Extending Drawings of Graphs to Arrangements of Pseudolines

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November 9, 2018

Abstract

A *pseudoline* is a homeomorphic image of the real line in the plane so that its complement is disconnected. An *arrangement of pseudolines* is a set of pseudolines in which every two cross exactly once. A drawing of a graph is *pseudolinear* if the edges can be extended to an arrangement of pseudolines. In the recent study of crossing numbers, pseudolinear drawings have played an important role as they are a natural combinatorial extension of rectilinear drawings. A characterization of the pseudolinear drawings of K_n was found recently. We extend this characterization to all graphs, by describing the set of minimal forbidden subdrawings for pseudolinear drawings. Our characterization also leads to a polynomial-time algorithm to recognize pseudolinear drawings and construct the pseudolines when it is possible.

*Supported by CONACYT
†ERC Advanced Grant GRACOL, project no. 320812
‡Supported by NSERC

1 Introduction

A *pseudoline* is an unbounded open arc in the plane whose complement is disconnected. In particular, lines are pseudolines, and any pseudoline is the image of a line under a homeomorphism of the plane into itself. An *arrangement of pseudolines* is a set of pseudolines in which every two intersect in exactly one point, and their intersection point is a crossing. A drawing of a graph G is *pseudolinear* if there is an arrangement of pseudolines consisting of a different pseudoline for each edge and each edge is contained in its pseudoline.

In this work we characterize pseudolinearity of a drawing of any graph, not just K_n: the drawing must be good (defined below) and not contain any of the configurations in Figure 1. Thomassen [15] already observed that many of the drawings in Figure 1 are obstructions for a drawing to be homeomorphic to a rectilinear drawing; they are also obstructions for pseudolinearity.

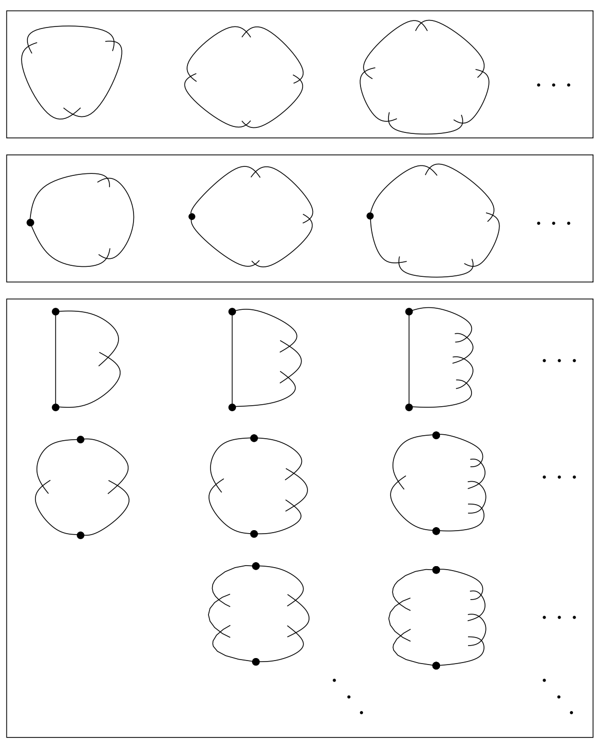

Figure 1: Obstructions to pseudolinear drawings.

We have been unable to find any literature that suggests even the possibility of a characterization of pseudolinearity. Moreover, informal conversations with colleagues seemed to be more along the lines of finding more obstructions.

A rectilinear drawing of a graph is one in which edges are drawn using straight line segments, and more generally, a stretchable drawing is one that is homeomorphic to a rectilinear drawing. Fáry's Theorem [6, 14, 16], a classic result in graph theory, asserts that drawings of simple graphs with no crossings between edges are stretchable.

In [15], Thomassen extended Fáry's Theorem by characterizing stretchable drawings of graphs in which every edge is crossed at most once: In addition to being a *good* drawing (that is, no edge self-intersects and no two edges have two points – either crossings or common endpoints – in common), there are two forbidden configurations, shown in Figure 2.

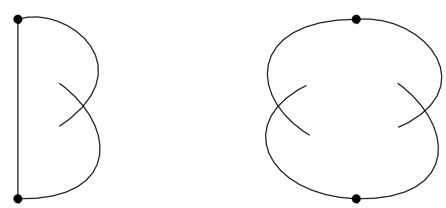

Figure 2: B and W configurations.

Thomassen's characterization is a partial answer to the general problem of determining which drawings are stretchable. There is not likely to be a complete characterization, as Mnëv [11, 12] showed that the closely related problem of stretchability of arrangements of pseudolines is NP-hard (in fact $\exists\mathbb{R}$-hard). This easily implies that stretchability of graph drawings is NP-hard.

The study of arrangements of pseudolines was initiated by Levi and Ringel [9, 13], and propagated by Grünbaum's popular monograph *Arrangements and spreads* [7].

Arrangements of pseudolines have played an important role in the study of the crossing number of K_n. A drawing of a graph G is *pseudolinear* if there is an arrangement of pseudolines consisting of a different pseudoline for each edge and each edge is contained in its pseudoline. The original, independent proofs by Ábrego and Fernández-Merchant [1] and by Lovász et al [10] that a rectilinear drawing of K_n has at least

$$\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

crossings in fact applies to pseudolinear drawings of K_n. The substantial progress on computing the rectilinear crossing number of K_n has continued this approach and has lead to further study of pseudolinear drawings [4, 8, 3, 2, 5]. Since the pseudolinear obstructions are also rectilinear obstructions, we wonder if this work might shed light on rectilinear drawings of graphs. For example, Thomassen characterizes when a drawing of a graph in which each edge has at most one crossing is homeomorphic to a rectilinear drawing. Our result shows that this is if and only if the drawing is pseudolinear. (Pseudolinearity is obviously necessary; that it is sufficient is a little surprising.)

There have been recent, independent characterizations of pseudolinear drawings of K_n [3, 2]. The simpler of the equivalent descriptions is that the drawing is good and that it does not contain the unique (up to homeomorphism) good drawing of K_4 having the crossing incident with the infinite face.

Our main theorem is best presented in the context of strings in the plane. A *string* σ is the image $f([0,1])$ of a continuous function $f : [0,1] \to \mathbb{R}^2$ that restricted to $(0,1)$ is injective; in other words, strings are arcs that are allowed to self-intersect only at their *ends* $f(0)$ and $f(1)$. If no such self-intersection exists, then σ is *simple*. Most of the time we will consider simple strings, although considering non-simple strings will come in handy for technical reasons.

A set of strings Σ is in *general position* if, for every two strings σ, $\sigma' \in \Sigma$ (i) $\sigma \cap \sigma'$ is a finite set of points in $\mathbb{R}^2$; and (ii) each point in $\sigma \cap \sigma'$ is either a crossing between σ and σ', or an end of either σ or σ'. For instance, the set of edge-arcs of a good drawing of a graph is a set of strings in general position, but not all the sets of strings in general position come in this fashion: a string might include end points of other strings in its interior.

For a set Σ of strings in general position, its *underlying plane graph* $G(\Sigma)$ is the plane graph obtained from Σ by replacing the crossings between strings and the end points of every string in Σ by vertices. Our main result below characterizes when a set of strings in general position can be extended to an arrangement of pseudolines.

Theorem 1.1. *A set of strings Σ in general position can be extended to an arrangement of pseudolines if and only if, for each cycle C in the underlying plane graph $G(\Sigma)$ of Σ, there are at least three vertices with the property that the edges incident to the vertex that are included in the closed disk bounded by C belong to distinct strings in Σ.*

For instance, let C be the unique cycle in the underlying plane graph in any of the drawings in Figure 1. There are at most two vertices of G in C, represented as black dots. The strings incident with such a vertex are distinct and contained in the closed disk bounded by C. The vertices represented as crossings do not satisfy this property: they are incident with four edges in the disk bounded by C, and these four edges consist of two strings that cross at this vertex. Theorem 1.1 implies that none of the drawings in Figure 1 is pseudolinear. Surprisingly, we will show, as a consequence of Theorem 1.1, that every non-pseudolinear drawing contains one of the configurations in Figure 1 as a subdrawing.

Theorem 1.2. *Let D be a non-pseudolinear good drawing of a graph H. Then there is a subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma\in S}\sigma'$ is one of the drawings in Figure 1.*

Cycles that have fewer than three vertices as in Theorem 1 are the *obstructions* of $G(\Sigma)$ (this definition will be made more precise at the beginning of Section 2). Showing that when $G(\Sigma)$ has obstructions, then Σ cannot be extended to an arrangement of pseudolines, is the first part of Section 2. The rest of Section 2 is devoted to show that if $G(\Sigma)$ has no obstructions, then Σ can be extended to an arrangement of pseudolines. The proof of two technical lemmas used in the proof of Theorem 1.1 are deferred to Section 3. In Section 4, we describe a simple algorithm that finds an obstruction in polynomial time. In Section 5, by applying Theorem 1.1, we prove that a drawing of a complete graph K_n is pseudolinear if and only if it does not contain the B configuration in Figure 2. This result is equivalent to the characterizations of pseudolinear drawings of K_n given in [2] and [3], but its proof is simpler. At the end, in Section 6, we show how Theorem 1.2 easily follows from Theorem 1.1, together with some concluding remarks.

