SOLUTIONS TO SUBLINEAR ELLIPTIC EQUATIONS WITH FINITE GENERALIZED ENERGY

ADISAK SEESANEA AND IGOR E. VERBITSKY

Abstract. We give necessary and sufficient conditions for the existence of a positive solution with zero boundary values to the elliptic equation

$$\mathcal{L}u = \sigma u^q + \mu \quad \text{in } \Omega,$$

in the sublinear case $0 < q < 1$, with finite generalized energy: $\mathbb{E}_\gamma[u] := \int_\Omega |\nabla u|^2 u^{\gamma-1} dx < \infty$, for $\gamma > 0$. In this case $u \in L^{\gamma+q}(\Omega, \sigma) \cap L^\gamma(\Omega, \mu)$, where $\gamma = 1$ corresponds to finite energy solutions.

Here $\mathcal{L}u := -\operatorname{div}(\mathcal{A}\nabla u)$ is a linear uniformly elliptic operator with bounded measurable coefficients, and σ, μ are nonnegative functions (or Radon measures), on an arbitrary domain $\Omega \subseteq \mathbb{R}^n$ which possesses a positive Green function associated with $\mathcal{L}$.

When $0 < \gamma \le 1$, this result yields sufficient conditions for the existence of a positive solution to the above problem which belongs to the Dirichlet space $\dot{W}_0^{1,p}(\Omega)$ for $1 < p \le 2$.

1. Introduction

We consider the elliptic equation

$$\mathcal{L}u = \sigma u^q + \mu \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

in the sublinear case $0 < q < 1$. Here Ω is an arbitrary domain (a nonempty open connected set) in $\mathbb{R}^n$, $n \ge 2$, which possesses a positive Green function, and σ, μ are nonnegative locally integrable functions, or more generally, nonnegative Radon measures in Ω. This class of (locally finite) measures is denoted by $\mathcal{M}^+(\Omega)$.

The operator $\mathcal{L}u := -\operatorname{div}(\mathcal{A}\nabla u)$ with bounded measurable coefficients is assumed to be uniformly elliptic, i.e., $\mathcal{A} : \Omega \to \mathbb{R}^{n\times n}$ is a

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real symmetric matrix-valued function on Ω, and there exist positive constants $m \le M$ such that

$$m|\xi|^2 \le \mathcal{A}(x)\xi \cdot \xi \le M|\xi|^2, \tag{1.2}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

Both homogeneous ($\mu \equiv 0$) and inhomogeneous equations ($\mu \not\equiv 0$) will be studied simultaneously. The latter case involves a nontrivial question regarding interaction between σ and μ (see Lemma 4.3 below).

We denote by $\dot{W}_0^{1,p}(\Omega)$ ($1 \le p < \infty$) the homogeneous Sobolev (or Dirichlet) space defined as the closure of $C_0^\infty(\Omega)$ with respect to the seminorm $\|u\|_{\dot{W}_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}$.

An indispensable tool in our study is the notion of the generalized energy. Suppose u is a positive Green potential, i.e., $u = \mathbf{G}\omega$ for $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$,

$$\mathbf{G}\omega(x) := \int_\Omega G(x,y)\, d\omega(y), \quad x \in \Omega,$$

where G is a positive Green function associated with $\mathcal{L}$ (see [11]).

As we will show below (see Theorem 3.1), the condition

$$\mathbb{E}_\gamma[u] := \int_\Omega |\nabla u|^2 u^{\gamma-1}\, dx < +\infty, \quad \gamma > 0 \tag{1.3}$$

is equivalent to the generalized Green energy $\mathcal{E}_\gamma[\omega]$ being finite,

$$\mathcal{E}_\gamma[\omega] := \int_\Omega (\mathbf{G}\omega)^\gamma\, d\omega = \int_\Omega u^\gamma\, d\omega < +\infty, \tag{1.4}$$

which is also equivalent to $u^{\frac{\gamma+1}{2}} \in \dot{W}_0^{1,2}(\Omega)$. In this case, we have

$$\int_\Omega u^\gamma\, d\omega = \gamma \int_\Omega (\mathcal{A}\nabla u \cdot \nabla u) u^{\gamma-1}\, dx. \tag{1.5}$$

This is well-known in the case $\gamma = 1$ for the Laplacian $\mathcal{L} = -\Delta$, see [14, 17]. Analogous integration by parts formulas with $\gamma > 0$ for functions u in certain Sobolev spaces with various extra restrictions on Ω and ω can be found in [16].

Two other key ingredients in our approach are weighted norm inequalities for Green potentials of the type $\mathbf{G} : L^r(\Omega, d\omega) \to L^s(\Omega, d\omega)$ for arbitrary $\omega \in \mathcal{M}^+(\Omega)$, in the non-classical case $0 < s < r$ and $r > 1$ (Theorem 2.4), along with iterated pointwise estimates for Green potentials (Theorem 2.3) discussed below.

Employing these tools, we establish *necessary and sufficient* conditions on σ and μ, in terms of their generalized Green energy (1.4), for the existence of a positive $\mathcal{A}$-superharmonic solution $u \in L^q_{loc}(\Omega, d\sigma)$ (see Definition 4.1) to (1.1) that satisfies (1.3).

In the case $\gamma = 1$, we also show that such a solution $u \in \dot{W}_0^{1,2}(\Omega)$ (the so-called finite energy solution) is unique. Notice that when $\mathcal{L} = -\Delta$, both the existence and uniqueness of a finite energy solution to (1.1) was obtained by the authors in [19] (see also [6] for $\Omega = \mathbb{R}^n$).

When $0 < \gamma < 1$, our result gives sufficient conditions for the existence of a positive solution $u \in \dot{W}_0^{1,p}(\Omega)$ to (1.1) where $1 < p < 2$.

Existence and uniqueness of *bounded* solutions to (1.1) on $\Omega = \mathbb{R}^n$ (with $\mu = 0$) was established by Brezis and Kamin in [3].

We state our main result and its consequences as follows.

Theorem 1.1. *Let $0 < q < 1$ and $\gamma > 0$. Suppose G is a positive Green function associated with $\mathcal{L}$ in $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$. Let $\sigma, \mu \in \mathcal{M}^+(\Omega)$ so that $\sigma \not\equiv 0$. Then there exists a positive solution $u \in L^q_{loc}(\Omega, d\sigma)$ to (1.1) which satisfies (1.3), or equivalently $u^{\frac{\gamma+1}{2}} \in \dot{W}_0^{1,2}(\Omega)$, if and only if the following conditions hold:*

$$\mathbf{G}\sigma \in L^{\frac{\gamma+q}{1-q}}(\Omega, d\sigma) \tag{1.6}$$

and

$$\mathbf{G}\mu \in L^{\gamma}(\Omega, d\mu). \tag{1.7}$$

When $\gamma = 1$, such a solution is unique in $\dot{W}_0^{1,2}(\Omega)$.

Corollary 1.2. *Under the assumptions of Theorem 1.1, suppose that $\frac{n}{n-1} < p \leq 2$, where $n \geq 3$. If both conditions (1.6) and (1.7) hold with $\gamma := \frac{p(n-1)-n}{n-p} \in (0, 1]$, i.e.,*

$$\mathbf{G}\sigma \in L^{r}(\Omega, d\sigma) \quad \textit{and} \quad \mathbf{G}\mu \in L^{s}(\Omega, d\mu), \tag{1.8}$$

where $r := \frac{p(n-2)}{(1-q)(n-p)} - 1$ and $s := \frac{p(n-1)-n}{n-p}$, then there exists a positive solution $u \in L^q_{loc}(\Omega, d\sigma) \cap \dot{W}_0^{1,p}(\Omega)$ to (1.1).

A sufficient condition for (1.8) is given by

$$\sigma \in L^{r_1}(\Omega) \quad \text{and} \quad \mu \in L^{s_1}(\Omega), \tag{1.9}$$

where $r_1 := \frac{n(\gamma+1)}{n(1-q)+2(\gamma+q)}$ and $s_1 := \frac{n(\gamma+1)}{n+2\gamma}$ (see Proposition 4.8 below). Thus, in light of Corollary 1.2, we have the following result.

Corollary 1.3. *Under the assumptions of Corollary 1.2, if*

$$\sigma \in L^{r_2}(\Omega) \quad \textit{and} \quad \mu \in L^{s_2}(\Omega), \tag{1.10}$$

where $r_2 := \frac{np}{(n-p)(1-q)+2p}$ and $s_2 := \frac{np}{n+p}$, then there exists a positive solution $u \in L^q_{loc}(\Omega, d\sigma) \cap \dot{W}_0^{1,p}(\Omega)$ to (1.1).

We observe that Corollary 1.3 in the case $\mu \equiv 0$ (for bounded domains Ω) is due to Boccardo and Orsina [4], with a different proof.

We sketch our method of proof of Theorem 1.1 in the inhomogeneous case $\mu \not\equiv 0$. (The homogeneous case $\mu \equiv 0$ is simpler since possible interaction between the nonlinear term involving σ and μ is omitted.)

We start with the corresponding integral equation

$$\tilde{u} = \mathbf{G}(\tilde{u}^q d\sigma) + \mathbf{G}\mu \quad \text{in } \Omega, \tag{1.11}$$

under some mild assumptions on the kernel G, which by the maximum principle are automatically satisfied by Green functions associated with elliptic operators (including $\mathcal{L}$) in Ω.

We find a crucial relation between σ and μ, which follows from conditions (1.6) and (1.7), and yields an important two-weight condition:

$$\mathbf{G}\mu \in L^{\gamma+q}(\Omega, d\sigma). \tag{1.12}$$

This supplementary fact allows us to construct a positive solution $\tilde{u} \in L^{\gamma+q}(\Omega, d\sigma) \cap L^{\gamma}(\Omega, d\mu)$ to (1.11) by using an iterative procedure, under assumptions (1.6) and (1.7).

