AN ENDPOINT ALEXANDROV BAKELMAN PUCCI ESTIMATE IN THE PLANE

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ABSTRACT. The classical Alexandrov-Bakelman-Pucci estimate for the Laplacian states

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c_{s,n} \operatorname{diam}(\Omega)^{2-\frac{n}{s}} \|\Delta u\|_{L^s(\Omega)}$$

where $\Omega \subset \mathbb{R}^n$, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $s > n/2$. The inequality fails for $s = n/2$. A Sobolev embedding result of Milman & Pustylink, originally phrased in a slightly different context, implies an endpoint inequality: if $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ is bounded, then

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c_n \|\Delta u\|_{L^{\frac{n}{2},1}(\Omega)},$$

where $L^{p,q}$ is the Lorentz space refinement of L^p. This inequality fails for $n = 2$ and we prove a sharp substitute result: there exists $c > 0$ such that for all $\Omega \subset \mathbb{R}^2$ with finite measure

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c \max_{x \in \Omega} \int_{y \in \Omega} \max\left\{1, \log\left(\frac{|\Omega|}{\|x-y\|^2}\right)\right\} |\Delta u(y)| \, dy.$$

This is somewhat dual to the classical Trudinger-Moser inequality – we also note that it is sharper than the usual estimates given in Orlicz spaces, the proof is rearrangement-free. The Laplacian can be replaced by any uniformly elliptic operator in divergence form.

1. Introduction and main results

1.1. **Introduction.** The Alexandrov-Bakelman-Pucci estimate [2, 3, 7, 27, 28] is one of the classical estimates in the study of elliptic partial differential equations. In its usual form it is stated for a second order uniformly elliptic operator

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u$$

with bounded measurable coefficients in a bounded domain $\Omega \subset \mathbb{R}^n$ and $c(x) \leq 0$. The Alexandrov-Bakelman-Pucci estimate then states that for any $u \in C^2(\Omega) \cap C(\overline{\Omega})$

$$\sup_{x \in \Omega} |u(x)| \leq \sup_{x \in \partial\Omega} |u(x)| + c \operatorname{diam}(\Omega) \|Lu\|_{L^n(\Omega)},$$

where c depends on the ellipticity constants of L and the L^n-norms of the b_i. It is a rather foundational maximum principle and discussed in most of the standard textbooks, e.g. Caffarelli & Cabré [13], Gilbarg & Trudinger [17], Han & Lin [19] and Jost [20]. The ABP estimate has inspired a very active field of research, we do not attempt a summary and refer to [11, 12, 13, 17, 33] and references therein. Alexandrov [4] and Pucci [28] showed that L^n can generically not be replaced by a smaller norm. However, for some elliptic operators operators it is possible to get estimates with L^p with $p < n$, see [6]. We will start our discussion with the special case of the Laplacian, where the inequality reads, for any $s > n/2$,

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c_{s,n} \operatorname{diam}(\Omega)^{2-\frac{n}{s}} \|\Delta u\|_{L^s(\Omega)}.$$

1.2. **Results.** The inequality is known to fail in the endpoint $s = n/2$. The purpose of our short paper is to note endpoint versions of the inequality. The first result is essentially due to Milman & Pustylink [22] (see also [23]), with an alternative proof due to Xiao & Zhai [34] (although ascribing it to anyone in particular is not an easy matter, one could reasonably argue that Talenti's seminal paper [31, Eq. 20] already contains the result without spelling it out).

2010 *Mathematics Subject Classification.* 28A75, 35A23, 35B50, 49Q20.
Key words and phrases. Alexandrov-Bakelman-Pucci estimate, maximum principle, Trudinger-Moser inequality.

Theorem 1. *Let $n \geq 3$, let $\Omega \subset \mathbb{R}^n$ be bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c_n \|\Delta u\|_{L^{\frac{n}{2},1}(\Omega)},$$

where c_n only depends on the dimension.