2 Proof of Theorem 1.1

In this section, we use Lemmas 2.4 and 2.5 (proved in the next section) to prove Theorem 1.1. As we enter into the subject, we need some notation that is useful in identifying an obstruction. Let C be a cycle of a plane graph G and let v be a vertex of C. The *rotation at v inside C* is the counterclockwise ordered list $e_0, e_1 ..., e_k$ of edges incident with v that are included in the closed disk bounded by C, with e_0 and e_k both in C. Likewise, the *rotation at v outside C* is defined as the counterclockwise ordered list e_k, $e_{k+1}, \ldots, e_0$ of edges incident with v included in the closure of the exterior of C.

(a) A reflecting vertex (b) A rainbow

Figure 3: A representation of reflecting and rainbow vertices, where each string in $G(\Sigma)$ has assigned a unique colour.

In the case $G = G(\Sigma)$ for some set Σ of strings in general position, a vertex v in a cycle C of $G(\Sigma)$ is *reflecting* in C if at least two edges in the rotation at v inside C belong to the same string (Figure 3a). The alternative is that v is a *rainbow*, in which case all the edges of its rotation inside C are in different strings (Figure 3b). In these terms, an *obstruction* is a cycle with at most two rainbows.

2.1 Sets of strings with obstructions are not extendible

The following observation will be used in this subsection and also in Theorem 5.1. If C is a cycle in $G(\Sigma)$, where Σ is a set of strings in general position, then $\delta(C)$ is the set of vertices in C for which their two incident edges in C belong to two distinct strings in Σ. Note that if $|\delta(C)| < 3$ for some cycle C, then either $|\delta(C)| = 2$ and two strings intersect more than once, or $|\delta(C)| \leq 1$ and some string is self-crossed. Both these possibilities are forbidden in good drawings.

Observation 2.1. *Let Σ be a set of simple strings in general position in which every two strings intersect at most once. Let:*

(a) C be an obstruction of $G(\Sigma)$ for which $|\delta(C)|$ is as small as possible;

(b) $x \in \delta(C)$;

(c) e be an edge in C incident to x;

(d) $\sigma \in \Sigma$ be the string containing e; and

(e) σ' be the component of $\sigma \setminus e$ containing x.

Then $\sigma' \cap C = \{x\}$.

Proof. By way of contradiction, suppose that $\sigma' \cap C$ includes a point distinct from x. This in particular implies that $\sigma' \neq \{x\}$, and, because $x \in \delta(C)$, the points of $\sigma' \setminus \{x\}$ near x are not in C. Let P be the path in $G(\Sigma)$ obtained by traversing σ', starting at x, and stopping the first time we encounter a point $y \in C \cap (\sigma' \setminus \{x\})$. Note that $y \in V(C)$ and that P is drawn in either the interior or the exterior of C.

First, suppose that P is drawn in the interior of C. Let C_1 and C_2 be the cycles obtained from the union of P and one of the two xy-subpaths in C. We may assume C_1 includes e. Each of $C_1 - P$ and $C_2 - P$ has a vertex in $\delta(C)$; otherwise one of C_1 or C_2 would be included in at most two strings, implying that a string is self-crossing or two strings intersect twice. Therefore $|\delta(C_1)|$ and $|\delta(C_2)|$ are strictly smaller than $|\delta(C)|$. Then, by assumption, C_1 and C_2 are not obstructions.

None of the vertices in $P - y$ is a rainbow for C_1 ($P \subseteq \sigma'$ and x is reflecting in C_1, so the interior rotations of the vertices in $P - y$ include two edges in σ). Since all the vertices in $C_1 - V(P)$ that are rainbow in C_1 are also rainbow in C, C_1 has at most two rainbows in $V(C_1) \setminus V(P)$. These last two observations and the fact that C_1 is not an obstruction, together imply that C_1 has three rainbows: two of them are in $V(C_1) \setminus V(P)$ and the other is y.

Now we look at the rainbows in C_2. Because C has two rainbows in $C_1 - V(P)$ and any rainbow in $V(C_2) \setminus V(P)$ for C_2 is rainbow for C, C_2 has no rainbow in $V(C_2) \setminus V(P)$. All the interior vertices of P are reflecting in C_2, so C_2 has at most two rainbows. This contradicts that C_2 is not an obstruction.

Secondly, suppose that P is drawn in the exterior of C. Let C_{out} be the cycle bounding the outer face of $C \cup P$. The cycle C_{out} is the union of P and one of the two xy-paths in C, and, in both cases, as $x \in \delta(C) \setminus \delta(C_{out})$ and $P - y \subset \sigma$, $|\delta(C_{out})| < |\delta(C)|$. Every vertex in $P - y$ is reflecting in C_{out} (this statement follows from the fact that the rotation of a vertex inside a cycle also includes the edges of the cycle incident with the vertex). Moreover, every vertex in $V(C_{out}) \setminus (V(P - y))$ that is a rainbow in C_{out} is also a rainbow in C. These two facts imply that C_{out} has at most as many rainbows as C; hence C_{out} is an obstruction. This contradicts the fact that C minimizes $|\delta|$. □

Next we show that, if a set of strings contains an obstruction, then it is not pseudolinear.

Observation 2.2. *If Σ is a set of strings in general position and $G(\Sigma)$ has an obstruction, then Σ cannot be extended to an arrangement of pseudolines.*

Proof. By way of contradiction, suppose that there is a set of strings Σ that can be extended to an arrangement of pseudolines and $G(\Sigma)$ has an obstruction C. Consider an extension of Σ to an arrangement of pseudolines, and then cut off the two infinite ends of each pseudoline to obtain a set of strings Σ' extending Σ, and in which every two strings in Σ' cross. In $G(\Sigma')$, there is a cycle C' that represents the same simple closed curve as C. Because C' is obtained from subdiving some edges of C, C' has fewer than three rainbows. Therefore, we may assume that $\Sigma = \Sigma'$ and $C = C'$. Now, the ends of every string in Σ are degree-one vertices in the outer face of $G(\Sigma)$.

As every string in Σ is simple, and no two strings intersect more than once, $|\delta(C)| \geq 3$. We will assume that C is chosen to minimize $|\delta(C)|$.

Since C is an obstruction, there is at least one vertex $x \in \delta(C)$ reflecting inside C. Let $e \in E(C)$ be an edge incident to x, and suppose that σ is the string including e. Traversing σ along e through x, we encounter another edge $e' \subseteq \sigma$ incident to x. Because $x \in \delta(C)$, e' is not in C. Suppose that e' is drawn in the outer face of C. As x is reflecting inside C, there exists a string $\bar{\sigma}$ that includes two edges in the rotation at x inside C. However, σ and $\bar{\sigma}$ tangentially intersect at x, contradicting that the strings in Σ are in general position. Therefore e' is drawn inside C.

Let y be the end of σ contained in the component of $\sigma \setminus e$ containing x. Since $|\delta(C)|$ is minimum, Observation 2.1 implies that the component of $\sigma \setminus e$

having x and y as ends have all its points, with the exception of x, in the inner face of C. However, y is drawn in the inner face of C, contradicting that the ends of all the strings in Σ are incident with the outer face of $G(\Sigma)$. □

2.2 Extending sets of strings with no obstructions

In this subsection we prove that a set of strings with no obstructions can be extended to an arrangement of pseudolines. We restate Theorem 1.1 using our new terminology.

Theorem 2.3. *A set of strings Σ in general position can be extended to an arrangement of pseudolines if and only if $G(\Sigma)$ has no obstructions.*

Proof. We showed in Observation 2.2 that if $G(\Sigma)$ has an obstruction, then Σ cannot be extended to an arrangement of pseudolines. For the converse, suppose that $G(\Sigma)$ has no obstructions.

We start by reducing the proof to the case in which the point set $\bigcup \Sigma$ is connected. If $\bigcup \Sigma$ is not connected, then we add a simple string to Σ, connecting two points in distinct components of $G(\Sigma)$, and so that it is included inside a face of $G(\Sigma)$. This operation: reduces the number of components; does not create obstructions; and ensures that any pseudolinear extension of the new set of strings shows the existence of one for Σ. We continue adding strings in this way until we obtain a connected set of strings and we redefine Σ to be this set. Thus, we may assume $\bigcup \Sigma$ is connected.

Our proof is algorithmic, and consists of repeatedly applying one of the three steps described below.

- **Disentangling Step.** If a string $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$, then we slightly extend the a-end of σ into one of the faces incident with a.

- **Face-Escaping Step.** If a string $\sigma \in \Sigma$ has an end a with degree 1 in $G(\Sigma)$, and is incident with an inner face, then we extend the a-end of σ until we intersect some point in the boundary of this face.

- **Exterior-Meeting Step.** Assuming that all the strings in Σ have their two ends in the outer face and these ends have degree 1 in $G(\Sigma)$, we extend the ends of two disjoint strings so that they meet in the outer face.

We can always perform at least one of these steps, unless the strings are pairwise intersecting and all of them have their ends in the outer face (in this case we extend their ends to infinity to obtain the desired arrangement of pseudolines). Each step increases the number of pairwise intersecting strings. Henceforth, our aim is to show that, as long as there is a pair of non-intersecting strings, then one of these three steps may be performed without adding an obstruction. The proof is now divided into three parts that can be read independently.

