

Spanning trees with at most 4 leaves in  $K_{1,5}$ -free graphs

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**Abstract**

In 2009, Kyaw proved that every  $n$ -vertex connected  $K_{1,4}$ -free graph  $G$  with  $\sigma_4(G) \geq n - 1$  contains a spanning tree with at most 3 leaves. In this paper, we prove an analogue of Kyaw's result for connected  $K_{1,5}$ -free graphs. We show that every  $n$ -vertex connected  $K_{1,5}$ -free graph  $G$  with  $\sigma_5(G) \geq n - 1$  contains a spanning tree with at most 4 leaves. Moreover, the degree sum condition " $\sigma_5(G) \geq n - 1$ " is best possible.

**Keywords:** spanning tree;  $K_{1,5}$ -free; degree sum

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# 1 Introduction

In this paper, we only consider finite simple graphs. Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . For any vertex  $v \in V(G)$ , we use  $N_G(v)$  and  $d_G(v)$  (or  $N(v)$  and  $d(v)$  if there is no ambiguity) to denote the set of neighbors of  $v$  and the degree of  $v$  in  $G$ , respectively. For any  $X \subseteq V(G)$ , we denote by  $|X|$  the cardinality of  $X$ . We define  $N(X) = \bigcup_{x \in X} N(x)$  and  $d(X) = \sum_{x \in X} d(x)$ . For  $k \geq 1$ , we let  $N_k(X) = \{x \in V(G) \mid |N(x) \cap X| = k\}$  and  $N_{\geq k}(X) = \{x \in V(G) \mid |N(x) \cap X| \geq k\}$ . We use  $G - X$  to denote the graph obtained from  $G$  by deleting the vertices in  $X$  together with their incident edges. The subgraph of  $G$  induced by  $X$  is denoted by  $G[X]$ . We define  $G - uv$  to be the graph obtained from  $G$  by deleting the edge  $uv \in E(G)$ , and  $G + uv$  to be the graph obtained from  $G$  by adding an edge  $uv$  between two non-adjacent vertices  $u$  and  $v$  of  $G$ . We write  $A := B$  to rename  $B$  as  $A$ .

A subset  $X \subseteq V(G)$  is called an *independent set* of  $G$  if no two vertices of  $X$  are adjacent in  $G$ . The maximum size of an independent set in  $G$  is denoted by  $\alpha(G)$ . For  $k \geq 1$ , we define  $\sigma_k(G) = \min\{\sum_{i=1}^k d(v_i) \mid \{v_1, \dots, v_k\} \text{ is an independent set in } G\}$ . For  $r \geq 1$ , a graph is said to be  $K_{1,r}$ -free if it does not contain  $K_{1,r}$  as an induced subgraph. A  $K_{1,3}$ -free graph is also called a *claw-free* graph. We use  $K_n$  to denote the complete graph on  $n$  vertices.

Let  $T$  be a tree. A vertex of degree one is a *leaf* of  $T$  and a vertex of degree at least three is a *branch vertex* of  $T$ . For two distinct vertices  $u, v$  of  $T$ , we denote by  $P_T[u, v]$  the unique path in  $T$  connecting  $u$  and  $v$  and denote by  $d_T[u, v]$  the distance between  $u$  and  $v$  in  $T$ . We define the *orientation* of  $P_T[u, v]$  is from  $u$  to  $v$ . We refer to [4] for terminology and notation not defined here.

There are several well-known conditions (such as the independence number conditions and the degree sum conditions) ensuring that a graph  $G$  contains a spanning tree with a bounded number of leaves or branch vertices (see the survey paper [12] and the references cited therein for details). Win [13] obtained a sufficient condition related to the independence number for  $k$ -connected graphs, which confirms a conjecture of Las Vergnas [9]. Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with at most  $m$  leaves.