In this procedure, we employ the fact established recently in [21] that condition (1.6) is equivalent to the weighted norm inequality for Green potentials,

$$\big\|\mathbf{G}(f d\sigma)\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)} \le C\|f\|_{L^{\frac{\gamma+q}{q}}(\Omega,\, d\sigma)}, \quad \forall f \in L^{\frac{\gamma+q}{q}}(\Omega, d\sigma), \tag{1.13}$$

where C is a positive constant independent of f. Therefore, either (1.6) or (1.13), together with (1.7), turns out to be necessary and sufficient for the existence of a positive solution to (1.1) satisfying (1.3).

When G is a positive Green function associated with $\mathcal{L}$ on Ω, the integral equation (1.11) is equivalent to problem (1.1). Appealing to our characterization of the generalized Green energy, (1.3) $\Longleftrightarrow$ (1.4) with $\omega := \sigma u^q + \mu$, we deduce that (1.6) and (1.7) are necessary and sufficient for the existence of a positive solution $u \in L^q_{loc}(\Omega, d\sigma)$ to (1.1) which satisfies (1.3).

This paper is organized as follows. In Sect. 2, we recall some background facts in potential theory and PDE, and collect useful results which are repeatedly referred to throughout this study. Sect. 3 is devoted to a characterization of the generalized Green energy. Our main result and its consequences are demonstrated in Sect. 4.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^n$ be a domain, and $\mathcal{L}u := -\text{div}\,(\mathcal{A}\nabla u)$, where $\mathcal{A} : \Omega \to \mathbb{R}^{n\times n}$ satisfies the uniform ellipticity condition (1.2).

2.1. **Function spaces.** Denote by $C_0^\infty(\Omega)$ the set of all smooth compactly supported functions on Ω. For $0 < p < \infty$ and $\omega \in \mathcal{M}^+(\Omega)$, we denote by $L^p(\Omega, d\omega)$ the usual Lebesgue space of all real-valued measurable functions u on Ω such that

$$\|u\|_{L^p(\Omega,\, d\omega)} := \left(\int_\Omega |u|^p \, d\omega \right)^{\frac{1}{p}} < +\infty.$$

The corresponding local space is denoted by $L^p_{loc}(\Omega, d\omega)$.

For $1 \le p < \infty$, the Sobolev space $W^{1,p}(\Omega)$ consists of all functions $u \in L^p(\Omega)$ such that $|\nabla u| \in L^p(\Omega)$, where ∇u is the vector of distributional partial derivatives of u of order 1, equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The corresponding local space is denoted by $W^{1,p}_{loc}(\Omega)$. The Sobolev space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. It is easy to see that $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$. The homogeneous version of $W_0^{1,p}(\Omega)$, called the homogeneous Sobolev space (or Dirichlet space), denoted by $\dot{W}_0^{1,p}(\Omega)$, is defined as the closure of $C_0^\infty(\Omega)$ with respect to the seminorm

$$\|u\|_{\dot{W}_0^{1,p}(\Omega)} := \|\nabla u\|_{L^p(\Omega)}.$$

That is, $\dot{W}_0^{1,p}(\Omega)$ is the set of all functions $u \in W^{1,p}_{loc}(\Omega)$ such that $|\nabla u| \in L^p(\Omega)$ for which there exists a sequence $\{\varphi_j\}_{j=1}^\infty \subset C_0^\infty(\Omega)$ such that $\|\nabla u - \nabla \varphi_j\|_{L^p(\Omega)} \to 0$ as $j \to \infty$. When $1 < p < n$, the dual space to $\dot{W}_0^{1,p}(\Omega)$ denoted by $\dot{W}^{-1,p'}(\Omega)$, is the space of distributions $\omega \in \mathcal{D}'(\Omega)$ such that

$$\|\omega\|_{\dot{W}^{-1,p'}(\Omega)} := \sup \frac{|\langle \omega, u \rangle|}{\|u\|_{\dot{W}_0^{1,p}(\Omega)}} < +\infty$$

where the supremum is taken over all nontrivial functions $u \in C_0^\infty(\Omega)$. Here $p' := \frac{p}{p-1}$ is the Hölder conjugate of p.

2.2. **$\mathcal{A}$-superharmonic functions.** A function $u \in W^{1,2}_{loc}(\Omega)$ is said to be $\mathcal{A}$-harmonic if u satisfies the equation

$$\mathcal{L}u = 0 \qquad \text{in } \Omega \tag{2.1}$$

in the distributional sense, i.e.,

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla \varphi \, dx = 0, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Every $\mathcal{A}$-harmonic function u has a continuous representative which coincides with u a.e. (see [9, Theorem 3.70]). We denote by $\mathcal{H}_{\mathcal{A}}(\Omega)$ the set of all continuous $\mathcal{A}$-harmonic functions in Ω.

A function $u : \Omega \to (-\infty, +\infty]$ is $\mathcal{A}$-superharmonic if u is lower semicontinuous in Ω, $u \not\equiv +\infty$ in each component of Ω, and whenever D is a relatively compact open subset of Ω and $h \in C(\overline{D}) \cap \mathcal{H}_{\mathcal{A}}(D)$, the inequality $h \le u$ on ∂D yields $h \le u$ on D. A function u in Ω is called $\mathcal{A}$-subharmonic if $-u$ is $\mathcal{A}$-superharmonic.

Every $\mathcal{A}$-superharmonic function u in Ω is quasicontinuous in Ω [9, Theorem 10.9], which means that for every $\epsilon > 0$, there is an open set $G \subset \Omega$ such that $\text{cap}(G) < \epsilon$ and the restriction $u_{|\Omega \setminus G}$ is continuous on $\Omega \setminus G$.

Here the capacity of an open set $G \subset \Omega$ is defined by

$$\text{cap}(G) := \sup\{\text{cap}(K) : \ K \subset \Omega \ \text{ compact}\},$$

where the capacity of a compact set $K \subset \Omega$ is given by

$$\text{cap}(K) := \inf\left\{\|\nabla u\|^2_{L^2(\Omega)} : \ u \ge 1 \ \text{ on } \ K, \quad u \in C_0^\infty(\Omega)\right\}.$$

For an arbitrary set $E \subset \Omega$,

$$\text{cap}(E) := \inf\{\text{cap}(G) : \ E \subseteq G \subset \Omega, \ \ G \text{ open}\}.$$

A statement is said to hold quasi-everywhere (q.e.) in Ω if it holds everywhere except for a set of capacity zero in Ω.

Denote by $\mathcal{M}^+_0(\Omega)$ the class of all measures $\omega \in \mathcal{M}^+(\Omega)$ which are absolutely continuous with respect to capacity, that is, $\omega(K) = 0$ whenever $\text{cap}(K) = 0$ for every compact subset K in Ω. It follows by Poincaré's inequality [17, Corollary 1.57] that Lebesgue measure is absolutely continuous with respect to the capacity.

Let u be an $\mathcal{A}$-superharmonic function in Ω. Then $u \in L^r_{loc}(\Omega)$ and $|\nabla u| \in L^s_{loc}(\Omega)$ whenever $0 < r < \frac{n}{n-2}$ and $0 < s < \frac{n}{n-1}$. In particular, $u \in W^{1,p}_{loc}(\Omega)$ whenever $1 \le p < \frac{n}{n-1}$. Moreover, there exists a unique measure $\omega \in \mathcal{M}^+(\Omega)$ such that

$$\mathcal{L}u = \omega \quad \text{in} \ \ \Omega \tag{2.2}$$

in the distributional sense, i.e.,

$$\int_\Omega \mathcal{A}\nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\omega, \quad \forall \varphi \in C_0^\infty(\Omega).$$

The measure ω is called the Riesz measure associated with u, often denoted by $\omega[u]$ (see [9, Theorem 7.46, Theorem 21.2]).

For a positive $\mathcal{A}$-superharmonic function u in Ω, we shall later use the fact that each truncation $u_k := \min(u, k)$, where $k \in \mathbb{N}$, is a positive $\mathcal{A}$-superharmonic function of the class $L^\infty(\Omega) \cap \dot{W}^{1,2}_{loc}(\Omega)$, and its Riesz measure $\omega[u_k]$ is locally in the dual of $W^{1,2}(\Omega)$. Moreover, $\omega[u_k] \to \omega[u]$ weakly in Ω as $k \to \infty$, see [9, 12].

2.3. **Kernels and potentials.** Let $G : \Omega \times \Omega \to (0, \infty]$ be a positive lower semicontinuous kernel. For $\omega \in \mathcal{M}^+(\Omega)$, the potential of ω is defined by

$$\mathbf{G}\omega(x) := \int_\Omega G(x, y)\, d\omega(y), \quad x \in \Omega.$$

Observe that $\mathbf{G}\omega(x)$ is lower semi-continuous on $\Omega \times \mathcal{M}^+(\Omega)$ if $G(x, y)$ is lower semi-continuous on $\Omega \times \Omega$, see [5].

A positive kernel G on $\Omega \times \Omega$ is said to satisfy the weak maximum principle (WMP) with constant $h \geq 1$ if for any $\omega \in \mathcal{M}^+(\Omega)$,

$$\sup\{\mathbf{G}\omega(x) : x \in \mathrm{supp}(\omega)\} \leq 1 \Longrightarrow \sup\{\mathbf{G}\omega(x) : x \in \Omega\} \leq h. \tag{2.3}$$

Here we use the notation $\mathrm{supp}(\omega)$ for the support of ω.

When $h = 1$ in (2.3), the kernel G is said to satisfy the strong maximum principle. It holds for Green functions associated with the classical Laplacian $-\Delta$, or more generally the linear uniformly elliptic operator in divergence form $\mathcal{L}$, as well as the fractional Laplacian $(-\Delta)^\alpha$ in the case $0 < \alpha \leq 1$, in every domain $\Omega \subset \mathbb{R}^n$ which possesses a Green function.

The WMP holds for Riesz kernels on $\mathbb{R}^n$ associated with $(-\Delta)^\alpha$ in the full range $0 < \alpha < \frac{n}{2}$, and more generally for all radially nonincreasing kernels on $\mathbb{R}^n$, see [2].

