The Obstacle Problem for Quasilinear Stochastic PDEs with Neumann boundary condition

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Abstract: We prove the existence and uniqueness of solution of the obstacle problem for quasilinear stochastic partial differential equations (OSPDEs for short) with Neumann boundary condition. Our method is based on the analytical technics coming from parabolic potential theory. The solution is expressed as a pair (u, ν) where u is a predictable continuous process which takes values in a proper Sobolev space and ν is a random regular measure satisfying minimal Skohorod condition.

Keywords and phrases: parabolic potential, regular measure, stochastic partial differential equations, Neumann boundary condition, obstacle problem, penalization method, Itô's formula, comparison theorem.
AMS 2000 subject classifications: Primary 60H15; 35R60; 31B150.

1. Introduction

Our aim is to prove the existence and uniqueness result for the following OSPDE with Neumann boundary condition that we write formally as:

$$
\begin{cases}
du_t(x) = \partial_i \left(a_{i,j}(x) \partial_j u_t(x) + g_i(t, x, u_t(x), \nabla u_t(x)) \right) dt + f(t, x, u_t(x), \nabla u_t(x)) dt \\
\qquad + \sum_{j=1}^{+\infty} h_j(t, x, u_t(x), \nabla u_t(x)) dB_t^j, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathcal{O}, \\
\sum_{i,j} (a_{i,j} \partial_j u_t(x) + g_i(t, x, u_t(x), \nabla u_t(x))) n_i(x) = l(t, x, u_t(x)), \quad \forall (t, x) \in \mathbb{R}^+ \times \partial \mathcal{O}, \\
u_t(x) \geq S_t(x), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathcal{O}, \\
u_0(x) = \xi(x), \quad \forall x \in \mathcal{O},
\end{cases}
\tag{1}
$$

where $\vec{n}(x) = (n_1(x), n_2(x), ..., n_d(x))$ is the outer normal to $\mathcal{O} \subset \mathbb{R}^d$ at $x \in \partial D$ and we also use the notation $\frac{\partial u_t}{\partial v_L} := \sum_{i,j} a_{i,j} \partial_j u_t(x) n_i(x)$. $a = (a_{i,j})_{1 \leq i,j \leq d}$ is a symmetric bounded measurable matrix which defines a second order operator on $\mathcal{O}$, with Neumann boundary condition. The initial condition is given as $u_0 = \xi$, a $L^2(\mathcal{O})-$valued random variable, and

1

f, $g = (g_1, ..., g_d)$ and $h = (h_1, ...h_i, ...)$ are non-linear random functions. Given an obstacle $S : \Omega \times [0, T] \times \mathcal{O} \to \mathbb{R}$, we study the OSPDE (1), i.e. we want to find a solution that satisfies "$u \geq S$" where the obstacle S is regular in some sense and controlled by the solution of a stochastic partial differential equation (SPDE for short).

The obstacle problems for SPDEs have been widely studied since Nualart and Pardoux obtained the existence and uniqueness of solution for the obstacle problem with the diffusion matrix $a = I$ ([20]). Then Donati-Martin and Pardoux [12] proved it for the general diffusion matrix. Various properties of the solutions were studied later in [8], [25], [26] etc.. As backward stochastic differential equations (BSDEs for short) were rapidly developed, obstacle problem associated with a nonlinear partial differential equation (PDE for short) with more general coefficients, and the properties of the solutions for OSPDEs were studied in the framework of BSDEs (see [13], [4], [18], etc.).

This article is motivated by a serial research of the obstacle problems ([11], [18], etc.). The approach used in [18] is based on the probabilistic interpretation of the solution presented by backward doubly stochastic differential equations (BDSDEs for short). In [11], the authors studied the OSPDE with null Dirichlet condition on an open domain in $\mathbb{R}^d$. Their method was based on the technics of parabolic potential theory developed by M. Pierre ([21], [22]). Inspired by this analytic method, we will continue the study on this obstacle problem but with Neumann boundary condition in this paper. The key point is to construct a solution which admits a quasi-continuous version defined outside a polar set and the regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets. Due to the Neumann boundary condition is considered, the boundary behavior of the solution should be estimated, and thanks to trace theorem, this term can be controlled, with leads to the contraction condition on the coefficients should be strengthened in some sense.

The remainder of this paper is organized as follows: in the second section, we set the assumptions then we recall in the third section some useful results on SPDEs with Neumann boundary condition. In Section 4, we introduce the notion of regular measures associated to parabolic potentials. The quasi-continuity of the solution for SPDE without obstacle is proved in the fifth section. The last section is devoted to proving the existence and uniqueness result, and Itô's formula and comparison theorem are also established.

2. Preliminaries

We consider a sequence $((B^i(t))_{t \geq 0})_{i \in \mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded open domain with a smooth boundary and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, which is an Hilbert space equipped with the scalar product and norm as follows

$$(u, v) = \int_{\mathcal{O}} u(x)v(x)dx, \qquad \|u\| = (\int_{\mathcal{O}} u^2(x)dx)^{1/2}.$$

We assume the boundary $\partial \mathcal{O}$ of $\mathcal{O}$ is smooth, and define a Hilbert space $L^2(\partial \mathcal{O})$ with the

scalar product and norm as follows

$$(u,v)_{L^2(\partial\mathcal{O})} = \int_{\partial\mathcal{O}} u(x)v(x)d\sigma(x), \qquad \|u\|_{L^2(\partial\mathcal{O})} = \left(\int_{\partial\mathcal{O}} u^2(x)d\sigma(x)\right)^{1/2},$$

where $d\sigma(x)$ is the Lebesgue measure in $\mathbb{R}^{d-1}$.

For the simplicity of the notations, we sometimes also denote $(\cdot,\cdot)_{L^2(\partial\mathcal{O})}$ and $\|\cdot\|_{L^2(\partial\mathcal{O})}$ by $(\cdot,\cdot)$ and $\|\cdot\|$. But it will be clear which space we are referring to from the context.

We assume that $a = (a_{i,j})_{1\leq i,j\leq d}$ is a smooth symmetric matrix defined on $\bar{\mathcal{O}}$ satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{d} a_{i,j}(x)\xi^i\xi^j \leq \Lambda|\xi|^2, \ \forall x \in \bar{\mathcal{O}}, \ \xi \in \mathbb{R}^d,$$

where λ and Λ are positive constants.

