A mixed multifractal formalism for finitely many non Gibbs Frostman-like measures

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Abstract

The multifractal formalism for measures hold whenever the existence of corresponding Gibbs-like measures supported on the singularities sets holds. In the present work we tried to relax such a hypothesis and introduce a more general framework of mixed (and thus single) multifractal analysis where the measures constructed on the singularities sets are not Gibbs but controlled by an extra-function allowing the multifractal formalism to hold. We fall on the classical case by a particular choice of such afunction.

Key words: Hausdorff and packing measures, Hausdorff and packing dimensions, Multifractal formalism, Mixed cases, Hölderian Measures.
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Contents

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1   Introduction

The multifractal analysis of a single measure passes through its local dimension or its Hölder exponent. For a measure μ eventually Borel and finite on $\mathbb{R}^d$ and $x \in \text{supp}(\mu)$, the local dimension of μ at the point x is defined by

$$\alpha_\mu(x) = \lim_{r\downarrow 0} \frac{\log(\mu(B(x,r)))}{\log r}$$

when such a limit exists. The next step concerns the geometric study of the α-singularity set of the measure μ defined by

$$X(\alpha) = \{\, x \in \text{supp}(\mu)\,;\ \alpha_\mu(x) = \alpha \,\}$$

by means of its Hausdorff dimension

$$d(\alpha) = \dim X(\alpha)$$

which defines the so-called spectrum of singularities. This means that the study of the behaviour of the measure is transformed into a study of sets where the focuses may somehow forget about the measure and its point-wise character and falls in set theory and the suitable coverings that permits the computation of the Hausdorff dimension. For a subset $E \subset \mathbb{R}^d$ and $\alpha \geq 0$, the α-Hausdorff measure is defined by

$$\mathcal{H}^\alpha(E) = \lim_{\varepsilon\downarrow 0} \left(\inf \sum_j (diam(U_j))^\alpha \right)$$

where the inf is taken over all coverings of E with subsets U_j, $j \in \mathbb{N}$ such that $diam(U_j) \leq \varepsilon$.

However, some geometric sets are essentially known by means of measures that are supported by them, i.e., given a set E and a measure μ, the quantity $\mu(E)$ may be computed as the maximum value $\mu(F)$ for all subsets $F \subset E$. So, contrarily to the previous idea, we mathematically forget the geometric

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set structure of E and focus instead on the properties of the measure μ. The set E is thus partitioned into α-level sets relatively to the regularity exponent of μ into subsets $X(\alpha)$.

This makes the including of the measure μ into the computation of the Hausdorff (or fractal) dimension and thus into the definition of the Hausdorff measure a necessity to understand more the geometry of the set simultaneously with the behaviour of the measure that is supported on. One step ahead in this direction has been conducted by Olsen in [11] where the author introduced multifractal generalisations of the fractal dimensins such as Hausdorff, packing and Bouligand ones by considering general variants of measures. For a Borel probability measure μ on $\mathbb{R}^d$, a nonempty set $E \subseteq \mathbb{R}^d$ and $q,\,t \in \mathbb{R}$, he considered the pre-mesure

$$\overline{\mathcal{H}}_\mu^{q,t}(E) = \lim_{\epsilon \downarrow 0} \Big(\inf \{ \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t \} \Big),$$

where the inf is taken over the set of all centered ϵ-coverings of E, and for the empty set, $\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(\emptyset) = 0$. This yields next the measure

$$\mathcal{H}_\mu^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_\mu^{q,t}(F).$$

Similarly, the following pre-measure is considered.

$$\overline{\mathcal{P}}_\mu^{q,t}(E) = \lim_{\epsilon \downarrow 0} \Big(\sup \{ \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t \} \Big),$$

where the sup is taken over the set of all centered ϵ-packings of E. For the empty set, we set as usual $\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(\emptyset) = 0$. This yields next the measure

$$\mathcal{P}_\mu^{q,t}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}_\mu^{q,t}(E_i).$$

In [11], it has been proved that the measures $\mathcal{H}_\mu^{q,t}$, $\mathcal{P}_\mu^{q,t}$ and the pre-measure $\overline{\mathcal{P}}_\mu^{q,t}$ assign in a usual way a dimension to every set $E \subseteq \mathbb{R}^d$ called respectively multifractal generalizations of the Hausdorff dimension $(\dim_\mu^q(E),)$, the packing dimension $(\mathrm{Dim}_\mu^q(E))$ and the logarithmic index $(\Delta_\mu^q(E))$ of the set E. These quantities satisfies the cut-off relations

$$\mathcal{H}_\mu^{q,t}(E) = +\infty \text{ for } t < \dim_\mu^q(E) \text{ and } 0 \text{ for } t > \dim_\mu^q(E).$$

$$\mathcal{P}_\mu^{q,t}(E) = +\infty \text{ for } t < \mathrm{Dim}_\mu^q(E) \text{ and } 0 \text{ for } t > \mathrm{Dim}_\mu^q(E).$$
$$\overline{\mathcal{P}}_\mu^{q,t}(E) = +\infty \text{ for } t < \Delta_\mu^q(E) \text{ and } 0 \text{ for } t > \Delta_\mu^q(E).$$

In [11], the characteristics of these functions such as monotony, convexity, lower and upper bounds have been studied. Next, to come back to the essential problem in multifractal formalism which consists in the computation of the

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spectrum of singularities $d(\alpha)$, the author proved that such generalizations may lead to a multifractal formalism but when some bad restriction for the single measure μ toke place. By assuming that μ belongs to the whole class of Gibbs-like measures the multifractal formalism has been proved to hold. This was one motivation that led us to develop the present paper where such a restriction has been avoided.

Next, a first step in the direction of the mixed multifractal analysis the same author already affected by the restriction on the single measure μ developed a mixed multifractal analysis for one very restrictive class of measures known as the self affine measures [13] dealing precisely with Rényi dimensions for finitely many self similar measures. It was one step ahead but in a restrictive case. This study has been the motivation of our paper [3] where we developed a general mixed analysis for vector valued measures by proving some results for general measures and some ones for special classes. However, we noticed that the hypothesis of Gibbs-like measures is somehow not possible to avoid and thus by contouring such hypothesis with some extra-hypothesis on the measure and by proving a general mixed large deviation formalism a mixed multifractal formalism has been proved there also. This study itself has been one motivation behind the one developed in [4] and [5] where a mixed multifractal analysis inspired from the one for measures has been developed in the functional case. By concentring a vector valued Gibbs-like measure on the singularities set of finitely and simultaneously many functions, we introduced a mixed multifractal formalism for functions. General results for almost all functions have been proved and a mixed multifractal formalism have been proved for self similar quasi self similar functions as well as their superpositions (which are not self similar neither quasi self similar). For more details and backgrounds on multifractal analysis as well as the mixed generalizations the readers may be referred also to the following essential references [1], [2], [12], [17], [18], [19], [20], [21], [22], [23].

In the present paper we are concerned with the introduction of a multifractal analysis in a mixed case (but which can already adapted to single cases) where the hypothesis of the existence of Gibbs-like and/or doubling measures supported by the singularities sets is relaxed. We aim to consider some cases of simultaneous behaviors of measures where the local Hölder behaviour is controlled by special and suitable function that allow the extra-hypothesis of Gibbs-like measures not to be necessary.

The present work will be organized as follows. The next section concerns the introduction of the new variant of the mixed multifractal generalizations of Hausdorff and packing measures and dimensions relatively to the control function φ. Section 3 is devoted to the mixed multifractal generalization of Bouligand-Minkowsky or Rényi dimension already with the control φ. In section 4, a mixed multifractal formalism associated to the mixed multifractal

generalizations of Hausdorff and packing measures and dimensions introduced in section 2 is proved in some case based on a generalization of the well known large deviation formalism and where no extra-hypothesis of Gibbs-like measures existence is assumed.

2 φ-mixed multifractal generalizations of Hausdorff and packing measures and dimensions

The purpose of this section is to present our ideas about mixed multifractal generalizations of Hausdorff and packing measures and dimensions. Consider a vector valued measure $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ composed of probability measures on $\mathbb{R}^d$. We aim to study the simultaneous scaling behavior of μ relatively to an exponential density function. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be such that

$$\varphi \text{ is non-decreasing and } \varphi(r) < 0 \text{ for } r \text{ small enough.} \tag{1}$$

The mixed generalized multifractal Hausdorff φ-measure is defined as follows. For $x \in \mathbb{R}^d$ and $r > 0$ we denote $B(x, r)$ the ball of radius r and center x. We denote next

$$\mu(B(x,r)) \equiv \Big(\mu_1(B(x,r)), \ldots, \mu_k(B(x,r)) \Big)$$

and the product

$$(\mu(B(x,r)))^q \equiv (\mu_1(B(x,r)))^{q_1} \ldots (\mu_k(B(x,r)))^{q_k}.$$

Let $E \subseteq \mathbb{R}^d$ be a nonempty set and $\epsilon > 0$. Let also $q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$ and $t \in \mathbb{R}$ and consider the quantity

$$\overline{\mathcal{H}}_{\mu,\varphi,\epsilon}^{q,t}(E) = \inf \{ \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \},$$

where the inf is taken over the set of all centered ϵ-coverings of E, and for the empty set, $\overline{\mathcal{H}}_{\mu,\epsilon}^{q,t}(\emptyset) = 0$. It consists of a non increasing function of the variable ε. We denote thus

$$\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathcal{H}}_{\mu,\varphi,\epsilon}^{q,t}(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_{\mu,\varphi,\epsilon}^{q,t}(E).$$

Let finally

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F).$$

Lemma 2.1 $\mathcal{H}_{\mu,\varphi}^{q,t}$ is an outer metric measure on $\mathbb{R}^d$.

Proof. We will prove firstly that $\mathcal{H}_{\mu,\varphi}^{q,t}$ is an outer measure. This means that

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i. $\mathcal{H}_{\mu,\varphi}^{q,t}(\emptyset) = 0$.

ii. $\mathcal{H}_{\mu,\varphi}^{q,t}$ is monotone, i.e.

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(F),$$

whenever $E \subseteq F \subseteq \mathbb{R}^d$.

iii. $\mathcal{H}_{\mu,\varphi}^{q,t}$ is sub-additive, i.e.

$$\mathcal{H}_{\mu,\varphi}^{q,t}(\bigcup_n A_n) \leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(A_n).$$

The first item is obvious. Let us prove (ii). Let $E \subseteq F$ be nonempty subsets of $\mathbb{R}^d$. We have

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) = \sup_{A \subseteq E} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(A) \leq \sup_{A \subseteq F} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(A) = \mathcal{H}_{\mu,\varphi}^{q,t}(F).$$

We next prove (iii). If the right hand term is infinite, the inequality is obvious. So, assume that it is finite. Let $(E_n)_n$ be a countable family of subsets $E_i \subseteq \mathbb{R}^d$ for which $\sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(E_n) < \infty$. Let also $\epsilon, \delta > 0$ and $(B(x_{ni}, r_{ni}))_i$ a centered ϵ-covering of E_n satisfying

$$\sum_i \mu(B(x_{ni}, r_{ni}))^q e^{t\varphi(r_{ni})} \leq \overline{\mathcal{H}}_{\mu,\varphi,\varepsilon}^{q,t}(E_n) + \frac{\delta}{2^n}.$$

The whole set $(B(x_{ni}, r_{ni}))_{n,i}$ is a centered ϵ-covering of the whole union $\bigcup_n E_n$. As a consequence,

$$
\begin{aligned}
\overline{\mathcal{H}}_{\mu,\varphi,\epsilon}^{q,t}(\bigcup_n E_n) &\leq \sum_n \sum_i (\mu(B(x_{ni}, r_{ni})))^q e^{t\varphi(r_{ni})} \\
&\leq \sum_n \left(\overline{\mathcal{H}}_{\mu,\varphi,\epsilon}^{q,t}(E_n) + \frac{\delta}{2^n} \right) \\
&\leq \sum_n \left(\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(E_n) + \frac{\delta}{2^n} \right) \\
&\leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(E_n) + \delta.
\end{aligned}
$$

Having ϵ and δ going towards 0, we obtain

$$\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(\bigcup_n E_n) \leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(E_n).$$

