SMOOTH COMPACTNESS FOR SPACES OF ASYMPTOTICALLY CONICAL SELF-EXPANDERS OF MEAN CURVATURE FLOW

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ABSTRACT. We show compactness in the locally smooth topology for certain natural families of asymptotically conical self-expanding solutions of mean curvature flow. Specifically, we show such compactness for the set of all two-dimensional self-expanders of a fixed topological type and, in all dimensions, for the set of self-expanders of low entropy and for the set of mean convex self-expanders with strictly mean convex asymptotic cones. From this we deduce that the natural projection map from the space of parameterizations of asymptotically conical self-expanders to the space of parameterizations of the asymptotic cones is proper for these classes.

1. INTRODUCTION

A *hypersurface*, i.e., a properly embedded codimension-one submanifold, $\Sigma \subset \mathbb{R}^{n+1}$, is a *self-expander* if

$$\mathbf{H}_\Sigma - \frac{\mathbf{x}^\perp}{2} = \mathbf{0}. \tag{1.1}$$

Here

$$\mathbf{H}_\Sigma = \Delta_\Sigma \mathbf{x} = -H_\Sigma \mathbf{n}_\Sigma = -\mathrm{div}_\Sigma(\mathbf{n}_\Sigma)\mathbf{n}_\Sigma$$

is the mean curvature vector, $\mathbf{n}_\Sigma$ is the unit normal, and $\mathbf{x}^\perp$ is the normal component of the position vector. Self-expanders arise naturally in the study of mean curvature flow. Indeed, Σ is a self-expander if and only if the family of homothetic hypersurfaces

$$\{\Sigma_t\}_{t>0} = \left\{\sqrt{t}\,\Sigma\right\}_{t>0}$$

is a *mean curvature flow* (MCF), that is, a solution to the flow

$$\left(\frac{\partial \mathbf{x}}{\partial t}\right)^\perp = \mathbf{H}_{\Sigma_t}.$$

Self-expanders are expected to model the behavior of a MCF as it emerges from a conical singularity [1]. They are also expected to model the long time behavior of the flow [8].

Throughout the paper $n, k \geq 2$ are integers and $\alpha \in (0,1)$. Let Γ be a $C^{k,\alpha}_*$-asymptotically conical $C^{k,\alpha}$-hypersurface in $\mathbb{R}^{n+1}$ and let $\mathcal{L}(\Gamma)$ be the link of the asymptotic cone of Γ. For instance, if $\lim_{\rho\to 0^+} \rho\Gamma = \mathcal{C}$ in $C^{k,\alpha}_{loc}(\mathbb{R}^{n+1}\setminus\{\mathbf{0}\})$, where $\mathcal{C}$ is a cone, then Γ is $C^{k,\alpha}_*$-asymptotically conical with asymptotic cone $\mathcal{C}$. For technical reasons, the actual definition is slightly weaker – see Section 3 of [3] for the details. We denote the space of $C^{k,\alpha}_*$-asymptotically conical $C^{k,\alpha}$-hypersurfaces in $\mathbb{R}^{n+1}$ by $\mathcal{ACH}^{k,\alpha}_n$.

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We now introduce the classes we will consider. First, for $g \geq 0$ and $e \geq 1$, let

$$\mathcal{E}^{k,\alpha}_{\mathrm{top}}(g,e) = \left\{ \Gamma \in \mathcal{ACH}^{k,\alpha}_2 \colon \Gamma \text{ satisfies (1.1) and } \Gamma \text{ is of genus } g \text{ with } e \text{ ends} \right\},$$

be the space of $C^{k,\alpha}_*$-asymptotically conical self-expanders in $\mathbb{R}^3$ with genus g and e ends. Similarly, for any $h_0 > 0$, let

$$\mathcal{E}^{k,\alpha}_{n,\mathrm{mc}}(h_0) = \left\{ \Gamma \in \mathcal{ACH}^{k,\alpha}_n \colon \Gamma \text{ satisfies (1.1)}, H_\Gamma > 0, H_{\mathcal{L}(\Gamma)} \geq h_0 \right\},$$

be the space of $C^{k,\alpha}_*$-asymptotically conical self-expanders in $\mathbb{R}^{n+1}$ which are strictly mean convex and have uniformly strictly mean convex asymptotic cones. Finally, for $1 < \Lambda_0 < 2$, let

$$\mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0) = \left\{ \Gamma \in \mathcal{ACH}^{k,\alpha}_n \colon \Gamma \text{ satisfies (1.1) and } \lambda[\Gamma] \leq \Lambda_0 \right\},$$

be the space of $C^{k,\alpha}_*$-asymptotically conical self-expanders in $\mathbb{R}^{n+1}$ which have entropy less than or equal to Λ_0. See Section 3.2 for the definition of entropy.

We prove the following smooth compactness result for the spaces $\mathcal{E}^{k,\alpha}_{\mathrm{top}}(g,e)$, $\mathcal{E}^{k,\alpha}_{n,\mathrm{mc}}(h_0)$ and, under suitable hypotheses on Λ_0, $\mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$.

Theorem 1.1. *The following holds:*

1. *If $\Sigma_i \in \mathcal{E}^{k,\alpha}_{\mathrm{top}}(g,e)$ and $\mathcal{L}(\Sigma_i) \to \sigma$ in $C^{k,\alpha}(\mathbb{S}^2)$, then there is a $\Sigma \in \mathcal{E}^{k,\alpha}(g,e)$ with $\mathcal{L}(\Sigma) = \sigma$ so that, up to passing to a subsequence, $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^3)$.*
2. *If $\Sigma_i \in \mathcal{E}^{k,\alpha}_{n,\mathrm{mc}}(h_0)$ and $\mathcal{L}(\Sigma_i) \to \sigma$ in $C^{k,\alpha}(\mathbb{S}^n)$, then there is a $\Sigma \in \mathcal{E}^{k,\alpha}_{n,\mathrm{mc}}(h_0)$ with $\mathcal{L}(\Sigma) = \sigma$ so that, up to passing to a subsequence, $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$.*
3. *If Assumption $(\star_{n,\Lambda})$ of Section 3.2 holds, $\Sigma_i \in \mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$ for $\Lambda_0 < \Lambda < 2$ and $\mathcal{L}(\Sigma_i) \to \sigma$ in $C^{k,\alpha}(\mathbb{S}^n)$, then there is a $\Sigma \in \mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$ with $\mathcal{L}(\Sigma) = \sigma$ so that, up to passing to a subsequence, $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$.*

In [3], the authors showed that the space $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$ – see (2.1) below – of asymptotically conical parameterizations of self-expanders modeled on Γ (modulo reparameterizations fixing the parameterization of the asymptotic cone) possesses a natural Banach manifold structure modeled on $C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$. They further showed that the map

$$\Pi \colon \mathcal{ACE}^{k,\alpha}_n(\Gamma) \to C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$$

given by $\Pi([\mathbf{f}]) = \mathrm{tr}^1_\infty[\mathbf{f}]$ is smooth and Fredholm of index 0. As such, by work of Smale [24], as long as Π is proper it possesses a well-defined mod 2 degree. In fact as shown in [2], when the map Π is proper it possesses an integer degree. These results are all analogs of work of White [26] who proved such results for a large class of variational problems for parameterizations from compact manifolds – see also [28].

In general, the map $\Pi \colon \mathcal{ACE}^{k,\alpha}_n(\Gamma) \to C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$ is not proper. However, using Theorem 1.1, we give several natural subsets of $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$ on which the restriction of Π is proper. This should be compared to [27]. As a first step, it is necessary to shrink the range of Π. To that end, for any $\Gamma \in \mathcal{ACH}^{k,\alpha}_n$, let

$$\mathcal{V}^{k,\alpha}_{\mathrm{emb}}(\Gamma) = \left\{ \varphi \in C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1}) \colon \mathscr{E}^{\mathrm{H}}_1[\varphi] \text{ is an embedding} \right\},$$

be the space of parameterizations of embedded cones. Here $\mathscr{E}^{\mathrm{H}}_1[\varphi]$ is the homogeneous degree-one extension of φ. This is readily seen to be an open subset of $C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$. It also follows from the definition of $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$ that $\Pi \colon \mathcal{ACE}^{k,\alpha}_n(\Gamma) \to \mathcal{V}^{k,\alpha}_{\mathrm{emb}}(\Gamma)$.

Theorem 1.2. *For any $\Gamma \in \mathcal{ACH}^{k,\alpha}_2$, $\Pi \colon \mathcal{ACE}^{k,\alpha}_2(\Gamma) \to \mathcal{V}^{k,\alpha}_{\mathrm{emb}}(\Gamma)$ is proper.*

Theorem 1.3. *For $\Gamma \in \mathcal{ACH}_n^{k,\alpha}$ and $\Lambda > 1$, let*

$$\mathcal{V}_{\mathrm{ent}}(\Gamma, \Lambda) = \left\{\varphi \in \mathcal{V}_{\mathrm{emb}}^{k,\alpha}(\Gamma) \colon \lambda[\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma))] < \Lambda\right\}$$

and

$$\mathcal{U}_{\mathrm{ent}}(\Gamma, \Lambda) = \left\{[\mathbf{f}] \in \mathcal{ACE}_n^{k,\alpha}(\Gamma) \colon \lambda[\mathbf{f}(\Gamma)] < \Lambda\right\}.$$

The following is true:

(1) *$\mathcal{U}_{\mathrm{ent}}(\Gamma, \Lambda)$ is an open subset of $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$.*
(2) *$\mathcal{V}_{\mathrm{ent}}(\Gamma, \Lambda)$ is an open subset of $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$.*
(3) *If $(\star_{n,\Lambda})$ holds for $\Lambda < 2$, then $\Pi|_{\mathcal{U}_{\mathrm{ent}}(\Gamma,\Lambda)} \colon \mathcal{U}_{\mathrm{ent}}(\Gamma, \Lambda) \to \mathcal{V}_{\mathrm{ent}}(\Gamma, \Lambda)$ is proper.*

Theorem 1.4. *For $\Gamma \in \mathcal{ACH}_n^{k,\alpha}$, let*

$$\mathcal{V}_{\mathrm{mc}}(\Gamma) = \left\{\varphi \in \mathcal{V}_{\mathrm{emb}}^{k,\alpha}(\Gamma) \colon H_\sigma > 0 \text{ where } \sigma = \mathcal{L}[\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma))]\right\}$$

and

$$\mathcal{U}_{\mathrm{mc}}(\Gamma) = \left\{[\mathbf{f}] \in \mathcal{ACE}_n^{k,\alpha}(\Gamma) \colon H_{\mathbf{f}(\Gamma)} > 0, \Pi([\mathbf{f}]) \in \mathcal{V}_{\mathrm{mc}}(\Gamma)\right\}.$$

The following is true:

(1) *$\mathcal{U}_{\mathrm{mc}}(\Gamma)$ is an open subset of $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$.*
(2) *$\mathcal{V}_{\mathrm{mc}}(\Gamma)$ is an open subset of $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$.*
(3) *$\Pi|_{\mathcal{U}_{\mathrm{mc}}(\Gamma)} \colon \mathcal{U}_{\mathrm{mc}}(\Gamma) \to \mathcal{V}_{\mathrm{mc}}(\Gamma)$ is a local diffeomorphism.*
(4) *$\Pi|_{\mathcal{U}_{\mathrm{mc}}(\Gamma)} \colon \mathcal{U}_{\mathrm{mc}}(\Gamma) \to \mathcal{V}_{\mathrm{mc}}(\Gamma)$ is proper.*

In particular, for each component $\mathcal{V}'$ of $\mathcal{V}_{\mathrm{mc}}(\Gamma)$, there is an integer $l' \geq 0$ so $\mathcal{U}' = \Pi^{-1}(\mathcal{V}') \cap \mathcal{U}_{\mathrm{mc}}(\Gamma)$ has l' components and for each component $\mathcal{U}''$ of $\mathcal{U}'$, $\Pi|_{\mathcal{U}''} \colon \mathcal{U}'' \to \mathcal{V}'$ is a (finite) covering map.