Consider the second order generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ with null Neumann boundary condition, where

$$\mathcal{D}(\mathcal{A}) = \{u \in L^2(\mathcal{O})|\mathcal{A}u \in L^2(\mathcal{O}) \text{ and } \sum_{i,j} a_{i,j}\partial_j u \, n_i = 0 \text{ on } \partial\mathcal{O}\}$$

and

$$\mathcal{A}u = div(a_{i,j}\nabla u) = \sum_{i,j} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial u}{\partial x_j}).$$

Let $(F, \mathcal{E})$ be the associated Dirichlet form given by

$$F := H^1(\mathcal{O}) := \{u \in L^2(\mathcal{O})|\nabla u \in L^2(\mathcal{O}) \text{ in weak sense}\}$$

and

$$\mathcal{E}(u,v) := \sum_{i,j=1}^{n}(a_{i,j}\partial_j u, \partial_j v) \text{ and } \mathcal{E}(u) = \mathcal{E}(u,u), \ \forall u,v \in F.$$

The semigroup associated with $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $L^2(\mathcal{O})$ is denoted by $\{P_t\}_{t\geq 0}$.

We have the following theorem (Theorem 7.26 in [14] and Theorem 5.22 in [1]):

Theorem 1. *Let $\mathcal{O}$ be a smooth bounded domain in $\mathbb{R}^d$. Then*

(1.) if $0 \leq m < k - \frac{d}{2} < m+1$, the space $H^k(\mathcal{O})$ is continuously imbedded in $C^{m,\alpha}(\bar{\mathcal{O}})$, with $\alpha = k - d/2 - m$;

(2.) The trace operator Tr admits a continuous extension from $H^1(\mathcal{O})$ to $L^2(\partial\mathcal{O})$.

Note that for any $l \in L^2(\partial O)$, one can associate it with an element $\tilde{l} \in H^{-1}(\mathcal{O})$ (the dual space of $H^1(\mathcal{O})$) by $\tilde{l}(u) = \int_{\partial\mathcal{O}} l(x)u(x)d\sigma(x)$, for any $u \in H^1(\mathcal{O})$. Due to the continuity of the trace operator, we have $\|\tilde{l}\|_{H^{-1}} \leq \|Tr\|\|l\|_{L^2(\partial\mathcal{O})}$.

We assume that we have predictable random functions

$$f : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R},$$
$$g = (g_1, ..., g_d) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d,$$
$$h = (h_1, ..., h_i, ...) : \mathbb{R}^+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*},$$
$$l : \mathbb{R}^+ \times \Omega \times \partial\mathcal{O} \times \mathbb{R} \to \mathbb{R}.$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm.

Assumption (H): There exist non-negative constants C, α, β, θ such that for almost all ω, the following inequalities hold for all $(x, y, z, t) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$:

1. $|f(t, \omega, x, y, z) - f(t, \omega, x, y', z')| \leq C(|y - y'| + |z - z'|)$,
2. $(\sum_{i=1}^{d} |g_i(t, \omega, x, y, z) - g_i(t, \omega, x, y', z')|^2)^{\frac{1}{2}} \leq C|y - y'| + \alpha|z - z'|$,
3. $(|h(t, \omega, x, y, z) - h(t, \omega, x, y', z')|^2)^{\frac{1}{2}} \leq C|y - y'| + \beta|z - z'|$,
4. $|l(t, \omega, x, y) - l(t, \omega, x, y')| \leq \theta|y - y'|$,
5. the contraction property: $2\alpha + \beta^2 + 2\|Tr\|^2\theta < 2\lambda$.

Remark 1. *The last contraction property ensures existence and uniqueness for the solution of the SPDE without obstacle. (see [10])*

Assumption (I): Moreover we assume that for any $T > 0$,

$$\xi \in L^2(\Omega \times \mathcal{O}) \text{ is an } \mathcal{F}_0\text{-measurable random variable};$$

$$f(\cdot, \cdot, \cdot, 0, 0) := f^0 \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R});$$

$$g(\cdot, \cdot, \cdot, 0, 0) := (g_1^0, ..., g_d^0) \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^d);$$

$$h(\cdot, \cdot, \cdot, 0, 0) := (h_1^0, ..., h_i^0, ...) \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*});$$

$$l(\cdot, \cdot, 0) := l^0 \in L^2([0, T] \times \Omega \times \partial\mathcal{O}; \mathbb{R}).$$

Now we introduce the notion of weak solution. For simplicity, we fix the terminal time $T > 0$. We denote by $\mathcal{H}_T$ the space of $H^1(\mathcal{O})$-valued predictable processes $(u_t)_{t \geq 0}$ which satisfies

$$E \sup_{t \in [0, T]} \|u_t\|^2 + E \int_0^T \mathcal{E}(u_t)dt < +\infty.$$

It is the natural space for solutions.

The space of test functions is denote by $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}^+) \otimes \mathcal{C}^2(\bar{\mathcal{O}})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ is the space of all real valued infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}^2(\bar{\mathcal{O}})$ the set of functions in $C^2(\mathcal{O})$ all of whose derivatives of order less than or equal to 2 have continuous extensions to $\bar{\mathcal{O}}$.

3. Linear SPDEs with Neumann boundary condition

In this section, we recall some results concerning the linear SPDEs with Neumann boundary condition. First, consider the equation with regular coefficients and null Neumann boundary condition:

$$\begin{cases} du_t(x) = \{\mathcal{A}u_t(x) + f(t, x)\}dt + h^i(t, x)dB_t^i, & \forall(t, x) \in \mathbb{R}^+ \times \mathcal{O}, \\ \sum_{i,j} a_{i,j}(x)\partial_j u_t(x)n_i(x) = 0, & \forall(t, x) \in \mathbb{R}^+ \times \partial\mathcal{O}, \\ u_0(x) = \xi, & \forall x \in \mathcal{O}, \end{cases} \tag{2}$$

with $f, h^i \in L^2([0, T] \times \Omega \times \mathcal{O}; \mathbb{R})$. According to Theorem 5.4 in [7], we know that (2) has a unique weak solution which can be written as:

$$u_t = P_t\xi + \int_0^t P_{t-s}f_s ds + \int_0^t P_{t-s}h_s^i dB_s^i, \qquad a.s..$$

Note that in this case, u belongs to $\mathcal{H}_T$.