We say that a positive kernel G on $\Omega \times \Omega$ is quasi-symmetric if there exists a constant $a \geq 1$ such that

$$a^{-1} G(y, x) \leq G(x, y) \leq a G(y, x), \quad x, y \in \Omega. \tag{2.4}$$

When $a = 1$ in (2.4), the kernel G is said to be symmetric. There are many kernels associated with elliptic operators that are quasi-symmetric and satisfy the WMP, see [1].

We summarize that the Green function G associated with $\mathcal{L}$ on Ω is a positive lower semicontinuous symmetric kernel, which satisfies the strong maximum principle [11, 15]. Further, for $\omega \in \mathcal{M}^+(\Omega)$, the Green potential $\mathbf{G}\omega$ is either $\mathcal{A}$-superharmonic or identically $+\infty$, in each component of Ω, see [7].

2.4. **Some known results.** We shall need the following weak continuity result established in [20].

Theorem 2.1 ([20])**.** *Suppose $\{u_j\}_{j=1}^\infty$ is a sequence of positive $\mathcal{A}$-superharmonic functions in Ω such that $u_j \to u$ a.e. as $j \to \infty$, where u is an $\mathcal{A}$-superharmonic function in Ω. Then $\omega[u_j]$ converges weakly to $\omega[u]$, that is,*

$$\lim_{j\to\infty} \int_\Omega \varphi\, d\omega[u_j] = \int_\Omega \varphi\, d\omega[u], \quad \forall \varphi \in C_0^\infty(\Omega).$$

The following theorem provides pointwise estimates for supersolutions to sublinear elliptic equations, see [8, Theorem 1.3].

Theorem 2.2 ([8])**.** *Let $0 < q < 1$, $\omega \in \mathcal{M}^+(\Omega)$, and let G be a positive lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP with constant $h \geq 1$. If $u \in L^q_{loc}(\Omega, d\omega)$ is a positive solution to the integral inequality*

$$u \geq \mathbf{G}(u^q d\omega) \quad \textit{in } \ \Omega, \tag{2.5}$$

then

$$u(x) \geq (1-q)^{\frac{1}{1-q}} h^{\frac{-q}{1-q}} [\mathbf{G}\omega(x)]^{\frac{1}{1-q}}, \quad \forall x \in \Omega. \tag{2.6}$$

The following pair of iterated pointwise inequalities plays an important role in this paper (see [8, Lemma 2.5]).

Theorem 2.3 ([8])**.** *Let $\omega \in \mathcal{M}^+(\Omega)$, and let G be a positive lower semicontinuous kernel on $\Omega\times\Omega$, which satisfies the WMP with constant $h \geq 1$. Then the following estimates hold:*

(i) If $s \geq 1$, then

$$(\mathbf{G}\omega)^s(x) \leq s h^{s-1}\, \mathbf{G}\left((\mathbf{G}\omega)^{s-1} d\omega\right)(x), \quad \forall x \in \Omega. \tag{2.7}$$

(ii) If $0 < s \leq 1$, then

$$(\mathbf{G}\omega)^s(x) \geq s h^{s-1}\, \mathbf{G}\left((\mathbf{G}\omega)^{s-1} d\omega\right)(x), \quad \forall x \in \Omega. \tag{2.8}$$

Our argument also relies on the following result established in [21, Theorem 1.1], which explicitly characterizes (p, r)-weighted norm inequalities

$$\left\|\mathbf{G}(f d\omega)\right\|_{L^r(\Omega,\, d\omega)} \leq C \|f\|_{L^p(\Omega,\, d\omega)}, \quad \forall f \in L^p(\Omega, d\omega), \tag{2.9}$$

where C is a positive constant independent of f, in the case $0 < r < p$ and $1 < p < \infty$, for arbitrary $\omega \in \mathcal{M}^+(\Omega)$, under certain assumptions on the kernel G.

Theorem 2.4 ([21])**.** *Let $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$, and let G be a positive quasi-symmetric lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP.*

(i) If $1 < p < \infty$ and $0 < r < p$, then the (p, r)-weighted norm inequality (2.9) holds if and only if

$$\mathbf{G}\omega \in L^{\frac{pr}{p-r}}(\Omega, d\omega). \tag{2.10}$$

(ii) If $0 < q < 1$ and $q < r < \infty$, then there exists a positive solution $u \in L^r(\Omega, d\omega)$ to the integral inequality (2.5) if and only if the weighted norm inequality (2.9) holds with $p = \frac{r}{q}$, that is,

$$\left\|\mathbf{G}(f d\omega)\right\|_{L^r(\Omega,\, d\omega)} \leq C \|f\|_{L^{\frac{r}{q}}(\Omega,\, d\omega)}, \quad \forall f \in L^{\frac{r}{q}}(\Omega, d\omega), \tag{2.11}$$

where C is a positive constant independent of f; or equivalently,

$$\mathbf{G}\omega \in L^{\frac{r}{1-q}}(\Omega, d\omega). \tag{2.12}$$

The following inequalities are often used in the theory of Schrödinger operators and potential theory. They can be found, for example, in [9, Theorem 7.48] and [10, Proposition 1.5], respectively.

Theorem 2.5 ([9, 10])**.** *Let $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$, and let G be a positive Green function associated with $\mathcal{L}$ on Ω. Suppose $u := \mathbf{G}\omega$ so that $u \not\equiv +\infty$. Then there exists a positive constant C which depends only on the ellipticity constants m, M such that*

$$\int_\Omega |\varphi|^2 \frac{|\nabla u|^2}{u^2} dx \le C \int_\Omega |\nabla \varphi|^2 \, dx \tag{2.13}$$

and

$$\int_\Omega |\varphi|^2 \frac{d\omega}{u} \le C \int_\Omega |\nabla \varphi|^2 \, dx, \tag{2.14}$$

for all (quasicontinuous representatives of) $\varphi \in \dot{W}_0^{1,2}(\Omega)$.

3. Generalized energy of measures

Let $\gamma > 0$ and $\omega \in \mathcal{M}^+(\Omega)$, and let G be a positive lower semicontinuous kernel on $\Omega \times \Omega$. Define the γ-energy of ω by

$$\mathcal{E}_\gamma[\omega] := \int_\Omega (\mathbf{G}\omega)^\gamma \, d\omega. \tag{3.1}$$

In the case $\gamma = 1$, we use the notation $\mathcal{E}[\omega] := \mathcal{E}_1[\omega]$. Observe that $\mathcal{E}_\gamma[\omega]$ is well-defined, even though it may be infinite.

When G is a quasi-symmetric kernel on $\Omega \times \Omega$ which satisfies the WMP, by the definition of $\mathcal{E}_\gamma[\omega]$ and Theorem 2.4 with $r := \frac{\gamma(1+\gamma)}{1+\gamma+\gamma^2}$ and $q := \frac{\gamma^2}{1+\gamma+\gamma^2}$, the following statements are equivalent:

(a) $\mathcal{E}_\gamma[\omega] < +\infty$.
(b) $\mathbf{G}\omega \in L^\gamma(\Omega, d\omega)$.
(c) The weighted norm inequality (2.11) is valid.
(d) There exists a positive solution $u \in L^r(\Omega, d\omega)$ to (2.5).

There is also a similar characterization of $\mathcal{E}_\gamma[\omega]$, in terms of weak-type and strong-type inequalities, for nondegenerate kernels $G \ge 0$, see [18].

We now consider $\mathcal{E}_\gamma[\omega]$ in the case where G is a positive Green function associated with $\mathcal{L}$ on Ω.

Theorem 3.1. *Let $\gamma > 0$ and $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$, and let G be a positive Green function associated with $\mathcal{L}$ on Ω. If $u := \mathbf{G}\omega$ then the condition*

$$\int_\Omega (\mathcal{A}\nabla u \cdot \nabla u)\, u^{\gamma-1}\, dx < +\infty \tag{3.2}$$

is equivalent to $u^{\frac{\gamma+1}{2}} \in \dot{W}_0^{1,2}(\Omega)$ as well as (a), (b), (c) and (d) above. In this case, we have

$$\mathcal{E}_\gamma[\omega] = \gamma \int_\Omega (\mathcal{A}\nabla u \cdot \nabla u)\, u^{\gamma-1}\, dx. \tag{3.3}$$

Remark 3.2. By uniform ellipticity condition (1.2), we see that (3.2) is equivalent to (1.3). Therefore, by our discussion above, it suffices to show that

$$\mathcal{E}_\gamma[\omega] < +\infty \Longleftrightarrow \mathbb{E}_\gamma[u] < +\infty \Longleftrightarrow u^{\frac{\gamma+1}{2}} \in \dot{W}_0^{1,2}(\Omega), \tag{3.4}$$

and also establish formula (3.3) whenever $\mathcal{E}_\gamma[\omega] < +\infty$.

We first prove an auxiliary fact which will be used in the proof of (3.4) when $0 < \gamma < 1$.

Lemma 3.3. *Let $0 < \gamma < 1$ and $\omega \in \mathcal{M}^+(\Omega)$, and let G be a positive Green function associated with $-\mathcal{L}$ on Ω. Suppose $u := \mathbf{G}\omega \not\equiv \infty$. Then $v := u^\gamma$ is a positive $\mathcal{A}$-superharmonic function on Ω, and $v = \mathbf{G}\mu$, where $\mu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of v. Moreover,*

$$\mathcal{E}_\gamma[\omega] < +\infty \Longleftrightarrow \mathcal{E}[\mu] < +\infty. \tag{3.5}$$

In this case, we have

$$\frac{\gamma+1}{2}\, \mathcal{E}_\gamma[\omega] \le \mathcal{E}[\mu] \le \frac{\gamma+1}{2\gamma}\, \mathcal{E}_\gamma[\omega]. \tag{3.6}$$

Proof. Notice that $u := \mathbf{G}\omega$ is a positive $\mathcal{A}$-superharmonic function on Ω since $\mathbf{G}\omega \not\equiv +\infty$, see [7]. Since $0 < \gamma < 1$, the map $x \longmapsto x^\gamma$, for $x \ge 0$, is concave and increasing; it follows that $v := u^\gamma$ is a positive $\mathcal{A}$-superharmonic function on Ω [9, Theorem 7.5]. In light of the Riesz decomposition theorem, $v = \mathbf{G}\mu + h$ where $\mu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of v, and h is the unique positive $\mathcal{A}$-harmonic function on Ω.