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Let next a set F covered with the countable set $(A_n)_n$. That is $F \subseteq \bigcup_n A_n$. We have

$$\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F) = \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}\left(\bigcup_n (A_n \cap F)\right)$$

$$\leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(A_n \cap F)$$

$$\leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(A_n).$$

Taking the sup on F, we obtain

$$\mathcal{H}_{\mu,\varphi}^{q,t}(F) \leq \sum_n \mathcal{H}_{\mu,\varphi}^{q,t}(A_n).$$

We now prove that $\mathcal{H}_{\mu,\varphi}^{q,t}$ is metric. Let A, B subsets of $\mathbb{R}^d$. We recall that the Hausdorff distance $d(A, B)$ is defined by

$$d(A, B) = inf\{|x - y|; \ x \in A \ y \in B\}.$$

Assume so that $d(A, B) > 0$, and that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A \cup B) < \infty.$$

Let next $0 < \delta < d(A, B)$, $\varepsilon > 0$, $F_1 \subseteq A$, $F_2 \subseteq B$ and $(B(x_i, r_i))_i$ a centered δ-covering of the set $F_1 \cup F_2$ and such that

$$\overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_1 \cup F_2) \leq \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \leq \overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_1 \cup F_2) + \varepsilon.$$

This is always possible from the definition of $\overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_1 \cup F_2)$. Denote next the index sets

$$I = \{i; \ B(x_i, r_i) \cap F_1 \neq \emptyset\} \quad \text{and} \quad J = \{i; \ B(x_i, r_i) \cap F_2 \neq \emptyset\}.$$

The countable sets $(B(x_i, r_i))_{i \in I}$ and $(B(x_i, r_i))_{i \in J}$ are centered δ-coverings of F_1 and F_2 respectively. Consequently,

$$\overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_1) + \overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_2) \leq \sum_{i \in I} (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)}$$

$$+ \sum_{i \in J} \mu(B(x_i, r_i))^q e^{t\varphi(r_i)}$$

$$= \sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t$$

$$\leq \overline{\mathcal{H}}_{\mu,\varphi,\delta}^{q,t}(F_1 \cup F_2) + \varepsilon.$$

As a result,

$$\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F_1) + \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F_2) \leq \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F_1 \cup F_2) + \varepsilon \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A \cup B) + \varepsilon.$$

When $\varepsilon \downarrow 0$ and taking the sup on the sets $F_1 \subseteq A$ and $F_2 \subseteq B$, we obtain

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A \cup B) \geq \mathcal{H}_{\mu,\varphi}^{q,t}(A) + \mathcal{H}_{\mu,\varphi}^{q,t}(B).$$

The inequality

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A \cup B) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A) + \mathcal{H}_{\mu,\varphi}^{q,t}(B).$$

follows from the sub-additivity property of the measure $\mathcal{H}_{\mu,\varphi}^{q,t}$.

Lemma 2.2 *Let $(A_n)_n$ be a non-decreasing sequence of subsets in $\mathbb{R}^d$ and denote $A = \bigcup_n A_n$. Assume further that $d(A_n, A \setminus A_{n+1}) > 0$ for all n. Then,*

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A) = \lim_{n \to +\infty} \mathcal{H}_{\mu,\varphi}^{q,t}(A_n).$$

Proof. The result is obvious if the limit above is infinite. So assume that it is finite and denote for $k \in \mathbb{N}$, $C_k = A_{k+1} \setminus A_k$. We then observe that

$$d(C_j, C_p) > 0, \ \forall j, p\,; \ |j - p| \geq 2 \quad \text{and} \quad A = A_n \cup \Big(\bigcup_{k \geq n+1} C_k \Big), \ \forall\, n.$$

Therefore,

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A_n) + \underbrace{\sum_{k \geq n+1} \mathcal{H}_{\mu,\varphi}^{q,t}(C_k)}_{R_n}.$$

Now, it is straightforward that for all n we have

$$\sum_{k=0}^{n} \mathcal{H}_{\mu,\varphi}^{q,t}(C_{2k}) = \mathcal{H}_{\mu,\varphi}^{q,t}\Big(\bigcup_{k=0}^{n} C_{2k} \Big) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A_{2n+2}) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A) < \infty$$

and similarly,

$$\sum_{k=0}^{n} \mathcal{H}_{\mu,\varphi}^{q,t}(C_{2k+1}) = \mathcal{H}_{\mu,\varphi}^{q,t}\Big(\bigcup_{k=0}^{n} C_{2k+1} \Big) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A_{2n+1}) \leq \mathcal{H}_{\mu,\varphi}^{q,t}(A) < \infty.$$

Hence, R_n is the rest of a convergent series, so it goes to 0 as n goes to infinity. Consequently,

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A) \leq \lim_{n \to +\infty} \mathcal{H}_{\mu,\varphi}^{q,t}(A_n).$$

The reciprocal inequality is obvious.

Lemma 2.3 *Borel sets are $\mathcal{H}_{\mu,\varphi}^{q,t}$-measurable.*

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Proof. Let B be a Borel subset of $\mathbb{R}^d$ (a closed subset for example), $E \subset \mathbb{R}^d$ and denote for $n \in \mathbb{N}$,

$$B_n = \{x \in E \,;\; d(x, B) \geq \frac{1}{n}\}.$$

It consists of a non-decreasing sequence of subsets of $\mathbb{R}^d$ satisfying further that

$$E \setminus B = \bigcup_n B_n \quad \text{and} \quad d(B_n, (E \setminus B) \setminus B_{n+1}) > 0, \ \forall n.$$

Hence, Lemma 2.2 yields that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E \setminus B) = \lim_{n \to +\infty} \mathcal{H}_{\mu,\varphi}^{q,t}(B_n).$$

Observe now that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) \geq \mathcal{H}_{\mu,\varphi}^{q,t}(E \cap B) + \mathcal{H}_{\mu,\varphi}^{q,t}(B_n), \ \forall n.$$

When $n \to \infty$, we get

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) \geq \mathcal{H}_{\mu,\varphi}^{q,t}(E \cap B) + \mathcal{H}_{\mu,\varphi}^{q,t}(E \setminus B).$$

Definition 2.1 *The restriction of $\mathcal{H}_{\mu,\varphi}^{q,t}$ on Borel sets is called the mixed generalized Hausdorff measure on $\mathbb{R}^d$.*

Now, we define the mixed generalized multifractal packing measure. We use already the same notations as previously. Let

$$\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(E) = \sup\{ \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \}$$

where the sup is taken over the set of all centered ϵ-packings of E. For the empty set, we set as usual $\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(\emptyset) = 0$. Next, we consider the limit as $\epsilon \downarrow 0$,

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E) = \lim_{\epsilon \downarrow 0} \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(E) = \inf_{\delta > 0} \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(E)$$

and finally,

$$\mathcal{P}_{\mu,\varphi}^{q,t}(E) = \inf_{E \subseteq \cup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E_i).$$

Lemma 2.4 $\mathcal{P}_{\mu,\varphi}^{q,t}$ *is an outer metric measure on $\mathbb{R}^d$.*

The proof of this lemma is more specific than Lemma 2.1 and uses the following result.

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A \cup B) = \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A) + \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(B), \quad \text{whenever } d(A, B) > 0. \tag{2}$$

Indeed, let $0 < \epsilon < \dfrac{1}{2} d(A, B)$ and $(B(x_i, r_i))_i$ be a centered ϵ-packing of the union $A \cup B$. It can be divided into two parts I and J,

$$(B(x_i, r_i))_i = \Big(B(x_i, r_i) \Big)_{i \in I} \bigcup \Big(B(x_i, r_i) \Big)_{i \in J}$$

where

$$\forall\, i \in I, \;\; B(x_i, r_i) \cap B = \emptyset \quad \text{and} \quad \forall\, i \in J, \;\; B(x_i, r_i) \cap A = \emptyset.$$

Therefore, $(B(x_i, r_i))_{i \in I}$ is a centered ϵ-packing of A and $(B(x_i, r_i))_{i \in J}$ is a centered ϵ-packing of the union B. Hence,

$$\sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} = \sum_{i \in I} (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)}$$
$$+ \sum_{i \in I} (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)}.$$

Now, it is straightforward that

$$\sum_{i \in I} (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \leq \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A)$$

and

$$\sum_{i \in I} (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} \leq \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(B).$$

Consequently,

$$\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A \cup B) \leq \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A) + \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(B)$$

and thus the limit for $\epsilon \downarrow 0$ gives

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A \cup B) \leq \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A) + \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(B).$$

The converse is more easier and it states that $\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}$ and next $\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}$ are sub-additive. Let $(B(x_i, r_i))_i$ be a centered ϵ-packing of A and $(B(y_i, r_i))_i$ be a centered ϵ-packing of B. The union $\Big(B(x_i, r_i) \Big)_i \bigcup \Big(B(y_i, r_i) \Big)_i$ is a centered ϵ-packing of $A \cup B$. So that

$$\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A \cup B) \geq \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)} + \sum_i (\mu(B(y_i, r_i)))^q e^{t\varphi(r_i)}.$$

Taking the sup on $(B(x_i, r_i))_i$ as a centered ϵ-packing of A and next the sup on $(B(y_i, r_i))_i$ as a centered ϵ-packing of B, we obtain

$$\overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A \cup B) \geq \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(A) + \overline{\mathcal{P}}_{\mu,\varphi,\epsilon}^{q,t}(B)$$

and thus the limit for $\epsilon \downarrow 0$ gives

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A \cup B) \geq \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(A) + \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(B).$$

Proof of Lemma 2.4. We shall prove as previously that

i. $\mathcal{P}^{q,t}_{\mu,\varphi}(\emptyset) = 0$.
ii. $\mathcal{P}^{q,t}_{\mu,\varphi}$ is monotone, i.e.

$$\mathcal{P}^{q,t}_{\mu,\varphi}(E) \leq \mathcal{P}^{q,t}_{\mu,\varphi}(F),$$

whenever $E \subseteq F \subseteq \mathbb{R}^d$.
iii. $\mathcal{P}^{q,t}_{\mu,\varphi}$ is sub-additive, i.e.

$$\mathcal{P}^{q,t}_{\mu,\varphi}(\bigcup_n A_n) \leq \sum_n \mathcal{P}^{q,t}_{\mu,\varphi}(A_n).$$

The first item is immediate from the definition of $\mathcal{P}^{q,t}_{\mu,\varphi}(\emptyset) = 0$. Let $E \subseteq F$ be subsets of $\mathbb{R}^d$. We have

$$\mathcal{P}^{q,t}_{\mu,\varphi}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i)$$

$$\leq \inf_{F \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i)$$

$$= \mathcal{P}^{q,t}_{\mu,\varphi}(F).$$

So is the item **ii**. Let next $(A_n)_n$ a countable set of subsets of $\mathbb{R}^d$, $\varepsilon > 0$ and for each n, $(E_{ni})_i$ be a covering of A_n such that

$$\sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_{ni}) \leq \mathcal{P}^{q,t}_{\mu,\varphi}(A_n) + \frac{\varepsilon}{2^n}.$$

It follows for all $\varepsilon > 0$ that

$$\mathcal{P}^{q,t}_{\mu,\varphi}(\bigcup_n A_n) \leq \sum_n \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_{ni})$$

$$\leq \sum_n \mathcal{P}^{q,t}_{\mu,\varphi}(A_n) + \varepsilon.$$

Hence,

$$\mathcal{P}^{q,t}_{\mu,\varphi}(\bigcup_n A_n) \leq \sum_n \mathcal{P}^{q,t}_{\mu,\varphi}(A_n).$$

So is the item **iii**. We now prove that $\mathcal{P}^{q,t}_{\mu,\varphi}$ is metric. Let A, B subsets of $\mathbb{R}^d$ be such that $d(A,B) > 0$. We shall prove that

$$\mathcal{P}^{q,t}_{\mu,\varphi}(A \cup B) = \mathcal{P}^{q,t}_{\mu,\varphi}(A) + \mathcal{P}^{q,t}_{\mu,\varphi}(B).$$

Since $\mathcal{P}^{q,t}_{\mu,\varphi}$ is an outer measure, it suffices to show that

$$\mathcal{P}^{q,t}_{\mu,\varphi}(A \cup B) \geq \mathcal{P}^{q,t}_{\mu,\varphi}(A) + \mathcal{P}^{q,t}_{\mu,\varphi}(B).$$

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Of course, if the left hand term is infinite, the inequality is obvious. So, suppose that it is finite. For $\varepsilon > 0$, there exists a covering $(E_i)_i$ of the union set $A \cup B$ such that

$$\sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E_i) \leq \mathcal{P}_{\mu,\varphi}^{q,t}(A \cup B) + \varepsilon.$$

By denoting $F_i = A \cap E_i$ and $H_i = B \cap E_i$, we get countable coverings $(F_i)_i$ of A and $(H_i)_i$ for B respectively. Furthermore, $F_i \cap H_j = \emptyset$ pour all i and j. Consequently,

$$\mathcal{P}_{\mu,\varphi}^{q,t}(A) + \mathcal{P}_{\mu,\varphi}^{q,t}(B) \leq \sum_i (\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(F_i) + \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(H_i)).$$

Since $d(A, B) > 0$, $F_i \subset A$ and $H_i \subset B$, it follows that $d(F_i, H_j) > 0$ for all i and j. Hence, claim 2 affirms that

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E_i) = \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(F_i \cup H_i) = \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(F_i) + \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(H_i).$$

Hence,

$$\mathcal{P}_{\mu,\varphi}^{q,t}(A) + \mathcal{P}_{\mu,\varphi}^{q,t}(B) \leq \sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E_i) \leq \mathcal{P}_{\mu,\varphi}^{q,t}(A \cup B) + \varepsilon$$

and the result is obtained by having $\varepsilon \downarrow 0$.