Finally, as an application of Theorem 1.4 and a result of Huisken [13], we have the following existence and uniqueness result for self-expanders of a given topological type asymptotic to cones that satisfy a natural pinching condition.

Corollary 1.5. *Let $\sigma \subset \mathbb{S}^n$ be a connected, strictly mean convex, $C^{k,\alpha}$-hypersurface. In addition, if $n \geq 3$, suppose that σ satisfies*

$$\begin{cases} |A_\sigma|^2 < \dfrac{1}{n-2} H_\sigma^2 + 2, & n \geq 4; \\ |A_\sigma|^2 < \dfrac{3}{4} H_\sigma^2 + \dfrac{4}{3}, & n = 3. \end{cases} \tag{1.2}$$

There exists a smooth self-expander $\Sigma \in \mathcal{ACH}_n^{k,\alpha}$ with $\mathcal{L}(\Sigma) = \sigma$, $H_\Sigma > 0$ and so Σ is diffeomorphic to $\mathbb{R}^n$. Moreover, if $\pi_0(\mathrm{Diff}^+(\mathbb{S}^{n-1})) = 0$, i.e., the group of orientation-preserving diffeomorphisms of $\mathbb{S}^{n-1}$ is path-connected, then Σ is the unique self-expander with these properties.

Remark 1.6. Hypothesis (1.2) is required only so that classical mean curvature flow can be used to show the space of admissible σ is path-connected.

Remark 1.7. By work of Cerf [4, 5] and Smale [22, 23] it is known, for $n \in \{2, 3, 4, 6\}$, that $\pi_0(\mathrm{Diff}^+(\mathbb{S}^{n-1})) = 0$.

Remark 1.8. Using only the mean convexity condition, a variational argument due to Ilmanen that is sketched in [17] and carried out by Ding in [7] gives the existence of a self-expanding solution with link σ. However, this method cannot directly say anything about the topology of the constructed self-expanders.

2. Notation and background

In this section we fix notation and also recall the main definitions from [3] we need. The interested reader should consult Sections 2 and 3 of [3] for specifics and further details.

2.1. **Basic notions.** Denote a (open) ball in $\mathbb{R}^n$ of radius R and center x by $B_R^n(x)$ and the closed ball by $\bar{B}_R^n(x)$. We often omit the superscript n when its value is clear from context. We also omit the center when it is the origin.

For an open set $U \subset \mathbb{R}^{n+1}$, a *hypersurface in* U, Γ, is a smooth, properly embedded, codimension-one submanifold of U. We also consider hypersurfaces of lower regularity and given an integer $k \geq 2$ and $\alpha \in (0,1)$ we define a $C^{k,\alpha}$*-hypersurface in* U to be a properly embedded, codimension-one $C^{k,\alpha}$ submanifold of U. When needed, we distinguish between a point $p \in \Gamma$ and its *position vector* $\mathbf{x}(p)$.

Consider the hypersurface $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, the unit n-sphere in $\mathbb{R}^{n+1}$. A *hypersurface in* $\mathbb{S}^n$, σ, is a closed, embedded, codimension-one smooth submanifold of $\mathbb{S}^n$ and $C^{k,\alpha}$*-hypersurfaces in* $\mathbb{S}^n$ are defined likewise. Observe, that σ is a closed codimension-two submanifold of $\mathbb{R}^{n+1}$ and so we may associate to each point $p \in \sigma$ its position vector $\mathbf{x}(p)$. Clearly, $|\mathbf{x}(p)| = 1$.

A *cone* is a set $\mathcal{C} \subset \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ that is dilation invariant around the origin. That is, $\rho\mathcal{C} = \mathcal{C}$ for all $\rho > 0$. The *link* of the cone is the set $\mathcal{L}[\mathcal{C}] = \mathcal{C} \cap \mathbb{S}^n$. The cone is *regular* if its link is a smooth hypersurface in $\mathbb{S}^n$ and $C^{k,\alpha}$*-regular* if its link is a $C^{k,\alpha}$-hypersurface in $\mathbb{S}^n$. For any hypersurface $\sigma \subset \mathbb{S}^n$ the *cone over* σ, $\mathcal{C}[\sigma]$, is the cone defined by

$$\mathcal{C}[\sigma] = \{\rho p \colon p \in \sigma, \rho > 0\} \subset \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}.$$

Clearly, $\mathcal{L}[\mathcal{C}[\sigma]] = \sigma$.

2.2. **Function spaces.** Let Γ be a properly embedded, $C^{k,\alpha}$ submanifold of an open set $U \subset \mathbb{R}^{n+1}$. There is a natural Riemannian metric, g_Γ, on Γ of class $C^{k-1,\alpha}$ induced from the Euclidean one. As we always take $k \geq 2$, the Christoffel symbols of this metric, in appropriate coordinates, are well-defined and of regularity $C^{k-2,\alpha}$. Let ∇_Γ be the covariant derivative on Γ. Denote by d_Γ the geodesic distance on Γ and by $B_R^\Gamma(p)$ the (open) geodesic ball in Γ of radius R and center $p \in \Gamma$. For R small enough so that $B_R^\Gamma(p)$ is strictly geodesically convex and $q \in B_R^\Gamma(p)$, denote by $\tau_{p,q}^\Gamma$ the parallel transport along the unique minimizing geodesic in $B_R^\Gamma(p)$ from p to q.

Throughout the rest of this subsection, let Ω be a domain in Γ, l an integer in $[0,k]$, $\beta \in (0,1)$ and $d \in \mathbb{R}$. Suppose $l + \beta \leq k + \alpha$. We first consider the following norm for functions on Ω:

$$\|f\|_{l;\Omega} = \sum_{i=0}^{l} \sup_\Omega |\nabla_\Gamma^i f|.$$

We then let

$$C^l(\Omega) = \left\{ f \in C^l_{loc}(\Omega) \colon \|f\|_{l;\Omega} < \infty \right\}.$$

We next define the Hölder semi-norms for functions f and tensor fields T on Ω:

$$[f]_{\beta;\Omega} = \sup_{\substack{p,q\in\Omega \\ q \in B_\delta^\Gamma(p)\setminus\{p\}}} \frac{|f(p)-f(q)|}{d_\Gamma(p,q)^\beta} \text{ and } [T]_{\beta;\Omega} = \sup_{\substack{p,q\in\Omega \\ q \in B_\delta^\Gamma(p)\setminus\{p\}}} \frac{|T(p)-(\tau_{p,q}^\Gamma)^* T(q)|}{d_\Gamma(p,q)^\beta},$$

where $\delta = \delta(\Gamma, \Omega) > 0$ so that for all $p \in \Omega$, $B_\delta^\Gamma(p)$ is strictly geodesically convex. We further define the norm for functions on Ω:

$$\|f\|_{l,\beta;\Omega} = \|f\|_{l;\Omega} + [\nabla_\Gamma^l f]_{\beta;\Omega},$$

and let

$$C^{l,\beta}(\Omega)=\left\{f\in C^{l,\beta}_{loc}(\Omega)\colon \|f\|_{l,\beta;\Omega}<\infty\right\}.$$

We also define the following weighted norms for functions on Ω:

$$\|f\|^{(d)}_{l;\Omega}=\sum_{i=0}^{l}\sup_{p\in\Omega}\left(|\mathbf{x}(p)|+1\right)^{-d+i}|\nabla^i_\Gamma f(p)|.$$

We then let

$$C^l_d(\Omega)=\left\{f\in C^l_{loc}(\Omega)\colon \|f\|^{(d)}_{l;\Omega}<\infty\right\}.$$

We further define the following weighted Hölder semi-norms for functions f and tensor fields T on Ω:

$$[f]^{(d)}_{\beta;\Omega}=\sup_{\substack{p,q\in\Omega\\ q\in B^\Gamma_{\delta_p}(p)\setminus\{p\}}}\left((|\mathbf{x}(p)|+1)^{-d+\beta}+(|\mathbf{x}(q)|+1)^{-d+\beta}\right)\frac{|f(p)-f(q)|}{d_\Gamma(p,q)^\beta},\text{ and,}$$

$$[T]^{(d)}_{\beta;\Omega}=\sup_{\substack{p,q\in\Omega\\ q\in B^\Gamma_{\delta_p}(p)\setminus\{p\}}}\left((|\mathbf{x}(p)|+1)^{-d+\beta}+(|\mathbf{x}(q)|+1)^{-d+\beta}\right)\frac{|T(p)-(\tau^\Gamma_{p,q})^*T(q)|}{d_\Gamma(p,q)^\beta},$$

where $\eta=\eta(\Omega,\Gamma)\in(0,\frac{1}{4})$ so that for any $p\in\Gamma$, letting $\delta_p=\eta(|\mathbf{x}(p)|+1)$, $B^\Gamma_{\delta_p}(p)$ is strictly geodesically convex. Finally, we define the norm for functions on Ω:

$$\|f\|^{(d)}_{l,\beta;\Omega}=\|f\|^{(d)}_{l;\Omega}+[\nabla^l_\Gamma f]^{(d-l)}_{\beta;\Omega},$$

and we let

$$C^{l,\beta}_d(\Omega)=\left\{f\in C^{l,\beta}_{loc}(\Omega)\colon \|f\|^{(d)}_{l,\beta;\Omega}<\infty\right\}.$$

We follow the convention that $C^{l,0}_{loc}=C^l_{loc}$, $C^{l,0}=C^l$ and $C^{l,0}_d=C^l_d$ and that $C^{0,\beta}_{loc}=C^\beta_{loc}$, $C^{0,\beta}=C^\beta$ and $C^{0,\beta}_d=C^\beta_d$. The notation for the corresponding norms is abbreviated in the same fashion.

2.3. **Homogeneous functions and homogeneity at infinity.** Fix a $C^{k,\alpha}$-regular cone $\mathcal{C}$ with its link $\mathcal{L}$. By our definition $\mathcal{C}$ is a $C^{k,\alpha}$-hypersurface in $\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}$. For $R>0$ let $\mathcal{C}_R=\mathcal{C}\setminus\bar{B}_R$. There is an $\eta=\eta(\mathcal{L},R)>0$ so that for any $p\in\mathcal{C}_R$, $B^{\mathcal{C}}_{\delta_p}(p)$ is strictly geodesically convex, where $\delta_p=\eta(|\mathbf{x}(p)|+1)$. We also fix an integer $l\in[0,k]$ and $\beta\in[0,1)$ with $l+\beta\leq k+\alpha$.

A map $\mathbf{f}\in C^{l,\beta}_{loc}(\mathcal{C};\mathbb{R}^M)$ is *homogeneous of degree* d if $\mathbf{f}(\rho p)=\rho^d\mathbf{f}(p)$ for all $p\in\mathcal{C}$ and $\rho>0$. Given a map $\varphi\in C^{l,\beta}(\mathcal{L};\mathbb{R}^M)$ the *homogeneous extension of degree* d of φ is the map $\mathscr{E}^{\mathrm{H}}_d[\varphi]\in C^{l,\beta}_{loc}(\mathcal{C};\mathbb{R}^M)$ defined by

$$\mathscr{E}^{\mathrm{H}}_d[\varphi](p)=|\mathbf{x}(p)|^d\varphi(|\mathbf{x}(p)|^{-1}p).$$

Conversely, given a homogeneous $\mathbb{R}^M$-valued map of degree d, $\mathbf{f}\in C^{l,\beta}_{loc}(\mathcal{C};\mathbb{R}^M)$, let $\varphi=\mathrm{tr}[\mathbf{f}]\in C^{l,\beta}(\mathcal{L};\mathbb{R}^M)$, the *trace* of $\mathbf{f}$, be the restriction of $\mathbf{f}$ to $\mathcal{L}$. Clearly, $\mathbf{f}$ is the homogeneous extension of degree d of φ.