Observe that $g := h^{\frac{1}{\gamma}}$ is a positive $\mathcal{A}$-subharmonic function on Ω since h is positive $\mathcal{A}$-harmonic and the map $x \longmapsto x^{\frac{1}{\gamma}}$ for $x \ge 0$, is convex [9, Theorem 7.5]. Therefore $-g$ is an $\mathcal{A}$-superharmonic function on Ω, and thus $-g = \mathbf{G}\nu + \tilde{h}$, where $\nu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of $-g$, and $\tilde{h}$ is the unique $\mathcal{A}$-harmonic function on Ω. Since

$$\mathbf{G}\omega = u = v^{\frac{1}{\gamma}} = (\mathbf{G}\mu + h)^{\frac{1}{\gamma}} \ge h^{\frac{1}{\gamma}} = g = -\mathbf{G}\nu - \tilde{h},$$

we deduce $\mathbf{G}(\omega+\nu) = u + \mathbf{G}\nu \geq -\tilde{h}$. In other words, $-\tilde{h}$ is a positive $\mathcal{A}$-harmonic minorant of the potential $\mathbf{G}(\omega+\nu)$. Consequently, $-\tilde{h} = 0$ and thus $g = -\mathbf{G}\nu \leq 0$. This yields $h = g^\gamma = 0$. We have shown that $v = \mathbf{G}\mu$ is a potential.

Suppose that $\mathcal{E}_\gamma[\omega] < +\infty$, and let $w := u^{\frac{\gamma+1}{2}}$. Since $\frac{\gamma+1}{2} \in (0,1)$, by a similar argument as above, w is a positive $\mathcal{A}$-superharmonic function on Ω, and $w = \mathbf{G}\mu$, where $\mu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of w. For each $k \in \mathbb{N}$, let $u_k := \min(u,k)$ and $w_k := \min(w, k^{\frac{\gamma+1}{2}})$. Using the same argument as above, we see that both u_k and w_k are potentials with the corresponding Riesz measures $\omega_k = \omega[u_k]$ and $\mu_k = \mu[w_k]$, respectively. Clearly, $\operatorname{supp}(\mu_k) \subset \{u \leq k\}$, thus both u_k and w_k are uniformly bounded (by k) $d\mu_k$-a.e. Using Fubini's theorem and the iterated inequality (2.8) with $\omega := \mu_k$, $s := \frac{2\gamma}{\gamma+1}$ and $h := 1$, we estimate

$$\begin{aligned}\int_\Omega \mathbf{G}\mu_k\, d\mu = \int_\Omega \mathbf{G}\mu\, d\mu_k &= \int_\Omega \mathbf{G}\omega\, (\mathbf{G}\mu)^{\frac{\gamma-1}{\gamma+1}} d\mu_k \leq \int_\Omega \mathbf{G}\omega\, (\mathbf{G}\mu_k)^{\frac{\gamma-1}{\gamma+1}} d\mu_k \\ &= \int_\Omega \mathbf{G}\Big((\mathbf{G}\mu_k)^{\frac{\gamma-1}{\gamma+1}} d\mu_k\Big)\, d\omega \leq \frac{\gamma+1}{2\gamma}\int_\Omega (\mathbf{G}\mu_k)^{\frac{2\gamma}{\gamma+1}} d\omega \\ &\leq \frac{\gamma+1}{2\gamma}\int_\Omega (\mathbf{G}\omega)^\gamma d\omega = \frac{\gamma+1}{2\gamma}\mathcal{E}_\gamma[\omega].\end{aligned}$$

Passing to the limit $k \to \infty$ and using the monotone convergence theorem, we deduce

$$\mathcal{E}[\mu] = \int_\Omega \mathbf{G}\mu\, d\mu \leq \frac{\gamma+1}{2\gamma}\mathcal{E}_\gamma[\omega] < +\infty$$

since $w_k = \mathbf{G}\mu_k \uparrow w = \mathbf{G}\mu$ in Ω.

Conversely, suppose $\mathcal{E}[\mu] < +\infty$. By using the same notation and argument as above, we estimate

$$\begin{aligned}\mathcal{E}[\mu] &= \int_\Omega \mathbf{G}\mu\, d\mu = \int_\Omega (\mathbf{G}\omega)^{\frac{\gamma+1}{2}}\, d\mu \geq \int_\Omega (\mathbf{G}\omega_k)^{\frac{\gamma+1}{2}}\, d\mu \\ &\geq \frac{\gamma+1}{2}\int_\Omega \mathbf{G}\Big((\mathbf{G}\omega_k)^{\frac{\gamma-1}{2}} d\omega_k\Big) d\mu \geq \frac{\gamma+1}{2}\int_\Omega \mathbf{G}\Big((\mathbf{G}\omega)^{\frac{\gamma-1}{2}} d\omega_k\Big) d\mu.\end{aligned}$$

In the above estimate, we have $u := \mathbf{G}\omega \leq k$ and $w = \mathbf{G}\mu \leq k$ $d\omega_k$-a.e. Applying Fubini's theorem yields

$$\begin{aligned}\int_\Omega \mathbf{G}\Big((\mathbf{G}\omega)^{\frac{\gamma-1}{2}} d\omega_k\Big)\, d\mu &= \int_\Omega (\mathbf{G}\omega)^{\frac{\gamma-1}{2}} \mathbf{G}\mu\, d\omega_k \\ &= \int_\Omega (\mathbf{G}\omega)^\gamma\, d\omega_k.\end{aligned}$$

Since $\gamma \in (0,1)$, we have $u^\gamma := (\mathbf{G}\omega)^\gamma$ is a potential, that is, $u^\gamma = \mathbf{G}\nu$, where $\nu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of u^γ. Applying Fubini's theorem and the monotone convergence theorem, we have

$$\int_\Omega (\mathbf{G}\omega)^\gamma \, d\omega_k = \int_\Omega \mathbf{G}\nu \, d\omega_k = \int_\Omega \mathbf{G}\omega_k \, d\nu \uparrow \int_\Omega \mathbf{G}\omega \, d\nu$$

since $\mathbf{G}\omega_k = u_k \uparrow u = \mathbf{G}\omega$ in Ω as $k \to \infty$. Hence,

$$\frac{\gamma+1}{2}\mathcal{E}_\gamma[\omega] = \frac{\gamma+1}{2}\int_\Omega \mathbf{G}\nu \, d\omega = \frac{\gamma+1}{2}\int_\Omega \mathbf{G}\omega \, d\nu \le \mathcal{E}[\mu] < +\infty.$$

This completes the proof of the lemma. ☐

We now establish (3.4), which yields the first part of Theorem 3.1.

Lemma 3.4. *Let $\gamma > 0$ and $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$, and let G be a positive Green function associated with $\mathcal{L}$ on Ω. If $u := \mathbf{G}\omega$ then (3.4) holds. In this case, there exists a positive constant C which depends only on m, M and γ such that*

$$C^{-1}\mathcal{E}_\gamma[\omega] \le \mathbb{E}_\gamma[u] \le C\,\mathcal{E}_\gamma[\omega]. \tag{3.7}$$

Proof. Without loss of generality, we may assume that $u \not\equiv +\infty$. It follows that u is a positive $\mathcal{A}$-superharmonic function in Ω. Moreover, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ whenever $1 \le p < \frac{n}{n-1}$, see [9]. Consider three cases as follows:

• Case $\gamma = 1$. This is completely analogous to the classical result shown in, for example, [14, Theorem 1.20], due to uniform ellipticity assumption (1.2). In this case we further have formula (3.3) using an approximation argument demonstrated below in Lemma 3.5.

• Case $0 < \gamma < 1$. In light of Lemma 3.3, we have $v := u^{\frac{\gamma+1}{2}}$ is a positive $\mathcal{A}$-superharmonic function in Ω, and $v = \mathbf{G}\mu$ where $\mu \in \mathcal{M}^+(\Omega)$ is the Riesz measure of v. Moreover,

$$\mathcal{E}_\gamma[\omega] < +\infty \Longleftrightarrow \mathcal{E}[\mu] < +\infty \Longleftrightarrow v \in \dot{W}^{1,2}_0(\Omega).$$

In this case, we have

$$\frac{\gamma+1}{2}\,\mathcal{E}_\gamma[\omega] \le \mathcal{E}[\mu] \le \frac{\gamma+1}{2\gamma}\,\mathcal{E}_\gamma[\omega].$$

On the other hand, notice that $\nabla v = \frac{\gamma+1}{2}u^{\frac{\gamma-1}{2}}\nabla u$ a.e. in Ω. Appealing to the previous case, we deduce

$$\mathcal{E}[\mu] = \int_\Omega \mathcal{A}\nabla v \cdot \nabla v \, dx \le M\Big(\frac{\gamma+1}{2}\Big)^2 \int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx$$

and similarly

$$\mathcal{E}[\mu] = \int_\Omega \mathcal{A}\nabla v \cdot \nabla v \, dx \ge m\Big(\frac{\gamma+1}{2}\Big)^2 \int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx.$$

This proves both assertions (3.4) and (3.7), respectively.