**Theorem 1.1** (Win [13]) *Let  $G$  be a  $k$ -connected graph and let  $m \geq 2$ . If  $\alpha(G) \leq k + m - 1$ , then  $G$  has a spanning tree with at most  $m$  leaves.*

**Theorem 1.2** (Broersma and Tuinstra [1]) *Let  $G$  be a connected graph with  $n$  vertices and let  $m \geq 2$ . If  $\sigma_2(G) \geq n - m + 1$ , then  $G$  has a spanning tree with at most  $m$  leaves.*

Kano et al. [6] presented a degree sum condition for a connected claw-free graph to have a spanning tree with at most  $m$  leaves, which generalizes a result of Matthews and Sumner [11] and a result of Gargano et al. [5]. Matsuda, Ozeki and Yamashita [10] and Chen, Li and Xu [2] considered the sufficient conditions for a connected claw-free graph to have a spanning tree with few branch vertices or few leaves, respectively.

**Theorem 1.3** (Kano et al. [6]) *Let  $G$  be a connected claw-free graph with  $n$  vertices and let  $m \geq 2$ . If  $\sigma_{m+1}(G) \geq n - m$ , then  $G$  has a spanning tree with at most  $m$  leaves.*

**Theorem 1.4** (Matsuda, Ozeki and Yamashita [10]) *Let  $G$  be a connected claw-free graph with  $n$  vertices. If  $\sigma_5(G) \geq n - 2$ , then  $G$  contains a spanning tree with at most one branch vertex.*

**Theorem 1.5** (Chen, Li and Xu [2]) *Let  $G$  be a  $k$ -connected claw-free graph with  $n$  vertices. If  $\sigma_{k+3}(G) \geq n - k$ , then  $G$  contains a spanning tree with at most 3 leaves.*

For connected  $K_{1,4}$ -free graphs, Kyaw [7, 8] obtained the following two sharp results.

**Theorem 1.6** (Kyaw [7]) *Let  $G$  be a connected  $K_{1,4}$ -free graph with  $n$  vertices. If  $\sigma_4(G) \geq n - 1$ , then  $G$  contains a spanning tree with at most 3 leaves.*

**Theorem 1.7** (Kyaw [8]) *Let  $G$  be a connected  $K_{1,4}$ -free graph with  $n$  vertices.*

(i) *If  $\sigma_3(G) \geq n$ , then  $G$  has a hamiltonian path.*

(ii) *If  $\sigma_{m+1}(G) \geq n - \frac{m}{2}$  for some integer  $m \geq 3$ , then  $G$  has a spanning tree with at most  $m$  leaves.*

Chen, Chen and Hu [3] considered the degree sum condition for a  $k$ -connected  $K_{1,4}$ -free graph to contain a spanning tree with at most 3 leaves.

**Theorem 1.8** (Chen, Chen and Hu [3]) *Let  $G$  be a  $k$ -connected  $K_{1,4}$ -free graph with  $n$  vertices and let  $k \geq 2$ . If  $\sigma_{k+3}(G) \geq n + 2k - 2$ , then  $G$  has a spanning tree with at most 3 leaves.*

In this paper, we further consider connected  $K_{1,5}$ -free graphs. We give a sufficient condition for a connected  $K_{1,5}$ -free graph to have a spanning tree with few leaves.

**Theorem 1.9** *Let  $G$  be a connected  $K_{1,5}$ -free graph with  $n$  vertices. If  $\sigma_5(G) \geq n - 1$ , then  $G$  contains a spanning tree with at most 4 leaves.*

It is easy to see that if a tree has at most  $k$  leaves ( $k \geq 2$ ), then it has at most  $k - 2$  branch vertices. Therefore, we immediately obtain the following corollary from Theorem 1.9.