A map $\mathbf{g}\in C^{l,\beta}_{loc}(\mathcal{C}_R;\mathbb{R}^M)$ is *asymptotically homogeneous of degree* d if

$$\lim_{\rho\to 0^+}\rho^d\mathbf{g}(\rho^{-1}p)=\mathbf{f}(p)\text{ in }C^{l,\beta}_{loc}(\mathcal{C};\mathbb{R}^M)$$

for some $\mathbf{f} \in C^{l,\beta}_{loc}(\mathcal{C}; \mathbb{R}^M)$ that is homogeneous of degree d. For such a $\mathbf{g}$ we define the *trace at infinity* of $\mathbf{g}$ by $\mathrm{tr}^d_\infty[\mathbf{g}] = \mathrm{tr}[\mathbf{f}]$. We define

$$C^{l,\beta}_{d,\mathrm{H}}(\mathcal{C}_R; \mathbb{R}^M) = \left\{ \mathbf{g} \in C^{l,\beta}_d(\mathcal{C}_R; \mathbb{R}^M) \colon \mathbf{g} \text{ is asymptotically homogeneous of degree } d \right\}.$$

It is straightforward to verify that $C^{l,\beta}_{d,\mathrm{H}}(\mathcal{C}_R; \mathbb{R}^M)$ is a closed subspace of $C^{l,\beta}_d(\mathcal{C}_R; \mathbb{R}^M)$ and that

$$\mathrm{tr}^d_\infty \colon C^{l,\beta}_{d,\mathrm{H}}(\mathcal{C}_R; \mathbb{R}^M) \to C^{l,\beta}(\mathcal{L}; \mathbb{R}^M)$$

is a bounded linear map. Finally, $\mathbf{x}|_{\mathcal{C}_R} \in C^{k;\alpha}_{1,\mathrm{H}}(\mathcal{C}_R; \mathbb{R}^{n+1})$ and $\mathrm{tr}^1_\infty[\mathbf{x}|_{\mathcal{C}_R}] = \mathbf{x}|_{\mathcal{L}}$.

2.4. **Asymptotically conical hypersurfaces.** A $C^{k,\alpha}$-hypersurface, $\Gamma \subset \mathbb{R}^{n+1}$, is $C^{k,\alpha}_*$-*asymptotically conical* if there is a $C^{k,\alpha}$-regular cone, $\mathcal{C} \subset \mathbb{R}^{n+1}$, and a homogeneous transverse section, $\mathbf{v}$, on $\mathcal{C}$ such that Γ, outside some compact set, is given by the $\mathbf{v}$-graph of a function in $C^{k,\alpha}_1 \cap C^k_{1,0}(\mathcal{C}_R)$ for some $R > 1$. Here a transverse section is a regularized version of the unit normal – see Section 2.4 of [3] for the precise definition. Observe, that by the Arzelà-Ascoli theorem one has that, for every $\beta \in [0, \alpha)$,

$$\lim_{\rho \to 0^+} \rho\Gamma = \mathcal{C} \text{ in } C^{k,\beta}_{loc}(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}).$$

Clearly, the asymptotic cone, $\mathcal{C}$, is uniquely determined by Γ and so we denote it by $\mathcal{C}(\Gamma)$. Let $\mathcal{L}(\Gamma)$ denote the link of $\mathcal{C}(\Gamma)$ and, for $R > 0$, let $\mathcal{C}_R(\Gamma) = \mathcal{C}(\Gamma) \setminus \bar{B}_R$. Denote the space of $C^{k,\alpha}_*$-asymptotically conical $C^{k,\alpha}$-hypersurfaces in $\mathbb{R}^{n+1}$ by $\mathcal{ACH}^{k,\alpha}_n$.

Finally, let K be a compact set of Γ and denote by $\Gamma' = \Gamma \setminus K$. By definition, we may choose K large enough so $\pi_{\mathbf{v}}$ – the projection of a neighborhood of $\mathcal{C}(\Gamma)$ along $\mathbf{v}$ – restricts to a $C^{k,\alpha}$ diffeomorphism of Γ' onto $\mathcal{C}_R(\Gamma)$. Denote its inverse by $\theta_{\mathbf{v};\Gamma'}$.

2.5. **Traces at infinity.** Fix an element $\Gamma \in \mathcal{ACH}^{k,\alpha}_n$. Let l be an integer in $[0, k]$ and $\beta \in [0, 1)$ such that $l+\beta < k+\alpha$. A map $\mathbf{f} \in C^{l,\beta}_{loc}(\Gamma; \mathbb{R}^M)$ is *asymptotically homogeneous of degree* d if $\mathbf{f} \circ \theta_{\mathbf{v};\Gamma'} \in C^{l,\beta}_{d,\mathrm{H}}(\mathcal{C}_R(\Gamma); \mathbb{R}^M)$ where $\mathbf{v}$ is a homogeneous transverse section on $\mathcal{C}(\Gamma)$ and $\Gamma', \theta_{\mathbf{v};\Gamma'}$ are introduced in the previous subsection. The *trace at infinity* of $\mathbf{f}$ is then

$$\mathrm{tr}^d_\infty[\mathbf{f}] = \mathrm{tr}^d_\infty[\mathbf{f} \circ \theta_{\mathbf{v};\Gamma'}] \in C^{l,\beta}(\mathcal{L}(\Gamma); \mathbb{R}^M).$$

Whether $\mathbf{f}$ is asymptotically homogeneous of degree d and the definition of tr^d_∞ are independent of the choice of homogeneous transverse sections on $\mathcal{C}(\Gamma)$. Clearly, $\mathbf{x}|_\Gamma$ is asymptotically homogeneous of degree 1 and $\mathrm{tr}^1_\infty[\mathbf{x}|_\Gamma] = \mathbf{x}|_{\mathcal{L}(\Gamma)}$.

We next define the space

$$C^{l,\beta}_{d,\mathrm{H}}(\Gamma; \mathbb{R}^M) = \left\{ \mathbf{f} \in C^{l,\beta}_d(\Gamma; \mathbb{R}^M) \colon \mathbf{f} \text{ is asymptotically homogeneous of degree } d \right\}.$$

One can check that $C^{l,\beta}_{d,\mathrm{H}}(\Gamma; \mathbb{R}^M)$ is a closed subspace of $C^{l,\beta}_d(\Gamma; \mathbb{R}^M)$, and the map

$$\mathrm{tr}^d_\infty \colon C^{l,\beta}_{d,\mathrm{H}}(\Gamma; \mathbb{R}^M) \to C^{l,\beta}(\mathcal{L}(\Gamma); \mathbb{R}^M)$$

is a bounded linear map. We further define the set $C^{l,\beta}_{d,0}(\Gamma; \mathbb{R}^M) \subset C^{l,\beta}_{d,\mathrm{H}}(\Gamma; \mathbb{R}^M)$ to be the kernel of tr^d_∞.

2.6. **Asymptotically conical embeddings.** Fix an element $\Gamma \in \mathcal{ACH}^{k,\alpha}_n$. We define the space of $C^{k,\alpha}_*$-asymptotically conical embeddings of Γ into $\mathbb{R}^{n+1}$ to be

$$\mathcal{ACH}^{k,\alpha}_n(\Gamma) = \left\{ \mathbf{f} \in C^{k,\alpha}_1 \cap C^k_{1,\mathrm{H}}(\Gamma; \mathbb{R}^{n+1}) \colon \mathbf{f} \text{ and } \mathscr{E}^{\mathrm{H}}_1 \circ \mathrm{tr}^1_\infty[\mathbf{f}] \text{ are embeddings} \right\}.$$

Clearly, $\mathcal{ACH}^{k,\alpha}_n(\Gamma)$ is an open set of the Banach space $C^{k,\alpha}_1 \cap C^k_{1,\mathrm{H}}(\Gamma; \mathbb{R}^{n+1})$ with the $\|\cdot\|^{(1)}_{k,\alpha}$ norm. The hypotheses on $\mathbf{f}$, $\mathrm{tr}^1_\infty[\mathbf{f}] \in C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$ ensure

$$\mathcal{C}[\mathbf{f}] = \mathscr{E}^{\mathrm{H}}_1 \circ \mathrm{tr}^1_\infty[\mathbf{f}] \colon \mathcal{C}(\Gamma) \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$$

is a $C^{k,\alpha}$ embedding. As this map is homogeneous of degree one, it parameterizes the $C^{k,\alpha}$-regular cone $\mathcal{C}(\mathbf{f}(\Gamma))$ – see [3, Proposition 3.3].

Finally, we introduce a natural equivalence relation on $\mathcal{ACH}^{k,\alpha}_n(\Gamma)$. First, say a $C^{k,\alpha}$ diffeomorphism $\phi\colon \Gamma \to \Gamma$ *fixes infinity* if $\mathbf{x}|_\Gamma \circ \phi \in \mathcal{ACH}^{k,\alpha}_n(\Gamma)$ and

$$\mathrm{tr}^1_\infty[\mathbf{x}|_\Gamma \circ \phi] = \mathbf{x}|_{\mathcal{L}(\Gamma)}.$$

Two elements $\mathbf{f}, \mathbf{g} \in \mathcal{ACH}^{k,\alpha}_n(\Gamma)$ are equivalent, written $\mathbf{f} \sim \mathbf{g}$, provided there is a $C^{k,\alpha}$ diffeomorphism $\phi\colon \Gamma \to \Gamma$ that fixes infinity so that $\mathbf{f} \circ \phi = \mathbf{g}$. Given $\mathbf{f} \in \mathcal{ACH}^{k,\alpha}_n(\Gamma)$ let $[\mathbf{f}]$ be the equivalence class of $\mathbf{f}$. Following [3] we define the space

$$\mathcal{ACE}^{k,\alpha}_n(\Gamma) = \left\{ [\mathbf{f}] \colon \mathbf{f} \in \mathcal{ACH}^{k,\alpha}_n(\Gamma) \text{ and } \mathbf{f}(\Gamma) \text{ satisfies (1.1)} \right\}. \tag{2.1}$$

3. Smooth compactness

In this section we prove Theorem 1.1. We first prove compactness in the asymptotic region and then treat the three special cases separately.

3.1. **Asymptotic regularity of self-expanders.** Fix a unit vector $\mathbf{e}$, a point $\mathbf{x}_0 \in \mathbb{R}^{n+1}$ and $r, h > 0$. Let

$$C_\mathbf{e}(\mathbf{x}_0, r, h) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \colon |(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}| < h, |\mathbf{x} - \mathbf{x}_0|^2 < r^2 + |(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}|^2 \right\}$$

be the solid open cylinder with axis $\mathbf{e}$ centered at $\mathbf{x}_0$ and of radius r and height $2h$.

Definition 3.1. Suppose that $l \geq 0$ is an integer and $\beta \in [0, 1)$. A hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ is a $C^{l,\beta}$ $\mathbf{e}$-graph of size δ on scale r at $\mathbf{x}_0$ if there is a function $f\colon B^n_r \subset P_\mathbf{e} \to \mathbb{R}$ with

$$\sum_{j=0}^{l} r^{-1+j} \|\nabla^j f\|_0 + r^{-1+l+\beta} [\nabla^l f]_\beta < \delta,$$

where $P_\mathbf{e}$ is the n-dimensional subspace of $\mathbb{R}^{n+1}$ normal to $\mathbf{e}$ and the last term on the left hand side will be dropped if $\beta = 0$, so that

$$\Sigma \cap C_\mathbf{e}(\mathbf{x}_0, r, \delta r) = \left\{ \mathbf{x}_0 + \mathbf{x}(x) + f(x)\mathbf{e} \colon x \in B^n_r \right\}.$$

Moreover, a hypersurface $\sigma \subset \mathbb{S}^n$ is a $C^{l,\beta}$ $\mathbf{e}$-graph of size δ on scale r at $\mathbf{x}_0$ if $\mathcal{C}[\sigma]$ is so. We omit $C^{l,\beta}$ in the above definitions when the hypersurface is of $C^{l,\beta}$ class or when it is clear from context.

