponding dual basis. Then we get

$$\frac{\partial}{\partial t}(\log h)=\sum_{i=1}^r(\frac{\mathrm{d}\lambda_i}{\mathrm{dt}})e_i\otimes e^i+\sum_{i\neq j}(\lambda_j-\lambda_i)\alpha_{ji}e_i\otimes e^j,$$

and

$$h^{-1}\frac{\partial h}{\partial t}=\sum_{i=1}^r(\frac{\mathrm{d}\lambda_i}{\mathrm{dt}})e_i\otimes e^i+\sum_{i\neq j}(e^{\lambda_j-\lambda_i}-1)\alpha_{ji}e_i\otimes e^j,$$

where $\frac{\mathrm{d}}{\mathrm{dt}}e_i=\alpha_{ij}e_j$. Since $(\lambda_i-\lambda_j)(e^{\lambda_i-\lambda_j}-1)\geq 0$ for all $\lambda_i,\lambda_j\in\mathbb{R}$, we have

$$\langle\frac{\partial}{\partial t}(\log h),h^{-1}\frac{\partial h}{\partial t}\rangle_H\geq 0.$$

Using the above formulas, we conclude that

$$\begin{aligned}(\frac{\partial}{\partial t}-\widetilde{\Delta})|\Phi_\varepsilon|_H^2&=-4\langle\sqrt{-1}\Lambda_\omega[\theta,[\theta^{*_H},\Phi_\varepsilon]],\Phi_\varepsilon\rangle_H-2|\partial_H\Phi_\varepsilon|_H^2-2|\bar{\partial}_E\Phi|_H^2\\&\quad+2\varepsilon\langle\frac{\partial}{\partial t}(\log h),\Phi_\varepsilon\rangle_H\\&\leq 0.\end{aligned}$$

□

We introduce the Donaldson's distance on the space of the Hermitian metrics as follows.

Definition 2.2. *For any two Hermitian metrics H and K on the bundle E, we define*

$$\sigma(H,K)=\mathrm{tr}(H^{-1}K)+\mathrm{tr}(K^{-1}H)-2\mathrm{rank}(E).$$

It is obvious that $\sigma(H,K)\geq 0$, with equality if and only if $H=K$. A sequence of metrics H_i converges to H in the usual C^0 topology if and only if $\sup_M\sigma(H_i,H)\to 0$.

Proposition 2.3. *Let $H(t)$, $K(t)$ be two solutions of the flow* (2.1)*, then*

$$(\widetilde{\Delta}-\frac{\partial}{\partial t})\sigma(H(t),K(t))\geq 0.$$

Proof. Setting $h(t)=K(t)^{-1}H(t)$, we have

$$\begin{aligned}
&(\widetilde{\Delta}-\frac{\partial}{\partial t})(\mathrm{tr}h+\mathrm{tr}h^{-1})\\
&=2\mathrm{tr}(-\sqrt{-1}\Lambda_\omega\bar{\partial}_E h h^{-1}\partial_K h)+2\mathrm{tr}(-\sqrt{-1}\Lambda_\omega\bar{\partial}_E h^{-1} h\partial_H h^{-1})\\
&\ +2\mathrm{tr}\{h(\sqrt{-1}\Lambda_\omega[\theta,\theta^{*_H}-\theta^{*_K}]))+2\mathrm{tr}(h^{-1}(\sqrt{-1}\Lambda_\omega[\theta,\theta^{*_K}-\theta^{*_H}])\}\\
&\ +2\varepsilon\mathrm{tr}\{h(\log(H_0^{-1}H)-\log(H_0^{-1}K))+h^{-1}(\log(H_0^{-1}K)-\log(H_0^{-1}H))\}\\
&\geq 0,
\end{aligned}$$

where we used

$$\mathrm{tr}\{h(\sqrt{-1}\Lambda_\omega[\theta,\theta^{*_H}-\theta^{*_K}])\}=|\theta h^{\frac{1}{2}}-h\theta h^{-\frac{1}{2}}|_K^2$$

and

$$\mathrm{tr}\{h^{-1}(\sqrt{-1}\Lambda_\omega[\theta,\theta^{*_K}-\theta^{*_H}])\}=|h^{-\frac{1}{2}}\theta-h^{\frac{1}{2}}\theta h^{-1}|_K^2.$$

It remains to show that

$$A:=\mathrm{tr}\{h(\log(H_0^{-1}H)-\log(H_0^{-1}K))+h^{-1}(\log(H_0^{-1}K)-\log(H_0^{-1}H))\}\geq 0.$$

Once we set $\log(H_0^{-1}H)=s_1$, $\log(H_0^{-1}K)=s_2$, we have

$$\begin{aligned}
A&=\mathrm{tr}\Big(e^{-s_2}e^{s_1}(s_1-s_2)+e^{-s_1}e^{s_2}(s_2-s_1)\Big)\\
&=\mathrm{tr}\Big(e^{-s_2}(e^{s_1}-e^{s_2})(s_1-s_2)+e^{-s_1}(e^{s_2}-e^{s_1})(s_2-s_1)\Big).
\end{aligned}$$

Hence we only need to show

$$\mathrm{tr}\Big(e^{-s_2}(e^{s_1}-e^{s_2})(s_1-s_2)\Big)\geq 0.$$

Choose unitary basis $\{e_\alpha\}_{\alpha=1}^r$ such that $s_2(e_\alpha)=\lambda_\alpha e_\alpha$. Similarly, $s_1(\widetilde{e}_\beta)=\widetilde{\lambda}_\beta\widetilde{e}_\beta$ under the unitary basis $\{\widetilde{e}_\beta\}_{\beta=1}^r$. We also assume that $e_\alpha=b_{\alpha\beta}\widetilde{e}_\beta$. Direct calculation yields

$$\begin{aligned}\operatorname{tr}\Big(e^{-s_2}(e^{s_1}-e^{s_2})(s_1-s_2)\Big)&=\sum_{\alpha=1}^r\langle e^{-s_2}(e^{s_1}-e^{s_2})(s_1-s_2)(e_\alpha),e_\alpha\rangle_{H_0}\\&=\sum_{\alpha=1}^r e^{-\lambda_\alpha}\langle\sum_{\beta=1}^r b_{\alpha\beta}(\widetilde{\lambda}_\beta-\lambda_\alpha)\widetilde{e}_\beta,\sum_{\gamma=1}^r b_{\alpha\gamma}(e^{\widetilde{\lambda}_\gamma}-e^{\lambda_\alpha})\widetilde{e}_\gamma\rangle_{H_0}\\&=\sum_{\alpha,\beta=1}^r e^{-\lambda_\alpha}b_{\alpha\beta}\overline{b_{\alpha\beta}}(\widetilde{\lambda}_\beta-\lambda_\alpha)(e^{\widetilde{\lambda}_\beta}-e^{\lambda_\alpha})\\&\geq 0.\end{aligned}$$

□

Corollary 2.4. *Let H, K be two Hermitian metrics satisfying* (1.3)*, then*

$$\widetilde{\Delta}\sigma(H,K)\geq 0.$$

At the end of this section, we give a proof of the identity (1.5). We first recall some notation. Set $\mathrm{Herm}(E,H_0)=\{\eta\in\mathrm{End}(E)\mid \eta^{*H_0}=\eta\}$. Given $s\in\mathrm{Herm}(E,H_0)$, we can choose a local unitary basis $\{e_\alpha\}_{\alpha=1}^r$ respect to H_0 and local functions $\{\lambda_\alpha\}_{\alpha=1}^r$ such that

$$s=\sum_{\alpha=1}^r\lambda_\alpha\cdot e_\alpha\otimes e^\alpha,$$

where $\{e^\alpha\}_{\alpha=1}^r$ denotes the dual basis of E^*. Let $\Psi\in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $A=\sum\limits_{\alpha,\beta=1}^r A^\alpha_\beta e_\alpha\otimes e^\beta\in\mathrm{End}(E)$. We define:

$$\Psi(\eta)(A)=\Psi(\lambda_\alpha,\lambda_\beta)A^\alpha_\beta e_\alpha\otimes e^\beta.$$

Let (M,g) be a compact Gauduchon manifold with non-empty smooth boundary ∂M. Let φ be a smooth function defined on M and satisfy the boundary condition $\varphi|_{\partial M}=t$, where t is a constant. By Stokes' formula, we have

(2.6)
$$\begin{aligned}\int_M|\mathrm{d}\varphi|^2\frac{\omega^n}{n!}&=2\int_M(t-\varphi)\sqrt{-1}\partial\bar{\partial}\varphi\wedge\frac{\omega^{n-1}}{(n-1)!}-2\int_M\sqrt{-1}\partial((t-\varphi)\bar{\partial}\varphi)\wedge\frac{\omega^{n-1}}{(n-1)!}\\&=\int_M(t-\varphi)\tilde{\Delta}\varphi\frac{\omega^n}{n!}+\int_M\sqrt{-1}\bar{\partial}((t-\varphi)^2\wedge\partial\frac{\omega^{n-1}}{(n-1)!})\\&\quad+\int_M\sqrt{-1}\partial(\bar{\partial}(t-\varphi)^2\wedge\frac{\omega^{n-1}}{(n-1)!})-\int_M\sqrt{-1}(t-\varphi)^2\bar{\partial}(\partial\frac{\omega^{n-1}}{(n-1)!})\\&=\int_M(t-\varphi)\tilde{\Delta}\varphi\frac{\omega^n}{n!}.\end{aligned}$$

Using (2.6), by the same argument as that in [29] (Lemma 5.2), we can obtain the following lemma.