Disentangling Step. *Suppose that $\sigma \in \Sigma$ has an end a with degree at least 2 in $G(\Sigma)$. Then we can extend the a-end of σ into one of the faces incident to a without creating an obstruction.*

Proof. An edge f of $G(\Sigma)$ incident with a is a *twin* if there exists another edge $f' \neq f$ incident with a such that both f and f' are part of the same string in Σ. Observe that the edge $e_0 \subseteq \sigma$ incident with a is not a twin.

The fact no pair of strings tangentially intersect at a tells us that if (f_1, f_1') and (f_2, f_2') are pairs of corresponding twins, then f_1, f_2, f_1', f_2' occur in this cyclic order for either the clockwise or counterclockwise rotation at a. Thus, we may assume that the twins at a are labeled as $f_1, \dots, f_t, f_1', \dots, f_t'$, and that this is their counterclockwise order occurrence when we follow the rotation at a starting at e_0. In such a case, (f_i, f_i') is a pair of corresponding twins for $i = 1, \dots, t$.

In order to avoid tangential intersections when twins are present, every valid extension of σ at a must cross into the angle between f_t and f_1' not containing e_0.

Let $(e_1, \dots, e_k)$ be the list of non-twin edges between f_t and f_1' in the counterclockwise rotation at a; this list might be empty. In the case there are no twins, we set f_t and f_1' both equal to e_0, so $(e_1, \dots, e_k)$ is all the edges incident with a other than e_0.

We consider all the feasible extensions for σ: for each $i \in \{1, ..., k-1\}$, we let Σ_i be the set of strings obtained from extending σ by adding a small bit of arc α_i starting at a, and continuing into the face between e_i and e_{i+1}. Let Σ_0 be the set of strings obtained by adding an arc α_0 in the face between f_t and e_1, and let Σ_k be obtained by adding an arc α_k in the face between e_k and f_1'.

Seeking a contradiction, suppose that, for each $i \in \{0, ..., k\}$, $G(\Sigma_i)$ contains an obstruction C_i. The cycle C_i does not include the bit of arc α_i as an edge, so C_i is a cycle in $G(\Sigma)$. This cycle is not an obstruction in $G(\Sigma)$,

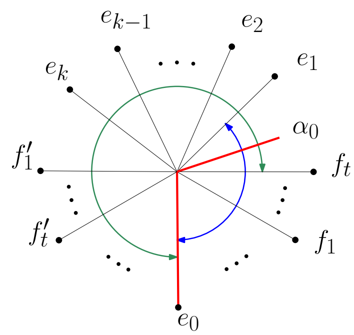

Figure 4: Substrings included in the disk bounded by C_0.

although it becomes one when we add α_i. The reason explaining this conversion is simple: in $G(\Sigma)$, C_i has exactly three vertices not reflecting, and one of them is a. After α_i is added, a is now reflecting in C_i (witnessed by σ).

Understanding how cycles with exactly three rainbows may behave in an obstruction-less set of strings is a crucial piece of the proof. In general, if v is a vertex in the underlying plane graph of a set of strings in general position, then a *near-obstruction at v* is a cycle with exactly three rainbows, and one of them is v. Each of the cycles C_0, C_1,...,C_k above is a near-obstruction at a in $G(\Sigma)$.

Both e_0 and α_0 are on the disk bounded by C_0, and since α_0 is not part of C_0, either $e_0, f_1, f_2, \dots, f_t, e_1$ are on the same side of C_0 (blue bidirectional arrow in Figure 4) or all of $f_t, e_1, \dots, e_k, f'_1, f'_2, \dots, f'_1, e_0$ are in the same side of C_0 (green bidirectional arrow in Figure 4). Because α_0 is the only edge between f_t and e_1, we see that e_1 belongs to the first sublist (the blue one) and f_t belongs to the second list (the green one). In the second case both f_t and f'_t are in the disk bounded by C_0, showing that a is not a rainbow for C_0 in Σ. Therefore, all of $e_0, f_1, f_2, \dots, f_t, e_1$ are in the disk bounded by C_0.

Regardless of the presence or absence of twins, we know that (e_0, e_1) occurs as a sublist of the rotation of a inside C_0. A symmetric argument shows that (e_k, e_0) occurs as a sublist of the rotation of a inside C_k.

Since (e_0, e_1) is a substring of the rotation at a inside C_0 and $(e_0, e_1, \dots, e_k, f'_1)$ is not inside C_k, there is a largest $i \in \{0, 1, \dots, k-1\}$ such that $(e_0, \dots, e_{i+1})$ is inside C_i. The choice of i implies $(e_{i+1}, \dots, e_k, e_0)$ is inside C_{i+1}.

The next lemma states that the existencce of such a pair of cycles C_i and C_{i+1} is impossible, completing the proof. □

Lemma 2.4. *Let Σ be a set of strings in general position. Suppose that C_1 and C_2 are cycles in $G(\Sigma)$ that are near-obstructions at v, so that the rotation at v inside C_1 includes (as a sublist) the rotation at v outside C_2, and that the rotation at v inside C_2 includes the rotation at v outside C_1. Then $G(\Sigma)$ has an obstruction.*

We defer the proof of Lemma 2.4 to Section 3 as it is technical and it deviates our attention from the proof of Theorem 1.1.

Face-Escaping Step. *Suppose that there is a string σ that has an end a with degree 1 in $G(\Sigma)$, and a is incident to an inner face F. Then there is an extension σ' of σ from its a-end to a point in the boundary of F such that the set $(\Sigma \setminus \{\sigma\}) \cup \{\sigma'\}$ has no obstruction.*

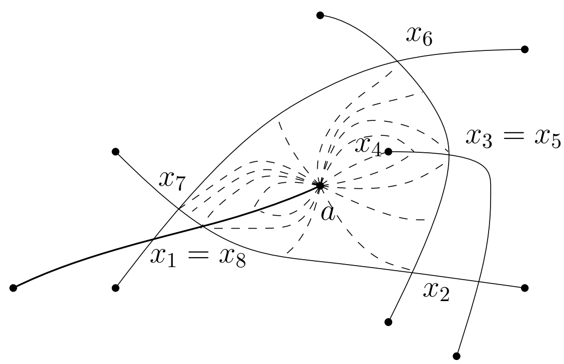

Figure 5: All possible extensions in the Face-Escaping Step.

Proof. Let W be the closed boundary walk $(x_0, e_1, \ldots, e_n, x_n)$ of F such that $x_0 = x_n = a$ and F is to the left as we traverse W. Let P denote the list of points $(m_1, x_1, m_2, x_2, \ldots, x_{n-1}, m_n)$. For each point p in P, let Σ_p be the set of strings obtained from Σ by extending the a-end of σ adding an arc α_p connecting a to p in F (see Figure 5).

Figure 6 shows the importance of considering extensions meeting points in the middle an edge in the boundary of F, as sometimes this is the only way for extending σ without creating an obstruction.

Let f_p be the edge $e_1 \cup \alpha_p$ in $G(\Sigma_p)$; it has ends x_1 and p. Also, let $\sigma^p = \sigma \cup \alpha_p$. The existence of obstructions in $G(\Sigma_p)$ is independent of how we draw α_p inside F. We will take advantage of this fact later on in the proof.

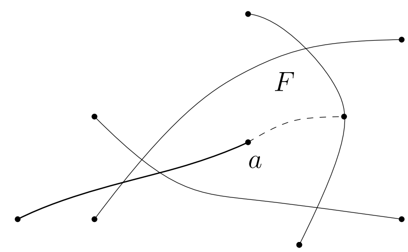

Figure 6: Face-Escaping Step.

Seeking a contradiction, suppose that each $G(\Sigma_p)$ has an obstruction. Our next claim gives two sufficient conditions on p that imply that all the obstructions in $G(\Sigma_p)$ contain f_p.

Claim 1. *Let $p \in P$ be either one of $m_1, \ldots, m_n$ or not in σ. Then every obstruction in $G(\Sigma_p)$ includes f_p.*

Proof. Let $p \in P$ be such that there is an obstruction C in $G(\Sigma_p)$ not including f_p.

First, we show that p is not a vertex in the middle of an edge in W. By contradiction, suppose that $p = m_i$ for some $i \in \{1, ..., n\}$. Since m_i is the only vertex whose rotation in $G(\Sigma)$ differs from its rotation in $G(\Sigma_{m_i})$, $m_i \in V(C)$. Consider the cycle C' of $G(\Sigma)$ obtained by replacing the subpath x_{i-1}, m_i, x_i of C by the edge $x_{i-1}x_i$. The inside rotation of each vertex in C' is the same as their rotation inside C. This shows that C' is an obstruction in $G(\Sigma)$, a contradiction.

Now suppose that p is not in the middle of an edge in W. Then C is a cycle in $G(\Sigma)$ and is not an obstruction in $G(\Sigma)$. The only vertex in $G(\Sigma_p)$ that has a rotation that is different from its rotation in $G(\Sigma)$ is p. Therefore p is a point in C that is reflecting inside C (witnessed by two edges included in σ^p), and is not reflecting in C with respect to $G(\Sigma)$. Exactly one of the two witnessing edges is in $G(\Sigma)$. So $p \in \sigma$. □

More can be said about the obstructions in $G(\Sigma_p)$ for each point in P, but for this we need some terminology. If we orient an edge e in a plane graph, then the *sides* of e are either the points near e that are to the right of e, or the points near e to the left of e. Our next lemma shows that if $p \in P$, then all the obstructions in $G(\Sigma_p)$ include the same side of f_p in its interior face. We defer its proof to Section 3 to keep the flow of the current proof. For the convenience of the reader, we provide all the hypotheses in the statement.