Let us summarize some elementary properties of this concept.

Proposition 3.2. *Let $l \geq 2$ be an integer and $\beta \in [0, 1)$. The following is true:*

1. *If Σ is a $C^{l,\beta}$-hypersurface in $\mathbb{R}^{n+1}$, then for every $\delta > 0$ and $p \in \Sigma$, there is an $r = r(\Sigma, p, \delta) > 0$ so that Σ is an $\mathbf{n}_\Sigma(p)$-graph of size δ on scale r at p.*
2. *If $\sigma \subset \mathbb{S}^n$ is an $\mathbf{e}$-graph of size δ on scale r at $\mathbf{x}_0$ and $\rho > 0$, then $\mathcal{C}[\sigma]$ is an $\mathbf{e}$-graph of size δ on scale ρr at $\rho \mathbf{x}_0$.*

(3) *Given a $C^{l,\beta}$-hypersurface $\sigma \subset \mathbb{S}^n$ and $\delta > 0$, there is an $r = r(\sigma, \delta) > 0$ so that σ is an $\mathbf{n}_{\mathcal{C}[\sigma]}(p)$-graph of size δ on scale r at every $p \in \sigma$.*
(4) *Suppose $\sigma_i \subset \mathbb{S}^n$ converges in $C^{l,\beta}(\mathbb{S}^n)$ to σ. If $p_i \in \sigma_i \to p \in \sigma$ and σ is an $\mathbf{n}_{\mathcal{C}[\sigma]}(p)$-graph of size δ on scale $2r$ at p, then there is an i_0 so that for $i \geq i_0$, σ_i is an $\mathbf{n}_{\mathcal{C}[\sigma_i]}(p_i)$-graph of size 2δ on scale r at p_i.*

The pseudo-locality property of mean curvature flow gives certain asymptotic regularity for self-expanders that are weakly asymptotic to a cone.

Proposition 3.3. *Let $l \geq 2$ be an integer and $\beta \in [0, 1)$. For each $\delta > 0$ and $r > 0$ there exist constants $\mathcal{R}, M, \gamma, \eta > 0$, depending only n, l, β, δ and r, so that if Σ is a self-expander in $\mathbb{R}^{n+1}$ with*

$$\lim_{\rho \to 0^+} \mathcal{H}^n \lfloor (\rho\Sigma) = \mathcal{H}^n \lfloor \mathcal{C}[\sigma]$$

for σ a $C^{l,\beta}$-hypersurface in $\mathbb{S}^n$, and σ is an $\mathbf{n}_{\mathcal{C}[\sigma]}(p)$-graph of size δ on scale r at every $p \in \sigma$, then

(1) *Σ is a $C^{l,\beta}$ $\mathbf{n}_{\mathcal{C}[\sigma]}(p)$-graph of size γ on scale $\eta|\mathbf{x}(p)|$ at every $p \in \mathcal{C}[\sigma] \setminus \bar{B}_{\mathcal{R}}$.*
(2) *There is a function $f \colon \mathcal{C}(\Sigma) \setminus \bar{B}_{\mathcal{R}} \to \mathbb{R}$ satisfying*

$$|f(p)| + |\nabla_{\mathcal{C}[\sigma]} f(p)| \leq M |\mathbf{x}(p)|^{-1}$$

and so

$$\Sigma \setminus \bar{B}_{2\mathcal{R}} = \left\{ \mathbf{x}(p) + f(p)\mathbf{n}_{\mathcal{C}[\sigma]}(p) \colon p \in \mathcal{C}[\sigma] \setminus \bar{B}_{\mathcal{R}} \right\} \setminus \bar{B}_{2\mathcal{R}}.$$

Proof. For simplicity, we set $\mathcal{C} = \mathcal{C}[\sigma]$. Consider the mean curvature flow (thought of as a space-time track)

$$\mathcal{S} = \bigcup_{t>0} \left(\sqrt{t}\,\Sigma \right) \times \{t\}.$$

Let $\bar{\mathcal{S}} = \mathcal{S} \cup (\mathcal{C} \times \{0\})$ so, by our hypothesis, $\bar{\mathcal{S}}$ is an integer Brakke flow (cf. [16, §2] and [15, §6]). Denote by $\bar{\mathcal{S}}_t$ the time t slice of $\bar{\mathcal{S}}$. For Item (1), it is sufficient to prove that there are constants $\tau, \gamma, \eta > 0$, depending only on n, l, β, δ and r, so that for all $t \in [0, \tau]$, $\bar{\mathcal{S}}_t$ is a $C^{l,\beta}$ $\mathbf{n}_{\mathcal{C}}(p)$-graph of size γ on scale η at every $p \in \sigma$.

By the pseudo-locality property for mean curvature flow (cf. [18, Theorem 1.5 and Remarks 1.6]) there is an $\epsilon \in (0, 1)$, depending on n, δ and r, so that for every $t \in [0, 16\epsilon^2)$ and $p \in \sigma$, $\bar{\mathcal{S}}_t \cap C_{\mathbf{n}_{\mathcal{C}}(p)}(p, 4\epsilon, 4\epsilon)$ is the graph of a function $\psi_p(t, x)$ over $T_p\mathcal{C}$ with

$$(4\epsilon)^{-1} \|\psi_p(t, \cdot)\|_0 + \|D_x \psi_p(t, \cdot)\|_0 \leq 1.$$

Moreover, as $\psi_p(t, x)$ satisfies

$$\frac{\partial \psi_p}{\partial t} = \sqrt{1 + |D_x \psi_p|^2}\, \mathrm{div} \left(\frac{D_x \psi_p}{\sqrt{1 + |D_x \psi_p|^2}} \right),$$

it follows from the Hölder estimate for quasi-parabolic equations (cf. [19, Theorem 1.1 of Chapter 6]) that for every $\alpha' \in (0, 1)$,

$$\sup_{t \in [0, 4\epsilon^2]} [D_x \psi_p(t, \cdot)]_{\alpha'; B^n_{2\epsilon}} + \sup_{x \in B^n_{2\epsilon}} [D_x \psi_p(\cdot, x)]_{\frac{\alpha'}{2}; [0, 4\epsilon^2]} \leq C(n, \alpha', \epsilon).$$

Furthermore, we appeal to the estimates of fundamental solutions and the Schauder theory (cf. [19, (13.1) and Theorem 5.1]) to get that

$$\sup_{t \in [0, \epsilon^2]} \|\psi_p(t, \cdot)\|_{l, \beta; B^n_\epsilon} \leq C'(n, l, \beta, \epsilon). \tag{3.1}$$

Using the equation of ψ_p and the fact that $\psi_p(0,0) = |D_x\psi_p(0,0)| = 0$, it follows from (3.1) that

$$|\psi_p(t,x)| \leq \tilde{C}\left(|x|^2 + t\right) \text{ and } |D_x\psi_p(t,x)| \leq \tilde{C}\left(|x| + \sqrt{t}\right), \tag{3.2}$$

where $\tilde{C} = \tilde{C}(n, C, C') > C'$. In particular, for $\rho \in (0,1)$ and for every $t \in [0, \rho^2\epsilon^2]$,

$$(\rho\epsilon)^{-1}\|\psi_p(t,\cdot)\|_{0;B^n_{\rho\epsilon}} + \|D_x\psi_p(t,\cdot)\|_{0;B^n_{\rho\epsilon}} \leq 4\tilde{C}\rho\epsilon.$$

This together with (3.1) further gives

$$\sum_{j=0}^{l}(\rho\epsilon)^{j-1}\|D^j_x\psi_p(t,\cdot)\|_{0;B^n_{\rho\epsilon}} + (\rho\epsilon)^{l+\beta-1}[D^l_x\psi_p(t,\cdot)]_{\beta;B^n_{\rho\epsilon}} \leq 5\tilde{C}\rho\epsilon.$$

Now we choose $\rho = (5\tilde{C}\epsilon)^{-1/2}$ so $5\tilde{C}\rho\epsilon < 4\rho^{-1}$. Hence, for each $t \in [0, \rho^2\epsilon^2]$, $\bar{\mathcal{S}}_t$ is a $C^{l,\beta}$ $\mathbf{n}_{\mathcal{C}}(p)$-graph of size $4\rho^{-1}$ on scale $\rho\epsilon$ at every $p \in \sigma$. The claim follows immediately with $\tau = \rho^2\epsilon^2, \gamma = 4\rho^{-1}$ and $\eta = \rho\epsilon$.

As $\bar{\mathcal{S}}$ is a mean curvature flow away from $(\mathbf{0}, 0)$, by comparing with shrinking spheres, one observes that

$$\Sigma \setminus \bar{B}_{2R} \subset \bigcup_{p \in \mathcal{C}\setminus\bar{B}_R} C_{\mathbf{n}_{\mathcal{C}}(p)}(p, \eta|\mathbf{x}(p)|, \gamma\eta|\mathbf{x}(p)|)$$

for some $R > \tau^{-1/2}$ depending on n, l, β, δ and r. Thus, invoking estimate (3.2) gives that for every $q \in \Sigma \setminus \bar{B}_{2R}$,

$$|\mathbf{x}(q) - \mathbf{x}(\pi(q))| + |\mathbf{n}_\Sigma(q) - \mathbf{n}_{\mathcal{C}}(\pi(q))| \leq \hat{C}(n, \tilde{C})|\mathbf{x}(q)|^{-1}.$$

Here π is the nearest point projection from $\Sigma \setminus \bar{B}_{2R}$ onto $\mathcal{C}$. Hence, Item (2) follows easily from this estimate and the implicit function theorem. □

Corollary 3.4. *If $\Sigma_i \in \mathcal{ACH}^{k,\alpha}_n$ are self-expanders and there is a $C^{k,\alpha}$-hypersurface in $\mathbb{S}^n$, σ, so $\mathcal{L}(\Sigma_i) \to \sigma$ in $C^2(\mathbb{S}^n)$, then there is an $\mathcal{R}' = \mathcal{R}'(n, k, \alpha, \sigma) > 0$ and a $C^{k,\alpha}_*$-asymptotically conical self-expanding end Σ in $\mathbb{R}^{n+1} \setminus \bar{B}_{\mathcal{R}'}$ with $\mathcal{L}(\Sigma) = \sigma$ so that, up to passing to a subsequence,*

$$\Sigma_i \to \Sigma \textit{ in } C^\infty_{loc}(\mathbb{R}^{n+1} \setminus \bar{B}_{\mathcal{R}'}).$$

Proof. By Items (3) and (4) of Proposition 3.2, there are $\delta, r > 0$ and i_0 so that for $i \geq i_0$ each $\mathcal{L}(\Sigma_i)$ is an $\mathbf{n}_{\mathcal{C}[\sigma]}(p)$-graph of size δ on scale r for all $p \in \mathcal{L}(\Sigma_i)$. In particular, we may apply Item (2) of Proposition 3.3 using these constants to obtain an $\mathcal{R}$ so that one has uniform graphical estimates for the Σ_i in $\mathbb{R}^{n+1} \setminus \bar{B}_{\mathcal{R}}$. It then follows from the Arzelà-Ascoli theorem and standard elliptic estimates (see [11, Theorems 6.17 and 8.24]) that there is a self-expanding end, Σ, in $\mathbb{R}^{n+1} \setminus \bar{B}_{\mathcal{R}}$ so that up to passing to a subsequence $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1} \setminus \bar{B}_{\mathcal{R}})$. In fact, by Item (2) of Proposition 3.3 applied to the Σ_i together with the Arzelà-Ascoli theorem, Σ is $C^{0,1}_*$-asymptotic to $\mathcal{C}[\sigma]$. Combining this fact with Item (1) of Proposition 3.3 applied to Σ gives that Σ is actually $C^{k,\alpha}_*$-asymptotic to $\mathcal{C}[\sigma]$ which completes the proof. □

3.2. **Entropy and smooth compactness of $\mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$.** We recall the notion of entropy introduced by Colding and Minicozzi [6] and use it to prove Item (3) of Theorem 1.1 as well as introduce several auxiliary results needed in other parts of the article.