For more general case,

$$\begin{cases} du_t(x) = \{\mathcal{A}u_t(x) + \partial_i g^i(t,x) + f(t,x)\}dt + h^i(t,x)dB_t^i, & \forall (t,x) \in \mathbb{R}^+ \times \mathcal{O}, \\ \sum_{i,j}(a_{i,j}(x)\partial_j u_t(x) + g_t^i(x))n_i(x) = l_t(x), & \forall (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}, \\ u_0(x) = \xi, & \forall x \in \mathcal{O}, \end{cases} \tag{3}$$

with $f, h^i, g^i \in L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$ and $l \in L^2([0,T] \times \Omega \times \partial\mathcal{O}; \mathbb{R})$. By the linearity of the coefficients, it is easy to see that the weak solution u can be decomposed into $u = u_1 + u_2$, where u_1 is the solution of (2) and u_2 is the solution of PDEs without random coefficients:

$$\begin{cases} du_t(x) = \{\mathcal{A}u_t(x) + \partial_i g^i(t,x)\}dt, & \forall (t,x) \in \mathbb{R}^+ \times \mathcal{O}, \\ \sum_{i,j}(a_{i,j}(x)\partial_j u_t(x) + g_t^i(x))n_i(x) = l_t(x), & \forall (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}, \\ u_0(x) = \xi, & \forall x \in \mathcal{O}. \end{cases} \tag{4}$$

Now we assume that the coefficients ξ, f, g and h are smooth and $h = \sum_k h_k 1_{[t_k, t_{k+1})}$ with h_k being $\mathcal{F}_{t_k}$-measurable and smooth in x. For the purpose of later discussion, we will show that the associated solution u has a continuous version in (t,x). First, let us recall some results of deterministic PDEs:

$$\begin{cases} du_t^2(x) = \{\mathcal{A}u_t^2(x) + \partial_i g^i(t,x)\}dt, & \forall (t,x) \in \mathbb{R}^+ \times \mathcal{O}, \\ \sum_{i,j}(a_{i,j}(x)\partial_j u_t^2(x) + g_t^i(x))n_i(x) = l_t(x), & \forall (t,x) \in \mathbb{R}^+ \times \partial\mathcal{O}, \\ u_0(x) = \xi, & \forall x \in \mathcal{O}. \end{cases} \tag{5}$$

According to Theorem 5.18 in [16], if $\partial_i g^i \in C_\alpha$, $l \in C_{1+\alpha}$ and $\xi \in C_{2+\alpha}$, then the solution of (5) belongs to $C_{2+\alpha}$, where C_α is the Hölder space defined in Section 4.1 of [16]. This implies that the solution u_2 of (4) is continuous for almost all ω. Due to the definition of h^i, it is easy to see that $P_s h_s^i(x)$ is continuous in space-time variables on any time interval $[t_k, t_{k+1})$. Hence, the Itô integral $\int_0^t P_{t-s} h_s^i(x) dB_s^i$ is continuous in space-time variables and belongs to $C_{2+\alpha}$ for any t. Moreover, if $f \in C_\alpha$, the solution u_1 of (2) belongs to $C_{2+\alpha}$, P-a.s.. Thus, we can deduce that u has a continuous version with respect to (t,x).

At last, Itô's formula for the solution of (3) is proved.

Lemma 1. *Let u be the solution of* (3) *and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial\Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded. Let u be a weak solution of* (3). *Then, P-a.s. for all $t \in [0,T]$,*

$$\int_\mathcal{O} \Phi(t, u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s, u_s), u_s)ds = \int_\mathcal{O} \Phi(0, \xi(x))dx + \int_0^t \int_\mathcal{O} \frac{\partial\Phi}{\partial s}(s, u_s(x))dxds$$

$$+ \int_0^t (\Phi'(s, u_s), f_s)ds - \sum_{i=1}^d \int_0^t \int_\mathcal{O} \Phi''(s, u_s(x))\partial_i u_s(x)g_s^i(x)dxds + \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s), h_j)dB_s^j$$

$$+ \int_0^t \int_{\partial\mathcal{O}} \Phi(s, u_s(x))l_s(x)d\sigma(x)ds + \frac{1}{2}\sum_{j=1}^{+\infty} \int_0^t \int_\mathcal{O} \Phi''(s, u_s(x))(h_s^j(x))^2 dxds.$$

Proof. The proof is similar to Lemma 7 in [10] and Theorem 4.17 in [7]. Thus we only give a sketch of it. Let $0 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = t$ be a partition of time interval $[0, t]$ and $\Delta := \max_{i=1,2,...,n} |t_k - t_{k-1}|$. Then, $P - a.s.$,

$$\int_{\mathcal{O}} \Phi(t, u_t(x))dx - \int_{\mathcal{O}} \Phi(0, \xi(x))dx = \sum_{k=1}^{n} \Big(\int_{\mathcal{O}} \Phi(t_k, u_{t_k}(x))dx - \int_{\mathcal{O}} \Phi(t_{k-1}, u_{t_{k-1}}(x))dx \Big)$$

$$= \sum_{k=1}^{n} \Big\{ \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(t_{k-1}, u_{t_{k-1}}(x))(t_k - t_{k-1})dx + \int_{\mathcal{O}} \Phi'(t_{k-1}, u_{t_{k-1}}(x))(u_{t_k}(x) - u_{t_{k-1}}(x))dx$$

$$+ \int_{\mathcal{O}} \frac{1}{2}\Phi''(t_{k-1}, u_{t_{k-1}}(x))(u_{t_k}(x) - u_{t_{k-1}}(x))^2 dx \Big\} + o(\Delta).$$

Since u is a weak solution of (3), we have

$$\int_{\mathcal{O}} \Phi'(t_{k-1}, u_{t_{k-1}}(x))(u_{t_k}(x) - u_{t_{k-1}}(x))dx = - \int_{t_{k-1}}^{t_k} \mathcal{E}(\Phi'(t_{k-1}, u_{t_{k-1}}), u_s)ds$$

$$+ \int_{t_{k-1}}^{t_k} (\Phi'(t_{k-1}, u_{t_{k-1}}), f_s)ds - \sum_{i=1}^{d} \int_{t_{k-1}}^{t_k} \int_{\mathcal{O}} \Phi''(t_{k-1}, u_{t_{k-1}}(x))\partial_i u_{t_{k-1}}(x)g_s^i(x)dxds$$

$$+ \sum_{j=1}^{+\infty} \int_{0}^{t} (\Phi'(t_{k-1}, u_{t_{k-1}}), h_s^j)dB_s^j + \int_{t_{k-1}}^{t_k} \int_{\partial \mathcal{O}} \Phi(t_{k-1}, u_{t_{k-1}}(x))l_s(x)d\sigma(x)ds.$$