Lemma 2.5 ([29, Lemma 5.2])**.** *Suppose (X,g) is a non-compact Gauduchon manifold admitting an exhaustion function ϕ with $\int_X|\widetilde{\Delta}\phi|\frac{\omega^n}{n!}<\infty$, and suppose η is a $(2n-1)$-form with $\int_X|\eta|^2\frac{\omega^n}{n!}<\infty$. Then if $\mathrm{d}\eta$ is integrable,*

$$\int_X \mathrm{d}\eta=0.$$

Proposition 2.6. *Let $(E,\bar{\partial}_E,\theta)$ be a Higgs bundle with a fixed Hermitian metric H_0 over a Gauduchon manifold (M,g). Let H be a Hermitian metric on E and $s:=\log(H_0^{-1}H)$. If one of the following two conditions is satisfied:*

(1)Suppose that M is a compact manifold with non-empty smooth boundary ∂M, and H is a Hermitian metric on E with the same boundary condition as that of H_0, i.e. $H|_{\partial M}=H_0|_{\partial M}$.

(2)Suppose that M is a non-compact manifold admitting an exhaustion function ϕ with $\int_M|\widetilde{\Delta}\phi|\frac{\omega^n}{n!}<+\infty$. Furthermore, we also assume that $|\mathrm{d}\omega^{n-1}|_g\in L^2(M)$, $s\in L^\infty(M)$ and $D^{1,0}_{H_0,\theta}s\in L^2(M)$.

Then we have the following identity:

$$\int_M \mathrm{tr}(\Phi(H_0,\theta)s)\frac{\omega^n}{n!}+\int_M\langle\Psi(s)(\overline{\partial}_\theta s),\overline{\partial}_\theta s\rangle_{H_0}\frac{\omega^n}{n!}=\int_M\mathrm{tr}(\Phi(H,\theta)s)\frac{\omega^n}{n!},\tag{2.7}$$

where $\overline{\partial}_\theta=\bar{\partial}_E+\theta$ and Ψ is the function which is defined in (1.6).

Proof. Set $h=H_0^{-1}H=e^s$. By the definition, we have

$$\mathrm{tr}((\Phi(H,\theta)-\Phi(H_0,\theta))s)=\langle\sqrt{-1}\Lambda_\omega(\bar{\partial}(h^{-1}\partial_{H_0}h)+[\theta,\theta^{*H}-\theta^{*H_0}]),s\rangle_{H_0}.\tag{2.8}$$

Using $\mathrm{tr}(h^{-1}(\partial_{H_0}h)s)=\mathrm{tr}(s\partial_{H_0}s)$, $\mathrm{tr}(s[\theta^{*H_0},s])=0$ and $\partial\bar{\partial}\omega^{n-1}=0$, we have

$$\begin{aligned}
&\int_M\langle\sqrt{-1}\Lambda_\omega(\bar{\partial}(h^{-1}\partial_{H_0}h)),s\rangle_{H_0}\frac{\omega^n}{n!}\\
=&\int_M\sqrt{-1}\bar{\partial}\mathrm{tr}(s\partial_{H_0}s)\wedge\frac{\omega^{n-1}}{(n-1)!}+\int_M\sqrt{-1}\mathrm{tr}(h^{-1}\partial_{H_0}h\bar{\partial}s)\wedge\frac{\omega^{n-1}}{(n-1)!}\\
=&\int_M\sqrt{-1}\mathrm{tr}(s\partial_{H_0}s)\wedge\overline{\partial}(\frac{\omega^{n-1}}{(n-1)!})+\int_M\sqrt{-1}\mathrm{tr}(h^{-1}\partial_{H_0}h\bar{\partial}s)\wedge\frac{\omega^{n-1}}{(n-1)!}\\
&+\int_M\sqrt{-1}\bar{\partial}(\mathrm{tr}(s\partial_{H_0}s)\wedge\frac{\omega^{n-1}}{(n-1)!})\\
=&\int_M\partial(\frac{\sqrt{-1}}{2}\mathrm{tr}(s^2)\wedge\overline{\partial}(\frac{\omega^{n-1}}{(n-1)!}))+\int_M\sqrt{-1}\bar{\partial}(\mathrm{tr}(s\partial_{H_0}s)\wedge\frac{\omega^{n-1}}{(n-1)!})\\
&+\int_M\sqrt{-1}\mathrm{tr}(h^{-1}\partial_{H_0}h\bar{\partial}s)\wedge\frac{\omega^{n-1}}{(n-1)!}\\
=&\int_M\partial(\frac{\sqrt{-1}}{2}\mathrm{tr}(s^2)\wedge\overline{\partial}(\frac{\omega^{n-1}}{(n-1)!}))+\int_M\sqrt{-1}\bar{\partial}(\mathrm{tr}(sD^{1,0}_{H_0,\theta}s)\wedge\frac{\omega^{n-1}}{(n-1)!})\\
&+\int_M\sqrt{-1}\mathrm{tr}(h^{-1}\partial_{H_0}h\bar{\partial}s)\wedge\frac{\omega^{n-1}}{(n-1)!}.
\end{aligned}\tag{2.9}$$

In condition (1), by using $s|_{\partial M}=0$ and Stokes formula, in condition (2), by using Lemma 2.5, we have

$$\int_M \langle \sqrt{-1}\Lambda_\omega(\bar{\partial}(h^{-1}\partial_{H_0}h)), s\rangle_{H_0}\frac{\omega^n}{n!} = \int_M \sqrt{-1}\mathrm{tr}(h^{-1}\partial_{H_0}h\bar{\partial}s)\wedge\frac{\omega^{n-1}}{(n-1)!}. \tag{2.10}$$

In [26, p.635], it was proved that

$$\mathrm{tr}\sqrt{-1}\Lambda_\omega(h^{-1}D^{1,0}_{H,\theta}h\overline{\partial}_\theta s) = \langle \Psi(s)(\overline{\partial}_\theta s), \overline{\partial}_\theta s\rangle_{H_0}, \tag{2.11}$$

and

$$\int_M \mathrm{tr}(\sqrt{-1}\Lambda_\omega[\theta, \theta^{*H}-\theta^{*H_0}]s)\frac{\omega^n}{n!} = \int_M \mathrm{tr}(\sqrt{-1}h^{-1}[\theta^{*H_0}, h][\theta, s])\frac{\omega^{n-1}}{(n-1)!}. \tag{2.12}$$

By (2.10), (2.11) and (2.12), we obtain

$$\int_M \langle \sqrt{-1}\Lambda_\omega(\bar{\partial}(h^{-1}\partial_{H_0}h)+[\theta,\theta^{*H}-\theta^{*H_0}]), s\rangle_{H_0}\frac{\omega^n}{n!} = \int_M \langle \Psi(s)(\overline{\partial}_\theta s), \overline{\partial}_\theta s\rangle_{H_0}\frac{\omega^n}{n!}. \tag{2.13}$$

Then (2.8) and (2.13) imply (2.7).

□

3. The related heat flow on Hermitian manifolds

In this section, we consider the existence of long-time solutions of the related heat flow (2.1). Let (M, g) be a compact Hermitian manifold (with possibly non-empty boundary), and $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle over M. If M is closed then we consider the following evolution equation:

$$\begin{cases} H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}]) - \lambda\cdot \mathrm{Id}_E + \varepsilon\log(H_0^{-1}H)), \\ H(0)=H_0. \end{cases} \tag{3.1}$$

If M is a compact manifold with non-empty smooth boundary ∂M, for given data $\widetilde{H}$ on ∂M, we consider the following Dirichlet boundary value problem:

$$\begin{cases} H^{-1}\frac{\partial H}{\partial t} = -2(\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}]) - \lambda\cdot \mathrm{Id}_E + \varepsilon\log(H_0^{-1}H)), \\ H(0)=H_0, \\ H|_{\partial M} = \widetilde{H}. \end{cases} \tag{3.2}$$

When $\varepsilon = 0$, (2.1) is just the Hermitian-Yang-Mills flow, the existence of long-time solutions of (3.1) and (3.2) on Hermitian manifolds was proved in [34]. It is easy to see that the flow (2.1) is strictly parabolic, so standard parabolic theory gives the short-time existence.