First of all, for a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ the *Gaussian surface area* of Σ is

$$F[\Sigma] = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|\mathbf{x}|^2}{4}} \, d\mathcal{H}^n.$$

Colding and Minicozzi [6] introduced the following notion of *entropy of a hypersurface*:

$$\lambda[\Sigma] = \sup_{\rho>0, \mathbf{y}\in\mathbb{R}^{n+1}} F[\rho\Sigma + \mathbf{y}].$$

By identifying Σ with $\mathcal{H}^n\lfloor\Sigma$, one extends F and λ in an obvious manner to Radon measures on $\mathbb{R}^{n+1}$. We first record a number of simple observations about the entropy of asymptotically conical self-expanders.

Lemma 3.5. *If $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$ is a self-expander, then*

$$\lambda[\Sigma] = \lambda[\mathcal{C}(\Sigma)].$$

Proof. On the one hand, for $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$,

$$\lim_{\rho\to 0^+} \rho\Sigma = \mathcal{C}(\Sigma) \text{ in } C^k_{loc}(\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}).$$

In particular,

$$\lim_{\rho\to 0^+} \mathcal{H}^n\lfloor(\rho\Sigma) = \mathcal{H}^n\lfloor\mathcal{C}(\Sigma).$$

By the lower semicontinuity and scaling invariance of entropy

$$\lambda[\Sigma] \geq \lambda[\mathcal{C}(\Sigma)].$$

On the other hand, we define $\mathcal{M} = \{\mu_t\}_{t\geq 0}$ to be a family of Radon measures on $\mathbb{R}^{n+1}$ given by

$$\mu_t = \begin{cases} \mathcal{H}^n\lfloor(\sqrt{t}\,\Sigma) & \text{if } t>0 \\ \mathcal{H}^n\lfloor\mathcal{C}(\Sigma) & \text{if } t=0, \end{cases}$$

so $\mathcal{M}$ is an integer Brakke flow (see [16, §2] and [15, §6]). Thus, the Huisken monotonicity formula [14] (see also [16, Lemma 7]) implies

$$\lambda[\Sigma] \leq \lambda[\mathcal{C}(\Sigma)],$$

which finishes the proof. □

Lemma 3.6. *There is a constant $\tilde{M} = \tilde{M}(n)$ so that if $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$ is a self-expander, then, for any $R>0$,*

$$\mathcal{H}^n(\Sigma\cap B_R) \leq \tilde{M}\lambda[\mathcal{C}(\Sigma)]R^n.$$

Proof. One computes,

$$R^{-n}\mathcal{H}^n(\Sigma\cap B_R) = \mathcal{H}^n\left((R^{-1}\Sigma)\cap B_1\right) \leq \tilde{M}(n)F[R^{-1}\Sigma] \leq \tilde{M}(n)\lambda[\Sigma]$$

and so the claim follows from Lemma 3.5. □

Lemma 3.7. *Fix any $\epsilon>0$. If $\mathcal{C}\subset\mathbb{R}^{n+1}$ is a C^2-regular cone and $\mathcal{L}[\mathcal{C}]$ is an $\mathbf{n}_{\mathcal{C}}(p)$-graph of size δ on scale r at every $p\in\mathcal{L}[\mathcal{C}]$, then there is an $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(n,\epsilon,\delta,r)$ so that either $\lambda[\mathcal{C}]\leq 1+\epsilon$ or $\lambda[\mathcal{C}] = F[\mathcal{C}+\mathbf{x}_0]$ for some $\mathbf{x}_0\in\bar{B}_{\tilde{\mathcal{R}}}$.*

Proof. First observe, that as $\mathcal{C}$ is invariant by homotheties one has

$$\lambda[\mathcal{C}] = \sup_{\mathbf{x}\in\mathbb{R}^{n+1}} F[\mathcal{C}+\mathbf{x}].$$

Next observe that an elementary covering argument gives an $A = A(n,\delta,r)$ so that

$$\mathcal{H}^{n-1}(\mathcal{L}[\mathcal{C}]) \leq A.$$

Hence, there is an $A' = A'(n,\delta,r)>0$ so for all $R>0$ and $\mathbf{x}\in\mathbb{R}^{n+1}$,

$$\mathcal{H}^n(\mathcal{C}\cap B_R(\mathbf{x})) \leq A'R^n.$$

A consequence of this is that there is an $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(n, \epsilon, \delta, r)$ so that for all $\mathbf{x} \notin \bar{B}_{\tilde{\mathcal{R}}}$,

$$F[\mathcal{C} + \mathbf{x}] \leq 1 + \epsilon.$$

The claim follows from this. □

Lemma 3.8. *Let σ_i and σ be C^2-hypersurfaces in $\mathbb{S}^n$. If $\sigma_i \to \sigma$ in $C^2(\mathbb{S}^n)$, then*

$$\lambda[\mathcal{C}[\sigma_i]] \to \lambda[\mathcal{C}[\sigma]].$$

Proof. As $\sigma_i \to \sigma$ in $C^2(\mathbb{S}^n)$,

$$\mathcal{H}^n \lfloor \mathcal{C}[\sigma_i] \to \mathcal{H}^n \lfloor \mathcal{C}[\sigma].$$

By the lower semicontinuity of entropy

$$\liminf_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \geq \lambda[\mathcal{C}[\sigma]].$$

By Lemma 3.7, given $\epsilon > 0$ there is an $R = R(n, \epsilon, \sigma) > 0$ so that for i sufficiently large, either $\lambda[\mathcal{C}[\sigma_i]] \leq 1 + \epsilon$ or $\lambda[\mathcal{C}[\sigma_i]] = F[\mathcal{C}[\sigma_i] + \mathbf{x}_i]$ for some $\mathbf{x}_i \in \bar{B}_R$. Observe, that there is an $A = A(n, \sigma)$ so that for i sufficiently large and for every $r > 0$ and $\mathbf{x} \in \mathbb{R}^{n+1}$,

$$\mathcal{H}^n(\mathcal{C}[\sigma_i] \cap B_r(\mathbf{x})) \leq A r^n.$$

Hence, we get

$$\limsup_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \leq \max\{1 + \epsilon, \lambda[\mathcal{C}[\sigma]]\}.$$

Passing ϵ to 0, as $\lambda[\mathcal{C}[\sigma]] \geq 1$, it follows that

$$\limsup_{i \to \infty} \lambda[\mathcal{C}[\sigma_i]] \leq \lambda[\mathcal{C}[\sigma]],$$

completing the proof. □

We are now ready to prove that $\mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$ is compact. In order to do so we introduce a necessary hypothesis about the entropy of minimal cones. Let $\mathcal{RMC}_n$ denote the space of *regular minimal cones* in $\mathbb{R}^{n+1}$, that is $\mathcal{C} \in \mathcal{RMC}_n$ if and only if it is a proper subset of $\mathbb{R}^{n+1}$ and $\mathcal{C} \setminus \{\mathbf{0}\}$ is a hypersurface in $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$ that is invariant under dilation about $\mathbf{0}$ and with vanishing mean curvature. Let $\mathcal{RMC}^*_n$ denote the set of non-flat elements of $\mathcal{RMC}_n$ – i.e., cones with non-zero curvature somewhere. For any $\Lambda > 0$, let

$$\mathcal{RMC}_n(\Lambda) = \{\mathcal{C} \in \mathcal{RMC}_n \colon \lambda[\mathcal{C}] < \Lambda\} \text{ and } \mathcal{RMC}^*_n(\Lambda) = \mathcal{RMC}^*_n \cap \mathcal{RMC}_n(\Lambda).$$

Now fix a dimension $n \geq 2$ and a value $\Lambda > 1$. Consider the following hypothesis:

$$(\star_{n,\Lambda}) \qquad \text{For all } 2 \leq l \leq n,\ \mathcal{RMC}^*_l(\Lambda) = \emptyset.$$

Observe that all regular minimal cones in $\mathbb{R}^2$ consist of unions of rays and so $\mathcal{RMC}^*_1 = \emptyset$. Likewise, as great circles are the only geodesics in $\mathbb{S}^2$, $\mathcal{RMC}^*_2 = \emptyset$ and so $(\star_{2,\Lambda})$ always holds. As a consequence of Allard's regularity theorem and a dimension reduction argument, there is always some $\Lambda_n > 1$ so that $(\star_{n,\Lambda_n})$ holds.

Proof of Item (3) *of Theorem 1.1.* First, by Corollary 3.4, there is an $R > 0$ so that, up to passing to a subsequence, $\Sigma_i \setminus \bar{B}_R$ converges in $C^\infty_{loc}(\mathbb{R}^{n+1} \setminus \bar{B}_R)$ to some hypersurface Σ' in $\mathbb{R}^{n+1} \setminus \bar{B}_R$. Moreover, Σ' is a $C^{k,\alpha}_*$-asymptotically conical self-expander in $\mathbb{R}^{n+1} \setminus \bar{B}_R$ and $\mathcal{L}(\Sigma') = \sigma$.

As $\lambda[\Sigma_i] \leq \Lambda_0 < \Lambda$, Lemma 3.6 and the standard compactness results, imply that, up to passing to a further subsequence, $\Sigma_i \cap B_{2R}$ converges in the sense of measures to some integral varifold, V, in B_{2R}. As such, there is an integral varifold, Σ, in $\mathbb{R}^{n+1}$ that agrees with V in B_{2R} and Σ' outside $\bar{B}_R$. In particular, Σ is smooth and properly embedded in $\mathbb{R}^{n+1} \setminus \bar{B}_R$. The lower semicontinuity of entropy gives $\lambda[\Sigma] \leq \Lambda_0 < \Lambda$, and so it follows from $(\star_{n,\Lambda})$, a dimension reduction theorem [29, Theorem 4] and Allard's regularity theorem (e.g., [21, Theorem 24.2]) that Σ is actually smooth and properly embedded in $\mathbb{R}^{n+1}$. That is, $\Sigma \in \mathcal{E}^{k,\alpha}_{n,\mathrm{ent}}(\Lambda_0)$. Finally, a further consequence of Allard's regularity theorem [30] is that $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$, finishing the proof. □

3.3. **Smooth compactness of $\mathcal{E}^{k,\alpha}_{\mathrm{top}}(g,e)$.** Combining the asymptotic compactness property, the area estimates from Lemma 3.6 and a result of White [25, Theorem 3 (4)] we can easily prove Item (1) of Theorem 1.1.