Thus, letting $\Delta \to +\infty$, we have, $P - a.s.$,

$$\sum_{k=1}^{n} \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(t_{k-1}, u_{t_{k-1}}(x))(t_k - t_{k-1})dx \to \int_{0}^{t} \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x))dxds,$$

$$\sum_{k=1}^{n} \Phi'(t_{k-1}, u_{t_{k-1}}(x))(u_{t_k}(x) - u_{t_{k-1}}(x))dx$$

$$\to - \int_{0}^{t} \mathcal{E}(\Phi'(s, u_s), u_s)ds + \int_{0}^{t} (\Phi'(s, u_s), f_s)ds - \sum_{i=1}^{d} \int_{0}^{t} \int_{\mathcal{O}} \Phi''(s, u_s(x))\partial_i u_s(x)g_s^i(x)dxds$$

$$+ \sum_{j=1}^{+\infty} \int_{0}^{t} (\Phi'(s, u_s), h_s^j)dB_s^j + \int_{0}^{t} \int_{\partial \mathcal{O}} \Phi(s, u_s(x))l_s(x)d\sigma(x)ds,$$

and

$$\sum_{k=1}^{n} \int_{\mathcal{O}} \frac{1}{2}\Phi''(t_{k-1}, u_{t_{k-1}}(x))(u_{t_k}(x) - u_{t_{k-1}}(x))^2 dx \to \frac{1}{2} \sum_{j=1}^{+\infty} \int_{0}^{t} \int_{\mathcal{O}} \Phi''(s, u_s(x))(h_s^j(x))^2 dxds.$$

Combine the three equalities above, we get the desired relation. $\square$

Remark 2. *As we have the Itô formula, the following estimate for the solution of (3) can be easily deduced:*

$$E \sup_t \|u_t\|^2 + E \int_0^T \mathcal{E}(u_t)dt \le CE \left(\|\xi\|^2 + \int_0^T \|f_t\|^2 + \||g_t|\|^2 + \||h_t|\|^2 + \|l_t\|^2 dt \right), \quad (6)$$

*where the constant C only depends on the constants in Assumption (**H**), T and d.*

4. Parabolic potential analysis

4.1. Parabolic capacity and potentials

In this section we will recall some important definitions and results concerning the obstacle problem for parabolic PDE in [21] and [22].
$\mathcal{K}$ denotes $L^\infty(0,T;L^2(\mathcal{O})) \cap L^2(0,T;H^1(\mathcal{O}))$ equipped with the norm:

$$
\begin{aligned}
\|v\|_{\mathcal{K}}^2 &= \|v\|_{L^\infty(0,T;L^2(\mathcal{O}))}^2 + \|v\|_{L^2(0,T;H^1(\mathcal{O}))}^2 \\
&= \sup_{t\in[0,T[} \|v_t\|^2 + \int_0^T \left(\|v_t\|^2 + \mathcal{E}(v_t)\right) dt.
\end{aligned}
$$

$\mathcal{C}$ denotes the space of continuous functions on compact support in $[0,T[\times\bar{\mathcal{O}}$ and finally:

$$
\mathcal{W} = \{\varphi \in L^2(0,T;H^1(\mathcal{O}));\ \frac{\partial\varphi}{\partial t} \in L^2(0,T;H^{-1}(\mathcal{O}))\},
$$

endowed with the norm $\|\varphi\|_{\mathcal{W}}^2 = \|\varphi\|_{L^2(0,T;H^1(\mathcal{O}))}^2 + \|\frac{\partial\varphi}{\partial t}\|_{L^2(0,T;H^{-1}(\mathcal{O}))}^2$.
It is known (see [17]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. Without ambiguity, we will also consider $\mathcal{W}_T = \{\varphi \in \mathcal{W}; \varphi(T) = 0\}$, $\mathcal{W}^+ = \{\varphi \in \mathcal{W}; \varphi \geq 0\}$, $\mathcal{W}_T^+ = \mathcal{W}_T \cap \mathcal{W}^+$.
We now introduce the notion of parabolic potentials and regular measures which permit to define the parabolic capacity.

Definition 1. *An element $v \in \mathcal{K}$ is said to be a **parabolic potential** if it satisfies:*

$$
\forall\varphi \in \mathcal{W}_T^+,\ \int_0^T -(\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt \geq 0.
$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 1. *(Proposition 1.1 in [22]) Let $v \in \mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\bar{\mathcal{O}}$, denoted by ν^v, such that:*

$$
\forall\varphi \in \mathcal{W}_T \cap \mathcal{C},\ \int_0^T (-\frac{\partial\varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt = \int_0^T \int_{\bar{\mathcal{O}}} \varphi(t,x)d\nu^v.
$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\hat{v}$ (resp. $\bar{v}$) : $[0,T] \mapsto L^2(\mathcal{O})$.
*Such a Radon measure, ν^v is called **a regular measure** and we write:*

$$
\nu^v = \frac{\partial v}{\partial t} - \mathcal{A}v.
$$

Remark 3. *As a consequence, we can also define for all $v \in \mathcal{P}$:*

$$
v_T = \lim_{t\uparrow T} \bar{v}_t \ \in L^2(\mathcal{O}).
$$

Definition 2. *Let $K \subset [0,T[\times\bar{\mathcal{O}}$ be compact, $v \in \mathcal{P}$ is said to be $\nu-$ superior than 1 on K, if there exists a sequence $v_n \in \mathcal{P}$ with $v_n \geq 1$ a.e. on a neighborhood of K converging to v in $L^2(0,T;H^1(\mathcal{O}))$.*

We denote:

$$\mathscr{S}_K = \{v \in \mathcal{P}; \ v \ is \ \nu - superior \ to \ 1 \ on \ K\}.$$

Proposition 2. *(Proposition 2.1 in [22]) Let $K \subset [0,T[\times\bar{\mathcal{O}}$ compact, then $\mathscr{S}_K$ admits a smallest $v_K \in \mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T \int_{\bar{\mathcal{O}}} d\nu_K^v = \inf_{v \in \mathcal{P}} \{ \int_0^T \int_{\bar{\mathcal{O}}} d\nu^v; \ v \in \mathscr{S}_K \}.$$

Definition 3. *(Parabolic Capacity)*

- *Let $K \subset [0,T[\times\bar{\mathcal{O}}$ be compact, we define $cap(K) = \int_0^T \int_{\bar{\mathcal{O}}} d\nu_K^v$;*
- *let $O \subset [0,T[\times\bar{\mathcal{O}}$ be open, we define $cap(O) = \sup\{cap(K); \ K \subset O \ compact\}$;*
- *for any borelian $E \subset [0,T[\times\bar{\mathcal{O}}$, we define $cap(E) = \inf\{cap(O); \ O \supset E \ open\}$.*

Definition 4. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Definition 5. *(Quasi-continuous)*
A function $u : [0,T[\times\bar{\mathcal{O}} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\bar{\mathcal{O}}$ with:

1. *for all n, the restriction of u to the complement of O_n is continuous;*
2. *$\lim_{n \to +\infty} cap(O_n) = 0$.*

We say that u admits a quasi-continuous version, if there exists $\tilde{u}$ quasi-continuous such that $\tilde{u} = u$ a.e..