• Case $\gamma > 1$. Suppose that $v := u^{\frac{\gamma+1}{2}} \in \dot{W}_0^{1,2}(\Omega)$. Therefore, $\mathbb{E}_\gamma[u] < +\infty$. For each $k \in \mathbb{N}$, let $u_k = \min(u, k)$, which is a positive $\mathcal{A}$-superharmonic function of the class $L^\infty(\Omega) \cap W^{1,2}_{loc}(\Omega)$. Denote the corresponding Riesz measure of each u_k by $\omega_k := \omega[u_k]$. Without loss of generality, we may suppose v is quasicontinuous. Applying inequality (2.14) with $\phi := v$ and $\omega := \omega_k$, we obtain

$$\begin{aligned}\int_\Omega u^\gamma d\omega_k \leq \int_\Omega v^2 \frac{d\omega_k}{\mathbf{G}\omega_k} &\leq C \int_\Omega |\nabla v|^2 \, dx \\ &= C\Big(\frac{\gamma+1}{2}\Big)^2 \int_\Omega |\nabla u|^2 u^{\gamma-1} dx.\end{aligned}$$

On the other hand, by Fubini's theorem and iterated estimate (2.7) with $\omega := \omega_k$ and $s := \gamma$, we have

$$\begin{aligned}\int_\Omega u^\gamma \, d\omega_k = \int_\Omega (\mathbf{G}\omega)^{\gamma-1} \mathbf{G}\omega \, d\omega_k &= \int_\Omega \mathbf{G}\Big((\mathbf{G}\omega)^{\gamma-1} d\omega_k\Big) \, d\omega \\ &\geq \int_\Omega \mathbf{G}\Big((\mathbf{G}\omega_k)^{\gamma-1} d\omega_k\Big) \, d\omega \geq \frac{1}{\gamma} \int_\Omega (\mathbf{G}\omega_k)^\gamma d\omega.\end{aligned}$$

Therefore,

$$\frac{1}{\gamma} \int_\Omega (\mathbf{G}\omega_k)^\gamma \, d\omega \leq C\Big(\frac{\gamma+1}{2}\Big)^2 \int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx.$$

Passing to the limit $k \to \infty$ and using the monotone convergence theorem, we obtain

$$\mathcal{E}_\gamma[\omega] = \int_\Omega u^\gamma \, d\omega \leq C\gamma\Big(\frac{\gamma+1}{2}\Big)^2 \int_\Omega |\nabla u|^2 u^{\gamma-1} dx < +\infty$$

since $u_k = \mathbf{G}\omega_k \uparrow u = \mathbf{G}\omega$ in Ω.

Conversely, suppose $\mathcal{E}_\gamma[\omega] < +\infty$. Write $\gamma = \frac{1+q}{1-q}$ where $0 < q < 1$, and consider the corresponding sublinear elliptic equation

$$\mathcal{L}w = \omega \, w^q \quad \text{in } \Omega. \tag{3.8}$$

Using a similar argument as in the proof of [19, Lemma 5.5], there exists a positive finite energy solution $w \in \dot{W}_0^{1,2}(\Omega)$ to (3.8), satisfying

$$||w||_{\dot{W}_0^{1,2}(\Omega)} \leq c \Big(\int_\Omega (\mathbf{G}\omega)^{\frac{1+q}{1-q}} \, d\omega \Big)^{\frac{1}{2}}, \tag{3.9}$$

where $c = c(m, M, q) > 0$. As usual, we may assume w is quasicontinuous. By Theorem 2.2, w obeys the lower bound

$$w \geq (1-q)^{\frac{1}{1-q}} (\mathbf{G}\omega)^{\frac{1}{1-q}} \quad \text{in } \Omega.$$

Therefore,

$$v := u^{\frac{\gamma+1}{2}} = (\mathbf{G}\omega)^{\frac{\gamma+1}{2}} \leq (1-q)^{-\frac{1}{1-q}} w \quad \text{in } \Omega. \tag{3.10}$$

From this, we deduce

$$\begin{aligned}\int_\Omega |\nabla u|^2 u^{\gamma-1}\,dx &= \int_\Omega v^2 \frac{|\nabla u|^2}{u^2}\,dx \\ &\le (1-q)^{-\frac{2}{1-q}} \int_\Omega w^2 \frac{|\nabla u|^2}{u^2}\,dx.\end{aligned} \tag{3.11}$$

Using inequality (2.13) with $\phi := w$, along with (3.9), we estimate

$$\int_\Omega w^2 \frac{|\nabla u|^2}{u^2}\,dx \le C\,||w||^2_{\dot{W}^{1,2}(\Omega)} \le Cc^2\,\mathcal{E}_\gamma[\omega]. \tag{3.12}$$

Hence, by (3.11) and (3.12), we arrive at

$$\int_\Omega |\nabla u|^2 u^{\gamma-1}\,dx \le Cc^2(1-q)^{-\frac{2}{1-q}}\,\mathcal{E}_\gamma[\omega] < +\infty.$$

Moreover, for $r = \frac{2n}{n-2}$ if $n \ge 3$, and any $r < \infty$ if $n = 2$, we have that $v \in L^r(\Omega)$, since the same is true for $w \in \dot{W}^{1,2}_0(\Omega)$. Recall that Ω is assumed to be a Green domain in the case $n = 2$. In other words, $v \in \dot{W}^{1,2}(\Omega)$, the corresponding Sobolev space equipped with the norm $||v||_{\dot{W}^{1,2}(\Omega)} = ||\nabla v||_{L^2(\Omega)} + ||v||_{L^r(\Omega)}$. The fact that $v \in \dot{W}^{1,2}_0(\Omega)$ follows from the Deny-Lions theorem (see [2, Sec. 9.12] and the references cited there). Notice that, for $z \in \partial\Omega$, the quasi-limit $\lim_{x\to z} v(x) = 0$ q.e. by (3.10), since the same is true for $w \in \dot{W}^{1,2}_0(\Omega)$ by [13], Corollary to Theorem 1. This finishes the proof of lemma. ☐

We now complete the proof of Theorem 3.1 by establishing formula (3.3) whenever $\mathcal{E}_\gamma[\omega] < +\infty$, using an approximation procedure.

Lemma 3.5. *Let $\gamma > 0$ and $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$, and let G be a positive Green function associated with $\mathcal{L}$ on Ω. If $u := \mathbf{G}\omega$ then formula (3.3) is valid whenever $\mathcal{E}_\gamma[\omega] < +\infty$.*

Proof. Suppose $\mathcal{E}_\gamma[\omega] < +\infty$. Therefore both (3.2) and (1.3) holds, in views of Lemma 3.4 and uniform ellipticity condition (1.2).

For each $k \in \mathbb{N}$, we set $u_k = \min(u, k)$. Notice that u_k is a positive $\mathcal{A}$-superharmonic function of the class $L^\infty(\Omega) \cap W^{1,2}_{loc}(\Omega)$. Denote the corresponding Riesz measure of u_k by $\omega_k := \omega[u_k]$.

Let $\{u_k^{(j)}\}_{j\ge k}$ be a sequence of mollified u_k, defined on $\Omega_j := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \frac{1}{j}\}$. Denote $\omega_k^{(j)} := \mathcal{L}u_k^{(j)}$. In addition, for $j \ge k$, select $\varphi_j \in C_0^\infty(\Omega)$ so that

$$0 \le \varphi_j \le 1, \quad \mathrm{supp}\,\varphi_j \subset \Omega_j, \quad \varphi_j \uparrow \chi_\Omega \text{ as } j \to \infty,$$
$$\int_\Omega |\nabla\varphi_j|^2\,dx \le \frac{1}{j^{2\gamma+2}}.$$

Using integration by parts, we obtain

$$\gamma \int_\Omega (\mathcal{A}\nabla u_k^{(j)} \cdot \nabla u_k^{(j)})(u_k^{(j)})^{\gamma-1}\varphi_j \, dx = \int_\Omega \nabla \left((u_k^{(j)})^\gamma \right) \cdot \mathcal{A}\nabla u_k^{(j)} \varphi_j \, dx$$
$$= \int_\Omega (u_k^{(j)})^\gamma \varphi_j \, d\omega_k^{(j)} - \int_\Omega (u_k^{(j)})^\gamma (\mathcal{A}\nabla u_k^{(j)} \cdot \nabla\varphi_j) \, dx.$$

Letting first $j \to \infty$, and then $k \to \infty$, we see that

$$\gamma \int_\Omega |\nabla u_k^{(j)}|^2 (u_k^{(j)})^{\gamma-1}\varphi_j \, dx \longrightarrow \gamma \int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx$$

and

$$\int_\Omega (u_k^{(j)})^\gamma \varphi_j \, d\omega_k^{(j)} \longrightarrow \int_\Omega u^\gamma \, d\omega$$

by means of mollification, the Lebesgue dominated convergence theorem, and weak continuity of $\mathcal{L}$ (Theorem 2.1). Moreover, by Schwarz's inequality, the construction of φ_j, and uniform ellipticity condition (1.2), we deduce

$$\left| \int_\Omega (u_k^{(j)})^\gamma (\mathcal{A}\nabla u_k^{(j)} \cdot \nabla\varphi_j) \, dx \right| \le M^{\frac{1}{2}} \left(\int_\Omega |\nabla\varphi_j|^2 \, dx \right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega |\nabla u_k^{(j)}|^2 (u_k^{(j)})^{2\gamma} \, dx \right)^{\frac{1}{2}}$$
$$\le \frac{M^{\frac{1}{2}}}{j^{\gamma+1}} \left(\int_\Omega |\nabla u_k^{(j)}|^2 (u_k^{(j)})^{\gamma-1} (u_k^{(j)})^{\gamma+1} \, dx \right)^{\frac{1}{2}}$$
$$\le \frac{M^{\frac{1}{2}} k^{\frac{\gamma+1}{2}}}{k^{\gamma+1}} \left(\int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx \right)^{\frac{1}{2}} = \frac{M^{\frac{1}{2}}}{k^{\frac{\gamma+1}{2}}} \left(\int_\Omega |\nabla u|^2 u^{\gamma-1} \, dx \right)^{\frac{1}{2}},$$

which converges to zero as $k \to \infty$. This proves (3.3). □

The following lemma shows, in particular, that if $\mathcal{E}_\gamma[\omega] < +\infty$ for some $\gamma > 0$, then $\omega \in \mathcal{M}_0^+(\Omega)$.

Lemma 3.6. *Let $\gamma > 0$ and $\omega \in \mathcal{M}^+(\Omega)$, and let G be a positive Green function associated with $\mathcal{L}$ on Ω. Suppose that $u := \mathbf{G}\omega \in L^\gamma_{loc}(\Omega, d\mu)$. Then for every compact set $K \subset \Omega$,*

$$\omega(K) \le [\mathrm{cap}(K)]^{\frac{\gamma}{1+\gamma}} \left(\int_K u^\gamma \, d\omega \right)^{\frac{1}{1+\gamma}}. \tag{3.13}$$

In particular, $\omega \in \mathcal{M}_0^+(\Omega)$.

