The Obstacle Problem for Quasilinear Stochastic Integral-Partial Differential Equations

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Abstract: We prove an existence and uniqueness result for the obstacle problem for quasilinear stochastic integral-partial differential equations. Our method is based on the probabilistic interpretation of the solution using backward doubly SDEs with jumps.

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1. Introduction

We consider the following stochastic partial differential equations with obstacles (OSPDEs for short)

$$\begin{cases} du_t(x) + [\frac{1}{2}\mathcal{A}u_t(x) + f_t(x, u_t(x), \nabla u_t(x))]\, dt + h_t(x, u_t(x), \nabla u_t(x)) \cdot \overleftarrow{dB}_t = 0,\ (t,x) \in [0,T] \times R^d; \\ u_t(x) \geq v_t(x), \quad (t,x) \in [0,T] \times R^d; \\ u_T(x) = \Phi(x), \quad x \in R^d. \end{cases} \tag{1}$$

The operator $\mathcal{A}$, which is non-local, is an symmetric infinitesimal generator of a Markov process with jumps. f and $h = \big(h_1, \cdots, h_{d^1}\big)$ are non-linear random functions. The differential term with $\overleftarrow{dB}_t$ refers to the backward stochastic integral with respect to a d^1-dimensional Brownian motion on a probability space $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, so that the doubly stochastic framework introduced by Pardoux and Peng [17] could be applied. Given an obstacle $v : \Omega \times [0,T] \times \mathcal{O} \to \mathbb{R}$, we study the OSPDE (1), i.e. we want to find a solution that satisfies "$u \geq v$" where the obstacle v is regular in some sense.

Nualart and Pardoux [16] have studied the obstacle problem for a nonlinear heat equation on the spatial interval $[0,1]$ driven by a space-time white noise with the diffusion matrix $a = I$. Then Donati-Martin and Pardox [10] proved it for the general diffusion matrix. Various properties of the solutions were studied later in [3], [20], [21] etc.. Denis, Matoussi and Zhang ([9]) study the OSPDE with null Dirichlet condition on an open domain in $\mathbb{R}^d$. Their method is based on the techniques of parabolic potential theory developed by M. Pierre ([18], [19]). The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying some minimal condition. The key point was to construct a solution u which admits a quasi-continuous version defined outside a polar set and the regular measures ν which in general are not absolutely continuous w.r.t. the Lebesgue measure.

As backward stochastic differential equations (BSDEs for short) was rapidly developed, obstacle problem associated with a non-linear partial differential equation (PDE for short) with more general coefficients, and the properties of the solutions for OSPDEs were studied in the framework of BSDEs ([1], [11], etc.). Matoussi and Stoica [15] have proved an existence and uniqueness result for the obstacle problem of backward quasilinear stochastic PDE on the whole space $\mathbb{R}^d$ and driven by a finite dimensional Brownian motion. The method is based on the probabilistic interpretation of the solution by using the backward doubly stochastic differential equation (BDSDE in short). The solution also is expressed as a pair (u, ν). It is essential to give the regular measure ν a probabilistic interpretation in term of the continuous increasing process K where (Y, Z, K) is the solution of a reflected generalized BDSDE.

The stochastic partial differential equations in (1) without the obstacle was studied in [7], which is the main motivation of this article. It inspires us to generalize the obstacle problem in [9] and consider a more general linear integral-differential operator. Reflected BSDE with jumps, which is a standard reflected BSDE driven by a Brownian motion and an independent Poisson point process, has been studied by Hamadène and Ouknine in [13]. After that, parabolic integro-differential partial equations with two obstacles are solved by Harraj, Ouknine and Turpin in [14]. But in the framework of BSDEs, they concern on viscosity solutions for obstacle problems.

Our aim is to study the OSPDE (1) with a non-negative, self-adjoint operator associated with a symmetric Lévy process by using the probabilistic method in [15] and prove the existence and uniqueness of the weak solution. The quasi-continuity of u makes it possible for us to find the regular measure ν satisfying $\nu(u > v) = 0$. In this paper, the difficulty mainly lies to the estimate on the jump part of the lévy process during the approximation so that the uniform convergence of the penalization sequence on the trajectories can be obtained. But in the present work, the model does not contain the term of divergence as in [15], since the Lévy process is considered here, and we will leave this problem into the future work.

The remainder of this paper is organized as follows: in the second section, we set the function spaces, probability space and introduce the notion of regular measures associated to parabolic potentials. The quasi-continuity of the solution for SPDE without obstacle is also proved in this section. The third section is devoted to proving the existence and uniqueness result.

2. Preliminaries

Let $L^2(\mathbb{R}^d)$ be the set of square integrable functions with respect to Lebesgue measure on $\mathbb{R}^d$. It is a Hilbert space equipped with the usual scalar product and norm as follows,

$$(u, v) = \int_{\mathbb{R}^d} u(x)v(x)dx, \quad \|u\|_2 = \left(\int_{\mathbb{R}^d} u^2(x)dx \right)^{\frac{1}{2}}.$$

We also use the notation

$$(u, v) = \int_{\mathbb{R}^d} u(x)v(x)\, dx,$$

where u, v are measurable functions defined in $\mathbb{R}^d$ and $uv \in L^1(\mathbb{R}^d)$. We will consider an evolution problem over a fixed time interval $[0, T]$ and define the norm for a function in $L^2([0, T] \times \mathbb{R}^d)$ as

$$\|u\|_{2,2} = \left(\int_0^T \int_{\mathbb{R}^d} |u(t, x)|^2 dxdt \right)^{\frac{1}{2}}.$$

Another Hilbert space that we use is the first order Sobolev space $H^1(\mathbb{R}^d)$. Its natural scalar product and norm are

$$(u,v)_{H^1(\mathbb{R}^d)} = (u,v) + (\nabla u, \nabla v)\,, \quad \|u\|_{H^1(\mathbb{R}^d)} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}},$$

where $\nabla u(t,x) = (\partial_1 u(t,x), \cdots, \partial_d u(t,x))$ denotes the gradient.
Of special interest is the subspace $\widetilde{F} \subset L^2([0,T]; H^1(\mathbb{R}^d))$ consisting of all functions $u(t,x)$ such that $t \mapsto u_t = u(t,\cdot)$ is continuous in $L^2(\mathbb{R}^d)$. The natural norm on $\widetilde{F}$ is

$$\|u\|_T = \sup_{0\le t\le T} \|u_t\|_2 + \left(\int_0^T \|\nabla u_t\|_2 dt\right)^{\frac{1}{2}}.$$

The space of test functions is denoted by $\mathcal{D}_T = \mathcal{C}^\infty\big([0,T]\big)\otimes \mathcal{C}_c^\infty\big(\mathbb{R}^d\big)$, where $\mathcal{C}^\infty\big([0,T]\big)$ denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of $[0,T]$ and $\mathcal{C}_c^\infty\big(\mathbb{R}^d\big)$ is the space of infinite differentiable functions with compact support in $\mathbb{R}^d$.

2.1. The corresponding Markov process

We consider a Dirichlet form $(\mathcal{E}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ as follows:

$$\mathcal{E}(u,v) = \mathcal{E}^1(u,v) + \mathcal{E}^2(u,v)$$

with

$$\mathcal{E}^1(u,v) = \frac{1}{2}(\nabla u, \nabla v) \quad \text{and} \quad \mathcal{E}^2(u,v) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Gamma} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{d+\alpha}} dxdy,$$

where Γ is the diagonal $\Gamma := \{(x,x) | x \in \mathbb{R}^d\}$ and $\alpha \in (0,2)$.
From classical probability theory, we know that such a Dirichlet form is related to a Hunt process, whose infinitesimal generator $\mathcal{A}$ is

$$\mathcal{A}u(x) = \frac{1}{2}\Delta u(x) + \int_{\mathbb{R}^d} \big(u(x+y) - u(x)\big)\frac{dy}{|y|^{d+\alpha}}.$$

Let $\Omega^1 = \mathcal{C}\left([0,\infty); \mathbb{R}^d\right)$ be the space of continuous trajectories. The canonical process $(W_t)_{t\ge 0}$ is defined by $W_t(\omega^1) = \omega^1(t)$, for any $\omega^1 \in \Omega^1$, $t \ge 0$ and the shift operator, $\theta_t : \Omega^1 \longrightarrow \Omega^1$, is defined by $\theta_t(\omega^1)(s) = \omega^1(t+s)$, for any s, $t \ge 0$. The canonical filtration $\mathcal{F}_t^1 = \sigma(W_s; s \le t)$ is completed by the standard procedure with respect to the probability measures produced by the transition function

$$Q_t(x,dy) = q_t(x-y)dy, \quad t > 0, \quad x \in \mathbb{R}^d,$$

where $q_t(x) = (2\pi t)^{-\frac{d}{2}} \exp\left(-|x|^2/2t\right)$ is the Gaussian density. Thus, we get a continuous Hunt process $\left(\Omega^1, W_t, \theta_t^1, \mathcal{F}^1, \mathcal{F}_t^1, \mathbb{P}_1^x\right)$. $\mathbb{P}_1^0$ is the Wiener measure, which is supported by the set $\Omega_0^1 = \{\omega^1 \in \Omega^1,\ w^1(0) = 0\}$. We set $\Pi_0(\omega^1)(t) = \omega^1(t) - \omega^1(0)$, $t \ge 0$, which defines a map $\Pi_0 : \Omega^1 \to \Omega_0^1$. Then $\Pi = (W_0, \Pi_0) : \Omega^1 \to \mathbb{R}^d \times \Omega_0^1$ is a bijection. For each probability measure on $\mathbb{R}^d$, the probability $\mathbb{P}^\mu$ of the Brownian motion started with the initial distribution μ is given by