Proof of Item (1) *of Theorem 1.1.* By Corollary 3.4 there is an $R > 0$ so that, up to passing to a subsequence,

$$\Sigma_i \setminus \bar{B}_R \to \Sigma' \text{ in } C^\infty_{loc}(\mathbb{R}^3 \setminus \bar{B}_R),$$

where Σ' is a $C^{k,\alpha}_*$-asymptotically conical self-expander in $\mathbb{R}^3 \setminus \bar{B}_R$, Σ' consists of e annuli and $\mathcal{L}(\Sigma') = \sigma$. In particular, for $r > R$ sufficiently large, ∂B_r meets Σ transversally and also meets each Σ_i transversally. As such, $\Sigma_i \cap B_r$ has genus g and $\Sigma_i \cap \partial B_r$ has e components. Thus, it follows that there is a constant $C = C(g,e)$ so

$$\int_{\partial B_r \cap \Sigma_i} \kappa < C$$

where κ denotes the geodesic curvature of the boundary curve. Moreover, by Lemma 3.8, for i sufficiently large, $\lambda[\mathcal{C}(\Sigma_i)] \leq \lambda[\mathcal{C}[\sigma]] + 1$ and so, by Lemma 3.6, there is a C' so

$$\mathcal{H}^2(\Sigma_i \cap B_r) \leq C'.$$

Hence, it follows from [25, Theorem 3 (4)] that, up to passing to a further subsequence,

$$\Sigma_i \cap B_r \to \Sigma'' \text{ in } C^\infty_{loc}(B_r)$$

where Σ'' is a self-expander in B_r and the convergence is with multiplicity one. It is clear that $\Sigma' = \Sigma''$ in $B_r \setminus \bar{B}_R$. Therefore the result follows with $\Sigma = \Sigma' \cup \Sigma''$. □

3.4. **Smooth compactness of $\mathcal{E}^{k,\alpha}_{n,\mathrm{mc}}(h_0)$.** We combine Lemma 3.6 with a curvature estimate for mean convex self-expanders in order to prove Item (2) of Theorem 1.1.

First we show a curvature estimate for strictly mean convex asympotitically conical self-expanders with strictly mean convex link. Our argument uses the maximum principle and is completely analogous to the one used in [6] for mean convex self-shrinkers.

Lemma 3.9. *Let $\Sigma \subset \mathbb{R}^{n+1}$, be an asymptotically conical self-expander. If Σ is strictly mean convex and whose asymptotic cone, $\mathcal{C}(\Sigma)$ is C^2-regular and strictly mean convex, then, for $p \in \Sigma$,*

$$|A_\Sigma(p)|^2 \leq K|\mathbf{x}(p)|^2$$

where

$$K = \frac{1}{4} \sup_{\mathcal{C}(\Sigma)} \frac{|A_{\mathcal{C}(\Sigma)}|^2}{H^2_{\mathcal{C}(\Sigma)}} = \frac{1}{4} \sup_{\mathcal{L}(\Sigma)} \frac{|A_{\mathcal{L}(\Sigma)}|^2}{H^2_{\mathcal{L}(\Sigma)}} < \infty.$$

Proof. By definition of asymptotic cones and scaling invariance

$$\lim_{R\to\infty} \sup_{\partial B_R \cap \Sigma} \frac{|A_\Sigma|^2}{H^2_\Sigma} = 4K.$$

Hence, for every $\epsilon > 0$, there is an R_ϵ so if $R > R_\epsilon$, then

$$\sup_{\partial B_R \cap \Sigma} \frac{|A_\Sigma|^2}{H^2_\Sigma} \leq 4K + \epsilon.$$

Computing as in [6] one has

$$L_\Sigma H_\Sigma = \left(\Delta_\Sigma + \frac{\mathbf{x}}{2}\cdot\nabla_\Sigma + |A_\Sigma|^2 - \frac{1}{2}\right) H_\Sigma = -H_\Sigma$$

which follows from a variant on Simons' identity, for the rough drift Laplacian,

$$(L_\Sigma A_\Sigma)_{ij} = -(A_\Sigma)_{ij}.$$

For details see the computations in (10.10)-(10.12) and (5.7)-(5.8) of [6]. These are carried out for self-shrinkers, however the formulas for self-expanders follow with a simple sign change.

Now consider the new operator

$$\mathscr{L}_{H_\Sigma^2} = \Delta_\Sigma + \frac{\mathbf{x}}{2}\cdot\nabla_\Sigma + 2(\nabla_\Sigma \log H_\Sigma)\cdot\nabla_\Sigma$$

We compute that

$$\left(\mathscr{L}_{H_\Sigma^2}\frac{A_\Sigma}{H_\Sigma}\right)_{ij} = 0$$

and so

$$\mathscr{L}_{H_\Sigma^2}\frac{|A_\Sigma|^2}{H_\Sigma^2} = 2\left|\nabla_\Sigma \frac{A_\Sigma}{H_\Sigma}\right|^2 \geq 0.$$

Hence, by the maximum principle, for any $R > R_\epsilon$,

$$\sup_{B_R\cap\Sigma}\frac{|A_\Sigma|^2}{H_\Sigma^2} \leq 4K + \epsilon.$$

Hence, letting $\epsilon \to 0$, gives

$$\sup_\Sigma \frac{|A_\Sigma|^2}{H_\Sigma^2} \leq 4K.$$

That is, for any $p \in \Sigma$,

$$|A_\Sigma(p)|^2 \leq 4KH_\Sigma^2(p) \leq K(\mathbf{x}(p)\cdot\mathbf{n}_\Sigma(p))^2 \leq K|\mathbf{x}(p)|^2.$$

This proves the claim. □

We are now ready to complete the proof of Theorem 1.1.

Proof of Item (2) *of Theorem 1.1.* By Corollary 3.4 there is an $R > 0$ so that, up to passing to a subsequence,

$$\Sigma_i \setminus \bar{B}_R \to \Sigma' \text{ in } C^\infty_{loc}(\mathbb{R}^{n+1}\setminus\bar{B}_R),$$

where Σ' is a $C^{k,\alpha}_*$-asymptotically conical self-expander in $\mathbb{R}^{n+1}\setminus\bar{B}_R$ and $\mathcal{L}(\Sigma') = \sigma$. The nature of the convergence, ensures that $H_\sigma \geq h_0$. Moreover, by Lemma 3.8, for i sufficiently large, $\lambda[\mathcal{C}(\Sigma_i)] \leq \lambda[\mathcal{C}[\sigma]] + 1$. Thus, by Lemma 3.6, there is a uniform C' so

$$\mathcal{H}^n(\Sigma_i \cap B_r) \leq C'.$$

As σ is strictly mean convex, for i sufficiently large each $\mathcal{L}(\Sigma_i)$ is strictly mean convex. Indeed, setting

$$K = \sup_\sigma \frac{|A_\sigma|^2}{H_\sigma^2} \in (0,\infty)$$

one has, after possibly throwing out a finite sequence of the Σ_i, that

$$\sup_{\mathcal{L}(\Sigma_i)} \frac{|A_{\mathcal{L}(\Sigma_i)}|^2}{H^2_{\mathcal{L}(\Sigma_i)}} \leq 4K.$$

Hence, by Lemma 3.9,

$$|A_{\Sigma_i}(p)|^2 \leq K|\mathbf{x}(p)|^2.$$

That is,

$$\sup_{\Sigma_i \cap B_{2R}} |A_{\Sigma_i}|^2 \leq 4KR^2.$$

Combining this with the area bound and the Arzelà-Ascoli theorem, gives that, up to passing to a subsequence, the Σ_i converge, possibly with multiplicities, in $C^\infty_{loc}(B_{2R})$ to a limit Σ''. As Σ'' is a smooth solution to (1.1) and there are no closed self-expanders, each component of Σ'' meets Σ'. In particular, as the Σ_i converge with multiplicity one to Σ' in $B_{2R}\setminus\bar{B}_R$, the Σ_i converge to Σ'' with multiplicity one in B_{2R}. Hence, setting $\Sigma = \Sigma'\cup\Sigma''$ one obtains a smooth asymptotically conical self-expander with $\Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$. As each Σ_i has positive mean curvature, Σ has non-negative mean curvature. However, as $\sigma = \mathcal{L}(\Sigma)$ has $H_{\mathcal{L}(\Sigma)} \geq h_0 > 0$ and $L_\Sigma H_\Sigma = -H_\Sigma \leq 0$, the strong maximum principle implies Σ has positive mean curvature completing the proof. □

4. PROPERNESS OF MAP Π

Before proving Theorems 1.2, 1.3 and 1.4 we need the following auxiliary proposition that relates sequential compactness in $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$ to locally smooth compactness in $\mathbb{R}^{n+1}$.

Proposition 4.1. *For $\Gamma \in \mathcal{ACH}^{k,\alpha}_n$, if $\varphi \in \mathcal{V}^{k,\alpha}_{\mathrm{emb}}(\Gamma)$, and $[\mathbf{f}_i] \in \mathcal{ACE}^{k,\alpha}_n(\Gamma)$ satisfy:*

(1) $\mathrm{tr}^1_\infty[\mathbf{f}_i] = \varphi_i \to \varphi$ in $C^{k,\alpha}(\mathcal{L}(\Gamma);\mathbb{R}^{n+1})$;
(2) $\mathbf{f}_i(\Gamma) = \Sigma_i \to \Sigma$ in $C^\infty_{loc}(\mathbb{R}^{n+1})$ for some hypersurface Σ,

then $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$ and is a self-expander. Moreover, there is a parameterization $\mathbf{f}\colon \Gamma \to \Sigma \subset \mathbb{R}^{n+1}$ so that

(1) $[\mathbf{f}] \in \mathcal{ACE}^{k,\alpha}_n(\Gamma)$;
(2) $\mathrm{tr}^1_\infty[\mathbf{f}] = \varphi$; and;
(3) $[\mathbf{f}_i] \to [\mathbf{f}]$ in the topology of $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$.

Proof. First observe that as each Σ_i satisfies (1.1), the nature of the convergence ensures that Σ does as well. Let

$$\mathcal{C}_i = \mathscr{E}^{\mathrm{H}}_1[\varphi_i](\mathcal{C}(\Gamma)) = \mathcal{C}(\Sigma_i) \text{ and } \mathcal{C} = \mathscr{E}^{\mathrm{H}}_1[\varphi](\mathcal{C}(\Gamma)).$$

Then $\mathcal{C}_i$ and $\mathcal{C}$ are $C^{k,\alpha}$-regular cones. By our hypothesis (1),

$$\mathcal{C}_i \to \mathcal{C} \text{ in } C^{k,\alpha}_{loc}(\mathbb{R}^{n+1}\setminus\{\mathbf{0}\}),$$

which further implies

$$\mathcal{L}(\Sigma_i) \to \mathcal{L}[\mathcal{C}] \text{ in } C^{k,\alpha}(\mathbb{S}^n).$$

Thus, by Corollary 3.4 and our hypothesis (2), we have $\Sigma \in \mathcal{ACH}^{k,\alpha}_n$ with $\mathcal{C}(\Sigma) = \mathcal{C}$.

As $\varphi \in \mathcal{V}^{k,\alpha}_{\mathrm{emb}}(\Gamma)$, the hypothesis (1) ensures that

$$\mathscr{E}^{\mathrm{H}}_1[\varphi_i] \circ \mathscr{E}^{\mathrm{H}}_1[\varphi]^{-1} \to \mathbf{x}|_{\mathcal{C}} \text{ in } C^{k,\alpha}_{loc}(\mathcal{C};\mathbb{R}^{n+1}).$$

Observe, that $\mathscr{E}^{\mathrm{H}}_1[\varphi_i]\circ\mathscr{E}^{\mathrm{H}}_1[\varphi]^{-1}$ are homogeneous of degree one, so we denote their traces at infinity by $\tilde{\varphi}_i$. Thus, letting $\mathcal{L}$ be the link of $\mathcal{C}$,

$$\tilde{\varphi}_i \to \mathbf{x}|_{\mathcal{L}} \text{ in } C^{k,\alpha}(\mathcal{L};\mathbb{R}^{n+1}).$$

Let $\mathbf{h}_i \in C^{k,\alpha}_1 \cap C^k_{1,\mathrm{H}}(\Sigma;\mathbb{R}^{n+1})$ be chosen so that

$$\mathscr{L}_\Sigma \mathbf{h}_i = \Delta_\Sigma \mathbf{h}_i + \frac{1}{2}\mathbf{x}\cdot\nabla_\Sigma \mathbf{h}_i - \frac{1}{2}\mathbf{h}_i = \mathbf{0} \text{ and } \mathrm{tr}^1_\infty[\mathbf{h}_i] = \tilde{\varphi}_i - \mathbf{x}|_{\mathcal{L}}.$$

By [3, Corollary 5.8] there is a unique such $\mathbf{h}_i$ and it satisfies the estimate

$$\|\mathbf{h}_i\|^{(1)}_{k,\alpha} \leq C'\|\tilde{\varphi}_i - \mathbf{x}|_{\mathcal{L}}\|_{k,\alpha}$$

for some constant $C' = C'(\Sigma, n, k, \alpha)$. We let

$$\mathbf{g}_i = \mathbf{x}|_\Sigma + \mathbf{h}_i \text{ and } \Upsilon_i = \mathbf{g}_i(\Sigma).$$

It is clear that for i sufficiently large, $\mathbf{g}_i \in \mathcal{ACH}^{k,\alpha}_n(\Sigma)$ and $\mathrm{tr}^1_\infty[\mathbf{g}_i] = \tilde{\varphi}_i$. Thus, by [3, Proposition 3.3], $\Upsilon_i \in \mathcal{ACH}^{k,\alpha}_n$ and $\mathcal{C}(\Upsilon_i) = \mathcal{C}_i$.