Here $L^{n/2,1}$ is the Lorentz space refinement of $L^{n/2}$. We note that its norm is slightly larger than $L^{n/2}$ and this turns out to be sufficient to establish an endpoint result in a critical space for which the geometry of Ω now longer enters into the inequality. We refer to Grafakos [18] for an introduction to Lorentz spaces. The proofs given in [22, 23, 24, 31] rely on rearrangement techniques. Theorem 1 fails for $n = 2$: the Lorentz spaces collapse to $L^{1,1} = L^1$ and the inequality is false in L^1 (see below). We obtain a sharp endpoint result in $\mathbb{R}^2$.

Theorem 2 (Main result)**.** *Let $\Omega \subset \mathbb{R}^2$ have finite measure and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c \max_{x \in \Omega} \int_{y \in \Omega} \max\left\{1, \log\left(\frac{|\Omega|}{\|x-y\|^2}\right)\right\} |\Delta u(y)|\, dy.$$

The result seems to be new. We observe that Talenti [31] is hinting at the proof of a slightly weaker result using rearrangement techniques (after his equation (22), see a recent paper of Milman [24] for a complete proof and related results). Ω need not be bounded, it suffices to assume that it has finite measure. We illustrate sharpness of the inequality with an example on the unit disk: define the radial function $u_\varepsilon(r)$ by

$$u(r) = \begin{cases} \frac{1}{2} - \log\varepsilon - \frac{1}{2}\varepsilon^{-2} r^2 & \text{if } 0 \leq r \leq \varepsilon \\ -\log r & \text{if } \varepsilon \leq r \leq 1. \end{cases}$$

We observe that $\Delta u_\varepsilon \sim \varepsilon^{-2} 1_{\{|x| \leq \varepsilon\}}$ and $\|u\|_{L^\infty} \sim \log(1/\varepsilon)$. This shows that the solution is unbounded as $\varepsilon \to 0$ while $\|\Delta u\|_{L^1} \sim 1$ remains bounded; in particular, no Alexandrov-Bakelman-Pucci inequality in L^1 is possible for $n = 2$. The example also shows Theorem 2 to be sharp: the maximum is assumed in the origin and

$$\int_{y \in \Omega} \max\left\{1, \log\left(\frac{|\Omega|}{\|y\|^2}\right)\right\} \varepsilon^{-2} 1_{\{|y| \leq \varepsilon\}} dy = \frac{1}{\varepsilon^2} \int_{B(0,\varepsilon)} \log\left(\frac{\pi}{\|y\|^2}\right) dy \sim \log\left(\frac{1}{\varepsilon}\right).$$

The proof will show that the constant $|\Omega|$ inside the logarithm is quite natural but can be improved if the domain is very different from a disk: indeed, we can get sharper result that recover some of the information that is lost in applying rearrangement type techniques and with a slight modification of the main argument we can obtain a slightly stronger result capturing more geometric information.

Corollary. *Let $\Omega \subset \mathbb{R}^2$ have finite measure and be simply connected and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \partial\Omega} |u(x)| + c \max_{x \in \Omega} \int_{y \in \Omega} \max\left\{1, \log\left(\frac{\text{inrad}(\Omega)^2}{\|x-y\|^2}\right)\right\} |\Delta u(y)|\, dy.$$

All results remain true if we replace the Laplacian $-\Delta$ by a uniformly elliptic operator in divergence form $-\text{div}(a(x) \cdot \nabla u)$ or replace $\mathbb{R}^n$ by a manifold as long as the induced heat kernel satisfies Aronson-type bounds [5].

1.3. **Related results.** There is a trivial connection between Alexandrov-Bakelman-Pucci estimates and second-order Sobolev inequalities that, to the best of our knowledge, has never been made explicit. After constructing

$$\begin{aligned} \Delta\phi &= 0 \qquad \text{in } \Omega \\ \phi &= u \qquad \text{on } \partial\Omega \end{aligned}$$

we may trivially estimate, using the maximum principle for harmonic functions,

$$\max_{x \in \Omega} |u(x)| \leq \max_{x \in \Omega} |\phi(x)| + \max_{x \in \Omega} |u(x) - \phi(x)| \leq \max_{x \in \partial\Omega} |u(x)| + \max_{x \in \Omega} |u(x) - \phi(x)|.$$