**Corollary 1.10** *Let  $G$  be a connected  $K_{1,5}$ -free graph with  $n$  vertices. If  $\sigma_5(G) \geq n - 1$ , then  $G$  contains a spanning tree with at most 2 branch vertices.*

We end this section by constructing an example to show that the degree sum condition “ $\sigma_5(G) \geq n - 1$ ” in Theorem 1.9 is sharp. For  $m \geq 1$ , let  $D_1, D_2, D_3, D_4, D_5$  be vertex-disjoint copies of  $K_m$  and let  $xy$  be an edge such that neither  $x$  nor  $y$  is contained in  $\bigcup_{i=1}^5 V(D_i)$ . Join  $x$  to all the vertices in  $D_1 \cup D_2 \cup D_3$  and join  $y$  to all the vertices in  $D_4 \cup D_5$ . The resulting graph is denoted by  $G$ . Then it is easy to check that  $G$  is a connected  $K_{1,5}$ -free graph with  $n = 5m + 2$  vertices and  $\sigma_5(G) = 5m = n - 2$ . However, every spanning tree of  $G$  contains at least 5 leaves.

## 2 Proof of the main result

In this section, we extend the idea of Kyaw in [7] to prove Theorem 1.9. For this purpose, we need the following lemma.

**Lemma 2.1** *Let  $G$  be a connected graph such that  $G$  does not have a spanning tree with at most 4 leaves, and let  $T$  be a maximal tree of  $G$  with 5 leaves. Then there does not exist a tree  $T'$  in  $G$  such that  $T'$  has at most 4 leaves and  $V(T') = V(T)$ .*

*Proof.* Suppose for a contradiction that there exists a tree  $T'$  in  $G$  with at most 4 leaves and  $V(T') = V(T)$ . Since  $G$  has no spanning tree with at most 4 leaves, we see that  $V(G) - V(T') \neq \emptyset$ . Hence there must exist two vertices  $v$  and  $w$  in  $G$  such that  $v \in V(T')$  and  $w \in N(v) \cap (V(G) - V(T'))$ . Let  $T_1$  be the tree obtained from  $T'$  by adding the vertex  $w$  and the edge  $vw$ .

If  $T_1$  has 5 leaves, then  $T_1$  contradicts the maximality of  $T$  (since  $|V(T_1)| = |V(T)| + 1 > |V(T)|$ ). So we may assume that  $T_1$  has at most 4 leaves. By repeating this process, we can recursively construct a set of trees  $\{T_i \mid i \geq 1\}$  in  $G$  such that  $T_i$  has at most 4 leaves and  $|V(T_{i+1})| = |V(T_i)| + 1$  for each  $i \geq 1$ . Since  $G$  has no spanning tree with at most 4 leaves and  $|V(G)|$  is finite, the process must terminate after a finite number of steps, i.e., there exists some  $k \geq 1$  such that  $T_{k+1}$  is a tree in  $T$  with 5 leaves. But this contradicts the maximality of  $T$ . So the lemma holds.  $\blacksquare$

**Proof of Theorem 1.9.** We prove the theorem by contradiction. Suppose to the contrary that  $G$  contains no spanning tree with at most 4 leaves. Then every spanning tree of  $G$  contains at least 5 leaves. We choose a maximal tree  $T$  of  $G$  with exactly 5 leaves. Let  $U = \{u_1, u_2, u_3, u_4, u_5\}$  be the set of leaves of  $T$ . By the maximality of  $T$ , we have  $N(U) \subseteq V(T)$ .

We consider three cases according to the number of branch vertices in  $T$ . (Note that  $T$  contains at most three branch vertices.)

*Case 1.*  $T$  contains two branch vertices.

Let  $s$  and  $t$  be the two branch vertices in  $T$  such that  $d_T(s) = 4$  and  $d_T(t) = 3$ . For each  $1 \leq i \leq 5$ , let  $B_i$  be the vertex set of the connected component of  $T - \{s, t\}$  containing  $u_i$  and let  $v_i$  be the unique vertex in  $B_i \cap N_T(\{s, t\})$ . Without loss of generality, we may assume that  $\{v_1, v_2, v_3\} \subseteq N_T(s)$  and  $\{v_4, v_5\} \subseteq N_T(t)$ . For each  $1 \leq i \leq 5$  and  $x \in B_i$ , we use  $x^-$  and  $x^+$  to denote the predecessor and the successor of  $x$  on  $P_T[s, u_i]$  or  $P_T[t, u_i]$ , respectively (if such a vertex exists). Let  $s^+$  and  $t^-$  be the successor of  $s$  and the predecessor of  $t$  on  $P_T[s, t]$ , respectively. Define  $P := V(P_T[s, t]) - \{s, t\}$ .