Now pick a transverse section $\mathbf{v}$ on Σ that, outside a compact set, equals $\mathscr{E}_\mathbf{w}[\mathbf{w}] = \mathbf{w} \circ \pi_\mathbf{w}|_\Sigma$ for $\mathbf{w}$ a chosen homogeneous transverse section on $\mathcal{C}$. For i sufficiently large, $\mathbf{v}_i = \mathbf{v} \circ \mathbf{g}_i^{-1}$ is an asymptotically homogeneous, transverse section on Υ_i. By Proposition 3.3, for i large, Σ_i lies in a $\mathbf{v}_i$-regular neighborhood of Υ_i and is transverse to $\mathscr{E}_{\mathbf{v}_i}[\mathbf{v}_i] = \mathbf{v}_i \circ \pi_{\mathbf{v}_i}$. In particular, $\pi_{\mathbf{v}_i}|_{\Sigma_i} : \Sigma_i \to \Upsilon_i$ is an element of $\mathcal{ACH}^{k,\alpha}_n(\Sigma_i)$. Thus, for i large, there is a unique $u_i \in C^{k,\alpha}_1 \cap C^k_{1,0}(\Sigma)$ so that Σ_i can be parametrized by the map

$$\tilde{\mathbf{f}}_i = (\pi_{\mathbf{v}_i}|_{\Sigma_i})^{-1} \circ \mathbf{g}_i = \mathbf{g}_i + u_i \mathbf{v}$$

which is an element of $\mathcal{ACH}^{k,\alpha}_n(\Sigma)$ by [3, Proposition 3.3]. Moreover, $\|u_i\|^{(1)}_{k,\alpha}$ is uniformly bounded and $\|u_i\|^{(1)}_1 \to 0$.

Observe, that there is a $\delta > 0$ (independent of i) so that for i sufficiently large

$$\inf_{p \in \Sigma} |\mathbf{v} \cdot (\mathbf{n}_{\Sigma_i} \circ \tilde{\mathbf{f}}_i)| > \delta.$$

As Σ_i is a self-expander, it follows from [3, Lemma 7.2] and direct calculations that

$$\mathscr{L}_\Sigma u_i = -\frac{\mathbf{n}_{\Sigma_i} \circ \tilde{\mathbf{f}}_i}{\mathbf{v} \cdot (\mathbf{n}_{\Sigma_i} \circ \tilde{\mathbf{f}}_i)} \cdot \left(2\nabla_\Sigma u_i \cdot \nabla_\Sigma \mathbf{v} + u_i \left(\mathscr{L}_\Sigma + \frac{1}{2} \right) \mathbf{v} + (g^{-1}_{\tilde{\mathbf{f}}_i} - g^{-1}_\Sigma)^{jl} (\nabla^2_\Sigma \tilde{\mathbf{f}}_i)_{jl} \right)$$

where $g_{\tilde{\mathbf{f}}_i}$ and g_Σ are the pull-back metrics of Euclidean one by $\tilde{\mathbf{f}}_i$ and $\mathbf{x}|_\Sigma$, respectively. One further uses [3, Proposition 3.1] to see that, for i large, the right hand side are elements of $C^{k-2,\alpha}_{-1}(\Sigma)$ with uniformly bounded $\|\cdot\|^{(-1)}_{k-2,\alpha}$ norm. Hence, by [3, Theorem 5.7 and Corollary 5.8], $u_i \in \mathcal{D}^{k,\alpha}(\Sigma)$ and $\|u_i\|^*_{k,\alpha}$ is uniformly bounded. Here

$$\mathcal{D}^{k,\alpha}(\Sigma) = \left\{ u \in C^{k,\alpha}_1 \cap C^{k-1,\alpha}_0 \cap C^{k-2,\alpha}_{-1}(\Sigma) \colon \mathbf{x} \cdot \nabla_\Sigma u \in C^{k-2,\alpha}_{-1}(\Sigma) \right\}$$

is a Banach space with norm

$$\|u\|^*_{k,\alpha} = \|u\|^{(-1)}_{k-2,\alpha} + \sum_{k-1 \leq i \leq k} \|\nabla^i_\Sigma u\|^{(1-k)}_\alpha + \|\mathbf{x} \cdot \nabla_\Sigma u\|^{(-1)}_{k-2,\alpha}.$$

As $\mathcal{D}^{k,\alpha}(\Sigma)$ is compactly embedded in $C^{k-1,\alpha}_1(\Sigma)$, we have $\|u_i\|^{(1)}_{k-1,\alpha} \to 0$. Thus it follows from [3, Lemma 7.5] that $\|u_i\|^*_{k,\alpha} \to 0$ and so $\|\tilde{\mathbf{f}}_i - \mathbf{x}|_\Sigma\|^{(1)}_{k,\alpha} \to 0$.

We pick a large integer I so that $\tilde{\mathbf{f}}_I$ is well-defined as above. Choose a representative $\mathbf{f}_I$ of $[\mathbf{f}_I]$. We define $\mathbf{f} = \tilde{\mathbf{f}}_I^{-1} \circ \mathbf{f}_I$, and it is clear that $\mathbf{f}(\Gamma) = \Sigma$. Moreover, by [3, Proposition 3.3], $\mathbf{f} \in \mathcal{ACH}^{k,\alpha}(\Gamma)$ and

$$\mathcal{C}[\mathbf{f}] = \mathcal{C}[\tilde{\mathbf{f}}_I]^{-1} \circ \mathcal{C}[\mathbf{f}_I] = (\mathscr{E}^{\mathrm{H}}_1[\varphi_I] \circ \mathscr{E}^{\mathrm{H}}_1[\varphi]^{-1})^{-1} \circ \mathscr{E}^{\mathrm{H}}_1[\varphi_I] = \mathscr{E}^{\mathrm{H}}_1[\varphi].$$

Thus, $[\mathbf{f}]$ represents a class in $\mathcal{ACE}^{k,\alpha}_n(\Gamma)$ which has $\Pi([\mathbf{f}]) = \mathrm{tr}^1_\infty[\mathbf{f}] = \varphi$.

Hence, to complete the proof it remains only to show that $[\mathbf{f}_i] \to [\mathbf{f}]$ in the topology of $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$. Observe that $\tilde{\mathbf{f}}_i \circ \mathbf{f}(\Gamma) = \Sigma_i$, and invoking [3, Proposition 3.3] again, $\tilde{\mathbf{f}}_i \circ \mathbf{f} \in \mathcal{ACH}_n^{k,\alpha}(\Gamma)$ and

$$\mathcal{C}[\tilde{\mathbf{f}}_i \circ \mathbf{f}] = \mathcal{C}[\tilde{\mathbf{f}}_i] \circ \mathcal{C}[\mathbf{f}] = (\mathscr{E}_1^{\mathrm{H}}[\varphi_i] \circ \mathscr{E}_1^{\mathrm{H}}[\varphi]^{-1}) \circ \mathscr{E}_1^{\mathrm{H}}[\varphi] = \mathscr{E}_1^{\mathrm{H}}[\varphi_i].$$

This gives that $\tilde{\mathbf{f}}_i \circ \mathbf{f}$ is an element of $[\mathbf{f}_i]$. Moreover, by [3, Proposition 3.1], $\tilde{\mathbf{f}}_i \circ \mathbf{f} \to \mathbf{f}$ in $C_1^{k,\alpha}(\Gamma; \mathbb{R}^{n+1})$. Therefore, $[\mathbf{f}_i] \to [\mathbf{f}]$ in the topology of $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$. ☐

The proofs of properness of Π now follow easily.

Proof of Theorem 1.2. The result follows directly from Item (1) of Theorem 1.1, Proposition 4.1 and an elementary topology fact, Lemma A.1. ☐

Proof of Theorem 1.3. First, by Lemma 3.8, $\mathcal{V}_{\mathrm{ent}}(\Gamma, \Lambda)$ is open in $C^{k,\alpha}(\mathcal{L}(\Gamma); \mathbb{R}^{n+1})$. Next, by Lemma 3.5,

$$\lambda[\mathbf{f}(\Gamma)] = \lambda[\mathcal{C}[\sigma]] \text{ where } \sigma = \Pi([\mathbf{f}])(\mathcal{L}(\Gamma)).$$

Thus, the continuity of Π and Lemma 3.8 imply that $\mathcal{U}_{\mathrm{ent}}(\Gamma, \Lambda)$ is open in $\mathcal{ACE}_n^{k,\alpha}(\Gamma)$. If $\mathcal{Z} \subset \mathcal{V}(\Gamma, \Lambda)$ is compact, then it follows from Lemma 3.8 that there is a $\Lambda_0 < \Lambda$ so that

$$\lambda[\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma))] \leq \Lambda_0$$

for all $\varphi \in \mathcal{Z}$. Hence, if $(\star_{n,\Lambda})$ holds for some $\Lambda < 2$, then the last claim follows from Item (3) of Theorem 1.1, Proposition 4.1 and Lemma A.1. ☐

Proof of Theorem 1.4. Items (1) and (2) are straightforward. For $[\mathbf{f}] \in \mathcal{U}_{\mathrm{mc}}(\Gamma)$ let $\Sigma = \mathbf{f}(\Gamma)$. As $H_\Sigma > 0$ and $L_\Sigma H_\Sigma = -H_\Sigma < 0$, L_Σ has a positive super-solution. Hence, it follows from the trick of Fischer-Colbrie and Schoen [10] that Σ is strictly stable. In particular, Σ admits no non-trivial Jacobi fields and so Item (3) follows from [3, Theorem 1.1 (4)]. Furthermore, if $\mathcal{Z} \subset \mathcal{V}_{\mathrm{mc}}(\Gamma)$ is compact, then there is an $h_0 > 0$ so $H_\sigma \geq h_0$, where σ is the link of the cone $\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma))$, for all $\varphi \in \mathcal{Z}$. As such, Item (4) follows from Item (2) of Theorem 1.1, Proposition 4.1 and Lemma A.1. It remains only to show the final remark. Observe, that by Item (4) and that $\mathcal{V}_{\mathrm{mc}}(\Gamma)$ is a compactly generated Hausdorff space, the map $\Pi|_{\mathcal{U}_{\mathrm{mc}}(\Gamma)}$ is a closed map. Hence, following the arguments in [20, Proposition 4.46], the final remark is an immediate consequence of Items (3) and (4). ☐

5. Existence of mean convex asymptotically conical self-expanders

We conclude by proving Corollary 1.5. We first show existence and uniqueness of mean convex self-expanders asymptotic to rotationally symmetric cones. This result is not new (see [1, 7]), but we include a proof for the sake of completeness.