Lemma 2.5 *Borel sets are $\mathcal{P}_{\mu,\varphi}^{q,t}$-measurable.*

The **Proof** is similar as in Lemma 2.5 and thus it is left to the reader.

Definition 2.2 *The restriction of $\mathcal{P}_{\mu,\varphi}^{q,t}$ on Borel sets is called the mixed generalized packing measure on $\mathbb{R}^d$.*

It holds as for the case of the multifractal analysis of a single measure that each of the measures $\mathcal{H}_\mu^{q,t}$, $\mathcal{P}_{\mu,\varphi}^{q,t}$ and the pre-measure $\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}$ assign a dimension to every set $E \subseteq \mathbb{R}^d$.

Proposition 2.1 *Given a subset $E \subseteq \mathbb{R}^d$,*

(1) There exists a unique number $\dim_{\mu,\varphi}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) = \begin{cases} +\infty & \text{for } t < \dim_{\mu,\varphi}^q(E) \\ 0 & \text{si } \ t > \dim_{\mu,\varphi}^q(E) \end{cases}$$

(2) There exists a unique number $Dim_{\mu,\varphi}^q(E) \in [-\infty, +\infty]$ such that

$$\mathcal{P}_{\mu,\varphi}^{q,t}(E) = \begin{cases} +\infty & \text{for } t < Dim_{\mu,\varphi}^q(E) \\ 0 & \text{for } t > Dim_{\mu,\varphi}^q(E) \end{cases}$$

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(3) There exists a unique number $\Delta_\mu^q(E) \in [-\infty, +\infty]$ such that

$$\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E) = \begin{cases} +\infty \ \ for \ t < \Delta_\mu^q(E) \\ 0 \quad \ \ for \ t > \Delta_\mu^q(E) \end{cases}$$

Definition 2.3 *The quantities $\dim_{\mu,\varphi}^q(E)$, $\mathrm{Dim}_{\mu,\varphi}^q(E)$ and $\Delta_\mu^q(E)$ define the so-called mixed multifractal generalizations of the Hausdorff dimension, the packing dimension and the logarithmic index of the set E.*

Remark that for $k = 1$ and φ the log function $\varphi(r) = \log(r)$, we come back to the classical definitions of the Hausdorff and packing measures and dimensions in their original forms (by taking $q = 0$) and their generalized multifractal variants for q being arbitrary. The mixed case studied here may be also applied for a single measure and thus the results and characterizations outpointed in the present work remains valid for a single measure. Indeed, denote $Q_i = (0, 0, ..., q_i, 0, ..., 0)$ the vector with zero coordinates except the ith one which equals q_i, we obtain the multifractal generalizations of the Hausdorff φ-measure and φ-dimension, the packing φ-dimension and the logarithmic φ-index of the set E for the single measure μ_i,

$$\dim_{\mu,\varphi}^{Q_i}(E) = \dim_{\mu_i,\varphi}^{q_i}(E),$$

$$\mathrm{Dim}_{\mu,\varphi}^{Q_i}(E) = \mathrm{Dim}_{\mu_i,\varphi}^{q_i}(E)$$

and

$$\Delta_{\mu,\varphi}^{Q_i}(E) = \Delta_{\mu_i,\varphi}^{q_i}(E).$$

Similarly, for the null vector of $\mathbb{R}^k$, we obtain

$$\dim_{\mu,\varphi}^0(E) = \dim_\varphi(E),$$

$$\mathrm{Dim}_{\mu,\varphi}^0(E) = \mathrm{Dim}_\varphi(E)$$

and

$$\Delta_{\mu,\varphi}^0(E) = \Delta_\varphi(E).$$

We may obtain further

$$\dim_{\mu,\log}^{Q_i}(E) = \dim_{\mu_i,\log}^{q_i}(E),$$

$$\mathrm{Dim}_{\mu,\log}^{Q_i}(E) = \mathrm{Dim}_{\mu_i,\log}^{q_i}(E)$$

and

$$\Delta_{\mu,\log}^{Q_i}(E) = \Delta_{\mu_i,\log}^{q_i}(E).$$

Similarly, for the null vector of $\mathbb{R}^k$, we obtain

$$\dim_{\mu,\log}^0(E) = \dim_{\log}(E) = \dim(E),$$

$$\mathrm{Dim}_{\mu,\log}^0(E) = \mathrm{Dim}_{\log}(E) = \mathrm{Dim}(E)$$

and
$$\Delta^0_{\mu,\log}(E) = \Delta_{\log}(E) = \Delta(E).$$

Proof of Proposition 2.1. We will sketch only the proof of the first point. The rest is analogous.

1. We claim that $\forall\, t \in \mathbb{R}$ such that $\mathcal{H}^{q,t}_{\mu,\varphi}(E) < \infty$ it holds that

$$\mathcal{H}^{q,t'}_{\mu,\varphi}(E) = 0\,, \quad \forall\, t' > t.$$

Indeed, let $0 < \epsilon < 1$, $F \subseteq E$ and $(B(x_i, r_i))_i$ be a centered ϵ-covering of F. We have

$$\overline{\mathcal{H}}^{q,t'}_{\mu,\epsilon}(F) \leq \sum_i (\mu(B(x_i, r_i)))^q e^{t'\varphi(r_i)}$$

$$\leq e^{(t'-t)\varphi(\varepsilon)} \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r_i)}.$$

Consequently,

$$\overline{H}^{q,t'}_{\mu,\varphi,\epsilon}(F) \leq e^{(t'-t)\varphi(\varepsilon)} \overline{H}^{q,t}_{\mu,\varphi,\epsilon}(F).$$

Hence, as $\varphi(\varepsilon) \to -\infty$ as $\varepsilon \to 0$, we obtain

$$\overline{\mathcal{H}}^{q,t'}_{\mu,\varphi}(F) = 0, \quad \forall\, F \subseteq E.$$

As a result, $\mathcal{H}^{q,t'}_{\mu,\varphi}(E) = 0$. We then set

$$\dim^q_{\mu,\varphi}(E) = \inf\{\, t \in \mathbb{R};\ \mathcal{H}^{q,t}_{\mu,\varphi}(E) = 0 \,\}.$$

One can proceed otherwise by claiming that $\forall\, t \in \mathbb{R}$ such that $\mathcal{H}^{q,t}_{\mu,\varphi}(E) > 0$ it holds that

$$\mathcal{H}^{q,t'}_{\mu,\varphi}(E) = +\infty\,, \quad \forall\, t' < t.$$

Indeed, proceeding as previously, we obtain for $\epsilon > 0$,

$$e^{(t'-t)\varphi(\epsilon)} \overline{H}^{q,t}_{\mu,\epsilon}(F) \leq \overline{H}^{q,t'}_{\mu,\epsilon}(F).$$

Hence, for the same reasons as above,

$$\overline{\mathcal{H}}^{q,t'}_{\mu,\varphi}(F) = +\infty, \quad \forall\, F \subseteq E.$$

As a result, $\mathcal{H}^{q,t'}_{\mu,\varphi}(E) = +\infty$. We then set

$$\dim^q_\mu(E) = \sup\{\, t \in \mathbb{R};\ \mathcal{H}^{q,t}_{\mu,\varphi}(E) = +\infty \,\}.$$

Next, we aim to study the characteristics of the mixed multifractal generalizations of dimensions. We now adapt the following notations. For $q = (q_1, ..., q_k) \in \mathbb{R}^k$,

$$b_{\mu,\varphi}(q, E) = \dim^q_{\mu,\varphi}(E),$$
$$B_{\mu,\varphi}(q, E) = \mathrm{Dim}^q_{\mu,\varphi}(E)$$

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and
$$\Lambda_{\mu,\varphi}(q, E) = \Delta_{\mu,\varphi}^q(E).$$

When $E = \mathrm{supp}(\mu)$ is the support of the measure μ, we will omit the indexation with E and denote simply

$$b_{\mu,\varphi}(q), \ B_{\mu,\varphi}(q) \text{ and } \Lambda_{\mu,\varphi}(q).$$

The following propositions resume the characteristics of these functions and extends the results of L. Olsen [11] to our case.

Proposition 2.2 *The following assertions hold.*

a. *$b_{\mu,\varphi}(q,.)$ and $B_{\mu,\varphi}(q,.)$ are non decreasing with respect to the inclusion property in $\mathbb{R}^d$.*
b. *$b_{\mu,\varphi}(q,.)$ and $B_{\mu,\varphi}(q,.)$ are σ-stable.*

Proof. a. Let $E \subseteq F$ be subsets of $\mathbb{R}^d$. We have

$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) = \sup_{A \subseteq E} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(A) \leq \sup_{A \subseteq F} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(A) = \mathcal{H}_{\mu,\varphi}^{q,t}(F).$$

So for the monotony of $b_{\mu,\varphi}(q,.)$.
b. Let $(A_n)_n$ be a countable set of subsets $A_n \subseteq \mathbb{R}^d$ and denote $A = \bigcup_n A_n$. It holds from the monotony of $b_{\mu,\varphi}(q,.)$ that

$$b_{\mu,\varphi}(q, A_n) \leq b_{\mu,\varphi}(q, A), \quad \forall n.$$

Hence,
$$\sup_n b_{\mu,\varphi}(q, A_n) \leq b_{\mu,\varphi}(q, A).$$

Next, for any $t > \sup_n b_{\mu,\varphi}(q, A_n)$, there holds that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(A_n) = 0, \quad \forall n.$$

Consequently, from the sub-additivity property of $\mathcal{H}_{\mu,\varphi}^{q,t}$, it holds that

$$\mathcal{H}_{\mu,\varphi}^{q,t}(\bigcup_n A_n) = 0, \quad \forall t > \sup_n b_{\mu,\varphi}(q, A_n).$$

Which means that

$$b_{\mu,\varphi}(q, A) \leq t, \quad \forall t > \sup_n b_{\mu,\varphi}(q, A_n).$$

Hence,
$$b_{\mu,\varphi}(q, A) \leq \sup_n b_{\mu,\varphi}(q, A_n).$$

Similar arguments permit to prove the properties of $B_{\mu,\varphi}(q,.)$.

Next, we continue to study the characteristics of the mixed generalized multifractal φ-dimensions. The following result is obtained.