The next proposition, whose proof may be found in [21] or [22] shall play an important role in the sequel:

Proposition 3. *Let $K \subset \bar{\mathcal{O}}$ a compact set, then $\forall t \in [0,T[$*

$$cap(\{t\} \times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T-}$ is also defined without ambiguity.

Remark 4. *The previous proposition applies if for example u is quasi-continuous.*

Proposition 4. *(Theorem III.1 in [22]) If $\varphi \in \mathcal{W}$, then it admits a unique quasi-continuous version that we denote by $\tilde{\varphi}$. Moreover, for all $v \in \mathcal{P}$, the following relation holds:*

$$\int_{[0,T[\times\bar{\mathcal{O}}} \tilde{\varphi} d\nu^v = \int_0^T \left(-\partial_t \varphi, v\right) + \mathcal{E}(\varphi, v) \, dt + \left(\varphi_T, v_T\right).$$

4.2. Applications to PDE's with obstacle

For any function $\psi : [0, T[\times\mathcal{O} \to \mathbb{R}$ and $u_0 \in L^2(\mathcal{O})$, following M. Pierre [21, 22], F. Mignot and J.P. Puel [19], we define

$$\kappa(\psi, u_0) = \operatorname{ess\,inf}\{u \in \mathcal{P}; \ u \geq \psi \ a.e., \ u(0) \geq u_0\}. \tag{7}$$

This lower bound exists and is an element in $\mathcal{P}$. Moreover, when ψ is quasi-continuous, this potential is the solution of the following reflected problem:

$$\kappa \in \mathcal{P}, \ \kappa \geq \psi, \ \frac{\partial \kappa}{\partial t} - \mathcal{A}\kappa = 0 \text{ and } \frac{\partial \kappa}{\partial v_L} = 0 \text{ on } \{u > \psi\}, \ \kappa(0) = u_0.$$

Mignot and Puel have proved in [19] that $\kappa(\psi, u_0)$ is the limit (increasingly and weakly in $L^2(0, T; H^1(\mathcal{O}))$) when ϵ tends to 0 of the solution of the following penalized equation

$$u_\epsilon \in \mathcal{W}, \ u_\epsilon(0) = u_0, \ \frac{\partial u_\epsilon}{\partial t} - \mathcal{A}u_\epsilon - \frac{(u_\epsilon - \psi)^-}{\epsilon} = 0, \frac{\partial u_\epsilon}{\partial v_L} = 0.$$

Let us point out that they obtain this result in the more general case where ψ is only measurable from $[0, T[$ into $L^2(\mathcal{O})$.

For given $f \in L^2(0, T; H^{-1}(\mathcal{O}))$ and $l \in L^2(0, T; L^2(\partial O))$, we denote by $\kappa_{u_0}^{f,l}$ the solution of the following problem:

$$\kappa \in \mathcal{W}, \ \kappa(0) = u_0, \ \frac{\partial \kappa}{\partial t} - \mathcal{A}\kappa = f, \frac{\partial \kappa}{\partial v_L} = l.$$

The next theorem ensures existence and uniqueness of the solution of parabolic PDE with obstacle, it is proved in [21], Theorem 1.1. The proof is based on a regularization argument of the obstacle, using the results of [6].

Theorem 2. *Let $\psi : [0, T[\times\mathcal{O} \to \mathbb{R}$ be quasi-continuous, suppose that there exists $\zeta \in \mathcal{P}$ with $|\psi| \leq \zeta$ a.e., $f \in L^2(0, T; H^{-1}(\mathcal{O})), l \in L^2(0, T; L^2(\partial\mathcal{O}))$, and the initial value $u_0 \in L^2(\mathcal{O})$ with $u_0 \geq \psi(0)$, then there exists a unique $u \in \kappa_{u_0}^{f,l} + \mathcal{P}$ quasi-continuous such that:*

$$u(0) = u_0, \ \tilde{u} \geq \psi, \ q.e.; \quad \int_0^T \int_{\bar{\mathcal{O}}} (\tilde{u} - \tilde{\psi}) d\nu^{u - \kappa_{u_0}^{f,l}} = 0$$

We end this section by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [22]):

Lemma 2. *If $v^n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2(0, T; H^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n \to +\infty} \int_0^T \int_{\bar{\mathcal{O}}} u d\nu^{v^n} = \int_0^T \int_{\bar{\mathcal{O}}} u d\nu^v.$$

Remark 5. *For the more general case one can see [22] Lemma 3.8.*

5. Quasi–continuity of the solution of SPDE without obstacle

Under assumptions **(H)** and **(I)**, SPDE (3) with Neumann boundary condition admits a unique solution in $\mathcal{H}_T$. We denote it by $\mathcal{U}(\xi, f, g, h, l)$.
The main theorem of this section is the following:

Theorem 3. *Under assumptions* **(H)** *and* **(I)**, *$u = \mathcal{U}(\xi, f, g, h, l)$ the solution of SPDE* (3) *admits a quasi-continuous version denoted by $\tilde{u}$ i.e. $u = \tilde{u}\ dP \otimes dt \otimes dx - a.e.$ and for almost all $w \in \Omega$, $(t, x) \to \tilde{u}_t(w, x)$ is quasi-continuous.*

Before giving the proof of this theorem, we need the following lemmas. The first one is proved in [22], Lemma 3.3:

Lemma 3. *There exists $C > 0$ such that, for all open set $\vartheta \subset [0, T[\times\bar{\mathcal{O}}$ and $v \in \mathcal{P}$ with $v \geq 1$ a.e. on ϑ:*

$$cap\vartheta \leq C\|v\|_{\mathcal{K}}^2.$$

Let $\kappa = \kappa(u, u^+(0))$ be defined by relation (7). One has to note that κ is a random function. From now on, we always take for κ the following measurable version

$$\kappa = \sup_n v^n,$$

where (v^n) is the non-decreasing sequence of random functions given by

$$
\begin{cases}
\dfrac{\partial v_t^n}{\partial t} = \mathcal{A}v_t^n + n(v_t^n - u_t)^-, \\
\dfrac{\partial v^n}{\partial v_L} = 0, \ v_0^n = u^+(0).
\end{cases}
\tag{8}
$$

Using the results recalled in Subsection 4, we know that for almost all $w \in \Omega$, $v^n(w)$ converges weakly to $v(w) = \kappa(u(w), u^+(0)(w))$ in $L^2(0, T; H^1(\mathcal{O}))$ and that $v \geq u$.