Proposition 5.1. *For $n \geq 2$ let $\mathcal{C} \subset \mathbb{R}^{n+1}$ be a connected non-flat rotationally symmetric cone. There is a unique smooth self-expander Σ that is smoothly asymptotic to $\mathcal{C}$. Moreover, Σ satisfies $H_\Sigma > 0$ and is an entire graph and so is diffeomorphic to $\mathbb{R}^n$.*

Proof. Without loss of generality we assume that $\mathbf{e}_{n+1}$ is the axis of symmetry of $\mathcal{C}$ and $\mathcal{C}$ lies in the half-space $\{x_{n+1} \geq 0\}$. First, we show the existence of a strictly mean convex self-expanders asymptotic to $\mathcal{C}$. Clearly, there is a $\tau > 0$ so $\mathcal{C}$ is the graph of a Lipschitz function $u_0 \colon \mathbb{R}^n \to \mathbb{R}$ given by $u_0(x) = \tau|x|$. Let $u_0^i(x) = \tau\sqrt{i^{-1} + |x|^2}$ so $u_0^i \colon \mathbb{R}^n \to \mathbb{R}$ is a sequence of smooth functions which are strictly convex, have Lipschitz constant at most τ and are asymptotic to u_0. Moreover, the u_0^i converge to u_0 uniformly. Let $\Sigma^i \subset \mathbb{R}^{n+1}$ be the graphs of the u_0^i. These are smooth convex hypersurfaces that are asymptotic to $\mathcal{C}$. Thus, Ecker-Huisken [8] and the strong maximum principle applied to the evolution equation for mean curvature [9] imply that there is a unique strictly mean convex, graphical solution, $\mathcal{M}^i = \left\{\Sigma_t^i\right\}_{t\geq 0}$ to mean curvature flow starting from Σ^i. Moreover, if we denote by $u^i(t,\cdot)$ the functions whose graphs are the Σ_t^i, then the $u^i(t,\cdot)$ have uniformly bounded Lipschitz constant.

Hence, by Brakke's compactness (cf. [15, §7]), the $\mathcal{M}^i$ converges as Brakke flows to $\mathcal{M} = \{\Sigma_t\}_{t\geq 0}$ with $\Sigma_0 = \mathcal{C}$. Furthermore, the interior estimates for graphical mean curvature flow [9] and the Arzelà-Ascoli theorem imply that for $t > 0$, Σ_t has nonnegative mean curvature and is given by the graph of a smooth function $u(t,\cdot)$ on $\mathbb{R}^n$ with a uniform Lipschitz constant. Moreover, as $t \to 0^+$, $\Sigma_t \to \mathcal{C}$ in $C^\infty_{loc}(\mathbb{R}^{n+1}\setminus\{\mathbf{0}\})$. By the parabolic maximum principle, such a solution $u(t,\cdot)$ for $t > 0$ is unique in the class of functions with at most linear growth. A consequence of this is that $u(t,x) = \sqrt{t}\,u(1, \frac{|x|}{\sqrt{t}})$. As such, $\Sigma = \Sigma_1$ is a graphical self-expanding hypersurface of revolution that is smoothly asymptotic to $\mathcal{C}$ and has nonnegative mean curvature. Furthermore, as $L_\Sigma H_\Sigma = -H_\Sigma \leq 0$ and $H_\mathcal{C} > 0$, the strong maximum principle implies $H_\Sigma > 0$.

Next we show the uniqueness of self-expanders asymptotic to the given cone $\mathcal{C}$. Suppose Σ' is another smooth self-expander smoothly asymptotic to $\mathcal{C}$. Let

$$s_0 = \inf\{s > 0 \colon \Sigma' \cap (\Sigma + s\mathbf{e}_{n+1}) = \emptyset\} \geq 0.$$

We claim $s_0 = 0$ and so Σ' lies below Σ. Suppose not, i.e., $s_0 > 0$. By Item (2) of Proposition 3.3, $\Sigma + s_0\mathbf{e}_{n+1}$ touches Σ' from above at some interior point. However, we observe that $H + \frac{1}{2}\mathbf{x}\cdot\mathbf{n} > 0$ on $\Sigma + s_0\mathbf{e}_{n+1}$ where $\mathbf{n}$ is chosen to be upward. This leads to a contradiction with the strong maximum principle. By similar arguments, Σ' also lies above Σ. Thus $\Sigma' = \Sigma$ proving the uniqueness. □

Proof of Corollary 1.5. Let $\{\sigma_t\}_{t\in[0,T)}$ be the solution of mean curvature flow in $\mathbb{S}^n$ with $\sigma_0 = \sigma$ and T the maximal time of existence. Pick a parameterization $\varphi_0 \colon \sigma \to \sigma_0$ and choose $\varphi_t \colon \sigma \to \sigma_t$ a corresponding evolution of parameterizations. As σ_0 is mean convex, the maximum principle implies that σ_t is as well for all $t \in (0,T)$. When $n = 2$, results of Grayson [12] and Zhu [31, Corollary 4.2] give that σ_t smoothly shrinks to a round point in finite time. When $n \geq 3$, by [13], the condition (1.2) is preserved, which, together with the mean convexity and connectedness, implies that $T < \infty$ and the flow disappears in a round point at time T.

Choose a rotationally symmetric hypersurface $\tilde{\sigma} \subset \mathbb{S}^n$ and a T_0 sufficiently close to T, so σ_{T_0} is a small normal graph over $\tilde{\sigma}$. Thus, there is a path of smooth, strictly mean convex embeddings, $\psi_t \colon \tilde{\sigma} \to \mathbb{S}^n$, $t \in [0,1]$, so $\psi_0(\tilde{\sigma}) = \sigma_{T_0}$ and $\psi_1 = \mathbf{x}|_{\tilde{\sigma}}$. Thus, setting

$$\tilde{\varphi}_t = \begin{cases} \varphi_t \circ \varphi_{T_0}^{-1} \circ \psi_0 & 0 \leq t \leq T_0 \\ \psi_{t-T_0} & T_0 \leq t \leq T_0 + 1 \end{cases}$$

one obtains a path of strictly mean convex $C^{k,\alpha}$-embeddings of $\tilde{\sigma}$ into $\mathbb{S}^n$ connecting $\tilde{\varphi}_{T_0+1} = \mathbf{x}|_{\tilde{\sigma}}$ to $\tilde{\varphi}_0 \colon \tilde{\sigma} \to \sigma$. By Proposition 5.1 there is a unique smooth self-expander Γ that is smoothly asymptotic to $\mathcal{C}[\tilde{\sigma}]$ and $H_\Gamma > 0$. Hence, Theorem 1.4 implies there is an $[\mathbf{f}_0] \in \mathcal{U}_{\mathrm{mc}}(\Gamma)$ with $\Pi([\mathbf{f}_0]) = \tilde{\varphi}_0$ and so $\Sigma = \mathbf{f}_0(\Gamma)$ is the desired element.

In what follows we restrict our discussions to those n such that $\pi_0(\mathrm{Diff}^+(\mathbb{S}^{n-1})) = 0$. For $n \geq 3$ we let

$$\mathcal{V} = \left\{\varphi \in \mathcal{V}_{\mathrm{mc}}(\Gamma) \colon \mathcal{L}[\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma))] \text{ satisfies (1.2)}\right\},$$

which is an open subset of $\mathcal{V}_{\mathrm{mc}}(\Gamma)$; for $n = 2$ let $\mathcal{V} = \mathcal{V}_{\mathrm{mc}}(\Gamma)$. By the previous discussions and the hypothesis that $\mathrm{Diff}^+(\mathcal{L}(\Gamma)) \simeq \mathrm{Diff}^+(\mathbb{S}^{n-1})$ is path-connected, we observe $\mathcal{V}$ has exactly two components. Let $\mathcal{V}_+$ be the component of $\mathcal{V}$ that contains $\mathbf{x}|_{\mathcal{L}(\Gamma)}$ and let $\mathcal{V}_-$ be the component containing $\mathbf{x}|_{\mathcal{L}(\Gamma)} \circ I$ where $I \in \mathrm{Diff}(\mathcal{L}(\Gamma))$ is an orientation-reversing involution. If $\mathcal{U}_\pm = \mathcal{U}_{\mathrm{mc}}(\Gamma) \cap \Pi^{-1}(\mathcal{V}_\pm)$, then it follows from Theorem 1.4 and the uniqueness of Γ that $\Pi|_{\mathcal{U}_\pm} : \mathcal{U}_\pm \to \mathcal{V}_\pm$ is a diffeomorphism.

Now suppose $\varphi, \psi \in \mathcal{V}_+$ so that $\mathscr{E}_1^{\mathrm{H}}[\varphi](\mathcal{C}(\Gamma)) = \mathscr{E}_1^{\mathrm{H}}[\psi](\mathcal{C}(\Gamma))$. If $[\mathbf{f}], [\mathbf{g}] \in \mathcal{U}_+$ with $\Pi([\mathbf{f}]) = \varphi$ and $\Pi([\mathbf{g}]) = \psi$, then we will show $\mathbf{f}(\Gamma) = \mathbf{g}(\Gamma)$. Observe that there is a path of homogeneous $\Phi_t \in \mathrm{Diff}^+(\mathcal{C}(\Gamma))$ with $\Phi_0 = \mathscr{E}_1^{\mathrm{H}}[\varphi]^{-1} \circ \mathscr{E}_1^{\mathrm{H}}[\psi]$ and $\Phi_1 = \mathbf{x}|_{\mathcal{C}(\Gamma)}$. Thus, Theorem 1.4 and the uniqueness of Γ imply that there is a path of $[\mathbf{h}_t] \in \mathcal{ACE}_n^{k,\alpha}(\Gamma)$ so that $\mathbf{h}_t(\Gamma) = \Gamma$ and $\Pi([\mathbf{h}_t]) = \mathrm{tr}_\infty^1[\Phi_t]$. In particular, $[\mathbf{f} \circ \mathbf{h}_0] \in \mathcal{U}_+$ and $\Pi([\mathbf{f} \circ \mathbf{h}_0]) = \psi$. As $\Pi|_{\mathcal{U}_+}$ is a diffeomorphism, $[\mathbf{g}] = [\mathbf{f} \circ \mathbf{h}_0]$ and so $\mathbf{g}(\Gamma) = \mathbf{f}(\Gamma)$. By symmetries, the same claim holds on $\mathcal{V}_-$. Finally, observe that as Γ is diffeomorphic to $\mathbb{R}^n$ there is an orientation-reversing diffeomorphism $\tilde{I} \in \mathcal{ACH}_n^{k,\alpha}(\Gamma)$ so that $\mathrm{tr}_\infty^1[\tilde{I}] = I$. Moreover, $[\mathbf{f}] \in \mathcal{U}_+$ if and only if $[\mathbf{f} \circ \tilde{I}] \in \mathcal{U}_-$. This concludes the proof of uniqueness.

□

Appendix A.

Lemma A.1. *Let X be a topological space. Suppose X has a countable cover, $\{A_i\}_{i\in\mathbb{N}}$, of closed subsets so that each A_i is metrizable. If K is a sequentially compact subspace of X, then it is compact.*

Proof. Let $\{U_\alpha\}$ be an arbitrary collection of open sets of X that covers K. As K is sequentially compact, so is every $K \cap A_i$. Since A_i is metrizable, there is a finite subcollection of $\{U_\alpha\}$ that covers $K \cap A_i$. Thus, there is a countable subcollection $\{U_{\alpha_i}\}_{i\in\mathbb{N}} \subset \{U_\alpha\}$ that covers K. We show that there is a finite subcollection of $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$ that covers K, implying the compactness of K. We argue by contradiction. Suppose not, then pick $x_j \in K \setminus (\bigcup_{i=1}^j U_{\alpha_i})$. Thus, up to passing to a subsequence, $x_j \to x \in K$. However, $x \in U_{\alpha_{i_0}}$ for some i_0, so for j sufficiently large $x_j \in U_{\alpha_{i_0}}$, which leads to a contradiction. □

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