Proof. Let K be a compact subset of Ω. By (2.14), we have

$$\int_K \frac{d\omega}{u} \le \mathrm{cap}(K). \tag{3.14}$$

On the other hand, by Hölder's inequality,

$$\omega(K) = \int_K u^{\frac{-\gamma}{1+\gamma}} u^{\frac{\gamma}{1+\gamma}} \, d\omega \leq \left(\int_K u^{-1} \, d\omega \right)^{\frac{\gamma}{1+\gamma}} \left(\int_K u^{\gamma} \, d\omega \right)^{\frac{1}{1+\gamma}}. \tag{3.15}$$

Thus, (3.13) follows from (3.14) and (3.15). ☐

4. Positive solutions to sublinear elliptic equations

In this section, we prove our main result stated in Theorem 1.1 using the argument outlined earlier in Sec. 1. Its consequences are discussed here as well.

Definition 4.1. Let $q > 0$ and $\sigma, \mu \in \mathcal{M}^+(\Omega)$. Let G be a positive Green function associated with $\mathcal{L}$ on Ω. A solution u to equation (1.1) is understood in the sense that u is an $\mathcal{A}$-superharmonic function on Ω such that $u \in L^q_{loc}(\Omega, d\sigma)$ with $u \geq 0$ $d\sigma$-a.e., and

$$u = \mathbf{G}(u^q d\sigma) + \mathbf{G}\mu \quad \text{in } \Omega. \tag{4.1}$$

If further $u \in \dot{W}^{1,2}_0(\Omega)$, it is called a finite energy solution to (1.1).

Our first theorem gives necessary and sufficient conditions for the existence of a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma)$ ($\gamma > 0$) to the integral equation (4.1) in the sublinear case $0 < q < 1$, under some mild assumptions on kernel G satisfied by the Green function associated with $\mathcal{L}$ on Ω.

Theorem 4.2. *Let $0 < q < 1$, $\gamma > 0$ and $\sigma, \mu \in \mathcal{M}^+(\Omega)$ with $\sigma \not\equiv 0$. Suppose G is a positive quasi-symmetric lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP. If (1.6) and (1.12) hold, then there exists a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma)$ to (4.1). The converse statement is valid without the quasi-symmetry assumption on G.*

Proof. Suppose (1.6) and (1.12) hold. In the homogeneous case $\mu \equiv 0$, we can construct a monotone increasing sequence of positive functions $\{u_j\}_{j=0}^{\infty} \subset L^{\gamma+q}(\Omega, d\sigma)$ by setting

$$u_0 := \kappa \, (\mathbf{G}\sigma)^{\frac{1}{1-q}} \quad \text{and} \quad u_{j+1} := \mathbf{G}(u_j^q d\sigma), \quad \text{for } j \in \mathbb{N}_0,$$

where $\kappa > 0$ is chosen to be sufficiently small. Then its pointwise limit $u := \lim_{j\to\infty} u_j$ is a positive solution of the class $L^{\gamma+q}(\Omega, d\sigma)$ to (4.1), by the monotone convergence theorem (see [21, Theorem 1.1 (ii)] for more details).

In the inhomogeneous case $\mu \not\equiv 0$, we set

$$u_0 := \mathbf{G}\mu, \qquad u_{j+1} := \mathbf{G}(u_j^q d\sigma) + \mathbf{G}\mu, \quad \text{for } j \in \mathbb{N}_0.$$

Observe that $u_0 > 0$ since $\mu \not\equiv 0$, and

$$u_1 = \mathbf{G}(u_0^q d\sigma) + u_0 \geq u_0.$$

Suppose $u_0 \leq u_1 \leq \ldots \leq u_j$ for some $j \in \mathbb{N}$. Then

$$u_{j+1} = \mathbf{G}(u_j^q d\sigma) + \mathbf{G}\mu \geq \mathbf{G}(u_{j-1}^q d\sigma) + \mathbf{G}\mu = u_j.$$

Hence, by induction, $\{u_j\}_{j=0}^{\infty}$ is an increasing sequence of positive functions. Further, each $u_j \in L^{\gamma+q}(\Omega, d\sigma)$. Notice that

$$u_0 = \mathbf{G}\mu \in L^{\gamma+q}(\Omega, d\sigma),$$

by the assumption (1.12). Suppose $u_0, \ldots, u_j \in L^{\gamma+q}(\Omega, d\sigma)$ for some $j \in \mathbb{N}$. Observe that

$$\begin{aligned} \|u_{j+1}\|_{L^{\gamma+q}(\Omega,\, d\sigma)} &= \big\|\mathbf{G}(u_j^q d\sigma) + \mathbf{G}\mu\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)} \\ &\leq c\big\|\mathbf{G}(u_j^q d\sigma)\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)} + c\big\|\mathbf{G}\mu\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)}, \end{aligned} \tag{4.2}$$

where $c = \max(1, 2^{\frac{1-\gamma-q}{\gamma+q}})$. In view of Theorem 2.4, the assumption (1.6) is equivalent to the weighted norm inequality (2.11) with $\omega = \sigma$ and $r = \gamma + q$. Therefore, we can estimate the first term on the right-hand side of (4.2) by applying (2.11) with $f = u_j^q \in L^{\frac{\gamma+q}{q}}(\Omega, d\sigma)$,

$$\begin{aligned} &\big\|\mathbf{G}(u_j^q d\sigma)\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)} \leq C\left(\int_\Omega u_j^{\gamma+q}\, d\sigma\right)^{\frac{q}{\gamma+q}} \\ &\leq C\left(\int_\Omega u_{j+1}^{\gamma+q}\, d\sigma\right)^{\frac{q}{\gamma+q}} = C\|u_{j+1}\|^q_{L^{\gamma+q}(\Omega,\, d\sigma)}. \end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we have

$$\|u_{j+1}\|_{L^{\gamma+q}(\Omega,\, d\sigma)} \leq Cc\|u_{j+1}\|^q_{L^{\gamma+q}(\Omega,\, d\sigma)} + c\big\|\mathbf{G}\mu\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)}. \tag{4.4}$$

We estimate the first term on the right-hand side of (4.4) using Young's inequality,

$$Cc\|u_{j+1}\|^q_{L^{\gamma+q}(\Omega,\, d\sigma)} \leq q\|u_{j+1}\|_{L^{\gamma+q}(\Omega,\, d\sigma)} + (1-q)(Cc)^{\frac{1}{1-q}}. \tag{4.5}$$

Hence, by (4.4) and (4.5), we obtain

$$\|u_{j+1}\|_{L^{\gamma+q}(\Omega,\, d\sigma)} \leq (Cc)^{\frac{1}{1-q}} + \frac{c}{1-q}\big\|\mathbf{G}\mu\big\|_{L^{\gamma+q}(\Omega,\, d\sigma)} < +\infty. \tag{4.6}$$

By induction, we have shown that each $u_j \in L^{\gamma+q}(\Omega, d\sigma)$. Finally, applying the monotone convergence theorem to the sequence $\{u_j\}_{j=0}^{\infty}$, we see that the pointwise limit $u := \lim_{j\to\infty} u_j$ exists so that $u > 0$, $u \in L^{\gamma+q}(\Omega, d\sigma)$, and satisfies (4.1).

Conversely, if there exists a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma)$ to (4.1), it is clear that (1.12) holds. Moreover, (1.6) follows from the global pointwise lower bound (2.6):

$$u(x) \geq c[\mathbf{G}\sigma(x)]^{\frac{1}{1-q}}, \quad \forall x \in \Omega,$$

which does not require quasi-symmetry of G (see [8]). □

An essential link between conditions (1.6), (1.7) and (1.12), will be obtained in the next lemma. It is an extension of [19, Lemma 5.3] in the case $\gamma = 1$; moreover, G does not need to be quasi-symmetric due to Theorem 2.3.

Lemma 4.3. *Let $0 < q < 1$, $\gamma > 0$, and $\sigma, \mu \in \mathcal{M}^+(\Omega)$. Suppose G is a positive lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP. Then conditions (1.6) and (1.7) imply (1.12).*

Proof. Without loss of generality, we may assume $\sigma, \mu \not\equiv 0$. Consider the following three cases:

• Case 1: $\gamma + q > 1$. Applying the iterated inequality (2.7) with $\omega = \mu$ and $s = \gamma + q$, together with Fubini's theorem and Hölder's inequality with the exponents $\frac{\gamma}{\gamma+q-1}$ and $\frac{\gamma}{1-q}$, we obtain

$$\begin{aligned}
\int_\Omega (\mathbf{G}\mu)^{\gamma+q} \, d\sigma &\leq c \int_\Omega \mathbf{G}\left((\mathbf{G}\mu)^{\gamma+q-1} \, d\mu\right) \, d\sigma \\
&= c \int_\Omega (\mathbf{G}\mu)^{\gamma+q-1} \mathbf{G}\sigma \, d\mu \\
&\leq c \left[\int_\Omega (\mathbf{G}\mu)^{\gamma} \, d\mu\right]^{\frac{\gamma+q-1}{\gamma}} \left[\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma}{1-q}} \, d\mu\right]^{\frac{1-q}{\gamma}}.
\end{aligned} \tag{4.7}$$

The second integral on the right-hand side of (4.7) is estimated by a similar argument as above. In fact, applying (2.7) again with $\omega = \sigma$ and $s = \frac{\gamma}{1-q}$, along with Fubini's theorem and Hölder's inequality with the exponents $\frac{\gamma+q}{\gamma+q-1}$ and $\gamma + q$, we deduce

$$\begin{aligned}
\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma}{1-q}} \, d\mu &\leq c \int_\Omega \mathbf{G}\left((\mathbf{G}\sigma)^{\frac{\gamma}{1-q}-1} d\sigma\right) \, d\mu \\
&= c \int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma+q-1}{1-q}} \, \mathbf{G}\mu \, d\sigma \\
&\leq c \left[\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma+q}{1-q}} \, d\sigma\right]^{\frac{\gamma+q-1}{\gamma+q}} \left[\int_\Omega (\mathbf{G}\mu)^{\gamma+q} \, d\sigma\right]^{\frac{1}{\gamma+q}}.
\end{aligned} \tag{4.8}$$

By (4.7) and (4.8), we have

$$\begin{aligned}\left[\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma\right]^{1-\frac{1-q}{\gamma(\gamma+q)}} &\leq c\left[\int_\Omega (\mathbf{G}\mu)^{\gamma}\, d\mu\right]^{\frac{\gamma+q-1}{\gamma}} \\ &\times\left[\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma+q}{1-q}}\, d\sigma\right]^{\frac{(\gamma+q-1)(1-q)}{\gamma(\gamma+q)}},\end{aligned} \tag{4.9}$$

which is finite by (1.6) and (1.7), and hence (1.12) holds.