For this case, we further choose  $T$  such that

(C1)  $d_T[s, t]$  is as small as possible, and

(C2) subject to (C1),  $\sum_{i=1}^3 |B_i|$  is as large as possible.

**Claim 2.2** *For all  $1 \leq i, j \leq 5$  and  $i \neq j$ , if  $x \in N(u_j) \cap B_i$ , then  $x \notin \{u_i, v_i\}$  and  $x^- \notin N(U - \{u_j\})$ .*

*Proof.* Suppose  $x \in \{u_i, v_i\}$ . Then  $T' := T - v_i v_i^- + x u_j$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So we have  $x \notin \{u_i, v_i\}$ .

Next, assume  $x^- \in N(U - \{u_j\})$ . Then there exists some  $k \in \{1, 2, 3, 4, 5\} - \{j\}$  such that  $x^- u_k \in E(G)$ . Now,  $T' := T - \{v_i v_i^-, x x^-\} + \{x u_j, x^- u_k\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , also contradicting Lemma 2.1. This proves Claim 2.2. ■

By Claim 2.2, we know that  $U$  is an independent set in  $G$ . Since  $G$  is  $K_{1,5}$ -free, we have  $N_5(U) = \emptyset$ .

**Claim 2.3**  $N(u_i) \cap P = \emptyset$  for each  $4 \leq i \leq 5$ .

*Proof.* Suppose the assertion of the claim is false. Then there exists some vertex  $x \in P$  such that  $x u_i \in E(G)$  for some  $i \in \{4, 5\}$ . Let  $T' := T - t v_i + x u_i$ , then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $s$  and  $x$ ,  $d_{T'}(s) = 4$ ,  $d_{T'}(x) = 3$  and  $d_{T'}[s, x] < d_T[s, t]$ . But this contradicts the condition (C1). So the claim holds. ■

**Claim 2.4** If  $P \neq \emptyset$ , then  $\sum_{i=1}^3 |N(u_i) \cap \{x\}| \leq 1$  for each  $x \in P$ .

*Proof.* Suppose to the contrary that there exists some vertex  $x \in P$  such that  $\sum_{i=1}^3 |N(u_i) \cap \{x\}| \geq 2$ . Then there exist two distinct  $j, k \in \{1, 2, 3\}$  such that  $x u_j, x u_k \in E(G)$ . Let  $T' := T - \{s v_j, s v_k\} + \{x u_j, x u_k\}$ , then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $x$  and  $t$ ,  $d_{T'}(x) = 4$ ,  $d_{T'}(t) = 3$  and  $d_{T'}[x, t] < d_T[s, t]$ , contradicting the condition (C1). This completes the proof of Claim 2.4. ■

**Claim 2.5** If  $P \neq \emptyset$ , then  $N(U) \cap \{s^+\} = \emptyset$ .

*Proof.* Suppose this is false. Then by Claim 2.3, there exists some  $i \in \{1, 2, 3\}$  such that  $s^+ u_i \in E(G)$ . Now,  $T' := T - s s^+ + s^+ u_i$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So the assertion of the claim holds. ■

**Claim 2.6**  $N(u_i) \cap \{s\} = \emptyset$  for each  $4 \leq i \leq 5$ .

*Proof.* Suppose  $s u_i \in E(G)$  for some  $i \in \{4, 5\}$ . If  $P = \emptyset$ , then we have  $st \in E(T)$  and  $T' := T - st + s u_i$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , contradicting Lemma 2.1. So we may assume that  $P \neq \emptyset$  and hence  $s^+ \neq t$ . By applying Claims 2.2 and 2.3, we deduce that  $N(u_i) \cap \{s^+, v_1, v_2, v_3\} = \emptyset$ .