Proposition 3.1. *For sufficiently small $T>0$, (3.1) and (3.2) have a smooth solution defined for $0\le t<T$.*

Next, following the arguments in [9, Lemma 19] and [29, Lemma 6.4], we will prove the long-time existence.

Lemma 3.2. *Suppose that a smooth solution $H(t)$ of (3.1) or (3.2) is defined for $0\le t< T<+\infty$. Then $H(t)$ converge in C^0-topology to some continuous non-degenerate metric H_T as $t\to T$.*

Proof. Given $\epsilon > 0$, by continuity at $t = 0$ we can find a δ such that

$$\sup_M \sigma(H(t_0), H(t_0')) < \epsilon$$

for $0 < t_0, t_0' < \delta$. Then Proposition 2.3 and the maximum principle imply that

$$\sup_M \sigma(H(t), H(t')) < \epsilon$$

for all $t, t' > T - \delta$. This implies that $H(t)$ are uniformly Cauchy and converge to a continuous limiting metric H_T. On the other hand, by Proposition 2.1, we know that

$$\sup_{M\times[0,T)} |\sqrt{-1}\Lambda_\omega(F_{H(t)} + [\theta, \theta^{*_{H(t)}}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H(t))|_{H(t)} < B,$$

where B is a uniform constant depending only on the initial data H_0. Then using

$$|\frac{\partial}{\partial t}(\log \mathrm{tr} h)|_H \leq 2|\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H)|_H,$$

and

$$|\frac{\partial}{\partial t}(\log \mathrm{tr} h^{-1})|_H \leq 2|\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H)|_H,$$

one can conclude that $\sigma(H(t), H_0)$ are bounded uniformly on $M \times [0, T)$, therefore H_T is a non-degenerate metric. □

For further consideration, we recall the following lemma.

Lemma 3.3 (Lemma 3.3 in [34])**.** *Let M be a compact Hermitian manifold without boundary (with non-empty boundary). Let $H(t)$, $0 \leq t < T$, be any one-parameter family of Hermitian metrics on the Higgs bundle E over M (and satisfying Dirichlet boundary condition), and suppose H_0 is the initial Hermitian metric. If $H(t)$ converge in the C^0 topology to some continuous metric H_T as $t \to T$, and if $\sup_M |\Lambda_\omega F_{H(t)}|_{H_0}$ is bounded uniformly in t, then $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < +\infty$) uniformly in t.*

Proposition 3.4. (3.1) *and* (3.2) *have a unique solution $H(t)$ which exists for $0 \leq t < +\infty$.*

Proof. Proposition 3.1 guarantees that a solution exists for a short time. Suppose that the solution $H(t)$ exists for $0 \leq t < T < +\infty$. By Lemma 3.2, $H(t)$ converges in C^0 to a non-degenerate continuous limit metric $H(T)$ as $t \to T$. Since $t < +\infty$, (2.3) implies $\sup_M |\Lambda_\omega F_{H(t)}|_{H_0}$ is bounded uniformly in $[0, T)$. Then by Lemma 3.3, $H(t)$ are bounded in C^1 and also bounded in L_2^p (for any $1 < p < +\infty$) uniformly in t. Since (3.1) and (3.2) is quadratic in the first derivative of H we can apply Hamilton's method [11] to deduce that $H(t) \to H(T)$ in C^∞, and the solution can be continued past T. Then (3.1) and (3.2) have a solution $H(t)$ defined for all time.

From Proposition 2.3 and the maximum principle, it is easy to conclude the uniqueness of the solution.

□

Proposition 3.5. *Suppose $H(t)$ is a long-time solution of the flow (2.1) on compact Hermitian manifold $\overline{M}$ (with nonempty smooth boundary ∂M). Set $h(t)=H_0^{-1}H(t)$ and assume that there exists a constant $\overline{C}_0$ such that*

$$\sup_{(x,t)\in\overline{M}\times[0,+\infty)}|\log h|_{H_0}\leq\overline{C}_0.$$

Then, for any compact subset $\Omega\subset\overline{M}$, there exists a uniform constant $\overline{C}_1$ depending only on $\overline{C}_0$, d^{-1} and the geometry of $\tilde{\Omega}$ such that

$$\sup_{(x,t)\in\Omega\times[0,+\infty)}|h^{-1}\partial_{H_0}h|_{H_0}\leq\overline{C}_1,\tag{3.3}$$

where d is the distance of Ω to ∂M and $\tilde{\Omega}=\{x\in\overline{M}|\mathrm{dist}(x,\Omega)\leq\frac{1}{2}d\}$.

Proof. We will follow the argument in [18, Lemma 2.4] to get local uniform C^1-estimate. Let $\mathcal{T}=h^{-1}\partial_{H_0}h$. Direct computations give us that

$$(\widetilde{\Delta}-\frac{\partial}{\partial t})\mathrm{tr}h\geq-2\mathrm{tr}(\sqrt{-1}\Lambda_\omega(\bar{\partial}hh^{-1}\partial_{H_0}h))+2\mathrm{tr}(h\Phi(H_0,\theta))+2\varepsilon\mathrm{tr}(h\log h),\tag{3.4}$$

$$\frac{\partial}{\partial t}\mathcal{T}=\partial_H(h^{-1}\frac{\partial}{\partial t}h),$$

and

$$\begin{aligned}(\widetilde{\Delta}-\frac{\partial}{\partial t})|\mathcal{T}|_H^2\geq{}&|\nabla_H\mathcal{T}|_H^2-\check{C}_1(|\Lambda_\omega F_H|_H+|F_{H_0}|_H+|\theta|_H^2+|Rm(g)|_g+|\nabla_gJ|_g^2+\varepsilon)|\mathcal{T}|_H^2\\&-\check{C}_2|\nabla_{H_0}(\Lambda_\omega F_{H_0})|_H|\mathcal{T}|_H-4|\nabla_{H_0}\theta|_H^2-\varepsilon|\log h|_H^2,\end{aligned}\tag{3.5}$$

where J is the complex structure on M and positive constants $\check{C}_1$, $\check{C}_2$ depend only on the dimension n and the rank r. By (3.5) and Proposition 2.1, we have

$$(\widetilde{\Delta}-\frac{\partial}{\partial t})|\mathcal{T}|_H^2\geq|\nabla_H\mathcal{T}|_H^2-\check{C}_3|\mathcal{T}|_H^2-\check{C}_3\tag{3.6}$$

on the domain $\tilde{\Omega}\times[0,+\infty)$, where $\check{C}_3$ is a uniform constant depending only on $\overline{C}_0$, $\max_{\tilde{\Omega}}|\theta|_{H_0}$ and the geometry of $\tilde{\Omega}$.

Setting $\overline{\Omega}=\{x\in\overline{M}|\mathrm{dist}(x,\Omega)\leq\frac{1}{4}d\}$. Let ψ_1, ψ_2 be non-negative cut-off functions satisfying:

$$\psi_1=\begin{cases}0, & x\in M\backslash\overline{\Omega},\\ 1, & x\in\Omega,\end{cases}$$

$$\psi_2=\begin{cases}0, & x\in M\backslash\tilde{\Omega},\\ 1, & x\in\overline{\Omega}.\end{cases}$$

and

$$|\mathrm{d}\psi_i|^2+|\widetilde{\Delta}\psi_i|\leq c,\quad i=1,2,$$

where $c=32d^{-2}$. Consider the following test function

$$f(\cdot,t)=\psi_1^2|\mathcal{T}|_H^2+W\psi_2^2\mathrm{tr}h,$$

where the constant W will be chosen large enough later. It follows from (3.4) and (3.6) that

$$(\widetilde{\Delta}-\frac{\partial}{\partial t})f \geq \psi_2^2(2We^{-\overline{C}_0}-\check{C}_3-18c-8e^{2\overline{C}_0})|\mathcal{T}|_H^2-\widetilde{C}_0,$$

where $\widetilde{C}_0$ is a positive constant depending only on $\overline{C}_0$. If we choose

$$W=\frac{1}{2}e^{-\overline{C}_0}(\check{C}_3+18c+8e^{2\overline{C}_0}+1),$$

then

$$(\widetilde{\Delta}-\frac{\partial}{\partial t})f \geq \psi_2^2|\mathcal{T}|_H^2-\widetilde{C}_0 \tag{3.7}$$

on $M\times[0,+\infty)$. Let $f(q,t_0)=\max_{M\times[0,+\infty)} f$. On the basis of the definition of ψ_i and the uniform C^0-bound of $h(t)$, we may assume that:

$$(q,t_0)\in\overline{\Omega}\times(0,+\infty).$$

Of course the inequality (3.7) yields

$$|\mathcal{T}(t_0)|_{H(t_0)}^2(q)\leq\widetilde{C}_0,$$

and then (3.3). □

In the next part of this section, we will consider the long-time existence of the heat flow (2.1) on some non-compact Hermitian manifold (X,g). In the following, we suppose that there exists a non-negative exhaustion function ϕ with $\sqrt{-1}\Lambda_\omega\partial\bar{\partial}\phi$ bounded, i.e. (X,g) satisfies the Assumption 2. Fix a number φ and let X_φ denote the compact space $\{x\in X|\phi(x)\leq\varphi\}$, with boundary ∂X_φ. Let H_0 be an initial metric on E over X. We consider the following Dirichlet boundary condition

$$H|_{\partial X_\varphi}=H_0|_{\partial X_\varphi}. \tag{3.8}$$

By Proposition 3.4, on every X_φ, the flow (2.1) with the above Dirichlet boundary condition and with the initial data H_0 admits a unique long-time solution $H_\varphi(t)$ for $0\leq t<+\infty$.