$$\mathbb{P}_1^\mu = \Pi^{-1}\left(\mu \otimes \mathbb{P}_1^0\right).$$

In particular, for the Lebesgue measure in $\mathbb{R}^d$, which we denote by $m = dx$, we have

$$\mathbb{P}_1^m = \Pi^{-1}\left(dx \otimes \mathbb{P}_1^0\right).$$

Denote $\Omega^2 := D([0,T];\mathbb{R}^d)$ as the Skorohod space. The canonical processs V_t and the shift operator θ_t^2 can be defined similarly to W_t and θ_t^1 given above. Hence, we get a Hunt process $(\Omega^2, V_t, \theta_t^2, \mathcal{F}^2, \mathcal{F}_t^2, \mathbb{P}_2^x)$ related to the Dirichlet form $\mathcal{E}^2$.
We consider the sample space $\Omega' = \Omega^1 \times \Omega^2$ and the process $(X_t)_{t\geq 0}$ defined by $X_t(\omega^1,\omega^2) = W_t(\omega^1) + V_t(\omega^2)$ for $t \geq 0$. The shift operator $\Theta_t : \Omega' \longleftarrow \Omega'$ is defined by $\Theta_t(w^1,w^2)(s) = (w^1(t+s), w^2(t+s))$, for any s, $t \geq 0$. The σ-field $\mathcal{F}$ and filtration $\mathcal{F}_t$ are given by $\mathcal{F} := \mathcal{F}^1 \times \mathcal{F}^2$ and $\mathcal{F}_t = \mathcal{F}_t^1 \times \mathcal{F}_t^2$. The family of probability measures $\{\mathbb{P}^x\}_x$ is defined by $\mathbb{P}^x := \mathbb{P}_1^x \times \mathbb{P}_2^0$. Therefore, we see that $(\Omega', X_t, \Theta_t, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x)$ is a homogeneous Markov process related to the Dirichlet form $(\mathcal{E}, H^1(\mathbb{R}^d))$. For the process X, we have the following decomposition:

$$X_t = W_t + \int_0^t \int_{|z|\geq 1} zN(dz,dt) + \int_0^t \int_{|z|<1} z\tilde{N}(dz,dt),$$

where $N(dz,dt)$ is the jumping measure of X with $v(dz)dt := \frac{1}{|z|^{d+\alpha}}dzdt$ as its predictable compensator and $\tilde{N}(dz,dt) := N(dz,dt) - v(dz)dt$ the associated compensated measure.
Denote by P_t the corresponding semigroup which is strongly continuous on $L^2(\mathbb{R}^d)$. It is easy to verify that the transition function is absolutely continuous with respect to the Lebseque measure:

$$P_t(x,dy) = p_t(y-x)dy,$$

where $p_t(x)$ is the density. It is easy to see that

$$\begin{aligned}\int_{\mathbb{R}^d} P_t f(x)dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)P_t(x,dy)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)p_t(x-y)dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+z)p_t(z)dzdx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x+z)p_t(z)dxdz \\ &= \int_{\mathbb{R}^d} f(x)dx.\end{aligned}$$

Thus, the Lebesgue measure is an invariant measure for the semigroup P_t.

Next, for future purposes, we give some results concerning the deterministic PDE with respect to $\mathcal{A}$. As the proofs are similar to that in [15], we omit them.

Lemma 1. *Let $f \in L^2\big([0,T]\times\mathbb{R}^d;\mathbb{R}\big)$ and denote by $(u^n)_{n\in\mathbb{N}}$ the sequence of solutions of the equations*

$$\big(\partial_t + \mathcal{A}\big)u^n - nu^n + f = 0, \quad \forall n \in \mathbb{N},$$

with final condition $u_T^n = 0$. Then we have

$$\int_0^T \mathcal{E}(u_t^n)dt \leq c\left[\frac{1}{n}\int_0^T \|f_t\|_2^2 dt + \int_0^T e^{-2n(T-t)}\|f_t\|_2^2 dt\right]. \tag{2}$$

Obviously, (2) implies that $\lim_{n\to\infty}\int_0^T \mathcal{E}(u_t^n)dt = 0$. We present a strengthened version of this relation in the next corollary.

Corollary 1. *Let f and f^n, for any $n\in\mathbb{N}$ be in $L^2([0,T]\times\mathbb{R}^d;\mathbb{R})$ such that $\lim_{n\to\infty}\int_0^T \|f_t^n - f_t\|_2^2 dt = 0$. Then, the solutions $(u^n)_{n\in\mathbb{N}}$ of the equations*

$$\big(\partial_t + \mathcal{A}\big)u^n - nu^n + f^n = 0,$$

with final condition $u_T^n = 0$, satisfy the relation $\lim_{n\to\infty}\int_0^T \mathcal{E}(u_t^n)dt = 0$.

2.2. Regular measures

In this section, we shall be concerned with some facts related to the time-space Markov process, with the state space $[0,T[\times\mathbb{R}^d$, corresponding to the generator $\partial_t+\mathcal{A}$. Its associated semigroup will be denoted by $(\widetilde{P}_t)_{t>0}$. We may express it in terms of the density the semigroup $(P_t)_{t>0}$ in the following way

$$\widetilde{P}_t\psi\left(s,x\right)=\begin{cases}\int_{\mathbb{R}^d}p_t\left(x,y\right)\psi\left(s+t,y\right)dy, & if \quad s+t<T,\\ 0, & otherwise,\end{cases}$$

where $\psi:[0,T[\times\mathbb{R}^d\to\mathbb{R}$ is a bounded Borel measurable function, $s\in[0,T[\,,x\in\mathbb{R}^d$ and $t>0$. So we may also write $\left(\widetilde{P}_t\psi\right)_s=P_t\psi_{t+s}$ if $s+t<T$. The corresponding resolvent has a density expressed in terms of the density p_t too, as follows

$$\widetilde{U}_\alpha\psi\left(t,x\right)=\int_t^T\int_{\mathbb{R}^d}e^{-\alpha(s-t)}p_{s-t}\left(x-y\right)\psi\left(s,y\right)dyds.$$

or

$$\left(\widetilde{U}_\alpha\psi\right)_t=\int_t^T e^{-\alpha(s-t)}P_{s-t}\psi_s\,ds.$$

In particular this ensures that the excessive functions with respect to the time-space Markov process are lower semicontinuous. In fact we will not use directly the time space process, but only its semigroup and resolvent. For related facts concerning excessive functions the reader is refered to [2], [12] and [5]. Some further properties of this semigroup are presented in the next lemma. The proof of it is almost the same to that of Lemma 2 in [15]. Thus we omit the proof.

Lemma 2. *The semigroup $(\widetilde{P}_t)_{t>0}$ acts as a strongly continuous semigroup of contractions on the spaces $L^2([0,T[\times\mathbb{R}^d)=L^2([0,T[\,;L^2(\mathbb{R}^d))$ and $L^2([0,T[\,;H^1(\mathbb{R}^d))$.*

Now we give the definition of the potentials belonging to $\widetilde{F}$, which appears in our obstacle problem.

Definition 1. *(i) A function $\psi:[0,T]\times\mathbb{R}^d\to\overline{\mathbb{R}}$ is called quasicontinuous provided that for each $\varepsilon>0$, there exists an open set, $D_\varepsilon\subset[0,T]\times\mathbb{R}^d$, such that ψ is finite and continuous on D_ε^c and*

$$\mathbb{P}^m\left(\{\omega'\in\Omega'|\exists\,t\in[0,T]\;\; s.t.\,(t,X_t\left(\omega'\right))\in D_\varepsilon\}\right)<\varepsilon.$$

(ii) A function $u:[0,T]\times\mathbb{R}^d\to[0,\infty]$ is called a regular potential, provided that its restriction to $[0,T[\times\mathbb{R}^d$ is excessive with respect to the time-space semigroup, it is quasicontinuous, $u\in\widetilde{F}$ and $\lim_{t\to T}u_t=0$ in $L^2(\mathbb{R}^d)$.

Observe that if a function ψ is quasicontinuous, then the process $(\psi_t(X_t))_{t\in[0,T]}$ is càdlàg and has only inaccessible jumps. Next we will present the basic properties of the regular potentials. Due to the expression of the semigroup $(\widetilde{P}_t)_{t>0}$ in terms of the density, it follows that two excessive functions which represent the same element in $\widetilde{F}$ should coincide.

Theorem 1. *Let $u\in\widetilde{F}$. Then u has a version which is a regular potential if and only if there exists a continuous increasing process $A=(A_t)_{t\in[0,T]}$ which is $(\mathcal{F}_t)_{t\in[0,T]}-$adapted and such that $A_0=0$, $\mathbb{E}^m[A_T^2]<\infty$ and*

$$u_t(X_t)=\mathbb{E}\left[A_T\,\middle|\mathcal{F}_t\right]-A_t,\;\mathbb{P}^m-a.s., \tag{i}$$

for each $t\in[0,T]$. The process A is uniquely determined by these properties. Moreover, the following relations hold:

$$u_t\left(X_t\right)=A_T-A_t-\sum_{i=1}^d\int_t^T\nabla u_s\left(X_{s-}\right)dW_s-\int_t^T\int_{\mathbb{R}^d}u_s(X_{s-}+z)-u_s(X_{s-})\tilde{N}(dz,ds),\;\mathbb{P}^m-a.s.\,, \tag{ii}$$

$$\|u_t\|_2^2 + \int_t^T \mathcal{E}(u_s, u_s)\, ds = \mathbb{E}^m \left(A_T - A_t\right)^2 , \tag{iii}$$

$$(u_0, \varphi_0) + \int_0^T \mathcal{E}(u_s, \varphi_s) + \big(u_s, \partial_s \varphi_s\big)\, ds = \int_0^T \int_{\mathbb{R}^d} \varphi\,(s,x)\, \nu\,(dsdx)\,, \tag{iv}$$

for each test function $\varphi \in \mathcal{D}_T$, where ν is the measure defined by

$$\nu\,(\varphi) = \mathbb{E}^m \int_0^T \varphi\,(t, X_t)\, dA_t,\ \ \varphi \in \mathcal{C}_c\left([0,T] \times \mathbb{R}^d\right). \tag{v}$$

Before proving Theorem 1, we recall a useful lemma.