Lemma 2.5. *Let Σ be a set of strings in general position. Let C_1 and C_2 be obstructions in $G(\Sigma)$ with $e \in E(C_1) \cap E(C_2)$. If C_1 and C_2 include distinct sides of e in their interior faces, then $G(\Sigma)$ has an obstruction not including e.*

The condition on the two cycles C_1 and C_2 containing distinct sides of e implies that e is incident with only interior faces of $C_1 \cup C_2$. The perspective of the cycles being on distinct sides of e is useful in the application, but what we really use in the proof of Lemma 2.5 is that e is not incident with the outer face of $C_1 \cup C_2$.

For each point $p \in P$, we will consider an obstruction C_p containing f_p; the choice of C_p will be more specific when $p \in \sigma$ (see below). For $p \in P$, we orient f_p from x_1 to p, so that we keep track of the side of f_p contained in the interior of C_p.

Observe that C_{x_1} contains the right of f_{x_1} while $C_{x_{n-1}}$ contains the left of $f_{x_{n-1}}$ (here we use the fact that F is bounded). This implies the existence of two consecutive vertices x_{i-1}, x_i in $W - a$, such that the interior of $C_{x_{i-1}}$ includes the right of $f_{x_{i-1}}$ and the interior of C_{x_i} includes the left of f_{x_i}.

Without loss of generality, suppose that the interior of C_{m_i} includes the left of f_{m_i} (otherwise we reflect our drawing in a mirror). To make the notation simpler, we let $x = x_{i-1}$ and $m = m_i$. We may assume that f_m is drawn near the left of f_x.

The next claim is the last ingredient to obtain a final contradiction.

Claim 2. *Exactly one of the following holds:*

(a) $x \in \sigma$ *and* $G(\Sigma_m)$ *has an obstruction containing* f_m *whose interior includes a side that is distinct from the side included by* C_m*; or*

(b) $x \notin \sigma$ *and* $G(\Sigma_x)$ *has an obstruction containing* f_x *whose interior includes a side of* f_x *that is distinct from the side included by* C_x*.*

Proof. First, suppose that $x \in \sigma$. For (2.a) we have two cases depending on whether $x_{i-1}x_i$ is an edge in C_x.

Case a.1 *$x_{i-1}x_i$ is not in C_x.*

In this case we consider the cycle C'_m obtained by replacing in C_x the edge f_x by the path $P = (x_1, f_m, m, mx, x)$. Since $x \in \sigma$, by the choice of C_x, all the edges in C_x are in σ^x. Therefore all the edges in C'_m, with the possible exception of mx, are in σ^m. Thus C'_m is an obstruction in $G(\Sigma_m)$.

It remains to show that the interior of C'_m includes the right side of f_m. Note that $C_x \cup P$ consists of three internally disjoint x_1x-paths, and because some points in P are near the left side of f_x, P is in the outer face of C_x. The face of $f_x \cup P$ that is to the right of f_m is included in the inner face F, so it is bounded. This implies the interior face of C'_m includes the right of f_m. Since the interior of C_m includes the left of f_m, C'_m and C_m are obstructions including distinct sides of f_m.

Case a.2. *$x_{i-1}x_i$ is in C_x.*

In this case, (x_1, f_x, x, xx_i, x_i) is a subpath of C_x. We let C'_m be the cycle obtained by replacing this path by $P = (x_1, f_m, m, mx_i, x_i)$. Since $x \in \sigma$, the way we choose C_x implies that all the edges in C_x are in σ^x. So all the edges in C'_m are in σ^m, and C'_m is an obstruction. An argument similar to the one given in the previous case shows that the interior of C'_m includes the right side of f_m. Thus the interior of C_m and C'_m include distinct sides of f_m.

Turning to (2.b), let us suppose that $p \notin \sigma$. We split the proof into two cases depending on whether x is in C_m.

Case b.1. *x is in C_m.*

First, we redraw f_x and f_m inside F so that $f_x \cap f_m = \{x_1\}$. Let T be the triangle bounded by f_x, f_m and xm. The interior face of T is to the left of f_x and to the right of f_m. Consider the mx-path P of C_m that does not include the edge f_m. Since the interior face of T is a subset of F, P is drawn in the closure of the exterior of T (possibly $P = (m, mx, x)$).

Let C be the simple closed curve bounded by $P \cup f_x \cup f_m$. We claim that the interior of C is on the left of f_x. In the alternative, suppose that the interior of C is on the right of f_x. Then $C' = P + xm$ is a cycle of $G(\Sigma_m)$ including f_x and f_m in its interior. The xx_1-path P' of C_m that does not include m, is an arc connecting x_1 to x inside C'. Thus, $V(C') \subseteq V(C_m)$ and the closed disk bounded by C' includes C_m. These two observations together imply that C' has at most as many rainbows as C_m, and hence, C' is an obstruction of $G(\Sigma_m)$ not including f_m. Claim 1 asserts that all the obstructions in $G(\Sigma_m)$ include f_m, a contradiction. Thus the interior of C is on the left of f_m.

From our last observation, it follows that P' is an arc connecting x_1 and x in the exterior of C. Because the interior of $C_m = P' \cup f_m \cup P$ is on the left of f_m, the interior of the cycle $C'_x = P' + f_x$ is on the left of f_x.

Now we show that C'_x is an obstruction. Note that $V(C'_x) \subseteq V(C_m)$ and that the closed disk bounded by C'_x includes C_m. Then, every rainbow in C'_x is a rainbow in C_m, and hence C'_x is an obstruction. The cycles C_x and C'_x are obstructions including distinct sides of f_x in their interiors, as claimed.

Case b.2. *x is not in C_m.*

In this case we let C'_x be the cycle obtained by replacing the path (x_1, f_m, m, mx_i, x_i) in C_m by the path $P = (x_1, f_x, x, xx_i, x_i)$ in $G(\Sigma_x)$. Let α be the subarc of P joining x_1 to m. As the points of α near x_1 are drawn on the left of f_m, and α is internally disjoint to C_m, α connects x_1 and m in the exterior of C_m. Since the interior face of $\alpha \cup f_m$ is on the left of f_x, the interior face of C'_x is on the left of f_x.

To show that C'_x is an obstruction, note that the disk bounded by C'_x includes C_m and that $V(C'_x) \setminus \{x\} \subseteq V(C_m)$. Thus all the rainbows of C'_x in $V(C'_x) \setminus \{x\}$ are also rainbows in C_m. The rotation of x inside C'_x is the list (xx_i, f_x), and, because $x \notin \sigma$, x is a rainbow in C'_x, and is not a vertex of C_m. To compensate, we note that m is a rainbow in C_m that is not in $V(C_x)$: if m is not rainbow, both f_m and xx_i are included in σ, implying that $x \in \sigma$. This shows that C'_x has at most as many rainbows as C_m. Thus C'_x is an obstruction. Again, the interiors of C_x and C'_x include distinct sides of f_x. □

By Claim 2, for some $p \in \{x, m\}$, $G(\Sigma_p)$ has obstructions including both sides of f_p (and when $p = x$, we can guarantee that $p \notin \sigma$). Lemma 2.5 implies that $G(\Sigma_p)$ has an obstruction not including f_p. Since either $p \notin \sigma$ or $p = m$, this last statement contradicts Claim 1. □

Exterior-Meeting Step. *Suppose that all the strings in Σ have their ends on the outer face of $G(\Sigma)$ and that all the ends have degree 1 in $G(\Sigma)$. Then either all the strings are pairwise intersecting, and then Σ can be extended to an arrangement of pseudolines, or we can extend two disjoint strings so that these strings intersect without creating an obstruction.*

Proof. We start by considering a simple closed curve $\mathcal{O}$ containing all the ends of the strings in Σ, and that is otherwise disjoint from $\bigcup \Sigma$. We construct this curve by connecting each pair of vertices with degree 1 that are consecutive in the boundary walk of the outer face. To connect these pairs we use an arc whose interior is included in the outer face, near the portion of the boundary walk between the two vertices.

Suppose σ_1, σ_2 are two disjoint strings in Σ. For $i = 1, 2$, let a_i, b_i be the ends of σ_i. Since σ_1 and σ_2 do not intersect inside $\mathcal{O}$, their ends do not alternate as we traverse $\mathcal{O}$ in counterclockwise order. We may assume, by relabeling if necessary, that the ends occur in the order a_1, b_1, b_2, a_2.

We extend the a_i-ends of σ_1 and σ_2 so that they meet in a point p in the outer face. We do this extension so that the two added segments are in the outer face, and, more importantly, so that the interior face of the simple closed curve bounded by the added segments and the a_2a_1-arc in $\mathcal{O}$ not containing $\{b_1, b_2\}$, does not include the inner face of $\mathcal{O}$. In Figure 7 we show the right and wrong way to extend, respectively.