This reduces the problem to studying functions $u \in C^2(\Omega)$ that vanish on the boundary and verifying the validity of estimates of the type

$$\|u\|_{L^\infty(\Omega)} \lesssim_\Omega \|\Delta u\|_X.$$

The Alexandroff-Bakelman-Pucci estimate is one such estimate. These objects have been actively studied for a long time, see e.g. [15, 16, 34] and references therein. Theorem 1 can thus be restated as second-order Sobolev inequality in the endpoint $p = \infty$ and requiring a Lorentz-space refinement; it can be equivalently stated as

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq c_n \|\Delta u\|_{L^{\frac{n}{2},1}(\mathbb{R}^n)} \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n),\ n \geq 3.$$

This inequality seems to have first been stated in the literature by Milman & Pustylink [22] in the context of Sobolev embedding at the critical scale. Xiao & Zhai [34] derive the inequality via harmonic analysis. The failure of the embedding of the critical Sobolev space into L^∞ is classical

$$W_0^{2,\frac{n}{2}}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

There are two natural options: one could either try to find a slightly larger space $Y \supset L^\infty(\Omega)$ to have a valid embedding or one could try to find a space slightly smaller than the Sobolev space to have a valid embedding. The result of Milman & Pustylink [22] deals with the second question. From the point of view of studying Sobolev spaces, the first question is quite a bit more relevant since it investigates extremal behavior of functions in a Sobolev space and has been addressed in many papers [1, 8, 10, 25, 22, 26]. We emphasize the Trudinger-Moser inequality [25, 32]: for $\Omega \subset \mathbb{R}^2$

$$\sup_{\|\nabla u\|_{L^2} \leq 1} \int_\Omega e^{4\pi |u|^2} dx \leq c|\Omega|.$$

Cassani, Ruf & Tarsi [14] prove a variant: the condition $\|\Delta u\|_{L^1} < \infty$ suffices to ensure that u has at most logarithmic blow-up. These results should be seen as somewhat dual to Theorem 2. Put differently, Theorem 2 is a natural converse to this result since it implies that any function with $\|\Delta u\|_{L^1} < \infty$ and logarithmic blow-up has a Laplacian Δu that concentrates its L^1-mass.

2. Proofs

The proofs are all based on the idea of representing a function $u : \Omega \to \mathbb{R}$ as the stationary solution of the heat equation with a suitably chosen right-hand side (these techniques have recently proven useful in a variety of problems [9, 21, 29, 30])

$$\begin{aligned} v_t + \Delta v &= \Delta u \qquad \text{in } \Omega \\ v &= u \qquad \text{on } \partial\Omega. \end{aligned}$$

The Feynman-Kac formula then implies a representation of $u(x) = v(t, x)$ as a convolution of the heat kernel and its values in a neighborhood to which standard estimates can be applied. We use $\omega_x(t)$ to denote Brownian motion started in $x \in \Omega$ at time t; moreover, in accordance with Dirichlet boundary conditions, we will assume that the boundary is sticky and remains at the boundary once it touches it. The Feynman-Kac formula then implies that for all $t > 0$

$$u(x) = \mathbb{E} u(\omega_x(t)) + \mathbb{E} \int_0^t (\Delta u)(\omega_x(t)) dt.$$

This representation will be used in all our proofs. The proof of Theorem 1 will be closely related in spirit to [34, Lemma 3.2.] phrased in a different language; this language turns out to be useful in the proof of Theorem 2 where an additional geometric argument is required.

2.1. A Technical Lemma.

The purpose of this section is to quickly prove a fairly basic inequality. The Lemma already appeared in a slightly more precise form in work of Lierl and the author [21]. We only need a special case and prove it for completeness of exposition.