Proposition 2.3 *The following assertions are true.*

a. *The functions $q \longmapsto B_{\mu,\varphi}(q)$ and $q \longmapsto \Lambda_{\mu,\varphi}(q)$ are convex.*
b. *For $i = 1, 2, ..., k$ and $\widehat{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_k)$ fixed, the functions $q_i \longmapsto b_{\mu,\varphi}(q)$, $q_i \longmapsto B_{\mu,\varphi}(q)$ and $q_i \longmapsto \Lambda_{\mu,\varphi}(q)$ are non increasing.*

Proof. a. We start by proving that $\Lambda_{\mu,,\varphi}(., E)$ is convex. Let p, $q \in \mathbb{R}^k$, $\alpha \in]0, 1[$ and let also

$$s > \Lambda_{\mu,\varphi}(p, E) \text{ and } t > \Lambda_{\mu,\varphi}(q, E).$$

Consider next a centered ϵ-packing $(B_i = B(x_i, r_i))_i$ of E. Applying Hölder's inequality, it holds that

$$\sum_i (\mu(B_i))^{\alpha q + (1-\alpha)p} e^{(\alpha t + (1-\alpha)s)\varphi(r_i)}$$

$$\leq \left(\sum_i (\mu(B_i))^q e^{t\varphi(r_i)} \right)^\alpha \left(\sum_i (\mu(B_i))^p e^{s\varphi(r_i)} \right)^{1-\alpha}.$$

Hence,

$$\overline{\mathcal{P}}_{\mu,\epsilon}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) \leq \left(\overline{\mathcal{P}}_{\mu,\epsilon}^{q,t}(E) \right)^\alpha \left(\overline{\mathcal{P}}_{\mu,\epsilon}^{p,s}(E) \right)^{1-\alpha}.$$

The limit on $\epsilon \downarrow 0$ gives

$$\overline{\mathcal{P}}_{\mu,\varphi}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) \leq \left(\overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E) \right)^\alpha \left(\overline{\mathcal{P}}_{\mu,\varphi}^{p,s}(E) \right)^{1-\alpha}.$$

Consequently,

$$\overline{\mathcal{P}}_{\mu,\varphi}^{\alpha q + (1-\alpha)p, \alpha t + (1-\alpha)s}(E) = 0, \quad \forall s > \Lambda_{\mu,E}(p) \text{ and } t > \Lambda_{\mu,E}(q).$$

It results that

$$\Lambda_{\mu\varphi}(\alpha q + (1 - \alpha)p, E) \leq \alpha \Lambda_{\mu,\varphi}(q, E) + (1 - \alpha)\Lambda_{\mu,\varphi}(p, E).$$

We now prove the convexity of $B_{\mu,\varphi}(., E)$. We set in this case

$$t = B_{\mu,\varphi}(q, E) \text{ and } s = B_{\mu,\varphi}(p, E).$$

We have

$$\mathcal{P}_{\mu,\varphi}^{q,t+\varepsilon}(E) = \mathcal{P}_{\mu,\varphi}^{p,s+\varepsilon}(E) = 0.$$

Therefore, there exists $(H_i)_i$ and $(K_i)_i$ coverings of the set E for which

$$\sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t+\varepsilon}(H_i) \leq 1 \qquad \text{et} \qquad \sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{p,s+\varepsilon}(K_i) \leq 1.$$

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Denote for $n \in \mathbb{N}$, $E_n = \bigcup_{1 \leq i,j \leq n} (H_i \cap K_j)$. Thus, $(E_n)_n$ is a covering of E. So that,

$$
\mathcal{P}_{\mu,\varphi}^{\alpha\,q+(1-\alpha)p,\alpha\,t+(1-\alpha)s+\varepsilon}(E_n)
$$

$$
\leq \sum_{i,j=1}^{n} \mathcal{P}_{\mu,\varphi}^{\alpha\,q+(1-\alpha)p,\alpha t+(1-\alpha)s+\varepsilon}(H_i \cap K_j)
$$

$$
\leq \sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu,\varphi}^{\alpha\,q+(1-\alpha)p,\alpha t+(1-\alpha)s+\varepsilon}(H_i \cap K_j)
$$

$$
\leq \left(\sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu,\varphi}^{q,t+\varepsilon}(H_i \cap K_j) \right)^{\alpha} \left(\sum_{i,j=1}^{n} \overline{\mathcal{P}}_{\mu,\varphi}^{p,s+\varepsilon}(H_i \cap K_j) \right)^{1-\alpha}
$$

$$
\leq n^{\alpha}\, n^{1-\alpha} = n < \infty.
$$

Consequently,

$$
B_{\mu,\varphi}(\alpha\,q + (1-\alpha)p, E_n) \leq \alpha\,t + (1-\alpha)s + \varepsilon, \quad \forall \varepsilon > 0.
$$

Hence,

$$
B_{\mu,\varphi}(\alpha\,q + (1-\alpha)p, E) \leq \alpha\,B_{\mu,\varphi}(q, E) + (1-\alpha)B_{\mu,\varphi}(p, E).
$$

b. For $i = 1, 2, \ldots, k$, let $\widehat{q}_i$ fixed and $p_i \leq q_i$ reel numbers. Denote next

$$
q = (q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_k) \text{ and } p = (q_1, \ldots, q_{i-1}, p_i, q_{i+1}, \ldots, q_k).
$$

Let finally $A \subseteq E$. For a centered ϵ-covering $(B(x_i, r_i))_i$ of A, we have immediately

$$
\mu(B(x_i, r_i))^q e^{t\varphi(r_i)} \leq \mu(B(x_i, r_i))^p e^{t\varphi(r_i)}, \quad \forall t \in \mathbb{R}.
$$

Hence,

$$
\overline{H}_{\mu,\varphi,\epsilon}^{q,t}(A) \leq \overline{H}_{\mu,\varphi,\epsilon}^{p,t}(A).
$$

When $\epsilon \downarrow 0$, we obtain

$$
\overline{H}_{\mu,\varphi}^{q,t}(A) \leq \overline{H}_{\mu,\varphi}^{p,t}(A).
$$

Therefore,

$$
\mathcal{H}_{\mu,\varphi}^{q,t}(E) = \sup_{A \subseteq E} \overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(A) \leq \sup_{A \subseteq E} \overline{\mathcal{H}}_{\mu,\varphi}^{p,t}(A) = \mathcal{H}_{\mu,\varphi}^{p,t}(E).
$$

This induces the fact that

$$
\mathcal{H}_{\mu,\varphi}^{q,t}(E) = 0, \quad \forall t > b_{\mu,\varphi}(p, E).
$$

Consequently

$$
b_{\mu,\varphi}(q, E) < t, \quad \forall t > b_{\mu,\varphi}(p, E).
$$

Hence,

$$
b_{\mu,\varphi}(q, E) \leq b_{\mu,\varphi}(p, E).
$$

The remaining part to prove the monotony $\Lambda_{\mu,\varphi}(.,E)$ and $B_{\mu,\varphi}(.,E)$ is analogous.

Proposition 2.4 *The following assertions are true.*

a. $0 \leq b_{\mu,\varphi}(q) \leq B_{\mu,\varphi}(q) \leq \Lambda_{\mu,\varphi}(q)$, *whenever* $q_i < 1$ *for all* $i = 1, 2, ..., k$.
b. $b_{\mu,\varphi}(\P_i) = B_{\mu,\varphi}(\P_i) = \Lambda_{\mu,\varphi}(\P_i) = 0$, *where* $\P_i = (0, 0, ..., 1, 0, ..., 0)$.
c. $b_{\mu,\varphi}(q) \leq B_{\mu,\varphi}(q) \leq \Lambda_{\mu,\varphi}(q) \leq 0$ *whenever* $q_i > 1$ *for all* $i = 1, 2, ..., k$.

The proof of this results reposes on the following intermediate ones.

Lemma 2.6 *There exists a constant* $\xi \in]0, +\infty[$ *satisfying for any* $E \subseteq \mathbb{R}^d$,

$$\mathcal{H}^{q,t}_{\mu,\varphi}(E) \leq \xi \mathcal{P}^{q,t}_{\mu,\varphi}(E) \leq \xi \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E), \qquad \forall\, q, t.$$

More precisely, ξ *is the number related to the Besicovitch covering theorem.*

Proof. It suffices to prove the first inequality. The second is always true for all $\xi > 0$. Let $F \subseteq \mathbb{R}^d$, $\epsilon > 0$ and $\mathcal{V} = \{ B(x, \frac{\epsilon}{2}); \quad x \in F \}$. Let next $\left((B_{ij})_j \right)_{1 \leq i \leq \xi}$ be the ξ sets of $\mathcal{V}$ obtained by the Besicovitch covering theorem. So that, $(B_{ij})_{i,j}$ is a centered ϵ-covering of the set F and for each i, $(B_{ij})_j$ is a centered ϵ-packing of F. Therefore,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\epsilon}(F) \leq \sum_{i=1}^{\xi} \sum_j \left(\mu(B_{ij}) \right)^q e^{t\varphi(r_{ij})} \leq \sum_{i=1}^{\xi} \overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\epsilon}(F) = \xi \overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\epsilon}(F).$$

Hence,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(F) \leq \xi \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(F).$$

Consequently, for $E \subseteq \bigcup_i E_i$, we obtain

$$\mathcal{H}^{q,t}_{\mu,\varphi}(E) = \mathcal{H}^{q,t}_{\mu,\varphi}(\bigcup_i (E_i \cap E))$$

$$\leq \sum_i \mathcal{H}^{q,t}_{\mu,\varphi}(E_i \cap E)$$

$$\leq \sum_i \sup_{F \subseteq E_i \cap E} \overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(F)$$

$$\leq \xi \sum_i \sup_{F \subseteq E_i \cap E} \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(F)$$

$$\leq \xi \sum_i \overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E_i).$$

So as Lemma 2.6.
Proof of Proposition 2.4. It follows from Proposition 2.2, Proposition 2.3 and Lemma 2.6.

3 φ-mixed multifractal generalization of Bouligand-Minkowski's dimensions

Let $k \geq 1$ be en integer and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. Denote as previously

$$\mu(B(x,r)) \equiv \Big(\mu_1(B(x,r)), \ldots, \mu_k(B(x,r))\Big)$$

and for $q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$,

$$(\mu(B(x,r)))^q \equiv (\mu_1(B(x,r)))^{q_1} \ldots (\mu_k(B(x,r)))^{q_k}.$$

Next, for a nonempty subset $E \subseteq \mathrm{supp}(\mu)$, $\delta > 0$ and $q \in \mathbb{R}$, we put

$$\mathcal{T}_{\mu,\delta}^q(E) = \inf \left\{ \sum_i \Big(\mu\big(B(x_i,\delta)\big)\Big)^q \right\}$$

where the inf is over the set of all centered δ-coverings $\big(B(x_i,\delta)\big)_i$ of the set E. The mixed multifractal generalized Bouligand-Minkowski φ-dimensions are

$$\overline{L}_{\mu,\varphi}^q(E) = \limsup_{\delta \downarrow 0} \frac{\log\Big(\mathcal{T}_{\mu,\delta}^q(E)\Big)}{-\varphi(\delta)}$$

for the upper one and

$$\underline{L}_{\mu,\varphi}^q(E) = \liminf_{\delta \downarrow 0} \frac{\log\Big(\mathcal{T}_{\mu,\delta}^q(E)\Big)}{-\varphi(\delta)}$$

for the lower. In the case of equality, the common value is denoted $L_{\mu,\varphi}^q(E)$ and is called the mixed multifractal generalized Bouligand-Minkowski φ-dimension of the set E.

Such dimensions may also be obtained via the δ-packings as follows. Indeed, for $\delta > 0$ and $q \in \mathbb{R}$, we set

$$\mathcal{S}_{\mu,\delta}^q(E) = \sup \left\{ \sum_i \Big(\mu\big(B(x_i,\delta)\big)\Big)^q \right\}$$

where the sup is taken over all the centered δ-packings $\big(B(x_i,\delta)\big)_i$ of the set E. The upper dimension is

$$\overline{C}_{\mu,\varphi}^q(E) = \limsup_{\delta \downarrow 0} \frac{\log\Big(\mathcal{S}_{\mu,\delta}^q(E)\Big)}{-\varphi(\delta)}$$

and the lower is

$$\underline{C}^q_{\mu,\varphi}(E) = \liminf_{\delta\downarrow0} \frac{\log\big(\mathcal{S}^q_{\mu,\delta}(E)\big)}{-\varphi(\delta)}$$

and similarly, when these are equal, the common value will be denoted $C^q_{\mu,\varphi}(E)$ and it defines the dimension of E.