Proposition 3.6. *Suppose $H_\varphi(t)$ is a long-time solution of the flow (2.1) on X_φ satisfying the Dirichlet boundary condition (3.8), then*

$$|\log h|_{H_0}(x,t)\leq\frac{1}{\varepsilon}\max_{X_\varphi}|\Phi(H_0,\theta)|_{H_0},\quad \forall(x,t)\in X_\varphi\times[0,+\infty). \tag{3.9}$$

where $h(t)=H_0^{-1}H_\varphi(t)$, $\overline{C}_0$ is a uniform constant depending only on ε^{-1} and the initial data $\max_{X_\varphi}|\Phi(H_0,\theta)|_{H_0}$.

Proof. By a direct calculation, we have

$$\begin{aligned}\langle H^{-1}\frac{\partial H}{\partial t},\log h\rangle_{H_0}&=-2\langle\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*_H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log h,\log h\rangle_{H_0}\\&=-2\langle\Phi(H_0,\theta)+\sqrt{-1}\Lambda_\omega(\bar{\partial}(h^{-1}\partial_{H_0}h)+[\theta,\theta^{*_H}-\theta^{*_{H_0}}])+\varepsilon\log h,\log h\rangle_{H_0}\\&\leq-2\langle\Phi(H_0,\theta)+\sqrt{-1}\Lambda_\omega\bar{\partial}(h^{-1}\partial_{H_0}h)+\varepsilon\log h,\log h\rangle_{H_0},\end{aligned}$$

where we have used the inequality ((2.6) in [26])

$$\langle \sqrt{-1}\Lambda_\omega[\theta, \theta^{*H} - \theta^{*H_0}], \log h\rangle_{H_0} \geq 0.$$

On the other hand, it is easy to check that

$$\langle H^{-1}\frac{\partial H}{\partial t}, \log h\rangle_{H_0} = \langle h^{-1}\frac{\partial h}{\partial t}, \log h\rangle_{H_0} = \frac{1}{2}\frac{\partial}{\partial t}|\log h|^2_{H_0}$$

and

$$\langle \sqrt{-1}\Lambda_\omega\bar{\partial}(h^{-1}\partial_{H_0}h), \log h\rangle_{H_0} \geq -\frac{1}{2}\widetilde{\Delta}(|\log h|^2_{H_0}).$$

Then

$$\frac{1}{4}(\frac{\partial}{\partial t} - \widetilde{\Delta})(|\log h|^2_{H_0}) \leq -\varepsilon|\log h|^2_{H_0} + |\Phi(H_0, \theta)|_{H_0}|\log h|_{H_0},$$

which together with the maximum principle implies (3.9).

□

Lemma 3.7 ([29, Lemma 6.7])**.** *Suppose u is a function on some $X_\varphi \times [0, T]$, satisfying*

$$(\widetilde{\Delta} - \frac{\partial}{\partial t})u \geq 0, \quad u|_{t=0} = 0,$$

and suppose there is a bound $\sup_{X_\varphi} u \leq C_1$. Then we have

$$u(x, t) \leq \frac{C_1}{\varphi}(\phi(x) + C_2 t),$$

where C_2 is the bound of $\widetilde{\Delta}\phi$ in Assumption 2.

In the following, we assume that there exists a constant C such that $\sup_X |\Phi(H_0, \theta)|_{H_0} \leq C$. For any compact subset $\Omega \subset X$, there exists a constant φ_0 such that $\Omega \subset X_{\varphi_0}$. Let $H_{\varphi_1}(t)$ and $H_{\varphi_2}(t)$ be the long-time solutions of the flow (2.1) satisfying the Dirichlet boundary condition (3.8) for $\varphi_0 < \varphi_1 < \varphi_2$. Let $u = \sigma(H_{\varphi_1}, H_{\varphi_2})$. Proposition 3.6 gives a uniform bound on u, and u is a subsolution for the heat operator with $u(0) = 0$. By Lemma 3.7, we have

$$\sigma(H_\varphi, H_{\varphi_1}) \leq C_1\frac{(\varphi_0 + C_2 T)}{\varphi} \tag{3.10}$$

on $X_{\varphi_0} \times [0, T]$. Then H_φ is a Cauchy sequence on $X_{\varphi_0} \times [0, T]$ for $\varphi \to \infty$. Proposition 3.6 and Proposition 3.5 give the uniform C^0 and local C^1 estimates of $H_\varphi(t)$. One can get the local uniform C^∞-estimate of $H_\varphi(t)$ by the standard Schauder estimate of the parabolic equation. It should be point out that by applying the parabolic Schauder estimate, one can only get the uniform C^∞-estimate of $h(t)$ on $X_\varphi \times [\tau, T]$, where $\tau > 0$ and the uniform estimate depends on τ^{-1}. To fix this, one can use the maximum principle to get a local uniform bound on the curvature $|F_H|_H$, then apply the elliptic estimates to get local uniform C^∞-estimates. We will omit this step here, since it is similar to [18, Lemma 2.5]. By choosing a subsequence $\varphi \to \infty$, we have that $H_\varphi(t)$ converge in C^∞_{loc}-topology to a long-time solution $H(t)$ of the heat flow (2.1) on X. So, we obtain the following theorem.

Theorem 3.8. *Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle with fixed Hermitian metric H_0 over a Hermitian manifold (X, g) satisfying the Assumptions 2. Suppose $\sup_X |\Phi(H_0, \theta)|_{H_0} < +\infty$, then, on the whole X, the flow (2.1) has a long-time solution $H(t)$ satisfying:*

$$\sup_{(x,t)\in X\times[0,+\infty)} |\log h|_{H_0}(x,t) \leq \frac{1}{\varepsilon}\sup_X |\Phi(H_0,\theta)|_{H_0}. \tag{3.11}$$

4. Poisson equations on the non-compact manifold

In this section, we are devoted to solve the equation $\widetilde{\Delta} f = \psi$ on a class of non-compact Gauduchon manifold. Since the difference of the complex Laplacian and the Beltrami-Laplcaian is given by a linear first order differential operator, the following proposition should be well known, it also can be proved in the same way as that in Theorem 5.1.

Proposition 4.1. *Let (M, g) be a compact Hermitian manifold with non-empty boundary ∂M. Suppose that $\psi \in C^\infty(M)$, then for any function $\widetilde{f}$ on the restriction to ∂M, there is a unique function $f \in C^\infty(M)$ which satisfies the equation $\widetilde{\Delta} f = \psi + \varepsilon f$ and $f = \widetilde{f}$ on ∂M for any $\varepsilon > 0$.*

Let (X, g) be a non-compact Gauduchon manifold with finite volume and a non-negative exhaustion function ϕ. By Proposition 4.1, we know that the following Dirichlet problem is solvable on X_φ, i.e.

$$\begin{cases} \widetilde{\Delta} f_\varphi - \varepsilon f_\varphi - \psi = 0, & \forall x \in X_\varphi, \\ f_\varphi(x)|_{\partial X_\varphi} = 0. \end{cases}$$

By simple calculations, we have

$$\widetilde{\Delta}|f_\varphi|^2 \geq 2|f_\varphi|(\varepsilon|f_\varphi| - |\psi|).$$

The maximum principle implies:

$$\max_{X_\varphi} |f_\varphi| \leq \frac{1}{\varepsilon}\sup_{X_\varphi}|\psi|.$$

By (2.6), we have

$$\begin{aligned} \int_{X_\varphi} |\mathrm{d} f_\varphi|^2 \frac{\omega^n}{n!} &= -\int_{X_\varphi} f_\varphi \tilde{\Delta} f_\varphi \frac{\omega^n}{n!} \\ &\leq \frac{1}{\varepsilon}\sup_{X_\varphi}|\psi|^2 \mathrm{Vol}(X_\varphi, g). \end{aligned}$$

Then, by using the standard elliptic estimates, we can prove that, by choosing a subsequence, f_φ converge in C^∞_{loc}-topology to a solution on whole X, i.e. we prove the following proposition.