Lemma. *Let* $n \in \mathbb{N}$*, let* $t > 0$*,* $c_1, c_2 > 0$ *and* $0 \neq x \in \mathbb{R}^n$*. We have*

$$\int_0^t \frac{c_1}{s} \exp\left(-\frac{\|x\|^2}{c_2 s}\right) ds \lesssim_{c_1,c_2} \left(1 + \max\left\{0, -\log\left(\frac{\|x\|^2}{c_2 t}\right)\right\}\right) \exp\left(-\frac{\|x\|^2}{c_2 t}\right).$$

and, for $n \geq 3$*,*

$$\int_0^\infty \frac{c_1}{s^{n/2}} \exp\left(-\frac{\|x\|^2}{c_2 s}\right) ds \lesssim_{c_1,c_2,n} \frac{1}{\|x\|^{n-2}}.$$

Proof. The substitutions $z = s/|x|^2$ and $y = 1/(c_2 z)$ show

$$\int_0^t \frac{c_1}{s} \exp\left(-\frac{|x|^2}{c_2 s}\right) ds \lesssim_{c_1,c_2} \int_{|x|^2/(c_2 t)}^\infty y^{-1} e^{-y} dy.$$

If $|x|^2/(c_2 d) \leq 1$ we have

$$\int_{|x|^2/(c_2 t)}^\infty y^{-1} e^{-y} dy \lesssim 1 + \int_{|x|^2/(c_2 t)}^1 y^{-1} e^{-y} dy \lesssim 1 + \int_{|x|^2/(c_2 t)}^1 y^{-1} dy \lesssim 1 - \log\left(\frac{|x|^2}{c_2 t}\right),$$

and if $|x|^2/(c_2 t) \geq 1$ we have

$$\int_{|x|^2/(c_2 t)}^\infty y^{-1} e^{-y} dy \leq \frac{c_2 d}{|x|^2} \int_{|x|^2/(c_2 t)}^\infty e^{-y} dy = \frac{c_2 t}{|x|^2} \exp\left(-\frac{|x|^2}{c_2 t}\right) \leq \exp\left(-\frac{|x|^2}{c_2 t}\right).$$

Summarizing, this establishes

$$\int_{|x|^2/(c_2 t)}^\infty \frac{1}{y} e^{-y} dy \lesssim \left(1 + \max\left\{0, -\log\left(\frac{|x|^2}{c_2 t}\right)\right\}\right) \exp\left(-\frac{|x|^2}{c_2 t}\right),$$

which is the desired statement for $n = 2$. The second statement, for $n \geq 3$, is trivial. □

2.2. **Proof of Theorem 1.**

Proof. We rewrite u as the stationary solution of the heat equation

$$\begin{aligned} v_t + \Delta v &= \Delta u \qquad \text{in } \Omega \\ v &= u \qquad \text{on } \partial\Omega. \end{aligned}$$

As explained above, the Feynman-Kac formula implies that for all $t > 0$

$$u(x) = v(t,x) = \mathbb{E} v(\omega_x(t)) + \mathbb{E} \int_0^t (\Delta u)(\omega_x(t)) dt.$$

Let x be arbitrary, we now let $t \to \infty$. The first term is quite simple since we recover the harmonic measure. Indeed, as $t \to \infty$, we have

$$\lim_{t\to\infty} \mathbb{E} v(\omega_x(t)) = \phi(x) \qquad \text{where} \quad \begin{cases} \Delta\phi = 0 \text{ inside } \Omega \\ \phi = u \text{ on } \partial\Omega. \end{cases}$$

This can be easily seen from the stochastic interpretation of harmonic measure. This implies

$$\lim_{t\to\infty} \mathbb{E} v(\omega_x(t)) \leq \max_{x \in \partial\Omega} u(x).$$

It remains to estimate the second term. We denote the heat kernel on Ω by $p_\Omega(t,x,y)$ and observe

$$\begin{aligned} \left| \mathbb{E} \int_0^t (\Delta u)(\omega_x(t)) dt \right| &\leq \mathbb{E} \int_0^t |\Delta u(\omega_x(t))| \, dt \\ &= \int_0^t \int_{y \in \Omega} p_\Omega(s,x,y) \, |\Delta u(y)| \, dy ds \\ &\leq \int_{y \in \Omega} \left(\int_0^\infty p_\Omega(s,x,y) ds \right) |\Delta u(y)| \, dy \end{aligned}$$