Definition 3.1 *For $E \subseteq supp(\mu)$ and $q = (q_1, q_2, \ldots, q_k) \in \mathbb{R}^k$, we will call*

a. $\overline{C}^q_{\mu,\varphi}(E)$ *and* $\overline{L}^q_{\mu,\varphi}(E)$ *the upper μ-mixed multifractal generalized Bouligand Minkowski φ-dimension of E.*
b. $\underline{C}^q_{\mu,\varphi}(E)$ *and* $\underline{L}^q_{\mu,\varphi}(E)$ *the lower μ-mixed multifractal generalized Bouligand Minkowski φ-dimension of E.*
c. $C^q_{\mu,\varphi}(E)$ *and* $L^q_{\mu,\varphi}(E)$ *the μ-mixed multifractal generalized Bouligand Minkowski φ-dimension of E.*

Remark 3.1 *As for the classical multifractal contexts, each of the quantities above defines in fact a mixed generalization that can be different from the other.*

Theorem 3.1 *(1) For all $q \in \mathbb{R}^k$, we have*

$$\underline{L}^q_{\mu,\varphi}(E) \leq \underline{C}^q_{\mu,\varphi}(E) \quad and \quad \overline{L}^q_{\mu,\varphi}(E) \leq \overline{C}^q_{\mu,\varphi}(E).$$

(2) For any $q \in \mathbb{R}^{*k}_-$, we have
   i. $b_{\mu,\varphi}(q, E) \leq \underline{L}^q_{\mu,\varphi}(E) = \underline{C}^q_{\mu,\varphi}(E)$.
   ii. $\overline{L}_{\mu,\varphi}(q, E) = \overline{C}^q_{\mu,\varphi}(E) = \Lambda_{\mu,\varphi}(q, E)$.
(3) For any $q \in \mathbb{R}^{*k}_+$, we have

$$\overline{L}_{\mu,\varphi}(q, E) \leq \overline{C}^q_{\mu,\varphi}(E) \leq \Lambda_{\mu,\varphi}(q, E).$$

Proof. 1. Using Besicovitch covering theorem we get

$$\mathcal{T}^q_{\mu,\varphi,\delta}(E) \leq C\mathcal{S}^q_{\mu,\varphi,\delta}(E),$$

with some constant C fixed. So **1.** is proved.
2. We firstly prove that

$$\underline{L}^q_{\mu,\varphi}(E) \geq \underline{C}^q_{\mu,\varphi}(E) \quad and \quad \overline{L}^q_{\mu,\varphi}(E) \geq \overline{C}^q_{\mu,\varphi}(E).$$

Indeed, let $\big(B(x_i, \delta)\big)_i$ be a centered δ-packing of E and $\big(B(y_i, \frac{\delta}{2})\big)$ be a centered $\frac{\delta}{2}$-covering of E. Consider for each i, the integer k_i such that $x_i \in B(y_{k_i}, \frac{\delta}{2})$. It is straightforward that for $i \neq j$ we have $k_i \neq k_j$. Consequently,

for $q \in \mathbb{R}_{-}^{*k}$, there holds that

$$\sum_{i}\big(\mu(B(x_i, \delta))\big)^q = \sum_{i}\left(\frac{\mu(B(x_i, \delta))}{\mu(B(y_{k_i}, \delta/2))}\right)^q \big(\mu(B(y_{k_i}, \frac{\delta}{2}))\big)^q$$
$$\leq \sum_{i}\big(\mu(B(y_i, \frac{\delta}{2}))\big)^q.$$

Which means that

$$\mathcal{S}_{\mu,\varphi,\delta}^{q}(E) \leq \mathcal{T}_{\mu,\varphi,\frac{\delta}{2}}^{q}(E)$$

and thus, for any $q \in \mathbb{R}_{-}^{*k}$,

$$\underline{L}_{\mu,\varphi}^{q}(E) \geq \underline{C}_{\mu,\varphi}^{q}(E) \quad \text{and} \quad \overline{L}_{\mu,\varphi}^{q}(E) \geq \overline{C}_{\mu,\varphi}^{q}(E)$$

Using assertion **1.**, we obtain the equalities

$$\underline{L}_{\mu,\varphi}^{q}(E) = \underline{C}_{\mu,\varphi}^{q}(E) \quad \text{and} \quad \overline{L}_{\mu,\varphi}^{q}(E) = \overline{C}_{\mu,\varphi}^{q}(E)$$

for all $q \in \mathbb{R}_{-}^{*k}$. Therefore, to prove **2.i.**, it remains to prove the inequality of the left hand side. So, let $t > \underline{L}_{\mu,\varphi}^{q}(E)$ and $F \subseteq E$. Consider next a sequence $(\delta_n)_n \subseteq]0,1[$ to be $\downarrow 0$, and satisfying

$$t > \frac{\log(\mathcal{T}_{\mu,\varphi,\delta_n}^{q}(E))}{-\varphi(\delta_n)}, \quad \forall n \in \mathbb{N}.$$

This means that for each $n \in \mathbb{N}$, there exists a centered δ_n-covering $\big(B(x_{ni}, \delta_n)\big)_i$ of E such that

$$\sum_{i}\big(\mu(B(x_{ni}, \delta_n))\big)^q < e^{-t\varphi(\delta_n)}.$$

There balls may be considered to be intersecting the set F. Next, for each i, choose an element $y_i \in B(x_{ni}, \delta_n) \cap F$. This results on a centered $2\delta_n$-covering $\big(B(y_i, 2\delta_n)\big)_i$ of F. Therefore,

$$\overline{\mathcal{H}}_{\mu,\varphi,2\delta_n}^{q,t}(F) \leq \sum_{i}\big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(2\delta_n)}$$
$$= C_t \sum_{i}\left(\frac{\mu(B(y_i, 2\delta_n))}{\mu(B(x_{ni}, \delta_n))}\right)^q \big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(\delta_n)}$$
$$\leq C_t \sum_{i}\big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(\delta_n)}$$
$$\leq C_t C_t'.$$

Hence,

$$\overline{\mathcal{H}}_{\mu,\varphi}^{q,t}(F) \leq C \quad \forall F \subseteq E, \ t > \underline{L}_{\mu,\varphi}^{q}(E).$$

So that,
$$\mathcal{H}_{\mu,\varphi}^{q,t}(E) \leq C < \infty, \quad \forall\, t > \underline{L}_{\mu,\varphi}^{q}(E).$$

Consequently,
$$b_{\mu,\varphi}(q,E) \leq t, \quad \forall\, t > \underline{L}_{\mu,\varphi}^{q}(E) \;\Rightarrow\; b_{\mu,\varphi}(q,E) \leq \underline{L}_{\mu,\varphi}^{q}(E).$$

The remaining part can be proved by following similar techniques.

Next we need to introduce the following quantities which will be useful later. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector valued measure composed of probability measures on $\mathbb{R}^d$. For $j = 1, 2, \ldots, k$, $a > 1$ and $E \subseteq \operatorname{supp}(\mu)$, denote

$$T_a^j(E) = \limsup_{r\downarrow 0}\left(\sup_{x\in E}\frac{\mu_j\big(B(x,ar)\big)}{\mu_j\big(B(x,r)\big)}\right)$$

and for $x \in \operatorname{supp}(\mu)$, $T_a^j(x) = T_a^j(\{x\})$. Denote also

$$P_0(\mathbb{R}^d, E) = \{\,\mu\,;\ \exists\, a > 1\,;\ \forall\, x \in E,\ T_a^j(x) < \infty,\ \forall\, j\,\},$$

$$P_1(\mathbb{R}^d, E) = \{\,\mu\,;\ \exists\, a > 1\,;\ T_a^j(E) < \infty,\ \forall\, j\,\},$$
$$P_0(\mathbb{R}^d) = P_0(\mathbb{R}^d, \operatorname{supp}(\mu)) \qquad \text{and} \qquad P_1(\mathbb{R}^d) = P_1(\mathbb{R}^d, \operatorname{supp}(\mu)).$$

Theorem 3.2 *(1) For $\mu \in P_0(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^{*k}$, there holds that*

$$b_{\mu,\varphi}(q,E) \leq \overline{L}_{\mu,\varphi}^{q}(E).$$

(2) For $\mu \in P_1(\mathbb{R}^d)$ and $q \in \mathbb{R}_+^{*k}$, there holds that
i. $\underline{L}_{\mu,\varphi}^{q}(E) = \underline{C}_{\mu,\varphi}^{q}(E)$.
ii. $\overline{L}_{\mu,\varphi}(q,E) = \overline{C}_{\mu,\varphi}^{q}(E) = \Lambda_{\mu,\varphi}(q,E)$.

Proof. 1. The vector valued measure $\mu \in P_0(\mathbb{R}^d)$ yields that

$$E = \bigcup_{m\in\mathbb{N}} E_m$$

where
$$E_m = \{\, x \in E\,;\ \frac{\mu_j(B(x_i,4r))}{\mu_j(B(x_i,r))} < m,\ 0 < r < \frac{1}{m},\ \forall\, j\,\}.$$

Next, remark that for $t > \overline{L}_{\mu,\varphi}^{q}(E)$ and $F \subseteq E_m$, there exists a sequence $(\delta_n)_n \downarrow 0$ for which

$$t < \frac{\log(\mathcal{T}_{\mu,\delta_n}^{q}(F))}{-\varphi(\delta_n)}, \qquad \forall\, n \in \mathbb{N}.$$

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Therefore, there exists a centered δ_n-covering $(B(x_{ni}, \delta_n))_i$ of F satisfying

$$\sum_i \big(\mu(B(x_{ni}, \delta_n))\big)^q < e^{-t\varphi(\delta_n)}.$$

Let next $y_{ni} \in B(x_{ni}, \delta_n)$. Then, $(B(x_{ni}, 2\delta_n))_i$ is a centered $2\delta_n$-covering of F. Hence,

$$\begin{aligned}
\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,2\delta_n}(F) &\le \sum_i \big(\mu(B(y_{ni}, 2\delta_n))\big)^q e^{t\varphi(2\delta_n)} \\
&\le C_t \sum_i \left(\frac{\mu(B(y_{ni}, 2\delta_n))}{\mu(B(x_{ni}, \delta_n))}\right)^q \big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(\delta_n)} \\
&\le C_t \sum_i \left(\frac{\mu(B(x_{ni}, 4\delta_n))}{\mu(B(x_{ni}, \delta_n))}\right)^q \big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(\delta_n)} \\
&\le C_t m^{|q|} \sum_i \big(\mu(B(x_{ni}, \delta_n))\big)^q e^{t\varphi(\delta_n)} \\
&\le C_t m^{|q|} C'_t
\end{aligned}$$

where $|q| = q_1 + q_2 + \cdots + q_k$. Thus,

$$\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(F) \le C_t C'_t m^{|q|}, \quad \forall m, \text{ and } F \subseteq E_m.$$

Which means that

$$\mathcal{H}^{q,t}_{\mu,\varphi}(E_m) \le C_t C'_t m^{|q|} < \infty, \quad \forall m, \text{ and } t > \underline{L}^q_{\mu,\varphi}(E).$$

Consequently,

$$b_{\mu,\varphi}(q, E_m) \le t, \quad \forall m \text{ and } t > \underline{L}^q_{\mu,\varphi}(E).$$

Using the σ-stability of $b_{\mu,\varphi}(q, .)$ (See Proposition 2.2. c.), it results that

$$b_{\mu,\varphi}(q, E) \le t, \quad \forall t > \underline{L}^q_{\mu,\varphi}(E).$$

As a result,

$$b_{\mu,\varphi}(q, E) \le \underline{L}^q_{\mu,\varphi}(E).$$

Assertion **2.** is left to the reader.