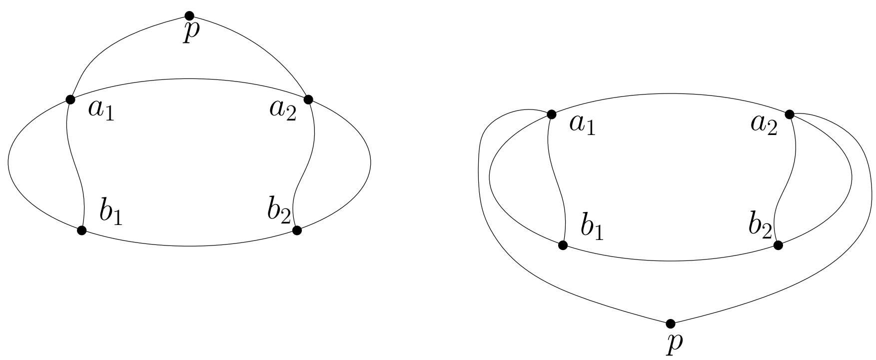

Figure 7: The right and wrong way to extend in the Exterior-Meeting Step.

We denote the new set of strings obtained as above by Σ'. To show that Σ' has no obstruction, we consider a cycle C in $G(\Sigma')$. If C does not contain p, then C is a cycle in $G(\Sigma)$, and so is not an obstruction in $G(\Sigma')$. Now suppose that p is in C.

The idea is to find three rainbows in C. To get the first one, we consider the path P_1 obtained by traversing C, starting at p, continuing along the path induced by σ_1, and stopping just before we reach a first vertex not in σ_1. Let c_1 be the last vertex in P_1, and let d_1 be the neighbour of c_1 in C that is not in P_1.

Claim 1. *The cycle C has a rainbow included in the disk Δ_1 bounded by σ_1 and the a_1b_1-arc of $\mathcal{O}$ not containing a_2.*

Proof. The vertex d_1 is in one of the two bounded faces of $\mathcal{O} \cup \sigma_1$. Suppose that d_1 is in the face F that is bounded by σ_1 and the a_1b_1-arc of $\mathcal{O}$ containing a_2 and b_2. The rotation at c_1 inside C does not include two edges in the same string σ, as otherwise σ and σ_1 tangentially intersect at c_1. Therefore, when $d_1 \in F$, c_1 is a rainbow of C in Δ_1.

Now suppose that d_1 is in Δ_1. Let P'_1 be the path of C starting at c_1 and the edge $c_1 d_1$, and ending at the first vertex we encounter that is in σ_1. The cycle C' enclosed by P'_1 and σ_1 is not an obstruction, so it has at least three rainbows. The vertices in $C' - V(P'_1)$ are reflecting inside C' because their rotations inside C' contain two edges in σ. Hence at least one internal vertex of P'_1 is a rainbow in C'. This vertex is also a rainbow in C, and is included in Δ_1. □

Considering σ_2 instead of σ_1, Claim 1 yields a second rainbow in C inside an analogous disk Δ_2. The third rainbow is p, showing that C is not an obstruction. □

Since the Disentangling Step, Face-Escaping Step and Exterior-Meeting Step can be performed without creating new obstructions, either: one of these steps can be performed to increase the number of pairwise intersecting strings in Σ; or the strings in Σ are pairwise intersecting and all of them have their ends in the outer face, which implies that Σ can be extended to an arrangement of pseudolines. □

3 Proof of Lemmas 2.4 and 2.5

We deferred the proofs of Lemmas 2.4 and 2.5, both essential in the proof of Theorem 1.1, to this section.

Our next observation follows immediately from the definition of rainbow, and it will be repeatedly used in the next proofs.

Useful Fact. *Let Σ be set of strings in general position. Let v be a vertex that is in both the cycles C and C' of $G(\Sigma)$ such that the rotation at v inside C includes the rotation at v inside C'. If v is a rainbow in C, then v is a rainbow in C'.*

Recall that a *near-obstruction* at v is a cycle C (in the underlying graph of a set of strings) that has precisely three rainbows, one of which is v. In Figure 8, we depict (up to symmetries) how two near-obstructions may intersect at v. In each of the nine diagrams, v is represented as a black dot, while the interiors of the near-obstructions are represented as dotted and dashed lines. In our next lemma, we will consider two near-obstructions at v that intersect only as in the last three diagrams, where every small open disk centered at v is included in the union of the disks bounded by the two near-obstructions. In the statement, an equivalent description is given in terms of the local rotation at v.

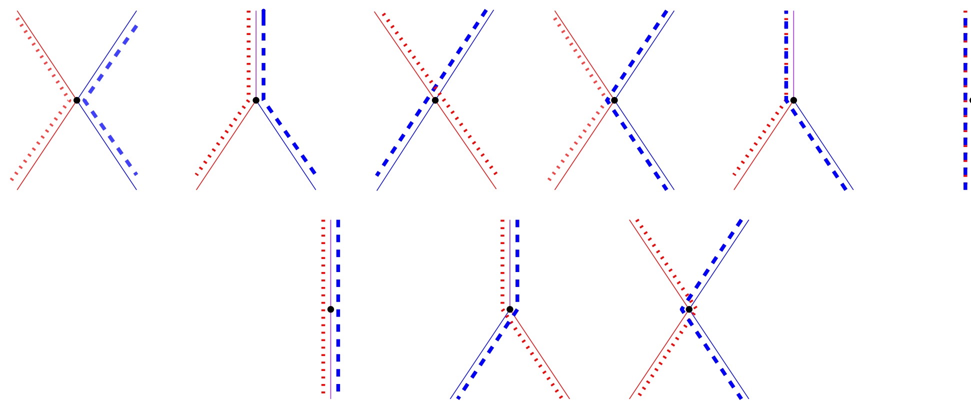

Figure 8: Two near obstuctions at v.

Lemma 2.4. *Let Σ be a set of strings in general position. Suppose that C_1 and C_2 are cycles in $G(\Sigma)$ that are near-obstructions at v, so that the rotation at v inside C_1 includes (as a sublist) the rotation at v outside C_2, and that the rotation at v inside C_2 includes the rotation at v outside C_1. Then $G(\Sigma)$ has an obstruction.*

Proof. In order to obtain a contradiction, suppose that $G(\Sigma)$ has no obstructions and that it contains such cycles C_1, C_2. The conditions on the rotation at v imply that every edge incident with v is in the interior of either C_1 or C_2. Thus, v is not incident with the outer face of $C_1 \cup C_2$.

Our next goal is to show that $C_1 \cap C_2$ has at least two vertices. If e is an edge of C_1 incident with v, then either e is an edge of C_2 or e is inside C_2. In the former case $|V(C_1) \cap V(C_2)| \geq 2$, thus we may assume that both edges of C_1 incident with v are inside C_2. If v is the only vertex in $V(C_1) \cap V(C_2)$, then $C_1 - v$ is in the interior of C_2, and hence the edges of C_2 incident with v are not in the rotation at v inside C_1, a contradiction. Thus $|V(C_1) \cap V(C_2)| \geq 2$. It follows that $C_1 \cup C_2$ is 2-connected; in particular, its outer face is bounded by a cycle C_{out}.

The Useful Fact applied to $C = C_{out}$ and to each $C' \in \{C_1, C_2\}$, shows that every vertex that is a rainbow in C_{out} is also a rainbow in each of the cycles in $\{C_1, C_2\}$ containing it. By assumption, C_{out} is not an obstruction, so it has at least three rainbows. The preceding two sentences imply that we may choose the labelling such that two of them, say p and q, are also rainbows in C_1. Neither p nor q is v and C_1 is a near-obstruction. Thus, p and q are the only rainbows of C_{out} that are in C_1.

Since $v \notin V(C_{out})$, C_1 has a subpath P_v containing v in which only the ends of P_v are in C_{out}. Since v is not in the outer face of C_{out}, P_v is included in the inner face of C_{out}. We let u and w be the ends of P_v, and let Q^1_{out}, Q^2_{out} be the uw-paths of C_{out}. The cycle C_1 is inside one of the two disks bounded by P_v and one of Q^1_{out} and Q^2_{out}. By symmetry, we may assume that C_1 is included in the disk bounded by $Q^1_{out} \cup P_v$. In this case Q^2_{out} is a subpath of C_2.

Our desired contradiction will be obtained by finding three rainbows in C_2 distinct from v. The first is relatively easy to find: if $C_1 - (P_v)$ is the uw path in C_1 distinct from P_v, we consider the cycle $(C_1 - (P_v)) \cup Q^2_{out}$. The disk bounded by $(C_1 - (P_v)) \cup Q^2_{out}$ contains the one bounded by C_1. Then the Useful Fact applied to $C = (C_1 - (P_v)) \cup Q^2_{out}$ and $C' = C_1$, implies that each vertex in $C_1 - (P_v)$ that is rainbow in $(C_1 - (P_v)) \cup Q^2_{out}$ is also rainbow in C_1. Since C_1 has at most two rainbows in $C_1 - (P_v)$, namely p and q, $(C_1 - (P_v)) \cup Q^2_{out}$ must have a third rainbow r_1 in the interior of Q^2_{out}. The interiors of the disks bounded by C_2 and $(C_1 - (P_v)) \cup Q^2_{out}$ are on the same side of Q^2_{out}; thus r_1 is a rainbow for C_2.

To find another rainbow in C_2, consider the edge e_u of C_2 incident to u and not in Q^2_{out}. We claim that either u is a rainbow in C_2 or that e_u is not included in the closed disk bounded by $P_v \cup Q^2_{out}$. Looking for a contradiction, suppose that u is reflecting in C_2 and that e_u is included in the disk. Then we can find two edges in the rotation at u, included in the disk bounded by $P_v \cup Q^2_{out}$, that belong to the same string σ. The vertex u is not reflecting in C_1, as else, we would find another pair of edges in the rotation at u inside $Q^1_{out} \cup P_v$, and included in a different string σ'; in this case, σ and σ' tangentially intersect at u, a contradiction. Therefore u is a rainbow in C_1, so u is one of p and q. This implies that u is a rainbow in C_{out}, and hence, a rainbow in C_2, a contradiction.