- Case 2: $\gamma+q<1$. Write

$$\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma = \int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, F^{a-1}F^{1-a}\, d\sigma,$$

where $0<a<1$ and F is a positive measurable function to be determined later. Applying Hölder's inequality with the exponents $\frac{1}{a}$ and $\frac{1}{1-a}$, we get

$$\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma \leq \left(\int_\Omega (\mathbf{G}\mu)^{\frac{\gamma+q}{a}}\, F^{\frac{a-1}{a}}\, d\sigma\right)^{a}\left(\int_\Omega F\, d\sigma\right)^{1-a}. \tag{4.10}$$

Setting $F=(\mathbf{G}\sigma)^{\frac{\gamma+q}{1-q}}$ and $a=\gamma+q$ in (4.10), we obtain

$$\begin{aligned}\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma &\leq \left(\int_\Omega \mathbf{G}\mu\,(\mathbf{G}\sigma)^{\frac{\gamma+q-1}{\gamma+q}}\, d\sigma\right)^{\gamma+q} \\ &\times\left(\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma+q}{1-q}}\, d\sigma\right)^{1-\gamma-q}.\end{aligned} \tag{4.11}$$

The first integral on the right-hand side of (4.11) is estimated by using Fubini's theorem, followed by inequality (2.8) with $\omega=\sigma$ and $s=\frac{\gamma}{1-q}$,

$$\begin{aligned}\int_\Omega \mathbf{G}\mu\,(\mathbf{G}\sigma)^{\frac{\gamma+q-1}{\gamma+q}}\, d\sigma &= \int_\Omega \mathbf{G}\left((\mathbf{G}\sigma)^{\frac{\gamma+q-1}{\gamma+q}}d\sigma\right)\, d\mu \\ &\leq c\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma}{1-q}}\, d\mu.\end{aligned} \tag{4.12}$$

As above, we deduce

$$\begin{aligned}\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma}{1-q}}\, d\mu &\leq \left(\int_\Omega \mathbf{G}\sigma(\mathbf{G}\mu)^{\gamma+q-1}\, d\mu\right)^{\frac{\gamma}{1-q}}\left(\int_\Omega (\mathbf{G}\mu)^{\gamma}\, d\mu\right)^{\frac{1-\gamma-q}{1-q}} \\ &= \left(\int_\Omega \mathbf{G}\left((\mathbf{G}\mu)^{\gamma+q-1}d\mu\right)\, d\sigma\right)^{\frac{\gamma}{1-q}}\left(\int_\Omega (\mathbf{G}\mu)^{\gamma}\, d\mu\right)^{\frac{1-\gamma-q}{1-q}} \\ &\leq c\left(\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma\right)^{\frac{\gamma}{1-q}}\left(\int_\Omega (\mathbf{G}\mu)^{\gamma}\, d\mu\right)^{\frac{1-\gamma-q}{1-q}}.\end{aligned}$$

Combining the preceding estimates, we have

$$\left(\int_\Omega (\mathbf{G}\mu)^{\gamma+q}\, d\sigma\right)^{1-\frac{\gamma(\gamma+q)}{1-q}} \leq c\left[\int_\Omega (\mathbf{G}\mu)^{\gamma}\, d\mu\right]^{\frac{(1-\gamma-q)(\gamma+q)}{1-q}} \times \left[\int_\Omega (\mathbf{G}\sigma)^{\frac{\gamma+q}{1-q}}\, d\sigma\right]^{1-\gamma-q}. \tag{4.13}$$

This proves (1.12), since both integrals on the right-hand side of (4.13) are finite by (1.6) and (1.7).

• Case 3: $\gamma+q=1$. Fix a positive number $\frac{1}{2-q}<a<1$. Applying Hölder's inequality with the exponents $\frac{1}{a}$ and $\frac{1}{1-a}$ we have

$$\begin{aligned}\int_\Omega \mathbf{G}\mu\, d\sigma &= \int_\Omega \mathbf{G}\mu (\mathbf{G}\sigma)^{\frac{a-1}{1-q}}(\mathbf{G}\sigma)^{\frac{1-a}{1-q}}\, d\sigma \\ &\leq \left(\int_\Omega (\mathbf{G}\mu)^{\frac{1}{a}}(\mathbf{G}\sigma)^{\frac{a-1}{a(1-q)}}\, d\sigma\right)^a \left(\int_\Omega (\mathbf{G}\sigma)^{\frac{1}{1-q}}\, d\sigma\right)^{1-a}.\end{aligned} \tag{4.14}$$

We estimate the first integral on the right-hand side of (4.14) using inequalities (2.7) and (2.8), together with Fubini's theorem and Hölder's inequality with the exponents $\frac{a(1-q)}{1-a}$ and $\frac{a(1-q)}{a(2-q)-1}$,

$$\begin{aligned}&\int_\Omega (\mathbf{G}\mu)^{\frac{1}{a}}(\mathbf{G}\sigma)^{\frac{a-1}{a(1-q)}}\, d\sigma \leq c\int_\Omega \mathbf{G}[(\mathbf{G}\mu)^{\frac{1-a}{a}}d\mu](\mathbf{G}\sigma)^{\frac{a-1}{a(1-q)}}\, d\sigma \\ &= c\int_\Omega (\mathbf{G}\mu)^{\frac{1-a}{a}}\mathbf{G}\left((\mathbf{G}\sigma)^{\frac{a-1}{a(1-q)}}\, d\sigma\right)\, d\mu \leq c\int_\Omega (\mathbf{G}\mu)^{\frac{1-a}{a}}(\mathbf{G}\sigma)^{\frac{a-1}{a(1-q)}+1}\, d\mu \\ &\leq c\left(\int_\Omega (\mathbf{G}\mu)^{1-q}\, d\mu\right)^{\frac{1-a}{a(1-q)}}\left(\int_\Omega \mathbf{G}\sigma\, d\mu\right)^{\frac{a(2-q)-1}{a(1-q)}} \\ &= c\left(\int_\Omega (\mathbf{G}\mu)^{1-q}\, d\mu\right)^{\frac{1-a}{a(1-q)}}\left(\int_\Omega \mathbf{G}\mu\, d\sigma\right)^{\frac{a(2-q)-1}{a(1-q)}}.\end{aligned}$$

Combining the preceding estimates, we deduce

$$\left(\int_\Omega \mathbf{G}\mu\, d\sigma\right)^{1-\frac{a(2-q)-1}{(1-q)}} \leq c\left(\int_\Omega (\mathbf{G}\mu)^{1-q}\, d\mu\right)^{\frac{1-a}{(1-q)}} \times \left(\int_\Omega (\mathbf{G}\sigma)^{\frac{1}{1-q}}\, d\sigma\right)^{1-a}, \tag{4.15}$$

which is finite by (1.6) and (1.7). Thus (1.12) holds. □

We are now prepared to show that conditions (1.6) and (1.7) are necessary and sufficient for the existence of a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma)\cap L^{\gamma}(\Omega, d\mu)$ to integral equation (4.1), under the same restrictions on the kernel G as above.

Theorem 4.4. *Let $0 < q < 1$, $\gamma > 0$, and let $\sigma, \mu \in \mathcal{M}^+(\Omega)$ with $\sigma \not\equiv 0$. Suppose G is a positive, lower semicontinuous, quasi-symmetric kernel on $\Omega\times\Omega$, which satisfies the WMP. Suppose (1.6) and (1.7) hold. Then there exists a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma) \cap L^{\gamma}(\Omega, d\mu)$ to (4.1). The converse statement is also valid without the quasi-symmetry assumption on G.*

Proof. Suppose (1.6) and (1.7) hold. Then, by Lemma 4.3, we see that (1.12) holds. Thus, in light of Theorem 4.2, there exists a positive solution $u \in L^{\gamma+q}(\Omega, d\sigma)$ to (4.1). We will show that $u \in L^{\gamma}(\Omega, d\mu)$ as well. By (1.7), it suffices to establish

$$\int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma \; d\mu < +\infty. \tag{4.16}$$

Without loss of generality, we may assume that G is symmetric, and $\mu \not\equiv 0$. Consider the following two cases:

• Case 1: $\gamma \geq 1$. Applying (2.8) with $\omega := \sigma u^q$ and $s := \gamma$, along with Fubini's theorem, and Hölder's inequality with the exponents $\frac{\gamma+q}{\gamma+q-1}$ and $\gamma + q$, we have

$$\begin{aligned}
&\int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma \; d\mu \leq c \int_\Omega \mathbf{G}\left((\mathbf{G}(u^q d\sigma))^{\gamma-1} u^q d\sigma \right) \; d\mu \\
&= c \int_\Omega (\mathbf{G}(u^q d\sigma))^{\gamma-1} (\mathbf{G}\mu)\, u^q d\sigma \\
&\leq c \left[\int_\Omega (\mathbf{G}(u^q d\sigma))^{\frac{(\gamma-1)(\gamma+q)}{\gamma+q-1}} u^{\frac{q(\gamma+q)}{\gamma+q-1}} \; d\sigma \right]^{\frac{\gamma+q-1}{\gamma+q}} \left[\int_\Omega (\mathbf{G}\mu)^{\gamma+q} \; d\sigma \right]^{\frac{1}{\gamma+q}} \\
&\leq c \left[\int_\Omega u^{\gamma+q} \; d\sigma \right]^{\frac{\gamma+q-1}{\gamma+q}} \left[\int_\Omega (\mathbf{G}\mu)^{\gamma+q} \; d\sigma \right]^{\frac{1}{\gamma+q}} \\
&\leq c \int_\Omega u^{\gamma+q} \; d\sigma < +\infty.
\end{aligned}$$