We now re-introduce the mixed multifractal generalization of the L^q-dimensions called also Renyi dimensions based on integral representations. See [13] for more details and other results. For $q \in \mathbb{R}^{*,k}$, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ and $\delta > 0$, we set

$$I^q_{\mu,\delta} = \int_{S_\mu} \big(\mu(B(t, \delta))\big)^q d\mu(t),$$

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where, in this case,

$$S_\mu = \mathrm{supp}(\mu_1) \times \mathrm{supp}(\mu_2) \times \ldots \times \mathrm{supp}(\mu_k),$$

$$\Big(\mu(B(t,\delta))\Big)^q = \Big(\mu_1(B(t_1,\delta))\Big)^{q_1} \Big(\mu_2(B(t_2,\delta))\Big)^{q_2} \ldots \Big(\mu_k(B(t_k,\delta))\Big)^{q_k}$$

and

$$d\mu(t) = d\mu_1(t_1)\,d\mu_2(t_2)\,\ldots\,d\mu_k(t_k).$$

The (μ,φ)-mixed multifractal generalizations of the Renyi dimensions are

$$\overline{T}^q_{\mu,\varphi} = \limsup_{\delta\downarrow 0} \frac{\log I^q_{\mu,\delta}}{-\varphi(\delta)}, \quad \text{and} \quad \underline{T}^q_{\mu,\varphi} = \liminf_{\delta\downarrow 0} \frac{\log I^q_{\mu,\delta}}{-\varphi(\delta)}.$$

We now propose to relate these dimensions to the quantities $\underline{C}^q_{\mu,\varphi}$, $\overline{C}^q_{\mu,\varphi}$, $\underline{L}^q_{\mu,\varphi}$, $\overline{L}^q_{\mu,\varphi}$ introduced previously.

Proposition 3.1 *The following results hold.*

a. $\forall\, q \in \mathbb{R}^{*,k}_-$,

$$\underline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) \geq \underline{T}^q_{\mu,\varphi} \quad and \quad \overline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) \geq \overline{T}^q_{\mu,\varphi}.$$

b. $\forall\, q \in \mathbb{R}^{*,k}_+$,

$$\underline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) \leq \underline{T}^q_{\mu,\varphi} \quad and \quad \overline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) \leq \overline{T}^q_{\mu,\varphi}.$$

c. $\forall\, q \in \mathbb{R}^{*,k}$, $\mu \in P_1(\mathbb{R}^d)$,

$$\underline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) = \underline{T}^q_{\mu,\varphi} \quad and \quad \overline{C}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) = \overline{T}^q_{\mu,\varphi}.$$

d. $\forall\, q \in \mathbb{R}^{*,k}_-$,

$$\underline{T}^q_{\mu,\varphi} \leq \underline{L}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)) \quad and \quad \overline{T}^q_{\mu,\varphi} \leq \overline{L}^{q+\mathbb{I}}_{\mu,\varphi}(supp(\mu)).$$

Proof. We only prove **a.** The remaining proofs of points **b.**, **c.** and **d.** follow the same ideas. For $\delta > 0$, let $\Big(B(x_i,\delta)\Big)_i$ be a centered δ-covering of $\mathrm{supp}(\mu)$ and let next $\Big(B(x_{ij},\delta)\Big)_j$, $1 \leq i \leq \xi$ the ξ sets defined in Besicovitch covering theorem. It holds that

$$\sum_{i,j} \Big(\mu(B(x_{ij},\delta))\Big)^{q+\mathbb{I}} = \sum_{i,j} \Big(\mu(B(x_{ij},\delta))\Big)^q \int_{B(x_{ij},\delta)^k} d\mu(t)$$

$$\geq \sum_{i,j} \int_{B(x_{ij},\delta)^k} \Big(\mu(B(t,2\delta))\Big)^q d\mu(t)$$

$$\geq \int_{S_\mu} \Big(\mu(B(t,2\delta))\Big)^q d\mu(t).$$

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As a results,
$$\xi \mathcal{S}_{\mu,\delta}^{q+\mathbb{I}}(\operatorname{supp}(\mu)) \geq I_{\mu,2\delta}^q.$$
Which implies that

$$\underline{C}_{\mu,\varphi}^{q+\mathbb{I}}(\operatorname{supp}(\mu)) \geq \underline{I}_\mu^q \quad \text{and} \quad \overline{C}_{\mu,\varphi}^{q+\mathbb{I}}(\operatorname{supp}(\mu)) \geq \overline{T}_\mu^q.$$

4   A φ-mixed multifractal formalism for vector valued measures

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be a vector valued probability measure on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $j = 1, 2, \ldots, k$, we denote

$$\underline{\alpha}_{\mu_j}^\varphi(x) = \liminf_{r\downarrow 0} \frac{\log(\mu_j(B(x,r)))}{\varphi(r)} \quad \text{and} \quad \overline{\alpha}_{\mu_j}^\varphi(x) = \limsup_{r\downarrow 0} \frac{\log(\mu_j(B(x,r)))}{\varphi(r)}$$

respectively the local lower φ-dimension and the local upper φ-dimension of μ_j at the point x and as usually the local dimension $\alpha_{\mu_j}^\varphi(x)$ of μ_j at x will be the common value when these are equal. Next for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}_+^k$, let

$$\underline{X}_\alpha(\varphi) = \{\, x \in \operatorname{supp}(\mu)\,;\, \underline{\alpha}_{\mu_j}^\varphi(x) \geq \alpha_j\,, \forall j = 1, 2, \ldots, k\,\},$$
$$\overline{X}^\alpha(\varphi) = \{\, x \in \operatorname{supp}(\mu)\,;\, \overline{\alpha}_{\mu_j}^\varphi(x) \leq \alpha_j\,, \forall j = 1, 2, \ldots, k\,\}$$

and

$$X(\alpha, \varphi) = \underline{X}_\alpha(\varphi) \cap \overline{X}^\alpha(\varphi).$$

The (μ, φ)-mixed multifractal spectrum of the vector valued measure μ is defined by

$$\alpha \longmapsto \dim X(\alpha, \varphi)$$

where dim stands for the Hausdorff dimension.

In this section, we propose to compute such a spectrum for some cases of measures that resemble to the situation raised by Olsen in [11] but in the mixed case. This will permit to describe better the simultaneous behavior of finitely many measures. We intend precisely to compute the mixed spectrum based on the mixed multifractal generalizations of the Haudorff and packing dimensions b_μ, B_μ and Λ_μ. We start with the following technic results.

Lemma 4.1 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be such that*

$$\varphi \text{ is non-decreasing and, } \varphi(r) = o(\log r) \text{ as } r \to 0. \qquad (3)$$

The following assertions hold.

1. $\forall \delta > 0$, $t \in \mathbb{R}$ *and* $q \in \mathbb{R}_+^k$, $\alpha \in \mathbb{R}^k$ *such that* $\langle \alpha, q \rangle + t \geq 0$, *we have*
i. $\mathcal{H}^{\langle \alpha,q \rangle + t + k\delta}(\overline{X}^\alpha) \leq C\mathcal{H}_{\mu,\varphi}^{q,t}(\overline{X}^\alpha).$
ii. $\mathcal{P}^{\langle \alpha,q \rangle + t + k\delta}(\overline{X}^\alpha) \leq C\mathcal{P}_{\mu,\varphi}^{q,t}(\overline{X}^\alpha).$

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2. $\forall \delta > 0$, $t \in \mathbb{R}$ and $q \in \mathbb{R}^k_-$, $\alpha \in \mathbb{R}^k$ such that $\langle \alpha, q \rangle + t \geq 0$, we have

i. $\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}(\underline{X}_\alpha) \leq C\mathcal{H}^{q,t}_{\mu,\varphi}(\underline{X}_\alpha)$.

ii. $\mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}(\underline{X}_\alpha) \leq C\mathcal{P}^{q,t}_{\mu,\varphi}(\underline{X}_\alpha)$.

where $C = C(\alpha, q, k, \delta) > 0$ is a generic constant.

Proof. **1. i**. We prove the first part. For $m \in \mathbb{N}^*$, consider the set

$$\overline{X}^\alpha_m = \left\{ x \in \overline{X}^\alpha; \; \frac{\log(\mu_j(B(x, r)))}{\varphi(r)} \leq \alpha_j + \frac{\delta}{q_j}; \; 0 < r < \frac{1}{m}, \; 1 \leq j \leq k \right\}.$$

Let next $0 < \eta < \dfrac{1}{m}$ and $(B(x_i, r_i))_i$ a centered η-covering of $\overline{X}^\alpha_m$. It holds that

$$(\mu(B(x_i, r_i)))^q \geq e^{(\langle \alpha, q \rangle + k\delta)\varphi(r)}.$$

Consequently, it holds from (3) that

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}_\eta(\overline{X}^\alpha_m) \leq \sum_i (2r_i)^{\langle \alpha, q \rangle + t + k\delta} \leq C \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r)}.$$

Hence, $\forall \eta > 0$, there holds that

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}_\eta(\overline{X}^\alpha_m) \leq C\overline{\mathcal{H}}^{q,t}_{\mu,\varphi,\eta}(\overline{X}^\alpha_m).$$

Which means that

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}^\alpha_m) \leq C\overline{\mathcal{H}}^{q,t}_{\mu,\varphi}(\overline{X}^\alpha_m) \leq C\mathcal{H}^{q,t}_{\mu,\varphi}(\overline{X}^\alpha_m).$$

Next, observing that $\overline{X}^\alpha = \bigcup_m \overline{X}^\alpha_m$, we obtain

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}^\alpha) \leq C\mathcal{H}^{q,t}_{\mu,\varphi}(\overline{X}^\alpha).$$

ii. For $q \in \mathbb{R}^{*,k}_+$ and $m \in \mathbb{N}^*$, consider the set $\overline{X}^\alpha_m$ defined previously and let $E \subseteq \overline{X}^\alpha_m$, $0 < \eta < \dfrac{1}{m}$ and $\left(B(x_i, r_i)\right)_i$ a centered η-packing of E. We have

$$\sum_i (2r_i)^{\langle \alpha, q \rangle + t + k\delta} \leq C \sum_i (\mu(B(x_i, r_i)))^q e^{t\varphi(r)} \leq C\overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\eta}(E).$$

Consequently, $\forall \eta > 0$,

$$\overline{\mathcal{P}}^{\langle \alpha, q \rangle + t + k\delta}_\eta(E) \leq C\overline{\mathcal{P}}^{q,t}_{\mu,\varphi,\eta}(E).$$

Hence, $\forall E \subseteq \overline{X}^\alpha_m$,

$$\overline{\mathcal{P}}^{\langle \alpha, q \rangle + t + k\delta}(E) \leq C\overline{\mathcal{P}}^{q,t}_{\mu,\varphi}(E).$$

Let next, $(E_i)_i$ be a covering of $\overline{X}_m^\alpha$. Thus,

$$\mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}_m^\alpha) = \mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}\left(\bigcup_i (\overline{X}_m^\alpha \cap E_i)\right)$$

$$= \sum_i \mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}\left(\overline{X}_m^\alpha \cap E_i\right)$$

$$\leq \sum_i \overline{\mathcal{P}}^{\langle \alpha, q \rangle + t + k\delta}\left(\overline{X}_m^\alpha \cap E_i\right)$$

$$\leq C \sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}\left(\overline{X}_m^\alpha \cap E_i\right)$$

$$\leq C \sum_i \overline{\mathcal{P}}_{\mu,\varphi}^{q,t}(E_i).$$

Hence, $\forall, m$,
$$\mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}_m^\alpha) \leq C\mathcal{P}_{\mu,\varphi}^{q,t}(\overline{X}_m^\alpha).$$

Consequently,
$$\mathcal{P}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}^\alpha) \leq C\mathcal{P}_{\mu,\varphi}^{q,t}(\overline{X}^\alpha).$$

2. i. and **ii**. follow similar arguments and techniques as previously.