Lemma 4. *We have the following estimate:*

$$E\|\kappa\|_{\mathcal{K}}^2 \leq C \left(E\|u_0^+\|^2 + E\|u_0\|^2 + E \int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2 dt \right),$$

where C is a constant depending only on the structure constants of the equation.

Proof. All along this proof, we shall denote by C or C_ϵ some constant which may change from line to line.
Consider the approximation $(v^n)_n$ defined by (8), P-almost surely, it converges weakly to $v = \kappa(u, u^+(0))$ in $L^2(0, T; H^1(\mathcal{O}))$. We remark that $v^n - u$ satisfies the following equation:

$$
\begin{cases}
d(v_t^n - u_t) - \mathcal{A}(v_t^n - u_t)dt = -f_t dt - \displaystyle\sum_{i=1}^d \partial_i g_t^i dt - \sum_{j=1}^{+\infty} h_t^j dB_t^j + n(v_t^n - u_t)^- dt, \\
\displaystyle\sum_{i,j} (a_{i,j}\partial_j(v_t^n - u_t) - g_t^i)n_i = -l_t.
\end{cases}
$$

Applying Itô's formula (Lemma 1) to $(v^n - u)^2$, we have almost surely, for all $t \in [0, T]$,

$$\|v_t^n - u_t\|^2 + 2\int_0^t \mathcal{E}(v_s^n - u_s)ds = \|u_0^+ - u_0\|^2 - 2\int_0^t (v_s^n - u_s, f_s)ds$$

$$+ 2\sum_{i=1}^d \int_0^t (\partial_i(v_s^n - u_s), g_s^i)ds + \int_0^t \||h_s|\|^2 ds - 2\int_0^t (l_s, v_s^n - u_s)_{L^2(\partial\mathcal{O})}ds \qquad (9)$$

$$- 2\sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j)dB_s^j + 2\int_0^t (n(v_s^n - u_s)^-, v_s^n - u_s)ds.$$

The last term in the right member of (9) is obviously non-positive and noting that

$$(l_s, v_s^n - u_s)_{L^2(\partial\mathcal{O})} \leq \|l_s\|_{L^2(\partial\mathcal{O})}\|v_s^n - u_s\|_{L^2(\partial\mathcal{O})} \leq \|Tr\|\|l_s\|_{L^2(\partial\mathcal{O})}\|v_s^n - u_s\|_{H^1(\mathcal{O})},$$

then, taking expectation and using Cauchy-Schwartz's inequality, we get

$$E\|v_t^n - u_t\|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds$$

$$\leq E\|u_0^+ - u_0\|^2 + CE\int_0^t \|v_s^n - u_s\|^2 ds + C_\epsilon E\int_0^t \|l(s, u_s)\|_{L^2(\partial\mathcal{O})}^2 ds$$

$$+ E\int_0^t \|f_s(u_s, \nabla u_s)\|^2 ds + C_\epsilon E\int_0^t \||g_s(u_s, \nabla u_s)|\|^2 ds + E\int_0^t \||h_s(u_s, \nabla u_s)|\|^2 ds.$$

Therefore, by using the Lipschitz conditions on the coefficients, we have

$$E\|v_t^n - u_t\|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds$$

$$\leq E\|u_0^+ - u_0\|^2 + CE\int_0^t \|v_s^n - u_s\|^2 ds + C\Big(E\int_0^t \|u_s\|^2 ds + E\int_0^t \mathcal{E}(u_s)ds\Big)$$

$$+ CE\int_0^t (\|f_s^0\|^2 + \||g_s^0|\|^2 + \||h_s^0|\|^2 + \|l_s^0\|_{L^2(\partial\mathcal{O})}^2)ds.$$

Combining with (6), this yields

$$E\|v_t^n - u_t\|^2 + (2 - \frac{\epsilon}{\lambda})E\int_0^t \mathcal{E}(v_s^n - u_s)ds \leq E\|u_0^+ - u_0\|^2 + CE\int_0^t \|v_s^n - u_s\|^2 ds$$

$$+ CE\Big(\|u_0\|^2 + \int_0^T (\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_s^0\|_{L^2(\partial\mathcal{O})}^2)dt\Big).$$

We take now ϵ small enough such that $(2 - \frac{\epsilon}{\lambda}) > 0$, then, with Gronwall's lemma, we obtain for each $t \in [0, T]$

$$E\|v_t^n - u_t\|^2 \leq Ce^{c'T}\Big(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2$$

$$+ E\int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2 dt\Big).$$

As we a priori know that P-almost surely, $(v^n)_n$ tends to κ strongly in $L^2([0, T] \times \mathcal{O})$, the previous estimate yields, thanks to the dominated convergence theorem, that $(v^n)_n$ converges to κ strongly in $L^2(\Omega \times [0, T] \times \mathcal{O})$ and

$$
\sup_{t \in [0,T]} E\|\kappa_t - u_t\|^2 \leq Ce^{c'T}\Big(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2
$$

$$
+ E \int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2 dt\Big).
$$

Moreover, as $(v^n)_n$ tends to κ weakly in $L^2([0, T]; H^1(\mathcal{O}))$ P-a.s., we have for all $t \in [0, T]$:

$$
E \int_0^T \mathcal{E}(\kappa_s - u_s) ds \leq \liminf_n E \int_0^T \mathcal{E}(v_s^n - u_s) ds
$$

$$
\leq TCe^{c'T}\Big(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E \int_0^T \|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2 dt\Big).
$$

Let us now study the stochastic term. Define the martingales

$$
M_t^n = \sum_{j=1}^{+\infty} \int_0^t (v_s^n - u_s, h_s^j) dB_s^j \quad \text{and} \quad M_t = \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j) dB_s^j.
$$