If u is a rainbow in C_2, then this is the desired second one. Otherwise, the preceding paragraph shows that e_u is not in the closed disk bounded by $P_v \cup Q^2_{out}$. In this latter case, e_u is in a path P_u that starts at u, ends at u' in P_v and is otherwise disjoint from P_v.

Note that $u' \neq w$, as otherwise $C_2 = P_u \cup Q^2_{out}$ and we have the contradiction that v is not in C_2. Let C_u be the cycle consisting of P_u and the uu'-subpath uP_vu' of P_v.

Claim 1. *If P_u does not have a rainbow of C_u in its interior, then:*

(a) C_u and C_2 are near-obstructions at v satisfying the conditions in Lemma 2.4; and

(b) the closed disk bounded by the outer cycle of $C_u \cup C_2$ contains fewer vertices than the disk bounded by C_{out}.

Proof. Suppose that all the rainbows of C_u are located in uP_vu'. Since C_u is not an obstruction, at least one of them is an interior vertex of uP_vu'. Each vertex in the interior of uP_vu' that is a rainbow in C_u, is also a rainbow in C_1. As v is the only vertex in the interior of P_v that is a rainbow in C_1, v is the only rainbow of C_u that is in the interior of uP_vu'. Since C_u is not an obstruction, u, u' and v are the only rainbows of C_u, and C_u is a near-obstruction at v. The rotation at v inside C_u is the same as inside C_1, so C_u and C_2 satisfy the conditions in Lemma 2.4.

Let C'_{out} be the outer cycle of $C_u \cup C_2$. Since $C_u \cup C_2 \subseteq C_1 \cup C_2$, the exterior of C_{out} is included in the exterior of C'_{out}. This shows that the disk bounded by C_{out} includes the disk bounded by C'_{out}.

If both p and q are in C_2, then p, q and r_1 are rainbows in C_2, and also distinct from v, contradicting that C_2 is a near-obstruction for v. Thus, we may assume $p \notin C_2$. Then p is not in $P_u \subseteq C_2$ and, since p is not an interior vertex of P_v, $p \notin V(C_u)$. Since p is in C_{out}, and p is not in $C_u \cup C_2$, p is in the outer face of $C_u \cup C_2$. Then p is in the disk bounded by C_{out} but not by C'_{out}, as required. □

The proof of the existence of the additional two rainbows in C_2 is by induction on the number of vertices in the closed disk bounded by C_{out}. If u is not a rainbow in C_2 and P_u does not have a rainbow of C_2 in its interior, then Claim 1 implies C_u and C_2 make a smaller instance and we are done. Thus, we may assume one of them yields the next additional rainbow.

In the same way, either the induction applies or the last rainbow comes by considering the edge of $C_2 - Q^2_{out}$ incident with w. It follows that v, r_1, and these two other vertices are four different rainbows in C_2, contradicting the fact that C_2 is a near-obstruction. □

Although the statements and proofs of Lemmas 2.4 and 2.5 are similar, some subtle differences make it hard to find a statement encapsulating both results. For instance, Lemma 2.5 assumes that $G(\Sigma)$ has obstructions, while finding an obstruction is the conclusion of Lemma 2.4. We sketch the proof of Lemma 2.5, emphasizing such differences. It would be interesting to find a common theory behind these two lemmas.

Lemma 2.5. *Let Σ be a set of strings in general position. Let C_1 and C_2 be obstructions in $G(\Sigma)$ with $e \in E(C_1) \cap E(C_2)$. If C_1 and C_2 include distinct sides of e in their interior faces, then $G(\Sigma)$ has an obstruction not including e.*

Sketch of the proof. We start assuming that such cycles exist and that every obstruction includes e.

By assumption, $C_1 \cap C_2$ has at least two vertices and, therefore, $C_1 \cup C_2$ is 2-connected. Thus, its outer face is bounded by a cycle C_{out}.

The Useful Fact shows that every rainbow in C_{out} is a rainbow in each of the cycles C_1 and C_2 containing it.

Since C_1 and C_2 include different sides of e, it follows that e is not in C_{out}. Therefore C_{out} is not an obstruction. Thus, C_{out} has at least three rainbows, and by our previous observation, we may choose the labelling such that two of them, say p and q, are also rainbows in C_1. Because C_1 is an obstruction, p and q are the only rainbows in C_1.

Then C_1 has a subpath P_e of containing e and in which only the ends u and w of P_e are in C_{out}. Let Q^1_{out} and Q^2_{out} be the uw-paths of C_{out}. We may assume that C_1 is drawn in the disk bounded by $Q^1_{out} \cup P_e$.

Let $C_1 - (P_e)$ be uw-path in C_1 that is not P_e. Note that p and q are the only vertices in $C_1 - (P_e)$ that are rainbows in $(C_1 - (P_e)) \cup Q^2_{out}$. Since $(C_1 - (P_e)) \cup Q^2_{out}$ is not an obstruction, the interior Q^2_{out} has a vertex r_1 that is a rainbow of $(C_1 - (P_e)) \cup Q^2_{out}$. This vertex r_1 is also a rainbow of C_2.

Let e_u be the edge incident to u in C_2 that is not in Q^2_{out}. As we did in Lemma 2.4, we can show that either u is a rainbow in C_2 or that e_u is not included in the disk bounded by $P_e \cup Q^2_{out}$. We assume the latter situation, as in the former we found our desired second rainbow in C_2.

Let P_u be the subpath of C_2 starting at u, continuing on e_u, and ending on the first vertex $u' \in V(P_e) \cap V(C_2)$ distinct from u. Note that $u' \neq w$, as otherwise $C_2 = P_u \cup Q^2_{out}$ and we have the contradiction that e is not in C_2. Let C_u be the cycle consisting of P_u and the uu'-subpath uP_eu' of P_e.

We claim that either P_u has an interior vertex that is a rainbow in C_2 or that there is a pair of cycles C_1' and C_2' satisfying the conditions in Lemma 2.5, but with fewer vertices in the closed disk bounded by the outer cycle of $C_1' \cup C_2'$ than in the disk bounded by C_{out}.

Suppose that none of the interior vertices in P_u is a rainbow in C_2. Because the interior of P_e has no vertices that are rainbows in C_1 (as p and q are the only rainbows of C_1), the interior of uP_eu' has no vertices that are rainbows in C_u. Therefore C_u is an obstruction, and $C_1' = C_u$ and $C_2' = C_2$ is a pair of obstructions including both sides of e. As $C_u \cup C_2 \subseteq C_1 \cup C_2$, the closed disk bounded by the C_{out} contains the closed disk bounded by the outer cycle of $C_u \cup C_2$. Not both of p and q are in the outer cycle of $C_u \cup C_2$, as both p and q would be part of C_2, concluding that C_2 has three rainbows p, q and r_1, and contradicting that C_2 is an obstruction.

From the previous paragraph, either C_u and C_2 is a smaller instance, and we are done by induction on the number of vertices in the closed disk bounded by C_{out}, or we found our second rainbow of C_2 in the interior of P_u.

In the same way, either the induction applies or the last rainbow comes by considering an edge of $C_2 - Q^2_{out}$ incident with w. It follows that r_1, and these other two vertices are three different rainbows in C_2, contradicting that C_2 is an obstruction. □

4 Finding obstructions in polynomial time

In this section we describe a polynomial-time algorithm that determines whether a set of strings has an obstruction. We will assume that our input is the underlying plane graph $G(\Sigma)$ of a set Σ of simple strings in general position, and that every string in Σ is identified as a path in $G(\Sigma)$ (see notation below).

The key idea behind the algorithm is simple: either find an obstruction in the outer boundary of $G(\Sigma)$ or find a vertex in the outer boundary whose removal reduces our problem into a smaller instance.

We start by describing the vertex removal operation. Suppose that x is a vertex of $G(\Sigma)$ incident to the outer face of $G(\Sigma)$. For each $\sigma \in \Sigma$, we consider the path P_σ of $G(\Sigma)$ representing σ. Let $P_\sigma - x$ be the plane graph obtained from P_σ by removing x and the edges of P_σ incident to x (if $x \notin P_\sigma$, then $P_\sigma - x = P_\sigma$). Each component of $P_\sigma - x$ is either a vertex that represents an end of σ, or a string. Let $S_{\sigma,x}$ be the set of string components of

$P_\sigma - x$ and let $\Sigma - x = \bigcup_{\sigma\in\Sigma} S_{\sigma,x}$. Note that $G(\Sigma - x)$ can be obtained from $G(\Sigma)$ by removing x and the edges incident to x, and then suppressing the degree-2 vertices whose incident edges belong to the same string in Σ, as well as removing remaining degree-0 vertices (Figure 9 illustrates this process).

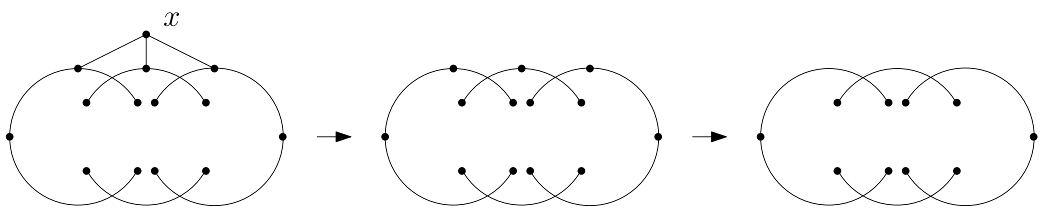

Figure 9: From Σ to $\Sigma - x$.