Proposition 4.1 *Let* $\alpha \in \mathbb{R}_+^k$, $q \in \mathbb{R}^k$ *and* φ *satisfying (3). The following assertions hold.*

a. *Whenever* $\langle \alpha, q \rangle + b_{\mu,\varphi}(q) \geq 0$, *we have*
 i. $dim\overline{X}^\alpha \leq \langle \alpha, q \rangle + b_{\mu,\varphi}(q), \quad \forall q\mathbb{R}_+^k$.
 ii. $dim\underline{X}_\alpha \leq \langle \alpha, q \rangle + b_{\mu,\varphi}(q), \quad \forall q\mathbb{R}_-^k$.
b. *Whenever* $\langle \alpha, q \rangle + B_{\mu,\varphi}(q) \geq 0$, *we have*
 i. $Dim\overline{X}^\alpha \leq \langle \alpha, q \rangle + B_{\mu,\varphi}(q), \quad \forall q\mathbb{R}_+^k$.
 ii. $Dim\underline{X}_\alpha \leq \langle \alpha, q \rangle + B_{\mu,\varphi}(q), \quad \forall q\mathbb{R}_-^k$.

Proof. a. i. It follows from Lemma 4.1, assertion **1. i.**,

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}(\overline{X}^\alpha) = 0, \qquad \forall t > b_{\mu,\varphi}(q), \ \delta > 0.$$

Consequently,

$$\dim\overline{X}^\alpha \leq \langle \alpha, q \rangle + t + k\delta, \qquad \forall t > b_{\mu,\varphi}(q), \ \delta > 0.$$

Hence,
$$\dim \overline{X}^\alpha \leq \langle \alpha, q \rangle + b_{\mu,\varphi}(q).$$

a. ii. It follows from Lemma 4.1, assertion **2. i.**, as previously, that

$$\mathcal{H}^{\langle \alpha, q \rangle + t + k\delta}(\underline{X}^\alpha) = 0, \qquad \forall t > b_{\mu,\varphi}(q), \ \delta > 0.$$

Hence,
$$\dim\underline{X}_\alpha \leq \langle \alpha, q \rangle + t + k\delta, \qquad \forall t > b_{\mu,\varphi}(q), \ \delta > 0$$

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and finally,
$$\dim \underline{X}_\alpha \leq \langle \alpha, q \rangle + b_{\mu,\varphi}(q).$$
b. i. observing Lemma 4.1, assertion **1. ii.**, we obtain
$$\mathcal{P}^{\langle \alpha,q \rangle + t + k\delta}(\overline{X}^\alpha), \qquad \forall\, t > B_{\mu,\varphi}(q), \;\; \delta > 0.$$

Consequently,
$$\mathrm{Dim}\, \overline{X}^\alpha \leq \langle \alpha, q \rangle + t + k\delta, \qquad \forall\, t > B_{\mu,\varphi}(q), \;\; \delta > 0.$$

Hence,
$$\mathrm{Dim}\, \overline{X}^\alpha \leq \langle \alpha, q \rangle + B_{\mu,\varphi}(q).$$
b. ii. observing Lemma 4.1, assertion **2. ii.**, we obtain
$$\mathcal{P}^{\langle \alpha,q \rangle + t + k\delta}(\underline{X}_\alpha) = 0, \qquad \forall\, t > B_{\mu,\varphi}(q), \;\; \delta > 0.$$

Hence,
$$\mathrm{Dim}\, \underline{X}_\alpha \leq \langle \alpha, q \rangle + t + k\delta, \qquad \forall\, t > B_{\mu,\varphi}(q), \;\; \delta > 0$$
and finally,
$$\mathrm{Dim}\, \underline{X}_\alpha \leq \langle \alpha, q \rangle + B_{\mu,\varphi}(q).$$

Lemma 4.2 $\forall\, q \in \mathbb{R}^k$ such that $\langle \alpha, q \rangle + b_{\mu,\varphi}(q) < 0$ or $\langle \alpha, q \rangle + B_{\mu,\varphi}(q) < 0$, we have $X(\alpha) = \emptyset$.

Proof. It is based on

Claim 1. For $q \in \mathbb{R}^k_-$ with $\langle \alpha, q \rangle + b_{\mu,\varphi}(q) < 0$ or $\langle \alpha, q \rangle + B_{\mu,\varphi}(q) < 0$, $\underline{X}_\alpha = \emptyset$.
Claim 2. For $q \in \mathbb{R}^k_+$ with $\langle \alpha, q \rangle + b_{\mu,\varphi}(q) < 0$ or $\langle \alpha, q \rangle + B_{\mu,\varphi}(q) < 0$, $\overline{X}^\alpha = \emptyset$.

Indeed, let $q \in \mathbb{R}^k_-$ and assume that $\underline{X}_\alpha \neq \emptyset$. This means that there exists at least one point $x \in \mathrm{supp}(\mu)$ for which $\underline{\alpha}_{\mu_j}(x) \geq \alpha_j$, for $1 \leq j \leq k$. Consequently, for all $\varepsilon > 0$, there is a sequence $(r_n)_n \downarrow 0$ and satisfying
$$0 < r_n < \frac{1}{n} \quad \text{and} \quad \mu_j(B(x,r_n)) < e^{(\alpha_j - \varepsilon)\varphi(r_n)}, \;\; 1 \leq j \leq k.$$

Hence,
$$\Big(\mu(B(x,r_n)) \Big)^q e^{t\varphi(r_n)} > C e^{(\langle (\alpha - \varepsilon \mathbb{I}), q \rangle + t)\varphi(r_n)}.$$

Choosing $t = \langle (\varepsilon \mathbb{I} - \alpha), q \rangle$, this induces that $\mathcal{H}^{q,t}_{\mu,\varphi}(\{x\}) > C > 0$ and consequently,
$$b_{\mu,\varphi}(q) \geq \dim^q_{\mu,\varphi}(\{x\}) \geq t, \quad \forall\, \varepsilon > 0.$$
Letting $\varepsilon \downarrow 0$, it results that $b_{\mu,\varphi}(q) \geq -\langle \alpha, q \rangle$ which is impossible. So as the first part of Claim 1. The remaining part as well as Claim 2 can be checked by similar techniques.

Theorem 4.1 *Let* $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ *be a vector-valued Borel probability measure on* $\mathbb{R}^d$ *and* $q \in \mathbb{R}^k$ *fixed. Let further* $t \in \mathbb{R}$, $\underline{K}, \overline{K} > 0$, ν *a Borel probability measure supported by* $supp(\mu)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}$ *satisfying* (3). *Let finally* $(r_n)_n \subset]0, 1[\downarrow 0$ *and satisfying*

$$\frac{\varphi(r_{n+1})}{\varphi(r_n)} \to 1 \quad and \quad \sum_n e^{\varepsilon \varphi(r_n)} < \infty, \ \forall \varepsilon > 0.$$

Assume next the following assumptions.

A1. $\forall x \in supp(\mu)$ *and* r *small enough,*

$$\underline{K} \leq \frac{\nu(B(x,r))}{\Big(\mu(B(x,r))\Big)^q e^{t\varphi(r)}} \leq \overline{K}.$$

A.2 $C(p) = \lim\limits_{n \to +\infty} C_n(p)$ *exists and finite for all* $p \in \mathbb{R}$, *where*

$$C_n(p) = \frac{-1}{\varphi(r_n)} \log\left(\int_{supp(\mu)} \Big(\mu(B(x,r_n))\Big)^p d\nu(x) \right).$$

Denote next $\alpha_-^0 = -\nabla_- C(0)$, $\alpha_+^0 = -\nabla_+ C(0)$ *and* $\Psi_q(a,b) = aq + b$, $\forall a, b$. *The following assertions hold.*

i. *For* $q \in \mathbb{R}_-^k$, *we have*

$$dim(\underline{X}_{\alpha_+^0} \cap \overline{X}^{\alpha_-^0}) \geq \Psi_q(\alpha_-^0, \Lambda_\mu(q)) \geq \Psi_q(\alpha_-^0, B_\mu(q)) \geq \Psi_q(\alpha_-^0, b_\mu(q)).$$

For $q \in \mathbb{R}_+^k$,

$$dim(\underline{X}_{\alpha_+^0} \cap \overline{X}^{\alpha_-^0}) \geq \Psi_q(\alpha_+^0, \Lambda_\mu(q)) \geq \Psi_q(\alpha_+^0, B_\mu(q)) \geq \Psi_q(\alpha_+^0, b_\mu(q)).$$

ii. *Whenever* C *is differentiable at 0, we have*

$$f_\mu(-\nabla C(0)) = b_\mu^*(-\nabla C(0)) = B_\mu^*(-\nabla C(0)) = \Lambda_\mu^*(-\nabla C(0)).$$

The proof of this result is based on the application of a large deviation formalism. This will permit to obtain a measure ν supported by $\underline{X}_{-\nabla_+ C(0)} \cap \overline{X}^{-\nabla_- C(0)}$. To do this, we re-formulate a mixed large deviation formalism to be adapted to the mixed multifractal formalism raised in our work.

Proof of Theorem 4.1. For $x \in supp(\mu)$, let

$$\underline{\alpha}_{\mu_j}^\varphi(x, r_n) = \liminf_n \frac{\log\Big[\mu_j(B(x,r_n))\Big]}{\varphi(r_n)}$$

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and

$$\overline{\alpha}_{\mu_j}^{\varphi}(x, r_n) = \limsup_{n} \frac{\log\left[\mu_j(B(x, r_n))\right]}{\varphi(r_n)}.$$

i. Using the hypothesis **A1.** and Lemma 2.6 we obtain

$$b_{\mu,\varphi}(q) = B_{\mu,\varphi}(q) = \Lambda_{\mu,\varphi}(q) = t.$$

Next, it is straightforward that the set

$$M = \left\{ x \in supp(\mu) \; ; \; -\nabla_+ C(0) \leq \underline{\alpha}_{\mu,\varphi}(x, r_n) \leq \overline{\alpha}_{\mu,\varphi}(x, r_n) \leq -\nabla_- C(0) \right\}$$

coincides with $\underline{X}_{-\nabla_+ C(0)} \cap \overline{X}^{-\nabla_- C(0)}$. Hence, by setting in the mixed large deviation formalism 5.2,

$$\Omega = supp(\mu), \quad \mathcal{A} = \mathcal{B}(supp(\mu)),$$

$$I\!P = \mu, \quad W_n(x) = \log(\mu(B(x, r_n)))$$

and

$$a_n = -\varphi(r_n),$$

it holds that

$$\underline{\alpha}_{\mu,\varphi}(x) \geq \begin{cases} -\nabla_- C(0)q + t \text{ for } q \leq 0 \\ -\nabla_+ C(0)q + t \text{ for } q \geq 0. \end{cases}$$

Finally, applying the famous Billingsley's Theorem [8], we obtain

$$\dim M \geq \begin{cases} -\nabla_- C(0)q + t \text{ for } q \leq 0 \\ -\nabla_+ C(0)q + t \text{ for } q \geq 0. \end{cases}$$

ii. Remark that if C is differentiable at 0, item **i.** states that

$$\dim M \geq -\nabla C(0)q + t \geq \Lambda_\mu^*(-\nabla C(0)) \geq B_\mu^*(-\nabla C(0)) \geq b_\mu^*(-\nabla C(0)).$$

In the other hand, since the set M is not empty, Lemma 4.2 implies that

$$-\nabla C(0)q + t \geq 0.$$

Hence, Proposition 4.1 yields that

$$\dim M \leq -\nabla C(0)q + t$$

for any $q \in \mathbb{R}^k$. Thus, taking the inf on q, we obtain

$$\dim M \leq b_\mu^*(-\nabla C(0)) \leq B_\mu^*(-\nabla C(0)) \leq \Lambda_\mu^*(-\nabla C(0)).$$

iii. We firstly claim that, there exists $\beta > 0$ such that, for all $x \in supp(\mu)$ and $0 < r <<< 1$, we have

$$\frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \beta.$$

So let $(B(x_{ij}, r_n))_{1 \le i\xi, j}$ the ξ sets relatively to Besicovitch theorem extracted from the set $(B(x_i, r_n))_i$. A careful computation yields that

$$|p + q - \mathbb{I}|I_\mu^{p+q-\mathbb{I}} = C_q(p) + t_q; \quad \forall p, q \in \mathbb{R}^k \tag{4}$$

where $|p+q-\mathbb{I}| = \sum_{i=1}^{k}(p_i+q_i-1)$. Theorem 3.1 and Proposition 3.1 guarantees that

$$|p + q - \mathbb{I}|I_\mu^{p+q-1} = C_\mu^{p+q}(supp(\mu)) = \Lambda_\mu(p + q).$$

Consequently,

$$C_q(p) = \Lambda_\mu(p + q) - \Lambda_\mu(p).$$

So, if Λ_μ is differentiable at q, C_q will be too at 0 and $\nabla C_q(0) = \nabla\Lambda_\mu(q)$. Thus, using the mixed large deviation formalism, we obtain

$$\alpha_\mu(x) = -\nabla C_q(0) \; ; \; \nu_q \text{ for almost all } x \in supp(\mu).$$

hence, finally, $\alpha_\mu(x) = -\nabla\Lambda_\mu(q)$.
iv. Let q be such that $\nabla\Lambda_\mu(q)$ exists. Then $\nabla C_q(0)$ exists too. So, item **ii.** states that

$$f_\mu(-\nabla C(0)) = \Lambda_\mu^*(-\nabla C(0)).$$

Which completes the proof.