Proposition 4.2. *Let (X, g) be a non-compact Gauduchon manifold with finite volume and a non-negative exhaustion function ϕ. Suppose that $\psi \in C^\infty(X)$ satisfies $\sup_X |\psi| < +\infty$. For any $\varepsilon > 0$, there is a function $f \in C^\infty(X)$ which satisfies the equation*

$$\widetilde{\Delta} f = \psi + \varepsilon f \tag{4.1}$$

with

$$\sup_X |f| \leq \frac{1}{\varepsilon}\sup_X |\psi| \tag{4.2}$$

and

$$\int_X |\mathrm{d}f|^2\frac{\omega^n}{n!} \leq \frac{1}{\varepsilon}(\sup_X |\psi|)^2 \mathrm{Vol}(X,g). \tag{4.3}$$

Now we are ready to solve the Poisson equation on the non-compact Gauduchon manifold.

Proposition 4.3. *Let (X,g) be a non-compact Gauduchon manifold satisfying Assumptions 1,2,3 and $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$. Suppose that $\psi \in C^\infty(X)$ satisfies $\int_X \psi = 0$ and $\sup_X |\psi| < +\infty$. Then there is a function $f \in C^\infty(X)$ which satisfies the Possion equation*

$$\widetilde{\Delta} f = \psi, \tag{4.4}$$

$$\int_X |\mathrm{d}f|^2\frac{\omega^n}{n!} < +\infty \tag{4.5}$$

and $\sup_X |f| < +\infty$.

Proof. By a direct calculation, we have

$$\widetilde{\Delta}\log(e^f + e^{-f}) \geq -|\widetilde{\Delta} f|.$$

On the other hand, it is easy to check that

$$|f| \leq \log(e^f + e^{-f}) \leq |f| + \log 2.$$

From Proposition 4.2, for any $\varepsilon > 0$, we have a solution f_ε of the equation (4.1) and f_ε satisfies (4.2). By Assumption 3, we have

$$\sup_X |f_\varepsilon| \leq \sup_X \log(e^{f_\varepsilon} + e^{-f_\varepsilon}) \leq \widetilde{C}_1 \int_X |f_\varepsilon| + \widetilde{C}_2,$$

where constants $\widetilde{C}_1$ and $\widetilde{C}_2$ depend only on $\sup_X |\psi|$ and $\mathrm{Vol}(X)$.

In the following, we will use a contradiction argument to prove that $\|f_\varepsilon\|_{C^0}$ is uniform bounded. If $\|f_\varepsilon\|_{C^0}$ is unbounded, then there exists a subsequence $\varepsilon \to 0$, such that $\|f_\varepsilon\|_{L^2} \to +\infty$. Set $u_\varepsilon = f_\varepsilon / \parallel f_\varepsilon \parallel_{L^2}$. It follows that

$$\|u_\varepsilon\|_{L^2} = 1 \quad \text{and} \quad \sup_X |u_\varepsilon| < \widetilde{C}_3 < +\infty,$$

where $\widetilde{C}_3$ is a uniform constant depending only on $\sup_X |\psi|$ and $\mathrm{Vol}(X)$. Using the conditions $\partial\bar{\partial}\omega^{n-1} = 0$, $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$, (4.2), (4.3), and Lemma 2.5, one can check that

$$\int_X f_\varepsilon \widetilde{\Delta} f_\varepsilon \frac{\omega^n}{n!} = -\int_X |\mathrm{d}f_\varepsilon|^2\frac{\omega^n}{n!}. \tag{4.6}$$

Substituting the perturbed equation into (4.6), we have

$$\int_X |\mathrm{d}u_\varepsilon|^2\frac{\omega^n}{n!} = -\varepsilon - \frac{1}{\|f_\varepsilon\|_{L^2}}\int_X u_\varepsilon \psi \frac{\omega^n}{n!}.$$

Then, by passing to a subsequence, we have that u_ε converges weakly to u_∞ in L_1^2 as $\varepsilon \to 0$, and u_∞ is constant almost everywhere. Note that for any relatively compact $Z \subset X$, $L_1^2 \to L^2(Z)$ is compact. So

$$\int_Z |u_\varepsilon|^2 \to \int_Z |u_\infty|^2.$$

Recalling $\sup_X |u_{\varepsilon_i}| < \widetilde{C}_3 < +\infty$ and X has finite volume, so for a small $\epsilon > 0$, we have

$$\int_{X\setminus Z} |u_\varepsilon|^2 < \epsilon,$$

when Z is big enough. Thus $1 \geq \int_Z |u_\infty|^2 \geq 1 - \epsilon$. So, we have

$$u_\infty = \text{const.} \neq 0 \quad a.e..$$

Using the conditions $\partial\bar{\partial}\omega^{n-1} = 0$, $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$, (4.2), (4.3) and Lemma 2.5, it is easy to check that

$$\int_X \widetilde{\Delta} f_\varepsilon \frac{\omega^n}{n!} = 0.$$

Then combining $\widetilde{\Delta} f_\varepsilon + \varepsilon f_\varepsilon + \psi = 0$ and $\int_X \psi = 0$, we have

$$\int_X f_\varepsilon \frac{\omega^n}{n!} = 0,$$

and

$$\int_X u_\varepsilon \frac{\omega^n}{n!} = 0.$$

Then, we can obtain

$$\int_X u_\infty \frac{\omega^n}{n!} = 0.$$

We get a contradiction, so we have proved that $\|f_\varphi\|_{C^0}$ is bounded uniformly when ε goes to zero. By standard elliptic estimates, we obtain, by choosing a subsequence f_ε must converge to a smooth function f_∞ in C^∞_{loc}-topology as $\varepsilon \to 0$, and f_∞ satisfies the equation (4.4). (4.6) implies (4.5). This completes the proof of Proposition 4.3. $\square$

5. Solvability of the perturbed equation

We first solve the Dirichlet problem for the perturbed equation, i.e. we obtain the following theorem.

Theorem 5.1. *Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle with fixed Hermitian metric H_0 over the compact Gauduchon manifold $\overline{M}$ with non-empty boundary ∂M. There is a unique Hermitian metric H on E such that*

$$\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*_H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H) = 0, \qquad H|_{\partial M} = H_0, \tag{5.1}$$

for any $\varepsilon \geq 0$. When $\varepsilon > 0$, we have

$$\max_{x\in\overline{M}} |s|_{H_0}(x) \leq \frac{1}{\varepsilon} \max_{\overline{M}} |\Phi(H_0, \theta)|_{H_0}. \tag{5.2}$$

and

$$\|D^{1,0}_{H_0,\theta}s\|_{L^2(\overline{M})}=\|\overline{\partial}_\theta s\|_{L^2(\overline{M})}\leq C(\varepsilon^{-1},\Phi(H_0,\theta),\mathrm{Vol}(M)),\tag{5.3}$$

where $s=\log(H_0^{-1}H)$. *Furthermore, if the initial metric* H_0 *satisfies the following condition*

$$\mathrm{tr}(\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E)=0,\tag{5.4}$$

then $\mathrm{tr}(s)=0$ *and* H *also satisfies the condition (5.4).*

Proof. Proposition 3.4 guaranteed the existence of long-time solution $H(t)$ of the heat equation (3.2). By Proposition 2.1, we have

$$(\widetilde{\triangle}-\frac{\partial}{\partial t})|\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^{*H(t)}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H(t))|_{H(t)}\geq 0.\tag{5.5}$$

If the initial metric H_0 satisfies the condition (5.4), by (2.2) and the maximum principle, we know that $H(t)$ must satisfy

$$\mathrm{tr}\{\sqrt{-1}\Lambda_\omega(F_{H(t)}+[\theta,\theta^{*H(t)}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H(t))\}=0.$$

Then, we have

$$\mathrm{tr}(\log H_0^{-1}H(t))=0$$

and $H(t)$ satisfies the condition (5.4) for all $t\geq 0$.

By [31, Chapter 5, Proposition 1.8], one can solve the following Dirichlet problem on M:

$$\widetilde{\triangle}v=-|\sqrt{-1}\Lambda_\omega(F_{H_0}+[\theta,\theta^{*H_0}])-\lambda\cdot\mathrm{Id}_E|_{H_0},\quad v|_{\partial M}=0.\tag{5.6}$$

Set $w(x,t)=\int_0^t|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H(x,\rho)\mathrm{d}\rho-v(x)$. From (5.5), (5.6), and the boundary condition satisfied by H implies that, for $t>0$, $|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H(x,t)$ vanishes on the boundary of M, it is easy to check that $w(x,t)$ satisfies

$$(\widetilde{\triangle}-\frac{\partial}{\partial t})w(x,t)\geq 0,\quad w(x,0)=-v(x),\quad w(x,t)|_{\partial M}=0.$$

By the maximum principle, we have

$$\int_0^t|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H(x,\rho)\mathrm{d}\rho\leq\sup_{y\in M}v(y),\tag{5.7}$$

for any $x\in M$, and $0<t<+\infty$.