Lemma 3. *([4], pp.202) Let $\{f_n\}$ be a decreasing sequence of positive càdlàg functions on $[0,\infty]$ which tend poitwisely to 0 as do their left limits. Then the sequence $\{f_n\}$ converges to 0 uniformly.*

Proof of Theorem 1: The uniqueness of the increasing process in the representation (i) comes from the uniqueness in the Doob-Meyer decomposition. Let us now assume that $\overline{u}$ is a regular potential which is a version of u. We will use an approximation of $\overline{u}$ constructed with the resolvent. By the resolvent equation one has

$$\alpha \widetilde{U}_\alpha \overline{u} = \alpha \widetilde{U}_0 \left(\overline{u} - \alpha \widetilde{U}_\alpha \overline{u}\right).$$

Set $f^n = n(\overline{u} - n\widetilde{U}_n \overline{u})$ and $u^n = n\widetilde{U}_n \overline{u} = \widetilde{U}_0 f^n$. Since $\overline{u}$ is excessive, one has $f^n \geq 0$ and $u^n, n \in \mathbb{N}^*$, is an increasing sequence of excessive functions with limit $\overline{u}$. In fact $u^n, n \in \mathbb{N}^*$, are potentials and their trajectories are continuous. Moreover, the trajectories $t \to \overline{u}_t(X_t)$ are càdlàg on $[0,T[$ by the quasi-continuity of $\overline{u}$. The process $(u_t(X_t))_{t\in[0,T[}$ is a super-martingale and, because $\lim_{t\to T} u_t = 0$ in L^2, it is a potential and the trajectories have null limits at T. By quasicontinuity of the functions and the fact that X is quasi-left-continuous, we have ${}^p(u_t^n(X_t)) = u_t^n(X_{t-})$ and ${}^p(u_t(X_t)) = u_t(X_{t-})$, where ${}^p(\cdot)$ denotes the predictable projection. Therefore, $(u^n - u)(X)$ decreasingly converges to 0 as do their left limits. Then, Lemma 3 implies that this approximation also holds uniformly on the trajectories, on the closed interval $[0,T]$,

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left|u_t^n\,(X_t) - \overline{u}_t\,(X_t)\right| = 0, \ \ \mathbb{P}^m - a.s..$$

The function u^n solves the equation $(\partial_t + \mathcal{A})\, u^n + f^n = 0$ with the condition $u_T^n = 0$ and its backward representation is

$$u_t^n(X_t) = \int_t^T f_s^n(X_s) ds - \sum_{i=1}^d \int_t^T \partial_i u_s^n(X_s) dW_s^i - \int_t^T \int_{\mathbb{R}^d} u_s^n(X_{s-}+z) - u_s^n(X_{s-}) \tilde{N}(dz,ds).$$

If we set $A_t^n = \int_0^t f_s^n\,(X_s)\, ds$, after conditioning, this representation gives

$$\begin{aligned} u_t^n\,(X_t) &= A_T^n - A_t^n - \sum_{i=1}^d \int_t^T \partial_i u_s^n\,(X_s)\, dW_s^i - \int_t^T \int_{\mathbb{R}^d} u_s^n(X_{s-}+z) - u_s^n(X_{s-}) \tilde{N}(dz,ds) \\ &= \mathbb{E}^m \left[A_T^n | \mathcal{F}_t\right] - A_t^n. \end{aligned} \tag{$*$}$$

By the relation $(*)$, it follows that

$$\begin{aligned} &\mathbb{E}^m \left(A_T^n - A_t^n\right)^2 \\ =& \mathbb{E}^m \Big(u_t^n(X_t) + \sum_{i=1}^d \int_t^T \partial_i u_s^n(X_s) dW_s^i + \int_t^T \int_{\mathbb{R}^d} u_s^n(X_{s-}+z) - u_s^n(X_{s-}) \tilde{N}(dz,ds)\Big)^2 \\ =& \|u_t^n\|_2^2 + \int_t^T \mathcal{E}(u_s^n) ds. \end{aligned} \tag{$**$}$$

A similar relation holds for differences. In particular, one has

$$\mathbb{E}^m (A_T^n - A_T^k)^2 = \|u_0^n - u_0^k\|^2 + 2\int_0^T \mathcal{E}(u_s^n - u_s^k) ds.$$

Moreover, the preceding lemma ensures that $\lim_{\alpha\to\infty} \alpha\widetilde{U}_\alpha = I$ in the space $L^2([0,T[;H^1(\mathbb{R}^d))$, which implies

$$\lim_{n\to 0} \int_0^T \mathcal{E}(u_s^n - u_s^k)\ ds = 0.$$

These last relations imply that there exists a limit $\lim_n A_T^n =: A_T$ in the sense of $L^2(\mathbb{P}^m)$.
Set $M_t^n := \mathbb{E}^m [A_T^n|\mathcal{F}_t], M_t := \mathbb{E}^m [A_T|\mathcal{F}_t]$. Then one has $\lim_{n\to\infty} M^n = M$ in $L^2(\mathbb{P}^m)$ and hence

$$\lim_{n\to\infty} \mathbb{E}^m \sup_{0\le t\le T} |M_t^n - M_t|^2 = 0.$$

Then the relation $u_t^n(X_t) = M_t^n - A_t^n$ shows that the processes $A^n, n\in\mathbb{N}^*$, also converge uniformly on the trajectories to a continuous process $A = (A_t)_{t\in[0,T]}$. The inequality

$$\sup_{0\le t\le T} |A_t^n - A_t| \le A_T + |A_T^n - A_T|,$$

ensure the conditins to pass to the limit and get

$$\lim_{n\to\infty} \mathbb{E}^m \sup_{0\le t\le T} |A_t^n - A_t|^2 = 0.$$

Passing to the limit in the relations $(*)$ and $(**)$ one deduces the relations (i), (ii) and (iii). The rest of the proof are almost the same as that of Theorem 2 in [15], so we omit it. □

The following lemma is concerning on the uniqueness of the potential related to a Randon measure via relation (iv). For its proof, one can refer to [15].

Lemma 4. *Let u be a regular potential and ν a Radon measure on $[0,T]\times\mathbb{R}^d$ such that relation (iv) holds. Then one has*

$$(\phi, u_t) = \int_t^T \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \phi(x)\, p_{s-t}(x-y)\, dx \right) \nu(dsdy),$$

for each $\phi \in L^2(\mathbb{R}^d)$ and $t\in[0,T]$.

We now introduce the class of measures which intervene in the notion of solution to the obstacle problem.

Definition 2. *A nonnegative Radon measure ν defined on $[0,T]\times\mathbb{R}^d$ is called regular if there exists a regular potential u such that the relation (iv) from the above theorem is satisfied.*

As a consequence of the preceding lemma we see that the regular measures are always represented as in the relation (v) of the theorem, with a certain increasing process. We also note the following properties of a regular measure, with the notation from the theorem.

1. A set $D\in\mathcal{B}([0,T]\times\mathbb{R}^d)$ satisfies the relation $\nu(D)=0$ if and only if $\int_0^T 1_D(t,X_t)\,dA_t = 0$, $\mathbb{P}^m-$a.s..
2. If a set $D\in\mathcal{B}(]0,T[\times\mathbb{R}^d)$ is polar, in the sense that

$$\mathbb{P}^m(\{\omega\in\Omega'|\exists t\in[0,T], (t, X_t(\omega))\in D\}) = 0,$$

then $\nu(D)=0$.

2.3. Hypotheses

Let $B = (B_t)_{t\geq 0}$ be a standard d^1-dimentional Brownian motion on a probability space $(\Omega, \mathcal{F}^B, \mathbb{P})$. Over the time interval $[0,T]$ we define the backward filtration $(\mathcal{F}^B_{s,T})_{s\in[0,T]}$ where $\mathcal{F}^B_{s,T}$ is the completion in $\mathcal{F}^B$ of $\sigma(B_r - B_s; s \leq r \leq T)$.
We denote by $\mathcal{H}_T$ the space of $H^1(\mathbb{R}^d)$-valued predictable and $\mathcal{F}^B_{t,T}$-adapted processes $(u_t)_{0\leq t\leq T}$ such that the trajectories $t \to u_t$ are in $\widetilde{F}$ a.s. and

$$\|u\|_T^2 < \infty.$$

In the remainder of this paper we assume that the final condition Φ is a given function in $L^2(\mathbb{R}^d)$ and the coefficients

$$\begin{aligned} f &: [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R} \, , \\ h &= (h_1, ..., h_{d^1}) \, : \, [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^{d^1} \end{aligned}$$

are random functions predictable with respect to the backward filtration $(\mathcal{F}^B_{t,T})_{t\in[0,T]}$. We set

$$f(\cdot,\cdot,\cdot,0,0) := f^0 \quad \text{and} \quad h(\cdot,\cdot,\cdot,0,0) := h^0 = (h^0_1, ..., h^0_{d^1}).$$

and assume the following hypotheses :

Assumption (H): There exist non-negative constants C, β such that

(i) $|f_t(\omega,x,y,z) - f(t,\omega,x,y',z')| \leq C\big(|y-y'| + |z-z'|\big)$

(ii) $\Big(\sum_{j=1}^{d_1} |h_{j,t}(\omega,x,y,z) - h_j(t,\omega,x,y',z')|^2\Big)^{\frac{1}{2}} \leq C\,|y-y'| + \beta\,|z-z'|,$

(iii) the contraction property (as in [7]) : $\beta^2 < 1$.