Then

$$
E[|M_T^n - M_T|^2] = E \int_0^T \sum_{j=1}^{+\infty} (\kappa_s - v_s^n, h_s)^2 \, ds \leq E \int_0^T \|\kappa_s - v_s^n\|^2 \||h_s|\|^2 ds.
$$

Using the strong convergence of (v^n) to κ we conclude that (M^n) tends to M in L^2 sense. Passing to the limit, we get, almost surely, for all $t \in [0, T]$,

$$
\|\kappa_t - u_t\|^2 + 2 \int_0^t \mathcal{E}(\kappa_s - u_s) ds \leq \|u_0^+ - u_0\|^2 - 2 \int_0^t (\kappa_s - u_s, f_s) ds
$$

$$
+ 2 \sum_{i=1}^d \int_0^t (\partial_i(\kappa_s - u_s), g_s^i) ds - 2 \int_0^t (\kappa_s - u_s, l_s)_{L^2(\partial\mathcal{O})} ds
$$

$$
- 2 \sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j) dB_s^j + \int_0^t \||h_s|\|^2 ds.
$$

As a consequence of Burkholder-Davies-Gundy's inequalities, we get

$$
E \sup_{t \in [0,T]} |\sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s^j) dB_s^j| \leq CE[\int_0^T \sum_{j=1}^{+\infty} (\kappa_s - u_s, h_s^j)^2 ds]^{1/2}
$$

$$
\leq CE[\sup_{t \in [0,T]} \|\kappa_t - u_t\|(\int_0^T \||h_t|\|^2 dt)^{1/2}]
$$

$$
\leq \epsilon E \sup_{t \in [0,T]} \|\kappa_t - u_t\|^2 + C_\epsilon E \int_0^T \||h_t|\|^2 dt.
$$

By Lipschitz conditions on h and (6) this yields

$$E \sup_{t \in [0,T]} |\sum_{j=1}^{+\infty} \int_0^t (\kappa_s - u_s, h_s) dB_s| \leq \epsilon E \sup_{t \in [0,T]} \|\kappa_t - u_t\|\|^2 + C\Big(E\|u_0\|^2$$
$$+ E \int_0^T \big(\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2\big)dt\Big).$$

Hence,

$$(1-\epsilon)E \sup_{t \in [0,T]} \|\kappa_t - u_t\|^2 + (2 - \frac{\epsilon}{\lambda})E \int_0^T \mathcal{E}(\kappa_t - u_t)dt$$
$$\leq C\Big(E\|u_0^+ - u_0\|^2 + E \int_0^T \big(\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2\big)dt\Big).$$

We can take ϵ small enough such that $1 - \epsilon > 0$ and $2 - \frac{\epsilon}{\lambda} > 0$, hence,

$$E \sup_{t \in [0,T]} \|\kappa_t - u_t\|^2 + E \int_0^T \mathcal{E}(\kappa_t - u_t)dt$$
$$\leq C\Big(E\|u_0^+ - u_0\|^2 + E\|u_0\|^2 + E \int_0^T \big(\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2\big)dt\Big).$$

Then, combining with (6), we get the desired estimate:

$$E \sup_{t \in [0,T]} \|\kappa_t\|^2 + E \int_0^T \mathcal{E}(\kappa_t)dt$$
$$\leq C\Big(E\|u_0^+\|^2 + E \int_0^T \big(\|f_t^0\|^2 + \||g_t^0|\|^2 + \||h_t^0|\|^2 + \|l_t^0\|_{L^2(\partial\mathcal{O})}^2\big)dt\Big).$$

$\square$

Proof of Theorem 3: Let $(u_0^n)_n$, $(f^n)_n$, $(g^n)_n$, $(h^n)_n$ and $(l^n)_n$ are sequences of smooth elements in $L^2(\Omega \times [0,T]; L^2(\mathcal{O}))$ and $L^2(\Omega \times [0,T]; L^2(\partial\mathcal{O}))$ respectively which converge respectively to u_0, f, g, h and l and $h = \sum_k h_k 1_{[t_k, t_{k+1})}$ with h_k being $\mathcal{F}_{t_k}$-measurable. For all $n \in \mathbb{N}^*$ we define $u^n = \mathcal{U}(u_0^n, f^n, g^n, h^n, l^n)$. From Section 3, we know that u^n is P-almost surely continuous in (t, x). Then we can do a similar argument as in [11] to end the proof of Theorem 3. $\square$

6. Existence and uniqueness result

6.1. Weak solution

Assumption (O): The obstacle S is assumed to be an adapted process, quasi-continuous, such that $S_0 \leq \xi$ P-almost surely and controlled by the solution of a SPDE, i.e. $\forall t \in [0,T]$,

$$S_t \leq S_t' \tag{10}$$

where S' is the solution of the linear SPDE

$$
\begin{cases}
dS'_t = \mathcal{A}S'_t dt + f'_t dt + \sum_{i=1}^{d} \partial_i g'_{i,t} dt + \sum_{j=1}^{+\infty} h'_{j,t} dB_t^j, \\
\sum_{i,j}(a_{i,j}\partial_j S'_t(x) + g'_i)n_i = l'(t,x), \quad S'(0) = S'_0,
\end{cases}
\tag{11}
$$

where $S'_0 \in L^2(\Omega \times \mathcal{O})$ is $\mathcal{F}_0$-measurable, f', g', h' and l' are adapted processes respectively in $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R})$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^d)$, $L^2([0,T] \times \Omega \times \mathcal{O}; \mathbb{R}^{\mathbb{N}^*})$ and $L^2([0,T] \times \Omega \times \partial\mathcal{O}; \mathbb{R})$.

Remark 6. *Here again, we know that S' uniquely exists and satisfies the following estimate:*

$$
E \sup_{t\in[0,T]} \|S'_t\|^2 + E \int_0^T \mathcal{E}(S'_t)dt \le CE\left[\|S'_0\|^2 + \int_0^T (\|f'_t\|^2 + \||g'_t\||^2 + \||h'_t\||^2 + \|l'_t\|^2_{L^2(\partial\mathcal{O})})dt \right].
\tag{12}
$$

Moreover, from Theorem 3, S' admits a quasi-continuous version.
Let us also remark that even if this assumption seems restrictive, since S' is driven by the same operator and Brownian motions as u, it encompasses a large class of examples.

We now are able to define rigorously the notion of solution to the problem with obstacle we consider.