However, using domain monotonicity $p_\Omega(t,x,y) \leq p_{\mathbb{R}^n}(t,x,y)$ as well as the explicit Gaussian form of the heat kernel on $\mathbb{R}^n$ and the Lemma we have, uniformly in $x,y \in \Omega$,

$$\int_0^\infty p_\Omega(s,x,y)ds \leq \int_0^\infty p_{\mathbb{R}^n}(s,x,y)ds \leq \frac{c_n}{\|x-y\|^{n-2}}.$$

The duality of Lorentz spaces

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^{\frac{n}{2},1}(\mathbb{R}^n)} \|g\|_{L^{\frac{n}{n-2},\infty}(\mathbb{R}^n)} \quad \text{and} \quad \frac{1}{\|x-y\|^{n-2}} \in L^{\frac{n}{n-2},\infty}(\mathbb{R}^n, dy)$$

then implies the desired result

$$\left|\mathbb{E}\int_0^t (\Delta u)(\omega_x(t))dt\right| \leq c_n \int_{y\in\Omega} \frac{|\Delta u|(y)}{\|x-y\|^{n-2}}dy \leq \left\| \frac{c_n}{\|x-y\|^{n-2}} \right\|_{L^{\frac{n}{n-2},\infty}} \|\Delta u\|_{L^{\frac{n}{2},1}}.$$

□

2.3. **Proof of Theorem 2.**

Proof. This argument requires a simple statement for Brownian motion: for all sets $\Omega \subset \mathbb{R}^2$ with finite volume $|\Omega| < \infty$ and all $x \in \Omega$,

$$\mathbb{P}\left(\exists\, 0 \leq t \leq \frac{|\Omega|}{8} : w_x(t) \notin \Omega\right) \geq \frac{1}{2}.$$

We start by bounding the probability from below: for this, we introduce the free Brownian motion $\omega_x^*(t)$ that also starts in x but moves freely through $\mathbb{R}^n$ without getting stuck on the boundary $\partial\Omega$. Continuity of Brownian motion then implies

$$\mathbb{P}\left(\exists\, 0 \leq t \leq \frac{|\Omega|}{8} : w_x(t) \notin \Omega\right) \geq \mathbb{P}\left(w_x^*\left(|\Omega|/8\right) \notin \Omega\right).$$

Moreover, we can compute

$$\mathbb{P}\left(w_x^*(|\Omega|/8) \notin \Omega\right) = \int_{\mathbb{R}^n\setminus\Omega} \frac{\exp\left(-2\|x-y\|^2/|\Omega|\right)}{(\pi|\Omega|/2)}dy.$$

We use the Hardy-Littlewood rearrangement inequality to argue that

$$\int_{\mathbb{R}^n\setminus\Omega} \frac{\exp\left(-2\|x-y\|^2/|\Omega|\right)}{(\pi|\Omega|/2)}dy \geq \int_{\mathbb{R}^n\setminus B} \frac{\exp\left(-2\|y\|^2/|B|\right)}{(\pi|B|/2)}dy,$$

where B is a ball centered in the origin having the same measure as Ω. However, assuming $|B| = R^2\pi$ this quantity can be computed in polar cordinates as

$$\int_{\mathbb{R}^n\setminus B} \frac{\exp\left(-2\|y\|^2/|B|\right)}{(\pi|B|/2)}dy = \int_R^\infty \frac{\exp\left(-2r^2/(R^2\pi)\right)}{R^2\pi^2/2} 2\pi r dr = e^{-\frac{2}{\pi}} > \frac{1}{2}.$$

We return to the representation, valid for all $t > 0$,

$$v(t,x) = \mathbb{E}v(\omega_x(t)) + \mathbb{E}\int_0^t (\Delta u)(\omega_x(t))dt.$$