Let $t_1\leq t\leq t_2$, and let $\bar{h}(x,t)=H^{-1}(x,t_1)H(x,t)$. It is easy to check that

$$\frac{\partial}{\partial t}\log\mathrm{tr}(\bar{h})\leq 2|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H.$$

By integration, we have

$$\begin{aligned}&\mathrm{tr}(H^{-1}(x,t_1)H(x,t))\\&\leq r\exp\big(2\int_{t_1}^t|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H\mathrm{d}\rho\big).\end{aligned}$$

We have a similar estimate for $\mathrm{tr}(H^{-1}(x,t)H(x,t_1))$. Combining them we have

$$\begin{aligned}&\sigma(H(x,t),H(x,t_1))\\ &\leq 2r(\exp(2\int_{t_1}^{t}|\sqrt{-1}\Lambda_\omega(F_H+[\theta,\theta^{*_H}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H)|_H\mathrm{d}\rho)-1).\end{aligned}\tag{5.8}$$

By (5.7) and (5.8), we have that $H(t)$ converge in the C^0 topology to some continuous metric H_∞ as $t\longrightarrow+\infty$. From Lemma 3.3, we know that $H(t)$ are bounded uniformly in C^1_{loc} and also bounded uniformly in $L^p_{2,loc}$ (for any $1<p<+\infty$) . On the other hand, we have known that $|H^{-1}\frac{\partial H}{\partial t}|$ is bounded uniformly. Then, the standard elliptic regularity implies that there exists a subsequence $H(t)\longrightarrow H_\infty$ in C^∞_{loc}-topology. From formula (5.7), we know that H_∞ is the desired Hermitian metric satisfying the boundary condition. By Corollary 2.4 and the maximum principle, it is easy to conclude the uniqueness of solution.

If $\varepsilon>0$, (3.9) in Proposition 3.6 implies (5.2). By the definition, it is easy to check

$$|\overline{\partial}_\theta s|^2_{H_0}\leq\tilde{C}\langle\Psi(s)(\overline{\partial}_\theta s),\overline{\partial}_\theta s\rangle_{H_0},$$

where $\tilde{C}$ is a positive constant depending only on the L^∞-bound of s. By the identity (2.7) in Proposition 2.6 and the equation (5.1), we have

$$\begin{aligned}\int_M|\overline{\partial}_\theta s|^2_{H_0}\frac{\omega^n}{n!}&\leq\tilde{C}\int_M\langle\Psi(s)(\overline{\partial}_\theta s),\overline{\partial}_\theta s\rangle_{H_0}\frac{\omega^n}{n!}\\ &=\tilde{C}\int_M(-\mathrm{tr}(\Phi(H_0,\theta)s)-\varepsilon|s|^2_{H_0})\frac{\omega^n}{n!}\\ &\leq\tilde{C}\frac{1}{\varepsilon}\cdot\sup_M|\Phi(H_0,\theta)|^2_{H_0}\cdot\mathrm{Vol}(M,g).\end{aligned}\tag{5.9}$$

Then (5.9) implies (5.3). □

Let X be a non-compact Gauduchon manifold, $\{X_\varphi\}$ an exhausting sequence of compact sub-domains of X. Suppose $(E,\bar{\partial}_E,\theta)$ is a Higgs bundle over X and H_0 is a Hermitian metric on E. By Theorem 5.1, we know that the following Dirichlet problem is solvable on X_φ, i.e. there exists a Hermitian metric $H_\varphi(x)$ such that

$$\begin{cases}\sqrt{-1}\Lambda_\omega(F_{H_\varphi}+[\theta,\theta^{*_{H_\varphi}}])-\lambda\cdot\mathrm{Id}_E+\varepsilon\log(H_0^{-1}H_\varphi)=0,\forall x\in X_\varphi,\\ H_\varphi(x)|_{\partial X_\varphi}=H_0(x).\end{cases}$$

In order to prove that we can pass to limit and eventually obtain a solution on the whole manifold X, we need some a priori estimates. The key is the C^0-estimate.

We denote $h_\varphi=H_0^{-1}H_\varphi$. Theorem 5.1 implies:

$$\sup_{x\in X_\varphi}|\log h_\varphi|_{H_0}(x)\leq\frac{1}{\varepsilon}\max_{X_\varphi}|\Phi(H_0,\theta)|_{H_0},$$

For any compact subset $\Omega\subset X$, we can choose a φ_0 such that $\Omega\subset X_{\varphi_0}$. By Proposition 3.5, we have the following local uniform C^1-estimates, i.e. for any $\varphi>\varphi_0$, there exists

$$\sup_{x\in\Omega}|h_\varphi^{-1}\partial_{H_0}h_\varphi|_{H_0}\leq\hat{C}_1,$$

where $\hat{C}_1$ is a uniform constant independent on φ. The perturbed equation (1.3) and standard elliptic theory give us uniform local higher order estimates. Then, by passing to a subsequence, H_φ converge in C^∞_{loc} topology to a metric H_∞ which is a solution of the perturbed equation (1.3) on the whole manifold X. Therefore we complete the proof of the following theorem.

Theorem 5.2. *Let $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle with fixed Hermitian metric H_0 over the non-compact Gauduchon manifold (X, g) with finite volume. Suppose there exists a non-negative exhaustion function ϕ on X and $\sup_X |\Phi(H_0,\theta)|_{H_0} < +\infty$, then for any $\varepsilon > 0$, there exists a metric H such that*

$$\sqrt{-1}\Lambda_\omega(F_H + [\theta, \theta^{*H}]) - \lambda \cdot \mathrm{Id}_E + \varepsilon \log(H_0^{-1}H) = 0,$$

$$\sup_{x\in X} |\log H_0^{-1}H|_{H_0}(x) \leq \frac{1}{\varepsilon} \sup_X |\Phi(H_0,\theta)|_{H_0}, \tag{5.10}$$

and

$$\|\overline{\partial}_\theta(\log H_0^{-1}H)\|_{L^2} \leq C(\varepsilon^{-1}, \Phi(H_0,\theta), \mathrm{Vol}(X)). \tag{5.11}$$

Furthermore, if the initial metric H_0 satisfies the condition (5.4) then $\mathrm{tr}\log(H_0^{-1}H) = 0$ *and H also satisfies the condition (5.4).*

6. Proof of the theorems

Let (X, g) be a non-compact Gauduchon manifold satisfying the Assumptions 1,2,3, and $|\mathrm{d}\omega^{n-1}|_g \in L^2(X)$, $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle over X. Fixing a proper background Hermitian metric K satisfying $\sup_X |\Lambda_\omega F_{K,\theta}|_K < +\infty$ on E. By Proposition 4.3, we can solve the following Poisson equation on (X, g):

$$\sqrt{-1}\Lambda_\omega \bar{\partial}\partial f = -\frac{1}{r}\mathrm{tr}(\sqrt{-1}\Lambda_\omega F_{K,\theta} - \lambda_{K,\omega}\cdot \mathrm{Id}_E),$$

where

$$\lambda_{K,\omega} = \frac{\sqrt{-1}\int_X \mathrm{tr}(\Lambda_\omega F_{K,\theta})\frac{\omega^n}{n!}}{\mathrm{rank}(E)\mathrm{Vol}(X)}.$$

By conformal change $\overline{K} = e^f K$, we can check that $\overline{K}$ satisfies

$$\mathrm{tr}(\sqrt{-1}\Lambda_\omega(F_{\overline{K}} + [\theta, \theta^{*\overline{K}}]) - \lambda_{K,\omega}\cdot \mathrm{Id}_E) = 0. \tag{6.1}$$

By the definition and properties of f, it is easy to check that if $(E, \bar{\partial}_E, \theta)$ is K-analytic stable then it must be $\overline{K}$-analytic stable. So, in the following we can assume that the initial metric K satisfies the condition (6.1).

From Theorem 5.2, we can solve the following perturbed equation

$$L_\varepsilon(h_\varepsilon) := \sqrt{-1}\Lambda_\omega(F_{H_\varepsilon} + [\theta, \theta^{*H_\varepsilon}]) - \lambda_{K,\omega}\cdot \mathrm{Id}_E + \varepsilon \log h_\varepsilon = 0, \tag{6.2}$$

where $h_\varepsilon = K^{-1}H_\varepsilon = e^{s_\varepsilon}$. Since the initial metric K satisfies the condition (6.1), then we have

$$\log\det(h_\varepsilon) = \mathrm{tr}(s_\varepsilon) = 0$$

and

$$\mathrm{tr}(\sqrt{-1}\Lambda_\omega(F_{H_\varepsilon}+[\theta,\theta^{*H_\varepsilon}])-\lambda_{K,\omega}\cdot\mathrm{Id}_E)=0.$$

Lemma 6.1.

$$\sup_X|\log h_\varepsilon|\le C_7\|\log h_\varepsilon\|_{L^2(X)}+C_8, \tag{6.3}$$

where C_7 and C_8 are positive constants independent on ε.