The next lemma is the key property used in the algorithm.

Lemma 4.1. *Let Σ be a set of simple strings in general position and let x be a vertex incident with the outer face. Then there is a $1-1$ correspondence between the obstructions in $G(\Sigma)$ not containing x and the obstructions of $G(\Sigma - x)$. Moreover, corresponding obstructions are the same simple closed curve.*

Proof. In general, there is a natural correspondence between cycles in $G(\Sigma)$ not containing x and cycles in $G(\Sigma - x)$: if C is a cycle in $G(\Sigma)$ not containing x, then every edge of C is not incident with x, and hence every edge is part of a string in $\Sigma - x$. Thus, there is a cycle C' in $G(\Sigma - x)$ that represents the same simple closed curve as C. Conversely, each cycle C' in $G(\Sigma - x)$ is a simple closed curve in $\bigcup(\Sigma - x) \subseteq \bigcup \Sigma$, and hence, there is a cycle C in $G(\Sigma)$ representing the same simple closed curve as C'.

To complete the proof it is enough to show that any two cycles C, C' that correspond as above have the same rainbows. Since $G(\Sigma - x)$ is obtained from suppressing and removing vertices in a subgraph of $G(\Sigma)$, $V(C') \subseteq V(C)$. Thus, $V(C) \setminus V(C')$ consists of suppressed and removed vertices in the process of converting $G(\Sigma)$ into $G(\Sigma - x)$. Since $x \notin V(C)$, if $v \in V(C)$ is suppressed, then the two edges of C incident to v belong to the same string in Σ. Therefore, none of the vertices in $V(C) \setminus V(C')$ is a rainbow in C.

Every rainbow in C is also a rainbow in C' because every two edges of $G(\Sigma - x)$ that are included in distinct strings of Σ are also included in distinct strings in $\Sigma - x$.

Conversely, suppose that $v \in V(C) \cap V(C')$ is reflecting in C. Let $\sigma \in \Sigma$ be a string including two edges of $G(\Sigma)$ in the rotation at v inside C. Since x is drawn in the exterior of C, these two edges are part of the same string in $\Sigma - x$, and hence v is reflecting inside C'.

Therefore every rainbow of C' is a rainbow of C, and thus, C and C' have the same rainbows. □

A vertex in $G(\Sigma)$ is an *outer-rainbow* if it is in the outer boundary and all the edges in its rotation belong to different strings. Note that every outer-rainbow is a rainbow for all the cycles in $G(\Sigma)$ that contain it.

An *outer cycle* is a cycle of $G(\Sigma)$ that has all its edges incident to the outer face of $G(\Sigma)$. For any graph $G(\Sigma)$, a *block* of $G(\Sigma)$ is a maximal connected subgraph of $G(\Sigma)$ with no cut-vertex. If $G(\Sigma)$ is connected with at least two vertices, then each block is either an edge or is 2-connected. In the latter case, the outer face of the block is bounded by a cycle of the block.

We find obstructions by solving an auxiliary problem: finding obstructions including one or two fixed outer-rainbows. The next subroutine (Algorithm 1) describes how to find an obstruction containing two fixed outer-rainbows. Below we discuss its correctness.

Algorithm 1: Finding obstructions through two fixed outer-rainbows.

Data: $G(\Sigma)$ and two outer-rainbows x and y.
Result: Either an obstruction containing x and y or that no such obstruction exists.

1 **repeat**
2   **if** *there is no cycle containing* x *and* y **then**
3   **return** $G(\Sigma)$ has no obstruction containing x and y;
4   **end**
5   Find the outer cycle C containing x and y;
6   **while** C *is not the outer boundary of* $G(\Sigma)$ **do**
7   Pick $w \in V(G(\Sigma)) \setminus V(C)$ incident with the outer face;
8         $\Sigma \longleftarrow \Sigma - w$
9   **end**
10   **if** C *has a rainbow* $z \notin \{x, y\}$ *in* $G(\Sigma)$ **then**
11         $\Sigma \longleftarrow \Sigma - z$;
12   **else**
13   **return** C;
14   **end**
15 **until** $V(G(\Sigma)) = \{x, y\}$;
16 **return** $G(\Sigma)$ has no obstruction containing x and y.

To see that Algorithm 1 is correct, observe that when Step 2 does not apply, then Step 5 can be performed: if there is a cycle containing x and y, then, as x and y are incident to the outer face of $G(\Sigma)$, the outer boundary of the block containing x and y is an outer cycle C containing x and y. Every obstruction $\mathcal{C}$ through x and y is drawn in the closed disk bounded by C. Lemma 4.1 guarantees that if we remove a vertex in the outer boundary that is not in C (Step 7) and we update Σ (Step 8), then $\mathcal{C}$ (or more precisely, the cycle in the new $G(\Sigma)$ that is the same simple closed curve as $\mathcal{C}$) is an obstruction through x and y.

In Step 10, if x and y are the only rainbows of C, then C is an obstruction returned in Step 13. Else, C has a rainbow $z \notin \{x, y\}$. Any obstruction $\mathcal{C}$ through x and y does not contain z, and hence removing z and updating Σ (Step 11) does not change the fact that $\mathcal{C}$ is an obstruction in the new $G(\Sigma)$. This algorithm terminates as the number of vertices in $G(\Sigma)$ is always decreasing.

We now turn to Algorithm 2, used as subroutine in the main algorithm.

Its correctness again easily follows from Lemma 4.1.

Algorithm 2: Finding obstructions through a fixed outer-rainbow.

Data: $G(\Sigma)$ and an outer-rainbow vertex x.

Result: Either an obstruction containing x or that no such obstruction exists.

```
repeat
    if there is no cycle containing x then
        return G(Σ) has no obstruction containing x;
    end
    Find an outer cycle C containing x;
    if C has an outer-rainbow y ≠ x then
        Run Algorithm 1 on (G(Σ), x, y);
        if G(Σ) has an obstruction D including x and y, then
            return D;
        end
        Σ ⟵ Σ − y;
    else
        return C;
    end
until V(G(Σ)) = {x};
return G(Σ) has no obstruction containing x.
```

Finally we present the algorithm to find obstructions, whose correctness also relies on Lemma 4.1.

Algorithm 3: Finding obstructions.

Data: $G(\Sigma)$.
Result: Either finds an obstruction or that no such obstruction exists.

repeat
  if $G(\Sigma)$ *has no cycles* **then**
    return $G(\Sigma)$ has no obstructions;
  end
  Find an outer cycle C;
  if C *has no rainbows* **then**
    return C;
  end
  Pick a rainbow x in C (x is outer-rainbow in $G(\Sigma)$);
  Run Algorithm 2 on $(G(\Sigma), x)$;
  if $G(\Sigma)$ *has an obstruction* D *including* x **then**
    return D;
  end
  $\Sigma \longleftarrow \Sigma - x$;
until $G(\Sigma) = \emptyset$;
return $G(\Sigma)$ *has no obstructions.*

5 Pseudolinear drawings of K_n

In this section we present a simple proof of a characterization of pseudolinear drawings of complete graphs (Theorem 5.1), equivalent to the ones given in [2] and [3].

Theorem 5.1. *A good drawing of a complete graph is pseudolinear if and only if it does not include the B configuration (see Figure 2).*

Proof. The unique cycle in a B configuration is an obstruction, so, by Theorem 1.1, no pseudolinear drawing of K_n can include it. Conversely, suppose that D is a good drawing of K_n that is not pseudolinear. Let $\Sigma = \{D[e] : e \in E(K_n)\}$ be the set of edge-arcs, and let $G(\Sigma)$ be its underlying plane graph. In order to avoid confusion between vertices and edges of K_n and $G(\Sigma)$, vertices in $G(\Sigma)$ are called *points*, and edges of $G(\Sigma)$ are *segments*. Because D is good, each point is either in $V(K_n)$ or a crossing.

For every cycle C in $G(\Sigma)$, we let $\delta(C)$ be the set of points in C for which their two incident segments in D belong to distinct edges in Σ. Theorem 1.1 implies that $G(\Sigma)$ has an obstruction C. We choose our obstruction C so that $|\delta(C)|$ is as small as possible.

Since D is good, $|\delta(C)| \geq 3$ and, because C is an obstruction, at most two vertices in $\delta(C)$ are rainbows in C. Consider a point $x \in \delta(C)$ that is reflecting inside C. Note that x is a crossing. Let σ_1 and σ_2 be the two edge-arcs in Σ crossed at x. We traverse σ_1, starting at x, continuing on the segment of σ_1 included in the interior of C, until an end $a_1 \in V(K_n)$ of σ_1 is reached. Likewise we define a_2 for σ_2. Henceforth, we refer to a_1 and a_2 as the *internal vertices* corresponding to the crossing x. The following claim explains why we call them "internal".