Assumption (HD2)

$$\mathbb{E}\left(\left\|f^0\right\|_{2,2}^2 + \left\|h^0\right\|_{2,2}^2\right) < +\infty.$$

Assumption (HO) : The obstacle $v(t,\omega,x)$ is a predictable random function w.r.t the backward filtration $(\mathcal{F}^B_{t,T})$. We also assume that $(t,x) \mapsto v(t,\omega,x)$ is $\mathbb{P}$-a.s. quasicontinuous on $[0,T]$ and satisfies $v(T,\cdot) \leq \Phi(\cdot)$.

We recall that a usual solution (non reflected one) of the equation (1) with final condition $u_T = \Phi$, is a processus $u \in \mathcal{H}_T$ such that for each test function $\varphi \in \mathcal{D}_T$ and any $t \in [0,T]$, we have a.s.

$$\int_t^T \big[(u_s, \partial_s\varphi_s) + \mathcal{E}(u_s,\varphi_s)\big]ds - \big(\Phi,\varphi_T\big) + \big(u_t,\varphi_t\big) = \int_t^T (f_s,\varphi_s)\,ds + \int_t^T (h_s,\varphi_s)\cdot \overleftarrow{dB}_s. \quad (3)$$

By Theorem 8 in [7] we have existence and uniqueness of the solution. Moreover, the solution belongs to $\mathcal{H}_T$. We denote by $\mathcal{U}(\Phi, f, h)$ this solution.

2.4. Quasi-continuity properties

In this section we are going to prove the quasi-continuity of the solution of the linear equation, i.e. when f and h do not depend of u and ∇u. To this end we first extend the double stochastic Itô's formula to our framework. We start by proving the following doubly representation theorem.

Theorem 2. *Let $u \in \mathcal{H}_T$ be a solution of the equation*

$$du_t + \mathcal{A}u_t dt + f_t dt + h_t \overleftarrow{dB}_t = 0,$$

where f, h are predictable processes such that

$$\mathbb{E}\int_0^T \left[\|f_t\|_2^2 + \|h_t\|_2^2\right] dt < \infty \quad \text{and} \quad \|\Phi\|_2^2 < \infty.$$

Then, for any $0 \le s \le t \le T$, one has the following stochastic representation, $\mathbb{P}^m$-a.s.,

$$\begin{aligned} u(t, X_t) - u(s, X_s) = & \sum_i \int_s^t \partial_i u(r, X_r)\, dW_r^i + \int_s^t \int_{\mathbb{R}^d} u_r(X_{r-} + z) - u_r(X_{r-})\tilde{N}(dz, dr) \\ & - \int_s^t f_r(X_r)\, dr - \int_s^t h_r(X_r) \cdot \overleftarrow{dB}_r. \end{aligned} \tag{4}$$

Noting that $\mathcal{F}_T$ and $\mathcal{F}_{0,T}^B$ are independent under $\mathbb{P} \otimes \mathbb{P}^m$ and therefore in the above formula the stochastic integrals with respect to dW_t act independently of $\mathcal{F}_{0,T}^B$ and similarly the integral with respect to $\overleftarrow{dB}_t$ acts independently of $\mathcal{F}_T$.

Proof. The proof will be similar to that of Proposition 11 in [7]. Thus we only give a sketch of it. The solution can be decomposed into two terms each corresponding to one of the coefficients f and h. It is enough to treat them separately.
a) In the case where $h = 0$, one can use Itô's formula to get the desired result. Thus we omit the proof.
b) In the case where $f = 0$, we first establish the formula for the elementary process,

$$h_t(\omega, x) = \chi_A(\omega)\chi_{[r_1, r_2)}(t)e(x),$$

where $0 \le r_1 \le r_2 \le T$, $A \in \mathcal{F}_{r_2,T}^B$ and $e \in \mathcal{D}(\mathcal{A}^{\frac{3}{2}}) \cap L^\infty$. In this case, let

$$v_t^n := \int_t^T P_{s-t} h_s^n d\overleftarrow{B}_s,$$

where $h_t^n = 2^n(B_{t_i^n} - B_{t_{i+1}^n})h_{t_{i+1}^n}$ for $t \in (t_i^n, t_{i+1}^n]$ and $t_i^n = \frac{i}{2^n}T$. Then, Lemma 10 in [7] implies that

$$\lim_n E \sup_{0 \le t \le T} \mathcal{E}(v_t^n - u_t) = 0.$$

We may have the following relation:

$$v_t^n(X_t) = -\int_t^T h^n(s, X_s)ds + \int_t^T \nabla v_s^n(X_s)dW_s + \int_t^T v_s^n(X_{s-} + z) - v_s^n(X_{s-})\tilde{N}(dz, ds).$$

Letting $n \to \infty$, one has

$$-\int_t^T h^n(s, X_s)ds \longrightarrow -\int_s^t h_r(X_r) \cdot \overleftarrow{dB}_r$$

$$\int_t^T \nabla v_s^n(X_s)dW_s \longrightarrow \int_t^T \nabla u_s(X_s)dW_s$$

$$\int_t^T \int_{\mathbb{R}^d} v_s^n(X_{s-} + z) - v_s^n(X_{s-})\tilde{N}(dz, ds) \longrightarrow \int_t^T \int_{\mathbb{R}^d} u_s(X_{s-} + z) - u_s(X_{s-})\tilde{N}(dz, ds)$$

Thus we proof the formula for the elementary processes. The general cases can be proved by approximation. $\square$

In particular the process $(u_t(X_t))_{t\in[0,T]}$ admits a càdlàg version which we usually denote by $Y = (Y_t)_{t\in[0,T]}$ and we introduce the notation $Z_t = \nabla u_t(X_t)$ and $U_t(z) = u_t(X_{t-}+z) - u_t(X_{t-})$. As a consequence of this theorem, with the application of Itô's formula and BDG's inequality, we shall have the following result.

Corollary 2. *Under the hypothesis of the preceding theorem one has the following stochastic representation for u^2, $\mathbb{P}\otimes\mathbb{P}^m$-a.e., for any $0\le t\le T$,*

$$\begin{aligned} u_t^2\left(X_t\right) - \Phi^2\big(X_T\big) = {}& 2\int_t^T \big[u_s f_s(X_s) - |Z_s|^2 - \int U_s^2(z)v(dz) + \frac{1}{2}|h_s|^2(X_s)\big]\,ds \\ & - 2\sum_i \int_t^T \big(u_r\partial_i u_r\big)\left(X_r\right)dW_r^i - \int_t^T\int_{\mathbb{R}^d} u_t^2(X_{t-}+z) - u_t^2(X_{t-})\tilde{N}(dz,dt) \\ & + 2\int_t^T \big(u_r h_r\big)\left(X_r\right)\cdot \overleftarrow{dB}_r. \end{aligned} \tag{5}$$

$$\mathbb{E}\mathbb{E}^m\big(\sup_{t\le s\le T}|u_s|^2\big) + \mathbb{E}\Big[\int_t^T \mathcal{E}(u_s)\,ds\Big] \le c\Big[\|\phi\|_2^2 + \mathbb{E}\int_t^T \big[\|f_s\|_2^2 + \|h_s\|_2^2\big]\,ds\Big], \tag{6}$$

for each $t\in[0,T]$.

Proof. Firstly, we may represent the solution in the form

$$\begin{aligned} u_t\left(X_t\right) - u_s\left(X_s\right) = {}& \sum_i \int_s^t \partial_i u_r\left(X_r\right)dW_r^i + \int_s^t\int_{\mathbb{R}^d} u_r(X_{r-}+z) - u_r(X_{r-})\tilde{N}(dz,dr) \\ & - \int_s^t f_r(X_r)\,dr - \int_s^t h_r(X_r)\cdot\overleftarrow{dB}_r. \end{aligned}$$

By similar proof in Lemma 1.3 of [17], it follows that

$$\begin{aligned} u_t^2\left(X_t\right) - u_s^2\left(X_s\right) = {}& -2\int_s^t \big[u_r f_r(X_r) - |\nabla u_r|^2(X_r) - |h_r|^2(X_r)\big]\,dr - 2\int_s^t \big(u_r h_r\big)\left(X_r\right)\cdot\overleftarrow{dB}_r \\ & + \int_s^t\int_{\mathbb{R}^d}(u_r(X_{r-}+z) - u_r(X_{r-}))^2 v(dz)dr + 2\sum_i\int_s^t \big(u_r\partial_i u_r\big)\left(X_r\right)dW_r^i \\ & + \int_s^t\int_{\mathbb{R}^d} u_r^2(X_{r-}+z) - u_r^2(X_{r-})\tilde{N}(dz,dr). \end{aligned}$$

Then the standard calculations of BSDE involving Young's inequality, B-D-G inequality and Gronwall's lemma give the estimate (6). □

From Corollary 2, one can easily obtain the following.

Lemma 5. *Let h and h^n, for $n\in\mathbb{N}$, be $L^2(\mathbb{R}^d;\mathbb{R}^{d^1})$-valued predictable processes on $[0,T]$ with respect to $(\mathcal{F}_{t,T}^B)_{t\ge0}$ and such that*

$$\mathbb{E}\int_0^T \|h_t\|_2^2dt < \infty, \quad \mathbb{E}\int_0^T \|h_t^n\|_2^2dt < \infty \quad \textit{and} \lim_{n\to\infty}\mathbb{E}\int_0^T \|h_t^n - h_t\|_2^2dt = 0.$$

Let $(u^n)_{n\in\mathbb{N}}$ be the solutions of the equations

$$du_t^n + [\mathcal{A}u_t^n - nu_t^n]\,dt + h_t^n\cdot\overleftarrow{dB}_t = 0,$$

with final condition $u_T^n = 0$, for each $n\in\mathbb{N}$. Then, one has $\lim_{n\to\infty}\int_0^T \mathcal{E}(u_t^n)dt = 0$.