Definition 6. *A pair (u,ν) is said to be a solution of OSPDE (1) if*

1. *$u \in \mathcal{H}_T$ and $u(t,x) \ge S(t,x)$, $dP \otimes dt \otimes dx - a.e.$ and $u_0(x) = \xi$, $dP \otimes dx - a.e.$;*
2. *ν is a random regular measure defined on $[0,T) \times \bar{\mathcal{O}}$;*
3. *the following relation holds almost surely, for all $t \in [0,T]$ and $\forall \varphi \in \mathcal{D}$,*

$$
(u_t, \varphi_t) - (\xi, \varphi_0) - \int_0^t (u_s, \partial_s\varphi_s)ds + \int_0^t \mathcal{E}(u_s, \varphi_s)ds + \sum_{i=1}^{d} \int_0^t (g_s^i, \partial_i\varphi_s)ds
$$

$$
= \int_0^t (f_s, \varphi_s)ds + \int_0^t \int_{\partial\mathcal{O}} l_s(x)\varphi_s(x)d\sigma(x)ds + \sum_{j=1}^{+\infty} \int_0^t (h_s^j, \varphi_s)dB_s^j + \int_0^t \int_{\bar{\mathcal{O}}} \varphi_s(x)\nu(dx,ds);
$$

4. *u admits a quasi-continuous version, $\tilde{u}$, and we have*

$$
\int_0^T \int_{\bar{\mathcal{O}}} (\tilde{u}(s,x) - S(s,x))\nu(dx,ds) = 0, \quad a.s..
$$

In order to apply the penalized method to obstacle problem (1), we need the following comparison theorem for the solution of (3), which can be easily obtained by Itô's formula.

Theorem 4. *Let ξ', f', l' be similar to ξ, f, l and let u be the solution of (3) corresponding to (ξ, f, g, h, l) and u' be the solution corresponding to (ξ', f', g, h, l'). Assume that the following conditions hold:*

1. *$\xi \le \xi'$, $dx \otimes dP - a.e.$;*
2. *$f(u, \nabla u) \le f'(u, \nabla u)$, $dt \otimes dx \otimes dP - a.e.$;*

3. $l(u) \leq l'(u)$, $dx \otimes dP - a.e..$

Then, for all most all ω, $u(t,x) \leq u'(t,x)$, q.e..

As we have Itô's formula and comparison theorem for the solution of SPDE (1) without obstacle, we can make a similar argument as Section 5 in [11]. More precisely, we first establish the result in the linear case. Then, we prove an Itô formula for the difference of two solutions of (linear) OSPDEs and finally conclude the following result thanks to a Picard iteration procedure:

Theorem 5. *Under assumptions* **(H)**, **(I)** *and* **(O)**, *there exists a unique weak solution of OSPDE* (1) *associated to (ξ, f, g, h, l, S). We denote by $\mathcal{R}(\xi, f, g, h, l, S)$ the solution of OSPDE* (1) *when it exists and is unique.*

6.2. Itô's formula

We can establish the following Itô formula for the solution of OSPDE (1). The proof is similar to that in [11].

Theorem 6. *Let u be the solution of OSPDE* (1) *and $\Phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. We denote by Φ' and Φ'' the derivatives of Φ with respect to the space variables and by $\frac{\partial \Phi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded. Then P-a.s. for all $t \in [0, T]$,*

$$\int_{\mathcal{O}} \Phi(t, u_t(x))dx + \int_0^t \mathcal{E}(\Phi'(s, u_s), u_s)ds = \int_{\mathcal{O}} \Phi(0, \xi(x))dx + \int_0^t \int_{\mathcal{O}} \frac{\partial \Phi}{\partial s}(s, u_s(x))dxds$$

$$+ \int_0^t (\Phi'(s, u_s), f_s)ds - \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))\partial_i u_s(x)g_i(x)dxds$$

$$+ \sum_{j=1}^{+\infty} \int_0^t (\Phi'(s, u_s), h_j)dB_s^j + \int_0^t \int_{\partial \mathcal{O}} \Phi(s, u_s(x))l_s(x)d\sigma(x)ds$$

$$+ \frac{1}{2} \sum_{j=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \Phi''(s, u_s(x))(h_{j,s}(x))^2 dxds + \int_0^t \int_{\mathcal{O}} \Phi'(s, \tilde{u}_s(x))\nu(dxds).$$

6.3. Comparison theorem

The above Itô formula naturally leads to a comparison theorem, the proof is similar to the Dirichlet case (see Theorem 8 in [11]). More precisely, consider $(u^1, \nu^1) = \mathcal{R}(\xi^1, f^1, g, h, l^1, S^1)$ the solution of the SPDE with obstacle

$$\begin{cases} du_t^1(x) = \mathcal{A}u_t^1(x)dt + f^1(t, x, u_t^1(x), \nabla u_t^1(x))dt + \sum_{i=1}^d \partial_i g_i(t, x, u_t^1(x), \nabla u_t^1(x))dt \\ \qquad + \sum_{j=1}^{+\infty} h_j(t, x, u_t^1(x), \nabla u_t^1(x))dB_t^j + \nu^1(x, dt), \\ \sum_{i,j}(a_{i,j}\partial_j u_t^1(x) + g_t^i(x))n_i = l^1(t, x), \\ u^1 \geq S^1, \ u_0^1 = \xi^1, \end{cases}$$

where we assume $(\xi^1, f^1, g, h, l^1, S^1)$ satisfy hypotheses **(H)**, **(I)** and **(O)**.
We consider another coefficient f^2 which satisfies the same assumptions as f^1, another obstacle S^2 which satisfies **(O)**, another initial condition ξ^2 belonging to $L^2(\Omega \times \mathcal{O})$ and $\mathcal{F}_0$ adapted such that $\xi^2 \geq S_0^2$ and another boundary coefficient l^2 satisfying the same assumption as l^1. We denote by $(u^2, \nu^2) = \mathcal{R}(\xi^2, f^2, g, h, l^2, S^2)$.

Theorem 7. *Assume that the following conditions hold*

1. $\xi^1 \leq \xi^2, \ dx \otimes dP - a.e.;$
2. $f^1(u^1, \nabla u^1) \leq f^2(u^1, \nabla u^1), \ dt \otimes dx \otimes dP - a.e.;$
3. $l^1(u) \leq l^2(u), \ dt \otimes dx \otimes dP - a.e.;$
4. $S^1 \leq S^2, \ dt \otimes dx \otimes dP - a.e..$

Then, for almost all $\omega \in \Omega$, $u^1(t,x) \leq u^2(t,x), \ q.e..$

Remark 7. *Applying the comparison theorem to the same obstacle gives another proof of the uniqueness of the solution.*

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