Claim 1. *Let $x \in \delta(C)$ be a point reflecting inside C. Then the two internal vertices corresponding to x are in the interior of C.*

Proof. Let a_1 be an internal vertex corresponding to x, and suppose σ_1 is the edge-arc including both x and a_1. Let σ_1' be the substring of σ, having x and a_1 as endpoints. Applying Observation 2.1 to our obstruction C, with $\sigma = \sigma_1$ and $\sigma' = \sigma_1'$, we obtain that $\sigma_1' \cap C = \{x\}$. Since points of σ_1' near x are in the interior face of C, $\sigma_1' \setminus \{x\}$ is included in the interior of C. In particular, a_1 is in such a face. □

Now we look at the points in $\delta(C)$ that are not reflecting inside C. If x is one of them, then x is a vertex or a crossing. Suppose that x is a crossing. Let σ_1, σ_2 be the edge-arcs crossing at x. Because x is not reflecting inside C, one of the two segments at x included in σ_1 is in the outer face of C. We traverse σ_1, starting in x, and continuing in the outer face until we reach an end b_1 of σ_1. Likewise we define b_2 for σ_2. These vertices b_1, b_2 are the *external* vertices corresponding to the crossing x.

Claim 2. *Let x be a crossing in $\delta(C)$ that is not reflecting inside C, and let σ be an edge-arc including x and an external vertex b of x. If σ' is the substring of σ connecting x to b, then $\sigma' \setminus \{x\}$ is included in the outer face of C.*

Proof. Applying Observation 2.1 to C, σ_1, and σ', we see that $\sigma' \cap C = \{x\}$. Since the points of σ_1' near x are in the outer face of C, $\sigma_1' \setminus \{x\}$ is included in the outer face of C. □

It is convenient, in the case when x is a vertex of K_n, to let x be its own external vertex.

Henceforth we refer to the vertices of K_n that are internal to some crossing in C as the *internal vertices* of C, and likewise, the *external vertices* of C are the vertices of K_n that are external to some crossing or to a vertex in C.

Claim 3. *Every segment in C is included in an edge-arc whose ends are either internal or external vertices of C.*

Proof. Any segment s of C is contained in a subpath P of C whose ends are in $\delta(C)$ but is otherwise disjoint from $\delta(C)$. This path P is part of an edge-arc $\sigma \in \Sigma$. Let $a \in V(K_n)$ be one of the ends of σ, and suppose that x is the first end of P that we encounter when we traverse σ from a to the other end of σ. If σ is reflecting at x, then a is internal. If σ is not reflecting at x, then a is external. Likewise, the other end of σ is internal or external. □

Suppose that K_n has a vertex y that is neither external nor internal to C. Then, by our previous claim, the underlying plane graph of $D[K_n - y]$ contains a cycle whose drawing is $D[C]$ and is an obstruction. Thus, $D[K_n - y]$ is not pseudolinear, and applying induction on n, we obtain that $D[K_n - y]$ has a B configuration. Henceforth we assume that all the vertices of K_n are either internal or external to C.

Claim 4. *Either the outer face of D is bounded by a cycle of K_n or D has a B configuration.*

Proof. Suppose that the outer face of D is not bounded by a cycle of K_n. Then the outer face is incident to a crossing $\times$ between two edge-arcs σ_1 and σ_2. Let K be the crossing K_4 induced by the ends of σ_1 and σ_2. The drawing $D[K]$ has exactly five faces, four of them incident to $\times$. Exactly one of the faces incident to $\times$ includes the outer face of D. Such a face of $D[K]$ is bounded by portions of σ_1, σ_2, and an edge e of K_n connecting an end of σ_1 to an end of σ_2. The drawing induced by σ_1, σ_2 and $D[e]$ is a B configuration. □

Claims 1 and 4 imply that the outer cycle of D consists of only external vertices of C. Every external vertex either is associated with a crossing that is not reflecting inside C, or is itself a vertex of K_n in C. Because C has at most two points not reflecting inside C, and each of them has at most two external vertices, there are at most four points in the outer cycle of D. Thus the outercycle is a 3- or 4-cycle of K_n.

As C has at least three external vertices (in the outer cycle), $\delta(C)$ has precisely two points p and q not reflecting inside C. The outer cycle of D has an edge uv, where u is external to p and v is external to q (possibly $u = p$ or $q = v$).

Consider the pq-path P in $G(\Sigma)$, starting at p, continuing on the edge-arc connecting p to u, then following the edge uv until we reach v, and ending by following the edge-arc connecting v to q. We finish our proof by considering two cases, depending on whether uv is a segment of C.

Case. *uv is not a segment of C.*

In this case, there exists a point $w \in D[uv] \setminus D[C]$. As $D[uv]$ is part of the outer cycle, it contains neither crossings nor vertices in its interior, so the arcs in $D[uv]$ connecting w to the ends u and v are internally disjoint from C. From Claim 2, it follows that the pu- and the qv-subpaths of P are internally disjoint from C. Thus P is an arc connecting p and q in the outer face of C.

Consider the cycle C' obtained from the union of P and the pq-path of C that lies in the outer face of $D[C \cup P]$.

We will show that C' is an obstruction by showing that u and v are the only rainbows of C'. If $p \neq u$, then the edge-arc σ connecting p and u shows that every point in $(P - u) \cap \sigma$ is reflecting inside C'. Analogously, if $q \neq v$, the points distinct from v in the edge-arc connecting q and v, are reflecting inside C'. Thus the internal points in P, with the exception of u and v, are not reflecting. The same holds for the points in $C' - P$, as these points are not reflecting inside C (recall that p and q are the only rainbows of C). Thus u and v are the only rainbows of C'.

Note that all the segments of C' are included in edges whose ends are u, v or interior points of C. So if y is a vertex in the outercycle of D distinct from u and v, $D[K_n - y]$ also includes $D[C']$ as an obstruction, implying that $D[K_n - y]$ is not pseudolinear. Again, by induction on n, we obtain that $K_n - y$ has a B configuration.

Case. *uv is a segment of C.*

In this case, as u, v are vertices of K_n in C, they are rainbows of C. Since p and q are the only rainbows, $p = u$, $q = v$, and $D[uv]$ is a segment of C. Then, all the segments of C are included in edge-arcs whose ends are u, v or interior points of C. Again, remove a vertex in the outer cycle of D distinct from u and v to obtain a non-pseudolinear drawing of K_{n-1} in which, by induction, we find a B configuration. □

6 Concluding remarks

In our initial attempts to formulate Theorem 1.1, we intended to characterize non-pseudolinear good drawings of graphs by means of having at least one of the configurations in Figure 1 as a subdrawing. We obtain this as an easy consequence of Theorem 1.1. We sketch its proof.

Theorem 1.2. *Let D be a non-pseudolinear good drawing of a graph H. Then there is a subset S of edge-arcs in $\{D[e] : e \in E(H)\}$, such that each $\sigma \in S$ has a substring $\sigma' \subseteq \sigma$ for which $\bigcup_{\sigma \in S} \sigma'$ is one of the drawings in Figure 1.*

Proof. Take C an obstruction of the underlying plane graph associated to D. We choose C so that $|\delta(C)|$ is as small as possible. Decompose C into a cyclic sequence of paths $P_0, \dots, P_m$, where P_i connects two points in $\delta(C)$ and it is otherwise disjoint from $\delta(C)$. By using Observation 2.1, one can show that $P_0, \dots, P_m$ belong to distinct edge-arcs $\sigma_0, \dots, \sigma_m$, respectively. For each P_i, we consider the string σ'_i, obtained by slightly extending the ends of P_i that are reflecting in C; we extend them along σ_i.

Let $x \in \delta(C)$ be an end shared by P_{i-1} and P_i. If x is reflecting in C, then x is a crossing between σ_{i-1} and σ_i. Moreover, the arcs added to P_{i-1} and P_i at x to obtain σ'_{i-1} and σ'_i are in the interior of C. If x is a rainbow in C, then P_i and P_{i-1} are not extended at x, and x acts as one of the black dots in Figure 1. The rest of the points in $\delta(C)$ are crossings in $\bigcup_{i=0}^{m} \sigma'_i$ facing the interior of C. Since C has at most two rainbows, $\bigcup_{i=0}^{m} \sigma'_i$ is one in Figure 1. □

There are pseudolinear drawings that are not stretchable. For instance, consider the Non-Pappus configuration in Figure 10. Nevertheless, as an immediate consequence of Thomassen's main result in [15], pseudolinear and stretchable drawings are equivalent, under the assumption that every edge is crossed at most once.

Corollary 6.1. *A drawing of a graph in which every edge is crossed at most once is stretchable if and only it is pseudolinear.*

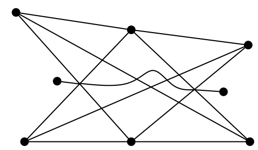

Figure 10: Non-Pappus configuration.

Proof. Let D be a drawing of a graph in which every edge is crossed at most once. If D is stretchable then clearly it is pseudolinear. To show the converse, suppose that D is pseudolinear. Then D does not contain any obstruction, and in particular, neither of the B and W configurations in Figure 2 occur in D. In [15], it was shown that not containing the B and W configurations is equivalent to being rectilinear. □

One can construct more general examples of pseudolinear drawings that are not stretchable by considering non-strechable arrangements of pseudolines. However, such examples seem to inevitably have edges crossing several times. This leads to two natural questions.

Question 1. *Is it true that if D is a pseudolinear drawing in which every edge is crossed at most twice, then D is stretchable?*

Question 2. *Is it true that if D is a pseudolinear drawing in which all the crossings involve a fixed edge, then D is stretchable?*

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