Thanks to estimate (6), we can do a similar proof as that of Proposition 1 in [15] and get the following proposition which concerns the quasicontinuity of the solution of an SPDE.

Proposition 1. *Under the hypothesis of Theorem 2, there exists a function* $\bar{u} : [0,T]\times\Omega\times\mathbb{R}^d \longrightarrow \mathbb{R}$ *which is a quasicontinuous version of* u*, in the sense that for each* $\epsilon > 0$*, there exits a predictable random set* $D^\epsilon \subset [0,T]\times\Omega\times\mathbb{R}^d$ *such that* $\mathbb{P}$*-a.s. the section* D^ϵ_ω *is open and* $\bar{u}(\cdot,\omega,\cdot)$ *is continuous on its complement* $(D^\epsilon_\omega)^c$ *and*

$$\mathbb{P}\otimes\mathbb{P}^m\left((\omega,\omega')\,\middle|\,\exists t\in[0,T]\ s.t.\ (t,\omega,X_t(\omega'))\in D^\epsilon\right)\leq\epsilon.$$

In particular the process $\left(\overline{u}_t(X_t)\right)_{t\in[0,T]}$ *has càdlàg trajectories,* $\mathbb{P}\otimes\mathbb{P}^m$*-a.s.*

We also need the quasicontinuity of the solution associated to a random regular measure, as stated in the next proposition. We first give the formal definition of this object.

Definition 3. *We say that* $u\in\mathcal{H}_T$ *is a random regular potential provided that* $u(\cdot,\omega,\cdot)$ *has a version which is regular potential,* $\mathbb{P}(d\omega)-a.s.$ *The random variable* $\nu : \Omega \longrightarrow \mathcal{M}\left([0,T]\times\mathbb{R}^d\right)$ *with values in the set of regular measures on* $[0,T]\times\mathbb{R}^d$ *is called a regular random measure, provided that there exits a random regular potential* u *such that the measure* $\nu(\omega)(dtdx)$ *is associated to the regular potential* $u(\cdot,\omega,\cdot)$*,* $\mathbb{P}(d\omega)$*-a.s.*

The relation between a random measure and its associated random regular potential is described by the following proposition, the proof of which results from approximation procedure used in the proof of Theorem 1.

Proposition 2. *Let* u *be a random regular potential and* ν *be the associated random regular measure. Let* $\overline{u}$ *be the excessive version of* u*, i.e.* $\overline{u}(\cdot,\omega,\cdot)$ *is a.s. an* $(\widetilde{P}_t)_{t>0}$ *-excessive function which coincides with* $u(\cdot,\omega,\cdot)$*,* $dtdx$*-a.e. Then we have the following properties:*

(i) For each $\varepsilon>0$*, there exists a* $(\mathcal{F}^B_{t,T})_{t\in[0,T]}$ *-predictable random set* $D^\varepsilon\subset[0,T]\times\Omega\times\mathbb{R}^d$ *such that* P *-a.s. the section* D^ε_ω *is open and* $\overline{u}(\cdot,\omega,\cdot)$ *is continuous on its complement* $(D^\varepsilon_\omega)^c$ *and*

$$\mathbb{P}\otimes\mathbb{P}^m\left((\omega,\omega')\,/\exists t\in[0,T]\quad s.t.\,(t,\omega,X_t(\omega'))\in D^\varepsilon_\omega\right)\leq\varepsilon.$$

In particular the process $(\overline{u}_t(X_t))_{t\in[0,T]}$ *has càdlàg trajectories,* $\mathbb{P}\otimes\mathbb{P}^m-a.s.$
(ii) There exists a continuous increasing process $A=(A_t)_{t\in[0,T]}$ *defined on* $\Omega\times\Omega'$ *such that* A_s-A_t *is measurable with respect to the* $\mathbb{P}\otimes\mathbb{P}^m$*-completion of* $\mathcal{F}^B_{t,T}\vee\sigma(W_r, r\in[t,s])$*, for any* $0\leq s\leq t\leq T$*, and such that the following relations are fulfilled a.s., with any* $\varphi\in\mathcal{D}$ *and* $t\in[0,T]$*,*

$$(a)\quad (u_t,\varphi_t)+\int_t^T\left(\frac{1}{2}(\nabla u_s,\nabla\varphi_s)+(u_s,\partial_s\varphi_s)\right)ds=\int_t^T\int_{\mathbb{R}^d}\varphi(s,x)\,\nu(dsdx),$$

$$(b)\quad u_t(X_t)=\mathbb{E}\left[A_T\,\middle|\mathcal{F}_t\vee\mathcal{F}^B_{t,T}\right]-A_t,$$

$$(c)\quad u_t(X_t)=A_T-A_t-\sum_{i=1}^d\int_t^T\partial_i u_s(X_s)\,dW^i_s+\int_0^t u_s(X_{s-}+z)-u_s(X_s)\tilde{N}(dz,ds),$$

$$(d)\quad \|u_t\|_2^2+\int_t^T\mathcal{E}(u_s)\,ds=\mathbb{E}^m\left(A_T-A_t\right)^2,$$

$$(e)\quad \nu(\varphi)=\mathbb{E}^m\int_0^T\varphi(t,X_t)\,dA_t.$$

3. Existence and uniqueness of the solution of the obstacle problem

3.1. The weak solution

We now precise the definition of the solution of our obstacle problem. We recall that the data satisfy the hypotheses of Section 2.3.

Definition 4. *We say that a pair (u,ν) is a weak solution of the obstacle problem for the SPDE (1) associated to (Φ, f, h, v), if*

(i) $u \in \mathcal{H}_T$ and $u(t,x) \geq v(t,x)$, $d\mathbb{P}\otimes dt \otimes dx - a.e.$ and $u(T,x) = \Phi(x)$, $d\mathbb{P}\otimes dx - a.e.$.

(ii) ν is a random regular measure on $(0,T)\times\mathbb{R}^d$.

(iii) for each $\varphi \in \mathcal{D}_T$ and $t\in[0,T]$,

$$\begin{aligned}\int_t^T \left[(u_s,\partial_s\varphi_s)+\mathcal{E}(u_s,\varphi_s)\right]ds-\big(\Phi,\varphi_T\big)+\big(u_t,\varphi_t\big)=\int_t^T\big(f_s\big(u_s,\nabla u_s\big),\varphi_s\big)\,ds\\+\int_t^T\left(h_s\left(u_s,\nabla u_s\right),\varphi_s\right)\cdot\overleftarrow{dB}_s+\int_t^T\int_{\mathbb{R}^d}\varphi_s(x)\,\nu(dx,ds).\end{aligned}\tag{7}$$

(iv) If $\overline{u}$ is a quasicontinuous version of u, then one has

$$\int_0^T\int_{\mathbb{R}^d}\left(\overline{u}_s\left(x\right)-v_s\left(x\right)\right)\nu\left(dsdx\right)=0,\ a.s.$$

We note that a given solution u can be writen as a sum $u=u_1+u_2$, where u_1 satisfies a linear equation $u_1=\mathcal{U}\big(\Phi,f(u,\nabla u),h(u,\nabla u)\big)$ with f,h determined by u, while u_2 is the random regular potential corresponding to the measure ν. By the Propositions 1 and 2, the conditions (ii) and (iii) imply that the process u always admits a quasicontinuous version, so that the condition (iv) makes sense. We also note that if $\overline{u}$ is a quasicontinuous version of u, then the trajectories of X do not visit the set $\{\overline{u}<v\}$, $\mathbb{P}\otimes\mathbb{P}^m$-*a.s.*
Here it is the main result of our paper

Theorem 3. *Assume that the assumptions **(H)**, **(HD2)** and **(HO)** hold. Then there exists a unique weak solution of the obstacle problem for the SPDE (1) associated to (Φ,f,h,v).*

In order to solve the problem we will use the backward stochastic differential equation technics. In fact, we shall follow the main steps of the second proof in [11], based on the penalization procedure. The uniqueness assertion of Theorem 3 results from the following comparison result which can be easily proved.

Theorem 4. *Let Φ',f',v' be similar to Φ,f,v and let (u,ν) be the solution of the obstacle problem corresponding to (Φ,f,h,v) and (u',ν') the solution corresponding to (Φ',f',h,v'). Assume that the following conditions hold*

(i) $\Phi\leq\Phi'$, $dx\otimes d\mathbb{P}$ -a.e.

(ii) $f\left(u,\nabla u\right)\leq f'\left(u,\nabla u\right)$, $dtdx\otimes\mathbb{P}$ -a.e.

(iii) $v\leq v'$, $dtdx\otimes\mathbb{P}$ -a.e.

Then one has $u\leq u'$, $dtdx\otimes\mathbb{P}$ -a.e.