• Case 2: $0 < \gamma < 1$. We write

$$\int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma \; d\mu = \int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma F^{a-1} F^{1-a} \, d\mu,$$

where $0 < a < 1$ and F is a positive measurable function to be determined later. Applying Hölder's inequality with the conjugate exponents $\frac{1}{a}$ and $\frac{1}{1-a}$ yields

$$\int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma \; d\mu \leq \left[\int_\Omega \mathbf{G}(u^q d\sigma) F^{\frac{a-1}{a}} \; d\mu \right]^a \left[\int_\Omega F \; d\mu \right]^{1-a}. \tag{4.17}$$

Setting $F := (\mathbf{G}\mu)^\gamma$ and $a := \gamma$ in (4.17), we get

$$\begin{aligned}(4.18)\qquad \int_\Omega [\mathbf{G}(u^q d\sigma)]^\gamma \, d\mu \le \left[\int_\Omega \mathbf{G}(u^q d\sigma)(\mathbf{G}\mu)^{\gamma-1} \, d\mu\right]^\gamma \\ \times \left[\int_\Omega (\mathbf{G}\mu)^\gamma \, d\mu\right]^{1-\gamma}.\end{aligned}$$

We estimate the first integral on the right-hand side of (4.18) using Fubini's theorem, followed by inequality (2.8) with $\omega = \mu$ and $s = \gamma$,

$$\begin{aligned}(4.19)\qquad &\int_\Omega \mathbf{G}(u^q d\sigma)(\mathbf{G}\mu)^{\gamma-1} \, d\mu = \int_\Omega \mathbf{G}\left((\mathbf{G}\mu)^{r-q-1} d\mu\right) u^q \, d\sigma \\ &\le c \int_\Omega (\mathbf{G}\mu)^\gamma u^q \, d\sigma \le c \int_\Omega u^{\gamma+q} \, d\sigma < +\infty.\end{aligned}$$

By (4.18) and (4.19), together with (1.7), this proves (4.16).

Conversely, since $u \in L^\gamma(\Omega, d\mu)$, it is clear that (1.7) holds. Further, by Theorem 4.2, condition (1.6) is valid since $u \in L^{\gamma+q}(\Omega, d\sigma)$. □

As an application of the preceding theorem, when G is a positive Green function associated with $\mathcal{L}$ in Ω, we deduce the first part of Theorem 1.1 by appealing to the characterization of the generalized Green energy obtained in Sec. 3.

Theorem 4.5. *Let $0 < q < 1$, $\gamma > 0$, and let $\sigma, \mu \in \mathcal{M}^+(\Omega)$ with $\sigma \not\equiv 0$. Let G be a positive Green function associated with $\mathcal{L}$ on Ω. Then there exists a positive solution $u \in L^q_{loc}(\Omega, d\sigma)$ to (1.1) satisfying (1.3) if and only if (1.6) and (1.7) hold.*

In this case, u is a minimal solution in the sense that $u \le v$ q.e. for any positive solution $v \in L^q_{loc}(\Omega)$ to (1.1) which satisfies (1.3).

Proof. This follows from Theorem 3.1 with $\omega := u^q d\sigma + \mu$, together with Theorem 4.4. Moreover, arguing by induction, we see that minimality of such a solution follows immediately from its construction in Theorem 4.2 (cf. [19, Lemma 5.5]). □

Remark 4.6. When $\gamma = 1$, uniqueness of such a solution in $u \in \dot{W}^{1,2}_0(\Omega)$, follows from a similar argument presented in [19, Sec. 6] (see also [6]), using its minimality together with a convexity property of the Dirichlet integral $\int_\Omega |\nabla u|^2 \, dx$, which is comparable to the expression $\int_\Omega \mathcal{A}\nabla u \cdot \nabla u \, dx$ due to the uniform ellipticity condition (1.2).

Applying the next theorem with $\omega := u^q d\sigma + \mu$ yields Corollary 1.2.

Theorem 4.7. *Let $n \ge 3$, and $\omega \in \mathcal{M}^+(\Omega)$ with $\omega \not\equiv 0$. Let G be a positive Green function associated with $\mathcal{L}$ on Ω. Suppose that $u := \mathbf{G}\omega$ satisfies (1.3) for some $0 < \gamma \le 1$. Then $u \in \dot{W}^{1,p}_0(\Omega)$ where*

$p = \frac{n(1+\gamma)}{n+\gamma-1}$. *If, in addition,* $|\Omega| < +\infty$, *then the assertion is valid for* $1 \le p \le \frac{n(1+\gamma)}{n+\gamma-1}$.

Proof. The classical case $\gamma = 1$ is known, so we can assume $\gamma \in (0,1)$. Observe that u is a positive $\mathcal{A}$-superharmonic function with zero boundary values, and $p = \frac{n(1+\gamma)}{n+\gamma-1} \in (\frac{n}{n-1}, 2)$. For each $k \in \mathbb{N}$, set $u_k = \min(u,k)$, which is a positive $\mathcal{A}$-superharmonic function of the class $L^\infty(\Omega) \cap W^{1,2}_{loc}(\Omega)$. Let $\{\Omega_k\}_{k=1}^\infty$ be an increasing sequence of relatively compact open subsets of Ω such that $\Omega = \bigcup_{k=1}^\infty \Omega_k$. Applying Hölder's inequality with the exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, followed by Sobolev's inequality [17, Theorem 1.56], we obtain

$$\begin{aligned}
&\|(\nabla u_k)\chi_{\Omega_k}\|^p_{L^p(\Omega)} = \int_{\Omega_k} |\nabla u_k|^p \, dx \\
&= \int_{\Omega_k} |\nabla u_k|^p u_k^{\frac{(\gamma-1)p}{2}} u_k^{\frac{(1-\gamma)p}{2}} \, dx \\
&\le \left(\int_{\Omega_k} |\nabla u_k|^2 u_k^{\gamma-1} \, dx \right)^{\frac{p}{2}} \left(\int_{\Omega_k} u_k^{\frac{(1-\gamma)p}{2-p}} \, dx \right)^{\frac{2-p}{2}} \\
&\le \left(\int_{\Omega} |\nabla u|^2 u^{\gamma-1} \, dx \right)^{\frac{p}{2}} \|u_k\|^{\frac{(1-\gamma)p}{2}}_{L^{\frac{(1-\gamma)p}{2-p}}(\Omega_k)} \\
&\le c \left(\int_{\Omega} |\nabla u|^2 u^{\gamma-1} \, dx \right)^{\frac{p}{2}} \|\nabla u_k\|^{\frac{(1-\gamma)p}{2}}_{L^p(\Omega_k)},
\end{aligned}$$

that is,

$$\|(\nabla u_k)\chi_{\Omega_k}\|^{p-\frac{(1-\gamma)p}{2}}_{L^p(\Omega)} \le c \left(\int_{\Omega} |\nabla u|^2 u^{\gamma-1} \, dx \right)^{\frac{p}{2}}, \tag{4.20}$$

where c is a positive constant independent of k. Since $p > \frac{(1-\gamma)p}{2}$, letting $k \to \infty$ in (4.20) yields the assertion by the monotone convergence theorem. This proves $u \in \dot{W}^{1,p}_0(\Omega)$, which is obviously true for all $1 \le p \le \frac{n(1+\gamma)}{n+\gamma-1}$ when $|\Omega| < +\infty$. □

The next proposition shows in particular that a pair of conditions in (1.9) is sufficient for both (1.6) and (1.7).

Proposition 4.8. *Let G be a positive lower semicontinuous kernel on $\Omega \times \Omega$ which satisfies*

$$G(x,y) \le c\, I_{2\alpha}(x-y), \quad \forall x, y \in \Omega, \tag{4.21}$$

where $I_{2\alpha}(\cdot) = |\cdot|^{2\alpha-n}$ is the Riesz kernel of order 2α $(0 < \alpha < \frac{n}{2})$ on $\mathbb{R}^n$, and c is a positive constant. Let $\beta > 0$. If $\omega \in L^s(\Omega)$ is a positive *function, where* $s = \frac{n(\beta+1)}{n+2\alpha\beta}$*, then*

$$\mathbf{G}\omega \in L^{\beta}(\Omega, d\omega). \tag{4.22}$$

Proof. Observe that $s = \frac{n(\beta+1)}{n+2\alpha\beta} > \frac{n(\beta+1)}{n+n\beta} = 1$. Then

$$\int_{\Omega} (\mathbf{G}\omega)^{\beta} \, d\omega \leq \left(\int_{\Omega} (\mathbf{G}\omega)^{\beta s'} \, dx \right)^{\frac{1}{s'}} \|\omega\|_{L^s(\Omega)}, \tag{4.23}$$

where $s' = \frac{s}{s-1}$ is the Hölder conjugate of s. Denote $\tilde{\omega}$ the zero extension of ω to $\mathbb{R}^n$. By (4.21) and the Hardy-Littlewood-Sobolev inequality, there is a positive constant $C \geq c$ such that

$$\|\mathbf{G}\omega\|_{L^{\beta s'}(\Omega)} \leq C \|\mathbf{I}_{2\alpha}\tilde{\omega}\|_{L^{\beta s'}(\mathbb{R}^n)} \leq C \|\tilde{\omega}\|_{L^s(\mathbb{R}^n)} = C \|\omega\|_{L^s(\Omega)}. \tag{4.24}$$

Combining (4.23) and (4.24) yields (4.22). □

Proposition 4.8 shows that (1.10) implies (1.8). Hence, Corollary 1.3 follows from Corollary 1.2.

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Department of Mathematics, Hokkaido University, Sapporo, Hokkaido 060-0810, Japan
E-mail address: seesanea@math.sci.hokudai.ac.jp

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: verbitskyi@missouri.edu