Proof. By [29, Lemma 3.1 (d)], we have

$$\widetilde{\Delta}\log(\mathrm{tr}h_\varepsilon+\mathrm{tr}h_\varepsilon^{-1})\ge-2(|\Lambda_\omega F_{H_\varepsilon,\theta}|_{H_\varepsilon}+|\Lambda_\omega F_{K,\theta}|_K).$$

From (5.10) and (6.2), it is easy to check that $|\Lambda_\omega F_{H_\varepsilon,\theta}|_{H_\varepsilon}$ is uniformly bounded. On the other hand, we have

$$\log(\frac{1}{2r}(\mathrm{tr}h_\varepsilon+\mathrm{tr}h_\varepsilon^{-1}))\le|\log h_\varepsilon|\le r^{\frac{1}{2}}\log(\mathrm{tr}h_\varepsilon+\mathrm{tr}h_\varepsilon^{-1}),$$

Then by Assumption 3, we have (6.3). □

Proof of Theorem 1.1

When $(E,\bar{\partial}_E,\theta)$ is K-stable, we will show that, by choosing a subsequence, H_ε converge to a Hermitian-Einstein metric H in C^∞_{loc} as $\varepsilon\to0$. By the local C^1-estimates in Proposition 3.5, the standard elliptic estimates and the identity (2.7) in Proposition 2.6, we only need to obtain a uniform C^0-estimate. By Lemma 6.1, the key is to get a uniform L^2-estimate for $\log h_\varepsilon$, i.e. there exists a constant $\hat{C}$ independent of ε, such that

$$\|\log h_\varepsilon\|_{L^2}=\int_X|\log h_\varepsilon|_{H_\varepsilon}\frac{\omega^n}{n!}\le\hat{C} \tag{6.4}$$

for all $0<\varepsilon\le1$. We prove (6.4) by contradiction. If not, there would exist a subsequence $\varepsilon_i\to0$ such that

$$\|\log h_{\varepsilon_i}\|_{L^2}\to+\infty.$$

Once we set

$$s_{\varepsilon_i}=\log h_{\varepsilon_i},\quad l_i=\|s_{\varepsilon_i}\|_{L^2},\quad u_{\varepsilon_i}=\frac{s_{\varepsilon_i}}{l_i},$$

we have

$$\mathrm{tr}(u_{\varepsilon_i})=0,\ \|u_{\varepsilon_i}\|_{L^2}=1.$$

Then combining Lemma 6.1, we also have

$$\sup_X|u_{\varepsilon_i}|\le\frac{1}{l_i}(C_7l_i+C_8)<C_{10}<+\infty. \tag{6.5}$$

• *Step* 1 We show that $\|u_{\varepsilon_i}\|_{L_1^2}$ are uniformly bounded. Since $\|u_{\varepsilon_i}\|_{L^2}=1$, we only need to prove $\|\overline{\partial}_\theta u_{\varepsilon_i}\|_{L^2}$ are uniformly bounded.

From Theorem 5.2 and Proposition 2.6, for each ε_i, we have

$$\int_X\mathrm{tr}(\Phi(K,\theta)u_{\varepsilon_i})\frac{\omega^n}{n!}+l_i\int_X\langle\Psi(l_iu_{\varepsilon_i})(\overline{\partial}_\theta u_{\varepsilon_i}),\overline{\partial}_\theta u_{\varepsilon_i}\rangle_K\frac{\omega^n}{n!}=-\varepsilon_il_i. \tag{6.6}$$

Consider the function

$$l\Psi(lx,ly)=\begin{cases} l, & x=y;\\ \frac{e^{l(y-x)}-1}{y-x}, & x\neq y.\end{cases}$$

From (6.5), we may assume that $(x,y)\in[-C_{10},C_{10}]\times[-C_{10},C_{10}]$. It is easy to check that

$$l\Psi(lx,ly)\to\begin{cases}(x-y)^{-1}, & x>y;\\ +\infty, & x\leq y,\end{cases} \tag{6.7}$$

increases monotonically as $l\to+\infty$. Let $\varsigma\in C^{\infty}(\mathbb{R}\times\mathbb{R},\mathbb{R}^{+})$ satisfying $\varsigma(x,y)<(x-y)^{-1}$ whenever $x>y$. From Eqs. (6.6), (6.7) and the arguments in [29, Lemma 5.4], we have

$$\int_X \operatorname{tr}(\Phi(K,\theta)u_{\varepsilon_i})\frac{\omega^n}{n!}+\int_X\langle\varsigma(u_{\varepsilon_i})(\overline{\partial}_\theta u_{\varepsilon_i}),\overline{\partial}_\theta u_{\varepsilon_i}\rangle_K\frac{\omega^n}{n!}\leq 0,\quad i\gg 0. \tag{6.8}$$

In particular, we take $\zeta(x,y)=\frac{1}{3C_{10}}$. It is obvious that when $(x,y)\in[-C_{10},C_{10}]\times[-C_{10},C_{10}]$ and $x>y$, $\frac{1}{3C_2}<\frac{1}{x-y}$. This implies that

$$\int_X \operatorname{tr}(\Phi(K,\theta)u_{\varepsilon_i})\frac{\omega^n}{n!}+\frac{1}{3C_{10}}\int_X|\overline{\partial}_\theta(u_{\varepsilon_i})|_K^2\frac{\omega^n}{n!}\leq 0,$$

for $i\gg 0$. Then we have

$$\int_X|\overline{\partial}_\theta(u_{\varepsilon_i})|_K^2\frac{\omega^n}{n!}\leq 3C_{10}^2\sup_X|\Phi(K,\theta)|_K\mathrm{Vol}(X).$$

Thus, u_{ε_i} are bounded in L_1^2. We can choose a subsequence $\{u_{\varepsilon_{i_j}}\}$ such that $u_{\varepsilon_{i_j}}\rightharpoonup u_\infty$ weakly in L_1^2, still denoted by $\{u_{\varepsilon_i}\}_{i=1}^{\infty}$ for simplicity. Noting that $L_1^2\hookrightarrow L^2$, we have

$$1=\int_X|u_{\varepsilon_i}|_K^2\to\int_X|u_\infty|_K^2.$$

This indicates that $\|u_\infty\|_{L^2}=1$ and u_∞ is non-trivial. Using (6.8) and following a similar discussion as in [29, Lemma 5.4], we have

$$\int_X \operatorname{tr}(\Phi(K,\theta)u_\infty)\frac{\omega^n}{n!}+\int_X\langle\varsigma(u_\infty)(\overline{\partial}_\theta u_\infty),\overline{\partial}_\theta u_\infty\rangle_K\frac{\omega^n}{n!}\leq 0. \tag{6.9}$$

• *Step* 2 Using Uhlenbeck and Yau's trick ([32]) and Simpson's argument ([29]), we construct a Higgs subsheaf which contradicts the stability of $(E,\bar{\partial}_E,\theta)$.

By (6.9) and the same argument in [29, Lemma 5.5], we conclude that the eigenvalues of u_∞ are constant almost everywhere. Let $\mu_1<\mu_2<\cdots<\mu_l$ be the distinct eigenvalues of u_∞. The facts that $\operatorname{tr}(u_\infty)=0$ and $\|u_\infty\|_{L^2}=1$ force $2\leq l\leq r$. For each $\mu_\alpha(1\leq\alpha\leq l-1)$, we construct a function $P_\alpha:\mathbb{R}\to\mathbb{R}$ such that

$$P_\alpha=\begin{cases}1, & x\leq\mu_\alpha;\\ 0, & x\geq\mu_{\alpha+1}.\end{cases}$$

Setting $\pi_\alpha=P_\alpha(u_\infty)$, from [29, p.887], we have: (i) $\pi_\alpha\in L_1^2$; (ii)$\pi_\alpha^2=\pi_\alpha=\pi_\alpha^{*_K}$; (iii) $(\mathrm{Id}_E-\pi_\alpha)\bar{\partial}\pi_\alpha=0$ and (iv) $(\mathrm{Id}_E-\pi_\alpha)[\theta,\pi_\alpha]=0$. By Uhlenbeck and Yau's regularity statement of L_1^2-subbundle [32], $\{\pi_\alpha\}_{\alpha=1}^{l-1}$ determine $l-1$ Higgs sub-sheaves of E. Set $E_\alpha = \pi_\alpha(E)$. From $\mathrm{tr}(u_\infty)=0$ and $u_\infty = \mu_l \cdot \mathrm{Id}_E - \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\pi_\alpha$, it holds

$$\mu_l \mathrm{rank}(E) = \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\mathrm{rank}(E_\alpha). \tag{6.10}$$