3.2. Approximation by the penalization method

For $n\in\mathbb{N}$, let u^n be a solution of the following SPDE

$$\begin{aligned}du_t^n(x)+\mathcal{A}u_t^n(x)dt+f(t,x,u_t^n(x),\nabla u_t^n(x))dt+n(u_t^n(x)-v_t(x))^-\,dt\\+h(t,x,u_t^n(x),\nabla u_t^n(x))\,\overleftarrow{dB}_t=0\end{aligned}\tag{8}$$

with final condition $u_T^n=\Phi$.
Now set $f_n(t,x,y,z)=f(t,x,y,z)+n(y-v_t(x))^-$ and $\nu^n(dt,dx):=n\big(u_t^n(x)-v_t(x)\big)^-dtdx$. Clearly, for each $n\in\mathbb{N}$, f_n is Lipschitz continuous in (y,z) uniformly in (t,x). For each $n\in\mathbb{N}$,

Theorem 8 in [7] ensures the existence and uniqueness of a weak solution $u^n \in \mathcal{H}_T$ of the SPDE (8) associated with the data (Φ, f_n, g, h). We denote by $Y_t^n = u^n(t, X_t)$, $Z_n = \nabla u^n(t, X_t)$,$U_t^n(z) = u_t^n(X_{t-}+z) - u_t^n(X_{t-})$ and $S_t = v(t, X_t)$. We shall also assume that u^n is quasicontinuous, so that Y^n is $\mathbb{P}\otimes\mathbb{P}^m$-*a.e.* càdlàg. Then $\big(Y^n, Z^n, U^n\big)$ solves the BSDE associated to the data (Φ, f_n, h)

$$\begin{aligned} Y_t^n =&\Phi(X_T) + \int_t^T f_r(X_r, Y_r^n, Z_r^n)dr + \int_t^T h_r(X_r, Y_r^n, Z_r^n)\cdot \overleftarrow{dB}_r \\ &+ n\int_t^T (Y_r^n - S_r^n)^- dr - \sum_i \int_t^T Z_{i,r}^n dW_r^i - \int_t^T \int_{\mathbb{R}^d} U_r^n(z)\tilde{N}(dz, dr)\,. \end{aligned} \tag{9}$$

We define $K_t^n = n\displaystyle\int_0^t (Y_s^n - S_s)^- ds$ and establish the following lemma.

Lemma 6. *The quadruple* (Y^n, Z^n, U^n, K^n) *satisfies the following estimates*

$$\begin{aligned} &\mathbb{E}\mathbb{E}^m\,|Y_t^n|^2 + \lambda_\epsilon \mathbb{E}\mathbb{E}^m \int_t^T |Z_r^n|^2 dr + \mathbb{E}\mathbb{E}^m \int_t^T \int_{\mathbb{R}^d} |U_s^n(z)|^2 v(dz)ds \\ \leq& \, c\,\mathbb{E}\mathbb{E}^m\big[|\Phi(X_T)|^2 + \int_t^T \big(|f_s^0(X_s)|^2 + |h_s^0(X_s)|^2\big)\,ds\big] + c_\epsilon\,\mathbb{E}\,\mathbb{E}^m \int_t^T |Y_r^n|^2\,dr \\ &+ c_\delta\,\mathbb{E}\,\mathbb{E}^m\,\big(\sup_{t\leq r\leq T}|S_r|^2\big) + \delta\,\mathbb{E}\,\mathbb{E}^m\,\big(K_T^n - K_t^n\big)^2 \end{aligned} \tag{10}$$

where $\lambda_\epsilon = 1 - \beta^2 - \epsilon$, c_ϵ, c_δ *are a positive constants and* $\epsilon > 0$, $\delta > 0$ *can be chosen small enough such that* $\lambda_\epsilon > 0$ *.*

Proof. Applying Itô 's formula to $(Y^n)^2$, it follows that

$$\begin{aligned} |Y_t^n|^2 &+ \int_t^T |Z_s^n|^2 ds + \int_t^T \int_{\mathbb{R}^d} |U_s^n(z)|^2 v(dz)ds = |\Phi(X_T)|^2 + 2\int_t^T Y_s f(s, X_s, Y_s^n, Z_s^n)ds \\ &+ 2\int_t^T Y_s dK_s^n + \int_t^T |h(s, X_s, Y_s^n, Z_s^n)|^2 ds + 2\int_t^T Y_s^n h_s(X_s, Y_s^n, Z_s^n)\cdot\overleftarrow{dB}_s \\ &- 2\int_t^T Y_s^n Z_s^n dW_s - \int_t^T \int_{\mathbb{R}^d} |Y_{s-}^n + z|^2 - |Y_{s-}^n|^2 \tilde{N}(dz, ds). \end{aligned} \tag{11}$$

Using assumption **(H)** and taking the expectation in the above equation under $\mathbb{P}\otimes\mathbb{P}^m$, we obtain

$$\begin{aligned} &\mathbb{E}\mathbb{E}^m\,|Y_t^n|^2 + \mathbb{E}\mathbb{E}^m \int_t^T |Z_s^n|^2\,ds + +\mathbb{E}\mathbb{E}^m \int_T^t \int_{\mathbb{R}^d} |U_s^n(z)|^2 v(dz)ds \\ \leq& \,\mathbb{E}\,|\Phi(X_T)|^2 + c_\varepsilon \mathbb{E}\mathbb{E}^m \int_t^T \big[|f_s^0(X_s)|^2 + |h_s^0(X_s)|^2\big]ds + c_\varepsilon\,\mathbb{E}\mathbb{E}^m \int_t^T |Y_s^n|^2\,ds \\ &+ \big(\beta^2 + \varepsilon\big)\,\mathbb{E}\mathbb{E}^m \int_t^T |Z_s^n|^2\,ds + \frac{1}{\gamma}\mathbb{E}\mathbb{E}^m[\sup_{t\leq s\leq T}|S_s|^2] + \gamma\mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \end{aligned}$$

where ε, γ are positive constants and c_ε is a constant which can be different from line to line. Finally Gronwall's lemma leads to the desired inequality. ∎

Lemma 7.

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \leq& \, c'\Big[\mathbb{E}\mathbb{E}^m\,|Y_t^n|^2 + \|\Phi\|_2^2\Big] + c_\varepsilon\Big[\mathbb{E}\int_t^T \big[\|f_s^0\|_2^2 + \|h_s^0\|_2^2\big]\,ds \\ &+ \mathbb{E}\mathbb{E}^m \int_t^T \big[\,|Y_s^n|^2 + |Z_s^n|^2 + \int_{\mathbb{R}^d} |U_s^n(z)|^2 v(dz)\big]ds\Big]. \end{aligned} \tag{12}$$

Proof. Let $(\widetilde{u}^n)_{n\in\mathbb{N}}$ be the weak solutions of the following linear type equations

$$d\widetilde{u}_t^n + \mathcal{A}\widetilde{u}_t^n + h_t\left(u_t^n, \nabla u_t^n\right) \cdot \overleftarrow{dB}_t = 0,$$

with final condition $\widetilde{u}_T^n = 0$. Set $\widetilde{Y}_t^n = \widetilde{u}^n(t, X_t)$, $\widetilde{Z}_t^n = \nabla \widetilde{u}^n(t, X_t)$ and $\widetilde{U}_t^n(z) = \widetilde{u}^n(t, X_{t-} + z) - \widetilde{u}^n(t, X_{t-})$. Then, by estimate (6), one has

$$\mathbb{E}\mathbb{E}^m\Big[\big|\widetilde{Y}_t^n\big|^2 + \int_0^T \big|\widetilde{Z}_s^n\big|\, ds + \int_0^T \int_{\mathbb{R}^d} |\widetilde{U}_t^n(z)|^2 v(dz)dt\Big] \le \tilde{c}\Lambda \tag{13}$$

where $\Lambda = \mathbb{E}\mathbb{E}^m \displaystyle\int_0^T |h_s(X_s, Y_s^n, Z_s^n)|^2 ds$. Since $u^n - \widetilde{u}^n$ verifies the equation

$$\partial_t(u_t^n - \widetilde{u}_t^n) + \mathcal{A}(u^n - \widetilde{u}_t^n) + f_t(u_t^n, \nabla u_t^n) + n(u_t^n - v_t)^-\, dt = 0,$$

we have the stochastic representation

$$\begin{aligned} Y_t^n - \widetilde{Y}_t^n = \Phi\left(X_T\right) + \int_t^T f_r\left(X_r, Y_r^n, Z_r^n\right) dr + K_T^n - K_t^n - \sum_i \int_t^T \left(Z_{i,r}^n - \widetilde{Z}_{i,r}^n\right) dW_r^i \\ - \int_t^T \int_{\mathbb{R}^d} \Big(U_s^n(z) - \tilde{U}_t^n(z)\Big)\, \tilde{N}(dz, ds), \end{aligned}$$

from which one obtains the estimate

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \le c\, \mathbb{E}\mathbb{E}^m\Big[\, |Y_t^n|^2 + |\widetilde{Y}_t^n|^2 + \big|\Phi\left(X_T\right)\big|^2 + \int_t^T \left(|f_s^0(X_s)|^2 + |Y_s^n|^2 + |Z_s^n|^2\right) ds \\ + \int_t^T |\widetilde{Z}_s^n|^2\, ds + \int_t^T \int_{\mathbb{R}^d} |U_t^n(z)|^2 v(dz)dt + \int_t^T \int_{\mathbb{R}^d} |\widetilde{U}_t^n(z)|^2 v(dz)dt\Big]. \end{aligned}$$