We will now work with finite values of t: the computation above implies that at time $t = |\Omega|$

$$|\mathbb{E}v(\omega_x(|\Omega|))| \leq \frac{1}{2}\max_{x\in\partial\Omega}|u(x)| + \frac{\max_{x\in\Omega} u(x)}{2}.$$

Arguing as above and employing the Lemma shows that

$$\begin{aligned}
\left|\mathbb{E}\int_0^{|\Omega|} (\Delta u)(\omega_x(t))dt\right| &\leq \int_{y\in\Omega} \left(\int_0^{|\Omega|} p(s,x,y)ds\right) |\Delta u(y)|\, dy \\
&\lesssim \|\Delta u\|_{L^1} + \int_{y\in\Omega} \max\left\{0, \log\left(\frac{|\Omega|}{\|x-y\|^2}\right)\right\} |\Delta u(y)|\, dy \\
&\lesssim \int_{y\in\Omega} \max\left\{1, \log\left(\frac{|\Omega|}{\|x-y\|^2}\right)\right\} |\Delta u(y)|\, dy.
\end{aligned}$$

We can now pick $x \in \Omega$ so that u assumes its maximum there and argue

$$\max_{x \in \Omega} u(x) = v(|\Omega|, x) = \mathbb{E}v(\omega_x(|\Omega|)) + \mathbb{E} \int_0^{|\Omega|} (\Delta u)(\omega_x(t)) dt$$
$$\leq \frac{1}{2} \max_{x \in \partial\Omega} |u(x)| + \frac{\max_{x \in \Omega} u(x)}{2} + c \max_{x \in \Omega} \int_{y \in \Omega} \max \left\{ 1, \log \left(\frac{|\Omega|}{\|x-y\|^2} \right) \right\} |\Delta u(y)| \, dy$$

which implies the desired statement. □

2.4. **Proof of the Corollary.**

Proof. The proof can be used almost verbatim, we only require the elementary statement that for all simply-connected domains $\Omega \subset \mathbb{R}^2$ and all $x_0 \in \Omega$

$$\mathbb{P}\left(\exists\ 0 \leq t \leq c \cdot \mathrm{inrad}(\Omega)^2 : w_{x_0}(t) \notin \Omega\right) \geq \frac{1}{100}.$$

The idea is actually rather simple: for any such x_0 there exists a point $\|x_0 - x_1\| \leq \mathrm{inrad}(\Omega)$ such that $y \notin \Omega$. Since Ω is simply connected, the boundary is an actual line enclosing the domain: in particular, the disk of radius $m\mathrm{inrad}(\Omega)$ centered around x_0 either already contains the entire domain Ω or has a boundary of length at least $(2m-2) \cdot \mathrm{inrad}(\Omega)$ (an example being close to the extremal case is shown in Figure 1).

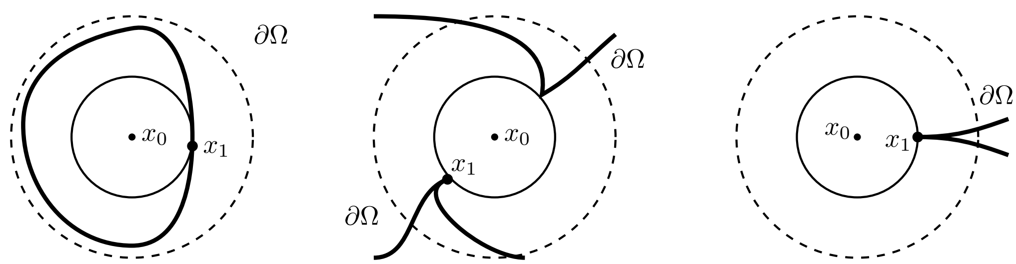

FIGURE 1. The point of maximum x_0, the circle with radius $d(x_0, \Omega)$, the circle with radius $2d(x_0, \Omega)$ (dashed) and the possible local geometry of $\partial\Omega$.

It turns out that $m = 2$ is already an admissible choice, the computations are carried out in earlier work of M. Rachh and the author [29].

□

Acknowledgement. The author is grateful to Mario Milman for discussions about the history of some of these results.

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