Suppose that  $s^+ v_j \in E(G)$  for some  $j \in \{1, 2, 3\}$ . Then  $T' := T - \{s s^+, s v_j\} + \{s u_i, s^+ v_j\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So we conclude that  $N(s^+) \cap \{v_1, v_2, v_3\} = \emptyset$ .

Now, assume there exists two distinct  $j, k \in \{1, 2, 3\}$  such that  $v_j v_k \in E(G)$ . Then by Claim 2.2, we see that  $u_k \neq v_k$ . Let  $T' := T - \{s v_j, t v_i\} + \{s u_i, v_j v_k\}$ , then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $s$  and  $v_k$ ,  $d_{T'}(s) = 4$ ,  $d_{T'}(v_k) = 3$  and  $d_{T'}[s, v_k] < d_T[s, t]$ , contradicting the condition (C1). Therefore,  $v_1, v_2$  and  $v_3$  are pairwise non-adjacent in  $G$ .

But then,  $\{s^+, u_i, v_1, v_2, v_3\}$  is an independent set and  $G[\{s, s^+, u_i, v_1, v_2, v_3\}]$  is an induced  $K_{1,5}$  of  $G$ , again a contradiction. This proves Claim 2.6. ■

**Claim 2.7** If  $\sum_{i=1}^5 |N(u_i) \cap \{t\}| \geq 3$ , then  $P \neq \emptyset$ .

*Proof.* Suppose for a contradiction that  $P = \emptyset$ . Then we have  $st \in E(G)$ . Since  $\sum_{i=1}^5 |N(u_i) \cap \{t\}| \geq 3$ , there exists some  $j \in \{1, 2, 3\}$  such that  $tu_j \in E(G)$ . Let  $T' := T - st + tu_j$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So the claim holds. ■

**Claim 2.8**  $N_4(U) = \emptyset$ .

*Proof.* Suppose to the contrary that there exists some vertex  $x \in N_4(U)$ . Then by Claims 2.3 and 2.6, we have  $x \in B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup \{t\}$ .

First, suppose  $x \in B_i$  for some  $1 \leq i \leq 5$ . By Claim 2.2, we know that  $x^- \notin N(U)$ . Then  $(N(x) \cap U) \cup \{x^-\}$  is an independent set and  $G[(N(x) \cap U) \cup \{x, x^-\}]$  is an induced  $K_{1,5}$  of  $G$ , contradicting the assumption that  $G$  is  $K_{1,5}$ -free.

So we may assume that  $x = t$ . Then by Claim 2.7, we conclude that  $P \neq \emptyset$  and hence  $t^- \neq s$ . It follows from Claim 2.3 that  $N(u_i) \cap \{t^-\} = \emptyset$  for each  $4 \leq i \leq 5$ . Suppose that  $t^-u_j \in E(G)$  for some  $j \in \{1, 2, 3\}$ . Since  $t \in N_4(U)$ , there exists some  $k \in \{1, 2, 3\} - \{j\}$  such that  $tu_k \in E(G)$ . Let  $T' := T - \{sv_j, tt^-\} + \{tu_k, t^-u_j\}$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. Therefore, we deduce that  $N(U) \cap \{t^-\} = \emptyset$ . But then,  $(N(t) \cap U) \cup \{t^-\}$  is an independent set and  $G[(N(t) \cap U) \cup \{t, t^-\}]$  is an induced  $K_{1,5}$  of  $G$ , again a contradiction. This completes the proof of Claim 2.8. ■

**Claim 2.9**  $(N_3(U) - N(u_i)) \cap B_i = \emptyset$  for each  $1 \leq i \leq 5$ .

*Proof.* Suppose this is false. Then there exists some vertex  $x \in (N_3(U) - N(u_i)) \cap B_i$  for some  $1 \leq i \leq 5$ . By applying Claim 2.2, we have  $x \notin \{u_i, v_i\}$  and  $x^-, x^+ \notin N(U - \{u_i\})$ .