Hence, using (13), we get

$$\begin{aligned} \mathbb{E}\mathbb{E}^m[(K_T^n - K_t^n)^2] \le c'\, \mathbb{E}\mathbb{E}^m\left[|Y_t^n|^2 + |\Phi(X_T)|^2\right] + c'_\varepsilon \mathbb{E}\mathbb{E}^m\Big[\int_t^T \left[|f_s^0(X_s)|^2 + |h_s^0(X_s)|^2\right]\, ds \\ + \int_t^T \left[\, |Y_s^n|^2 + |Z_s^n|^2 + \int_{\mathbb{R}^d} |U_t^n(z)|^2 v(dz)\right] ds\,\Big], \end{aligned}$$

which gives our assertion. □

Lemma 8. *The quadruple* (Y^n, Z^n, U^n, K^n) *satisfies the following estimate*

$$\begin{aligned} &\mathbb{E}\mathbb{E}^m\big(\sup_{0\le s\le T} |Y_s^n|^2\big) + \mathbb{E}\mathbb{E}^m \int_0^T |Z_s^n|^2\, ds + \mathbb{E}\mathbb{E}^m\left(K_T^n\right)^2 + \mathbb{E}\,\mathbb{E}^m \int_0^T \int_{\mathbb{R}^d} |U_t^n(z)|^2 v(dz)dt \\ &\le c\left[\|\Phi\|_2^2 + \mathbb{E}\mathbb{E}^m\big(\sup_{0\le s\le T} |S_s|^2\big) + \mathbb{E}\int_0^T \left[\|f_s^0\|_2^2 + \|h_s^0\|_2^2\,\right] ds\right] \end{aligned}$$

where $c > 0$ *is a constant.*

Proof. From (10) and (12) we get

$$\begin{aligned} &(1 - \delta c')\, \mathbb{E}\mathbb{E}^m\, |Y_s^n|^2 + \left(1 - \beta^2 - \epsilon - \delta c'_\varepsilon\right) \mathbb{E}\mathbb{E}^m \int_s^T |Z_r^n|^2\, dr + \mathbb{E}\mathbb{E}^m \int_t^T \int_{\mathbb{R}^d} |U_r^n(z)|^2 v(dz)dr \\ &\le (1 + c'\delta)\, \|\Phi\|_2^2 + (c_\epsilon + \delta c'_\varepsilon)\, \Lambda + (c_\epsilon + \delta c'_\epsilon)\, \mathbb{E}\mathbb{E}^m \int_s^T |Y_r^n|^2\, ds + c_\delta\, \mathbb{E}\,\mathbb{E}^m\big(\sup_{t\le r\le T} |S_r|^2\big), \end{aligned}$$

where $\Lambda=\mathbb{E}\mathbb{E}^m\int_t^T\left[|f_s^0(X_s)|^2+|h_s^0(X_s)|^2\right]ds$. It then follows from Gronwall's lemma that

$$\sup_{0\le s\le T}\mathbb{E}\mathbb{E}^m\left(|Y_s^n|^2\right)+\mathbb{E}\mathbb{E}^m\int_s^T|Z_r^n|^2\,dr+\mathbb{E}\mathbb{E}^m\int_0^T\int_{\mathbb{R}^d}|U_t^n(z)|^2v(dz)dt+\mathbb{E}\mathbb{E}^m\left(K_T^n\right)^2$$
$$\le c_1\left[\|\Phi\|_2^2+\mathbb{E}\mathbb{E}^m\big(\sup_{0\le r\le T}|S_r|^2\big)+\mathbb{E}\int_s^T\left[\|f_r^0\|_2^2+\|h_r^0\|_2^2\right]dr\right].$$

Coming back to the equation (9) and using Bukholder-Davis-Gundy's inequality and the last estimates we get our statement. □

In order to prove the strong convergence of the sequence (Y^n,Z^n,U^n,K^n) we shall need the following essential result.

Lemma 9.

$$\lim_{n\to\infty}\mathbb{E}\mathbb{E}^m\left[\sup_{0\le t\le T}\left((Y_t^n-S_t)^-\right)^2\right]=0. \tag{14}$$

Proof. Let $(u^n)_{n\in\mathbb{N}}$ be the sequence of solutions of the penalized SPDE defined in (8). From Lemma 8, it follows that the sequence $(f(u^n,\nabla u^n),h(u^n,\nabla u^n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\times\Omega\times\mathbb{R}^d;\mathbb{R}^{1+d_1})$. By double extraction argument, we may choose a subsequence which is weakly convergent to a system of predictable processes $(\bar f,\bar h)$. Moreover, we can construct sequences $(\hat f^k)_k$ and $(\hat h^k)_k$ of convex combinations of elements of the form

$$\hat f^k=\sum_{i\in I_k}\alpha_i^kf(u^i,\nabla u^i),\quad \hat h^k=\sum_{i\in I_k}\alpha_i^kh(u^i,\nabla u^i)$$

converging strongly to $\bar f$ and $\bar h$ respectively in $L^2([0,T]\times\Omega\times\mathbb{R}^d;\mathbb{R})$ and $L^2([0,T]\times\Omega\times\mathbb{R}^d;\mathbb{R}^{d_1})$. For $i\ge n$, we denote by $u^{i,n}$ the solution of the equation

$$du_t^{i,n}+[\mathcal{A}u_t^{i,n}-nu_t^{i,n}+nv_t+f_t(u^i,\nabla u^i)]dt+h_t(u^i,\nabla u^i)\cdot\overleftarrow{dB}_t=0 \tag{15}$$

with final condition $u_T^{i,n}=v_T$. By comparison theorem, we know that $u^{i,n}\le u^i$. We set $\hat u^k=\sum_{i\in I_k}\alpha_i^ku^{i,n_k}$, where $n_k=\inf I_k$ and deduce that

$$\hat u^k\le\sum_{i\in I_k}\alpha_i^ku^i\le\lim_{n\to\infty}u^n, \tag{16}$$

where the last inequality comes from the monotonicity of the sequence u^n. Moreover, it is clear that $\hat u^k$ is a solution of the equation

$$d\hat u_t^k+[\mathcal{A}\hat u_t^k-n_k\hat u_t^k+n_kv_t+\hat f_t^k\,]\,dt+\hat h_t^k\cdot\overleftarrow{dB}_t=0 \tag{17}$$

with final condition $\hat u_T^k=v_T$.
Now we are going to take the advantage of the fact that the equations satisfied by the sequence of solutions $\hat u^k$ have strongly convergent coefficients. Let us denote by $\widehat{Y}^k$ the càdlàg version on $[0,T]$ of the process $(\hat u^k(X_t))_{t\in[0,T]}$, for any $k\in\mathbb{N}$. We will prove now that there exists a subsequence such that, for any stopping time $\tau\le T$

$$\lim_{k\to\infty}\widehat{Y}_\tau^k-S_\tau=0,\ \ \mathbb{P}\otimes\mathbb{P}^m\text{-}a.s. \tag{18}$$

Since equation (17) is linear, the solution decomposes as a sum of three terms each corresponding to one of the coefficients $\hat f^k,\hat h^k,v$. So it is enough to treat separately each term.

a) In the case where $f\equiv0$ and $h\equiv0$, one obtains the term corresponding to ν. The equation becomes:

$$\left(\partial_t+\mathcal{A}\right)\hat u^k-k\hat u^k+kv=0.$$

It is well known that $\widehat{Y}^k$ has the following representation:

$$\widehat{Y}^k_\tau = \mathbb{E}^m\Big[e^{-k(T-\tau)}S_T + k\int_\tau^T e^{-k(r-\tau)}S_r dr\Big|\mathcal{G}_\tau\Big].$$

Then, it is easy to get that $\lim_{k\to\infty}\widehat{Y}^k_\tau - S_\tau = 0$.

b) In the case where $v \equiv 0$, $h \equiv 0$, the representation of $\widehat{Y}^k$ is given by

$$\begin{aligned}\widehat{Y}^k_t = &\int_t^T e^{-n_k(s-t)}\hat{f}^k_s(X_s)ds - \int_t^T e^{-n_k(s-t)}\nabla\hat{u}^k_s(X_{s-})\,dW_s \\ &-\int_t^T\int_{\mathbb{R}^d} e^{-n_k(s-t)}\big(\hat{u}^k_s(X_{s-}+z) - \hat{u}^k_s(X_{s-})\big)\tilde{N}(dz,ds).\end{aligned} \tag{19}$$

Since

$$\Big|\int_t^T e^{-n_k(s-t)}\hat{f}^k_s(X_s)ds\Big| \le \frac{1}{\sqrt{2n_k}}\Big(\int_t^T(\hat{f}^k_s(X_s))^2ds\Big)^{1/2},$$

then $\lim_{k\to\infty}\sup_{0\le t\le T}\Big|\int_t^T e^{-n_k(s-t)}\hat{f}^k_s(X_s)ds\Big| = 0$, $\mathbb{P}\otimes\mathbb{P}^m$-*a.s.*. For the second and the third term in the expression of $\hat{Y}^k$ we make an integration by parts formula and get

$$\begin{aligned}&\int_t^T e^{-n_k(s-t)}\nabla\hat{u}^k_s(X_{s-})\,dW_s + \int_t^T\int_{\mathbb{R}^d} e^{-n_k(s-t)}\big(\hat{u}^k_s(X_{s-}+z) - \hat{u}^k_s(X_{s-})\big)\tilde{N}(dz,ds) \\ &= e^{-n_k(T-t)}U^k_T - U^k_t + n_k\int_t^T U^k_s e^{-n_k(s-t)}ds\end{aligned}$$

where $U^k_t = \int_0^t\nabla\hat{u}^k_s(X_{s-})\,dW_s + \int_0^t\int_{\mathbb{R}^d}(\hat{u}^k_s(X_{s-}+z) - \hat{u}^k_s(X_{s-})\tilde{N}(dz,ds)$. According to Corollary 1, we know that the martingales U^k, $k\in\mathbb{N}$ converges to zero in L^2, and hence on a subsequence we have $\lim_{k\to\infty}\sup_{0\le t\le T}|U^k_t| = 0$, $\mathbb{P}\otimes\mathbb{P}^m-$a.s.. Therefore, the desired result (18) holds also in this case.

c) In the case where $f\equiv 0$, $v\equiv 0$, the representation of $\widehat{Y}^k$ is given by

$$\begin{aligned}\widehat{Y}^k_t = &\int_t^T e^{-n_k(s-t)}\hat{h}^k_s(X_s)\cdot\overleftarrow{dB}_s - \int_t^T e^{-n_k(s-t)}\nabla\hat{u}^k_s(X_{s-})\,dW_s \\ &-\int_t^T\int_{\mathbb{R}^d} e^{-n_k(s-t)}\big(\hat{u}^k_s(X_{s-}+z) - \hat{u}^k_s(X_{s-})\big)\tilde{N}(dz,ds)\,.\end{aligned}$$

On account of Lemma 5, the same arguments used in the previous cases work again.