Theorem 4.2 *Assume that the hypotheses of Theorem 4.1 are satisfied for all $q \in \mathbb{R}^k$. Then, the following assertions hold.*

i. $\alpha_\mu = -B_\mu, \quad \nu_q$ *a.s, whenever B_μ is differentiable at q.*
ii. $Dom(B) \subseteq \alpha_\mu(supp(\mu))$ *and* $f_\mu = B_\mu^*$ *on* $Dom(B)$.

Proof.
i. Using the same notations as in Theorem 4.1, we obtain C differentiable at 0, $B_{\mu,\varphi}$ differentiable at q, and $\nabla C(0) = \nabla B_{\mu,\varphi}(q)$. In the other hand, we obtain also

$$\alpha_\mu^\varphi(x) = \alpha_\mu^\varphi(x, r_n) = \lim_n \frac{W_n(x)}{-a_n} = -\nabla C(0) = \nabla B_\mu^\varphi(q), \quad \nu \text{ a.s.}$$

ii. follows immediately from **i.** and Theorem 4.1.

5 Appendix

5.1 Besicovitch covering theorem

Theorem 5.1 *There exists a constant $\xi \in \mathbb{N}$ satisfying: For any $E \in \mathbb{R}^d$ and $(r_x)_{x \in E}$ a bounded set of positive real numbers, there exists ξ sets B_1, B_2, ..., B_ξ, that are finite or countable composed of balls $B(x, r_x)$, $x \in E$ such that*

- $E \subseteq \bigcup_{1 \leq i \leq \xi} \bigcup_{B \in B_i} B.$
- *each B_i is composed of disjoint balls.*

5.2 A mixed large deviation theorem

To do this, we re-formulate a mixed large deviation formalism to be adapted to the mixed multifractal formalism raised in our work.

Theorem 5.2 *Consider a sequence $(W_n = (W_{n,1}, W_{n,2}, \ldots, W_{n,k}))_n$ of vector-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(a_n)_n \subset]0, +\infty[$ with $\lim_{n \to +\infty} a_n = +\infty$. Let next the function*

$$C_n : \mathbb{R}^k \to \overline{\mathbb{R}}$$
$$t \mapsto C_n(t) = \frac{1}{a_n} \log \Big(E(\exp(\langle t, W_n \rangle)) \Big).$$

Assume that

A1. *$C_n(t)$ is finite for all n and t.*
A2. *$C(t) = \lim_{n \to +\infty} C_n(t)$ exists and is finite for all t.*

There holds that

i. *The function C is convex.*
ii. *If $\nabla_- C(t) \leq \nabla_+ C(t) < \alpha$, for some $t \in \mathbb{R}^k$, then*

$$\limsup_{n \to +\infty} \frac{1}{a_n} \log \left(e^{-a_n C(t)} E \left(\exp(\langle t, W_n \rangle) 1_{\{\frac{W_n}{a_n} \geq \alpha\}} \right) \right) < 0.$$

iii. *If $\sum_n e^{-\varepsilon a_n} < \infty$ for all $\varepsilon > 0$, then*

$$\limsup_{n \to +\infty} \frac{W_n}{a_n} \leq \nabla_+ C(0) \qquad \mathbb{P} \ a.s.$$

iv. If $\alpha < \nabla_- C(t) \leq \nabla_+ C(t)$, for some $t \in \mathbb{R}^k$, then

$$\limsup_{n \to +\infty} \frac{1}{a_n} \log \left(e^{-a_n C(t)} E \left(\exp(\langle t, W_n \rangle) 1_{\{\frac{W_n}{a_n} \leq \alpha\}} \right) \right) < 0.$$

v. If $\sum_n e^{-\varepsilon a_n}$ is finite for all $\varepsilon > 0$, then

$$\nabla_- C(0) \leq \limsup_{n \to +\infty} \frac{W_n}{a_n} \qquad \mathbb{P} \ a.s.$$

Proof.

i. It follows from Holder's inequality.

ii. Let $h \in \mathbb{R}_+^{*,k}$ be such that $C(t) + \langle \alpha, h \rangle - C(t + h) > 0$. We have

$$\frac{1}{a_n} \log \left[e^{-a_n C(t)} \mathbb{E} \left(\exp(\langle t, W_n \rangle) 1_{\{\frac{W_n}{a_n} \geq \alpha\}} \right) \right]$$

$$= \frac{1}{a_n} \log \left[e^{-a_n C(t)} \int_{\{\frac{W_n}{a_n} \geq \alpha\}} e^{\langle t, W_n \rangle} d\mathbb{P} \right]$$

$$= \frac{1}{a_n} \log \left[e^{-a_n (C(t) + \langle \alpha, h \rangle)} \int_{\{\frac{W_n}{a_n} \geq \alpha\}} e^{\langle t, W_n \rangle + a_n \langle \alpha, h \rangle} d\mathbb{P} \right]$$

$$\leq \frac{1}{a_n} \log \left[e^{-a_n (C(t) + \langle \alpha, h \rangle)} \int_{\{\frac{W_n}{a_n} \geq \alpha\}} e^{\langle t+h, W_n \rangle} d\mathbb{P} \right]$$

$$\leq \frac{1}{a_n} \log \left[e^{-a_n (C(t) + \langle \alpha, h \rangle)} \mathbb{E}(\exp(\langle t + h, W_n \rangle)) \right]$$

$$= \frac{1}{a_n} \log \left[e^{-a_n (C(t) + \langle \alpha, h \rangle - C_n(t+h))} \right]$$

$$= -(C(t) + \langle \alpha, h \rangle - C_n(t + h)).$$

Next, by taking the limsup as $n \longrightarrow +\infty$, the result follows immediately.

iii. Denote for $n, m \in \mathbb{N}$,

$$T_{n,m} = \{ \frac{W_n}{a_n} \geq \nabla_+ C(0) + \frac{1}{m} \}.$$

By choosing in item **ii.** $t = 0$ and $\alpha = \nabla_+ C(0) + \frac{1}{m}$, and observing that $C(0) = 0$, we obtain

$$\limsup_{n \to +\infty} \frac{1}{a_n} \log \left(E \left(1_{\{\frac{W_n}{a_n} \geq \nabla_+ C(0) + \frac{1}{m}\}} \right) \right) < 0.$$

which means that

$$\limsup_n \frac{1}{a_n} \log \mathbb{P}(T_{n,m}) < 0.$$

Consequently, for some $\varepsilon > 0$ and n large enough, there holds that

$$\limsup_n \frac{1}{a_n} \log \mathbb{P}(T_{n,m}) < -\varepsilon.$$

Thus,

$$\mathbb{P}(T_{n,m}) < e^{-\varepsilon a_n}$$

which implies the convergence of the series $\sum_n \mathbb{P}(T_{n,m})$. Hence, using Borel-Cantelli theorem, we obtain

$$\mathbb{P}(\limsup_n T_{n,m}) = 0, \ \forall m.$$

Therefore,

$$\mathbb{P}\left(\limsup_n \frac{W_n}{a_n} > \nabla_+ C(0) \right) = \mathbb{P}(\bigcup_m \limsup_n T_{n,m}) = 0$$

and finally,

$$\limsup_n \frac{W_n}{a_n} \leq \nabla_+ C(0), \ \mathbb{P} \text{ a.s.}$$

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References

[1] A. Ben Mabrouk, A note on Hausdorff and packing measures, Interna. J. Math. Sci., 8(3-4) (2009), 135-142.

[2] A. Ben Mabrouk, A higher order multifractal formalism, Stat. Prob. Lett. 78 (2008), 1412-1421.

[3] A. Ben Mabrouk, M. Ben Slimane and J. Aouidi, A wavelet multifractal fromalism for simultaneous singularities of functions. International Journal of Wavelets, Multiresolution and Information Processing, 12(1) (2014), 14 pages.

[4] A. Ben Mabrouk, Mohamed Menceur and Kamel Betina, The Multifractal Formalism For Measures, Review and Extension to Mixed Cases. Analysis in Theory and Application, 32 (1) (2016), 77-106.

[5] A. Ben Mabrouk, Mourad Ben Slimane and Jamil Aouidi, Mixed multifractal analysis for functions: general upper bound and optimal results for vectors of self-similar or quasi-self-similar of functions and their superpositions. Fractals, 24(4) (2016), 12 pages.

[6] F. Ben Nasr, I. Bhouri and Y. Heurteaux, The validity of the multifractal formalism: results and examples, Adv. Math. 165 (2002), 264-284.

[7] I. Bhouri, On the projections of generalized upper L^q-spectrum, Chaos, Solitons and Fractals, 42 (2009), 14511462

[8] P. Billingsley, Ergodic theory and information, J. Wiley & Sons Inc., New York, 1965.

[9] M. Dai and Z. Liu, The Quantization Dimension and Other Dimensions of Probability Measures. International Journal of Nonlinear Science, 5 (2008), 267-274.

[10] M. Menceur, A. Ben Mabrouk and K. Betina, The Multifractal Formalism for Measures, Review and Extension to Mixed Cases, Anal. Theory Appl., 32(1) (2016), 77–106

[11] L. Olsen, A multifractal formalism, Adv. Math., 116 (1995), 82-196.

[12] L. Olsen, Dimension inequalities of multifractal Hausdorff measures and multifractal packing measures, Math. Scand., 86 (2000), 109-129.

[13] L. Olsen, Mixed generalized dimensions of self-similar measures, J. Math. Anal. and Appl., 306 (2005), 516-539.

[14] Y. Pesin, Dimension Theory in Dynamical Systems, University of Chicago Press, 1997.

[15] S. J. Taylor, The fractal analysis of Borel measures in $\mathbb{R}^d$, J. Fourier. Anal. and Appl., Kahane Special Issue, (1995), 553-568.

[16] X. Wang, M. Dai, Mixed Quantization Dimension Function and Temperature Function for Conformal Measures, International Journal of Nonlinear Science, 10(1) (2010), 24-31.

[17] Sh. Xu and W. Xu, Note on the Paper "An Negative Answer to a Conjecture on the Self-similar Sets Satisfying the Open Set Condition", Anal. Theory Appl., 28 (2012), 49-57.

[18] Sh. Xu, W. Xu and D. Zhong, Some New Iterated Function Systems Consisting of Generalized Contractive Mappings, Anal. Theory Appl., 28 (2012), 269-277.

[19] Y.-L. Ye, Self-similar vector-valued measures, Adv. Appl. Math., 38 (2007), 7196.

[20] Y. Yuan, Spectral Self-Affine Measures on the Generalized Three Sierpinski Gasket, Anal. Theory Appl., 31 (2015), 394-406.

[21] Ch. Zeng, D. Yuan and Sh. Xui, The Hausdorff Measure of Sierpinski Carpets Basing on Regular Pentagon, Anal. Theory Appl., 28 (2012), 27-37.

[22] Z. Zhou and L. Feng, A Theoretical Framework for the Calculation of Hausdorff Measure Self-similal Set Satisfying OSC, Anal. Theory Appl., 27 (2011), 387-398.

[23] Z. Zhu and Z. Zhou, A Local Property of Hausdorff Centered Measure of Self-Similar Sets, Anal. Theory Appl., 30 (2014), 164-172.