Suppose that  $x^-x^+ \in E(G)$ . Since  $x \in N_3(U) - N(u_i)$ , there must exist two distinct  $j, k \in \{1, 2, 3, 4, 5\} - \{i\}$  such that  $xu_j, xu_k \in E(G)$ . Then  $T' := T - \{v_jv_j^-, xx^-, xx^+\} + \{xu_j, xu_k, x^-x^+\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , contradicting Lemma 2.1. Hence  $x^-x^+ \notin E(G)$ .

Now,  $(N(x) \cap U) \cup \{x^-, x^+\}$  is an independent set and  $G[(N(x) \cap U) \cup \{x, x^-, x^+\}]$  is an induced  $K_{1,5}$  of  $G$ , giving a contradiction. So the assertion of the claim holds. ■

**Claim 2.10**  $N(u_j) \cap B_i = \emptyset$  for all  $4 \leq i \leq 5$  and  $1 \leq j \leq 3$ . In particular,  $N_3(U) \cap N(u_i) \cap B_i = \emptyset$  for each  $4 \leq i \leq 5$ .

*Proof.* Suppose the assertion of the claim is false. Then there exists some vertex  $x \in B_i$  such that  $xu_j \in E(G)$  for some  $i \in \{4, 5\}$  and  $j \in \{1, 2, 3\}$ . By Claim 2.2, we have  $x \notin \{u_i, v_i\}$ . Let  $T' := T - xx^- + xu_j$ , and let  $B'_k$  be the vertex set of the connected component of  $T' - \{s, t\}$  containing  $u_k$  for each  $1 \leq k \leq 3$ . It is easy to check that  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $s$  and  $t$ ,  $d_{T'}(s) = 4$ ,  $d_{T'}(t) = 3$ ,  $d_{T'}[s, t] = d_T[s, t]$  and  $\sum_{k=1}^3 |B'_k| = \sum_{k=1}^3 |B_k| + |V(P_T[x, u_i])| > \sum_{k=1}^3 |B_k|$ . But this contradicts the condition (C2). This proves Claim 2.10. ■

**Claim 2.11**  $|N_3(U) \cap N(u_i) \cap B_i| \leq 1$  for each  $1 \leq i \leq 3$ .

*Proof.* Suppose for a contradiction that there exist two distinct vertices  $x, y \in N_3(U) \cap N(u_i) \cap B_i$  for some  $i \in \{1, 2, 3\}$ . Without loss of generality, we may assume that  $x \in V(P_T[s, y])$ . By Claim 2.2, we have  $x, y \notin \{u_i, v_i\}$ ,  $x^- \notin N(U)$  and  $x^+ \notin N(U - \{u_i\})$ . In particular,  $x^+ \neq y$ . Since  $x, y \in N_3(U) \cap N(u_i)$ , there exist two distinct  $j, k \in \{1, 2, 3, 4, 5\} - \{i\}$  such that  $xu_j, yu_k \in E(G)$ . We may assume that  $x^-x^+, x^+u_i \notin E(G)$ ; for otherwise,

$$T' := \begin{cases} T - \{sv_i, xx^-, xx^+, yy^+\} + \{xu_i, xu_j, x^-x^+, yu_k\}, & \text{if } x^-x^+ \in E(G), \\ T - \{sv_i, xx^+, yy^-\} + \{xu_j, x^+u_i, yu_k\}, & \text{if } x^+u_i \in E(G), \end{cases}$$

is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. But then,  $(N(x) \cap U) \cup \{x^-, x^+\}$  is an independent set and  $G[(N(x) \cap U) \cup \{x, x^-, x^+\}]$  is an induced  $K_{1,5}$  of  $G$ , again a contradiction. So the claim holds.  $\blacksquare$

**Claim 2.12** *For each  $1 \leq i \leq 3$ , if  $u_i v_i \in E(G)$ , then  $N_3(U) \cap N(u_i) \cap B_i = \emptyset$ .*