Since $\widehat{Y}^n$ is increasing and bounded, it converges to a limit process Y. Due to (16) and (18), we see that $Y_\tau \ge S_\tau$ for any stopping time τ. From this and the section theorem ([4], pp.220), we deduce that $\forall t,\ Y_t\ge S_t,\ \mathbb{P}\otimes\mathbb{P}^m-$a.s.. Hence, $\forall t,\ (Y^n_t - S_t)^-\searrow 0,\ \mathbb{P}\otimes\mathbb{P}^m-$a.s.. Since $Y^n\nearrow Y$, if we denote by ${}^pX$ the predictable projection of X, then ${}^pY^n\nearrow{}^pY$ and ${}^pY\ge{}^pS$. But the jumping times of Y^n are all inaccessible. Henceforth ${}^pY^n = Y^n_-$, where Y^n_- is the left limit process of Y^n. In the same way, we have ${}^pS = S_-$. So we have proved that ${}^pY^n\nearrow{}^pY\ge{}^pS$, which implies that $(Y^n_{t-} - S_{t-})^-\searrow 0, \mathbb{P}\otimes\mathbb{P}^m-$a.s. for any t. Consequencely from Lemma 3, we deduce that $\sup_{0\le t\le T}(Y^n_t - S_t)^-\longrightarrow 0$, a.s.. Finally, we finish the proof of the lemma, due to dominated convergence theorem. □

Then, we have the following result.

Lemma 10. *There exists a predictable measurable set of processes* $(Y_t, Z_t, U_t, K_t)_{t\in[0,T]}$ *such that*

$$\mathbb{E}\mathbb{E}^m\Big[\sup_{0\le s\le T}|Y_t^n - Y_t|^2 + \int_0^T |Z_t^n - Z_t|^2 dt + \sup_{0\le t\le T}|K_t^n - K_t|^2 + \int_0^T\int_{\mathbb{R}^d}|U_t^n(z) - U_t(z)|^2 v(dz)dt\Big] \longrightarrow 0 \quad \text{as } n\to\infty. \tag{20}$$

Moreover, we have that $(Y_t, Z_t, U_t, K_t)_{t\in[0,T]}$ *satisfies* $Y_t \ge S_t,\ \forall\, t\in[0,T]$ *and* $\int_0^T (Y_s - S_s)dK_s = 0$, $\mathbb{P}\otimes\mathbb{P}^m -$*a.e..*

Proof. From the monotonicity of the sequence $(f_n)_{n\in\mathbb{N}}$ and comparison theorem (see Theorem 4), we get that $u^n(t,x) \le u^{n+1}(t,x)$, $dtdx\otimes\mathbb{P}$-a.e., therefore one has $Y_t^n \le Y_t^{n+1}$, for all $t\in[0,T]$, $\mathbb{P}\otimes\mathbb{P}^m$ -a.s. Thus, there exists a predictable real valued process $Y = (Y_t)_{t\in[0,T]}$ such that $Y_t^n \uparrow Y_t$, for all $t\in[0,T]$ a.s. and by Lemma 8 and Fatou's lemma , one gets

$$\mathbb{E}\mathbb{E}^m\Big(\sup_{0\le s\le T}|Y_t|^2\Big) \le c.$$

Moreover, from the dominated convergence theorem one has

$$\mathbb{E}\mathbb{E}^m\int_0^T |Y_t^n - Y_t|^2 dt \longrightarrow 0 \quad \text{as } n\to\infty. \tag{21}$$

Moreover, for $n\ge p$,

$$\begin{aligned}
&|Y_t^n - Y_t^p|^2 + \int_t^T |Z_s^n - Z_s^p|^2 ds + \int_t^T\int_{\mathbb{R}^d}|U_s^n(z) - U_s^p(z)|^2 v(dz)ds\\
=&2\int_t^T (Y_s^n - Y_s^p)[f_s(X_s, Y_s^n, Z_s^n) - f_s(X_s, Y_s^p, Z_s^p)]ds + 2\int_t^T (Y_s^n - Y_s^p)d(K_s^n - K_s^p)\\
&+\int_t^T |h_s(X_s, Y_s^n, Z_s^n) - h_s(X_s, Y_s^p, Z_s^p)|^2 ds - 2\sum_i\int_t^T (Y_s^n - Y_s^p)(Z_{i,s}^n - Z_{i,s}^p)dW_s^i\\
&+2\int_t^T (Y_s^n - Y_s^p)[h_s(X_s, Y_s^n, Z_s^n) - h_s(X_s, Y_s^p, Z_s^p)]\cdot \overleftarrow{dB}_s\\
&+\int_t^T\int_{\mathbb{R}^d}|Y_{s-}^n + U_s^n - Y_{s-}^p - U_s^p|^2 - |Y_{s-}^n - Y_{s-}^p|^2\tilde{N}(dz, ds).
\end{aligned} \tag{22}$$

By standard calculation one deduces that

$$\begin{aligned}
&\mathbb{E}\mathbb{E}^m\int_t^T |Z_s^n - Z_s^p|^2 ds + \mathbb{E}\mathbb{E}^m\int_t^T\int_{\mathbb{R}^d}|U_s^n(z) - U_s^p(z)|^2 v(dz)ds\\
\le& c\,\mathbb{E}\mathbb{E}^m\int_t^T |Y_s^n - Y_s^p|^2 + 4\mathbb{E}\mathbb{E}^m\int_t^T (Y_s^n - S_s)^-\, dK_s^p + 4\mathbb{E}\mathbb{E}^m\int_t^T (Y_s^p - S_s)^-\, dK_s^n
\end{aligned} \tag{23}$$

Therefore , by Lemma 9, (21) and (23), it follows that

$$\begin{aligned}
&\mathbb{E}\mathbb{E}^m\int_0^T |Y_t^n - Y_t^p|^2 dt + \mathbb{E}\mathbb{E}^m\int_0^T |Z_t^n - Z_t^p|^2 dt + \mathbb{E}\mathbb{E}^m\int_0^T\int_{\mathbb{R}^d}|U_t^n(z) - U_t^p(z)|^2 v(dz)dt\\
&\longrightarrow 0 \quad \text{as } n, p\to\infty.
\end{aligned} \tag{24}$$

Then we can do a same argument as in Hamadène and Ouknine [13] and obtain (10). It is obvious that $(K_t)_{t\in[0,T]}$ is an increasing continuous process.

Moreover, Lemma 9 implies that

$$Y_t \geq S_t, \quad \forall\, t \in [0,T], \quad \mathbb{P}\otimes\mathbb{P}^m{-}a.s., \tag{25}$$

which yields that $\int_0^T (Y_s - S_s)dK_s \geq 0$.
Finally we also have $\int_0^T (Y_s - S_s)dK_s = 0$ since on the other hand the sequences $(Y^n)_{n\geq 0}$ and $(K^n)_{n\geq 0}$ converge uniformly (at least for a subsequence) respectively to Y and K and

$$\int_0^T (Y_s^n - S_s)dK_s^n = -n\int_0^T ((Y_s^n - S_s)^-)^2 ds \leq 0.$$

□

Proof of Theorem 3: Since

$$\int_0^T \|u_t^n - u_t^p\|_2^2 + \mathcal{E}(u_t^n - u_t^p)dt = \mathbb{E}^m\int_0^T [|Y_t^n - Y_t^p|^2 + |Z_t^n - Z_t^p|^2 + \int_{\mathbb{R}^d} |U_t^n(z) - U_t^p(z)|^2 v(dz)]dt,$$

by the preceding lemma, one deduces that the sequence $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega\times[0,T];H^1(\mathbb{R}^d))$ and hence has a limit u in this space. Also from the preceding lemma it follows that dK_t^n weakly converges to dK_t, $\mathbb{P}\otimes\mathbb{P}^m - a.e.$. This implies that

$$\lim_n \int_0^T \int_{\mathbb{R}^d} n(u_t^n - v_t)^- \varphi(t,x)\, dtdx = \lim_n \mathbb{E}^m\int_0^T \varphi_t(X_t)dK_t^n = \int_0^T\int_{\mathbb{R}^d}\varphi(t,x)\nu(dt,dx),$$

where ν is the regular measure defined by

$$\int_0^T\int_{\mathbb{R}^d}\varphi(t,x)\,\nu\big(dtdx\big) = E^m\int_0^T \varphi_t(X_t)\,dK_t.$$

Writing the equation (8) in the weak form and passing to the limit one obtains the equation (7) with u and this ν. The arguments we have explained after Definition 4 ensure that u admits a quasicontinuous version $\bar{u}$. Then one deduces that $(\bar{u}_t(X_t))_{t\in[0,T]}$ should coincide with $(Y_t)_{t\in[0,T]}$, $\mathbb{P}\otimes\mathbb{P}^m{-}$a.e.. Therefore, the inequality $Y_t \geq S_t$ implies $u \geq v$, $dt\otimes\mathbb{P}\otimes dx{-}$a.e. and the relation $\int_0^T (Y_t - S_t)dK_t = 0$ implies the relation (iv) of Definition 4. □

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