Construct

$$\nu = \mu_l \deg(E,K) - \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\deg(E_\alpha,K).$$

Substituting Eq. (6.10) into ν,

$$\nu = \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\mathrm{rank}(E_\alpha)\Big(\frac{\deg(E,K)}{\mathrm{rank}(E)} - \frac{\deg(E_\alpha,K)}{\mathrm{rank}(E_\alpha)}\Big). \tag{6.11}$$

On the other hand, substituting Eq. (1.1) into ν we have

$$\begin{aligned}
\nu &= \mu_l \int_X \sqrt{-1}\mathrm{tr}(\Lambda_\omega F_{K,\theta}) - \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\Big\{\int_X \sqrt{-1}\mathrm{tr}(\pi_\alpha \Lambda_\omega F_{K,\theta}) - |\overline{\partial}_\theta \pi_\alpha|^2\Big\} \\
&= \int_X \mathrm{tr}\Big\{\Big(\mu_l \cdot \mathrm{Id}_E - \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\pi_\alpha\Big)(\sqrt{-1}\Lambda_\omega F_{K,\theta})\Big\} + \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\int_X |\overline{\partial}_\theta \pi_\alpha|^2 \\
&= \int_X \mathrm{tr}(u_\infty \sqrt{-1}\Lambda_\omega F_{K,\theta}) + \Big\langle \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)(\mathrm{d}P_\alpha)^2(u_\infty)(\overline{\partial}_\theta u_\infty), \overline{\partial}_\theta u_\infty \Big\rangle_K,
\end{aligned}$$

where the function $\mathrm{d}P_\alpha : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is defined by

$$\mathrm{d}P_\alpha(x,y) = \begin{cases} \frac{P_\alpha(x)-P_\alpha(y)}{x-y}, & x \neq y; \\ P'_\alpha(x), & x = y. \end{cases}$$

One can easily check that,

$$\sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)(\mathrm{d}P_\alpha)^2(\mu_\beta,\mu_\gamma) = |\mu_\beta - \mu_\gamma|^{-1},$$

if $\mu_\beta \neq \mu_\gamma$. Then using (6.9), we have

$$\begin{aligned}
\nu &= \int_X \mathrm{tr}(u_\infty \sqrt{-1}\Lambda_\omega F_{K,\theta}) + \Big\langle \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)(\mathrm{d}P_\alpha)^2(u_\infty)(\overline{\partial}_\theta u_\infty), \overline{\partial}_\theta u_\infty \Big\rangle_K \\
&\leq 0.
\end{aligned} \tag{6.12}$$

Combining (6.11) and (6.12), we have

$$\sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\mathrm{rank}(E_\alpha)\Big(\frac{\deg(E,K)}{\mathrm{rank}(E)} - \frac{\deg(E_\alpha,K)}{\mathrm{rank}(E_\alpha)}\Big) \leq 0,$$

which contradicts the stability of E.

□

In the following, we will prove that the semi-stability implies the existence of approximate Hermitian-Einstein structure.

Proof of Theorem 1.2

We only need to prove the following claim.

Claim If $(E, \bar{\partial}_E, \theta)$ is semi-stable, then it holds

$$\lim_{\varepsilon\to 0}\sup_X |\sqrt{-1}\Lambda_\omega F_{H_\varepsilon,\theta} - \lambda_{K,\omega}\cdot \mathrm{Id}_E|_{H_\varepsilon} = \lim_{\varepsilon\to 0}\varepsilon \sup_X |\log h_\varepsilon|_{H_\varepsilon} = 0.$$

Proof. If the claim does not hold, then there exist $\delta > 0$ and a subsequence $\varepsilon_i \to 0, i \to +\infty$, such that

$$\sup_X |\sqrt{-1}\Lambda_\omega F_{H_{\varepsilon_i},\theta} - \lambda_{K,\omega}\cdot \mathrm{Id}_E|_{H_{\varepsilon_i}} = \varepsilon_i \sup_X |\log h_{\varepsilon_i}|_{H_{\varepsilon_i}} \geq \delta, \tag{6.13}$$

for any ε_i, and

$$\|\log h_{\varepsilon_i}\|_{L^2} \to +\infty.$$

Setting

$$s_{\varepsilon_i} = \log h_{\varepsilon_i}, \quad l_i = \|s_{\varepsilon_i}\|_{L^2}, \quad u_{\varepsilon_i} = \frac{s_{\varepsilon_i}}{l_i},$$

we have

$$\mathrm{tr}(u_{\varepsilon_i}) = 0, \ \|u_{\varepsilon_i}\|_{L^2} = 1.$$

By (6.13) and Lemma 6.1, we have

$$l_i \geq \frac{\delta}{\varepsilon_i C_7} - \frac{C_8}{C_7} \tag{6.14}$$

and

$$\sup_X |u_{\varepsilon_i}| \leq \frac{1}{l_i}(C_7 l_i + C_8) < C_{10} < +\infty.$$

By (6.6) and (6.14), we have

$$\frac{\delta}{C_7} + \int_X \mathrm{tr}(\Phi(K,\theta)u_{\varepsilon_i})\frac{\omega^n}{n!} + l_i\int_X \langle \Psi(l_i u_{\varepsilon_i})(\overline{\partial}_\theta u_{\varepsilon_i}), \overline{\partial}_\theta u_{\varepsilon_i}\rangle_K \frac{\omega^n}{n!} \leq \varepsilon_i \frac{C_8}{C_7}. \tag{6.15}$$

By (6.15) and the arguments in [29, Lemma 5.4], we have

$$\frac{\delta}{2C_7} + \int_X \mathrm{tr}(\Phi(K,\theta)u_{\varepsilon_i})\frac{\omega^n}{n!} + \int_X \langle \varsigma(u_{\varepsilon_i})(\overline{\partial}_\theta u_{\varepsilon_i}), \overline{\partial}_\theta u_{\varepsilon_i}\rangle_K \frac{\omega^n}{n!} \leq 0, \quad i \gg 0. \tag{6.16}$$

By the same argument as that in *Step* 1 in the proof of Theorem 1.1, we can prove that $\|\overline{\partial}_\theta u_{\varepsilon_i}\|_{L^2}$ are uniformly bounded. By choosing a subsequence, we have $u_{\varepsilon_i} \rightharpoonup u_\infty$ weakly in L_1^2, and $\|u_\infty\|_{L^2} = 1$. Using Eq. (6.16) and following a similar discussion as in [29, Lemma 5.4, Lemma 5.5], we have

$$\frac{\delta}{2C_7} + \int_X \mathrm{tr}(\Phi(K,\theta)u_\infty)\frac{\omega^n}{n!} + \int_X \langle \varsigma(u_\infty)(\overline{\partial}_\theta u_\infty), \overline{\partial}_\theta u_\infty\rangle_K \frac{\omega^n}{n!} \leq 0. \tag{6.17}$$

and $u_\infty = \mu_l \cdot \mathrm{Id}_E - \sum\limits_{\alpha=1}^{l-1}(\mu_{\alpha+1} - \mu_\alpha)\pi_\alpha$, where $\mu_1 < \mu_2 < \cdots < \mu_l$, $\{\pi_\alpha\}_{\alpha=1}^{l-1}$ determine $l-1$ Higgs sub-sheaves $\{E_\alpha\}_{\alpha=1}^{l-1} := \{\pi_\alpha(E)\}_{\alpha=1}^{l-1}$ of E.

By (6.17) and the same arguments in [17, p.793-794], we have

$$\begin{aligned}\nu &= \sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)\mathrm{rank}(E_\alpha)(\frac{\deg(E,K)}{\mathrm{rank}(E)}-\frac{\deg(E_\alpha,K)}{\mathrm{rank}(E_\alpha)})\\ &= \int_X \mathrm{tr}(u_\infty\sqrt{-1}\Lambda_\omega F_{K,\bar{\partial}_E,\theta})+\langle\sum_{\alpha=1}^{l-1}(\mu_{\alpha+1}-\mu_\alpha)(\mathrm{d}P_\alpha)^2(u_\infty)(\overline{\partial}_\theta u_\infty),\overline{\partial}_\theta u_\infty\rangle_K\\ &\leq -\frac{\delta}{2C_7},\end{aligned}$$

which contradicts the semi-stability of $(E,\bar{\partial}_E,\theta)$. This completes the proof of the claim. □

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Chuanjing Zhang, Pan Zhang and Xi Zhang
School of Mathematical Sciences, University of Science and Technology of China
Anhui 230026, P.R. China
Email:chjzhang@mail.ustc.edu.cn; panzhang@mail.ustc.edu.cn; mathzx@ustc.edu.cn