*Proof.* Suppose to the contrary that  $u_i v_i \in E(G)$  and there exists some vertex  $x \in N_3(U) \cap N(u_i) \cap B_i$  for some  $i \in \{1, 2, 3\}$ . By Claim 2.2, we have  $x \neq v_i$ . Since  $x \in N_3(U) \cap N(u_i)$ , there exists some  $j \in \{1, 2, 3, 4, 5\} - \{i\}$  such that  $xu_j \in E(G)$ . Let  $T' := T - \{sv_i, xx^-\} + \{u_i v_i, xu_j\}$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , contradicting Lemma 2.1. This completes the proof of Claim 2.12.  $\blacksquare$

**Claim 2.13** *For each  $1 \leq i \leq 3$ , if  $su_i \in E(G)$ , then  $N_3(U) \cap N(u_i) \cap B_i = \emptyset$ .*

*Proof.* For the sake of convenience, we may assume by symmetry that  $i = 1$ . Suppose the assertion of the claim is false. Then there exists some vertex  $x \in N_3(U) \cap N(u_1) \cap B_1$ . By applying Claims 2.2 and 2.12, we know that  $x \notin \{u_1, v_1\}$  and  $N(u_1) \cap \{v_1, v_2, v_3\} = \emptyset$ .

Suppose  $v_1 v_j \in E(G)$  for some  $j \in \{2, 3\}$ . Then  $T' := T - \{sv_1, sv_j\} + \{su_1, v_1 v_j\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So we have  $v_1 v_2, v_1 v_3 \notin E(G)$ .

Next, assume that  $v_2 v_3 \in E(G)$ . Then  $u_2 \neq v_2$  and  $u_3 \neq v_3$  by Claim 2.2. If there exists some  $j \in \{2, 3\}$  such that  $xu_j \in E(G)$ , then  $T' := T - \{sv_2, sv_3\} + \{v_2 v_3, xu_j\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , contradicting Lemma 2.1. Hence  $xu_2, xu_3 \notin E(G)$ . Since  $x \in N_3(U) \cap N(u_1)$ , we conclude that  $xu_4, xu_5 \in E(G)$ . Let  $T' := T - \{sv_2, tt^-, xx^-\} + \{su_1, v_2 v_3, xu_4\}$ . If  $P = \emptyset$ , then  $t^- = s$ , and  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , giving a contradiction. So we deduce that  $P \neq \emptyset$ . But then,  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $s$  and  $v_3$ ,  $d_{T'}(s) = 4$ ,  $d_{T'}(v_3) = 3$  and  $d_{T'}[s, v_3] < d_T[s, t]$ , contradicting the condition (C1). Therefore,  $v_1, v_2$  and  $v_3$  are pairwise non-adjacent in  $G$ .

We now consider the vertex  $s^+$ . We will show that  $N(s^+) \cap \{u_1, v_1, v_2, v_3\} = \emptyset$ .

We first prove that  $s^+ u_1 \notin E(G)$ . Suppose this is false. Then by Claim 2.5, we see that  $P = \emptyset$  and hence  $s^+ = t$ . Let  $T' := T - st + tu_1$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1.

We then have  $s^+ v_1 \notin E(G)$ ; for otherwise,  $T' := T - \{ss^+, sv_1\} + \{su_1, s^+ v_1\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , also contradicting Lemma 2.1.

Finally, we show that  $s^+ v_2, s^+ v_3 \notin E(G)$ . Suppose not, and let  $s^+ v_j \in E(G)$  for some  $j \in \{2, 3\}$ . If there exists some  $k \in \{4, 5\}$  such that  $xu_k \in E(G)$ , then  $T' := T - \{ss^+, sv_j\} + \{s^+ v_j, xu_k\}$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1.

Therefore, we have  $xu_4, xu_5 \notin E(G)$ . Since  $x \in N_3(U) \cap N(u_1)$ , we deduce that  $xu_2, xu_3 \in E(G)$ . Let  $T' := T - \{ss^+, sv_j, xx^-, xx^+\} + \{su_1, s^+v_j, xu_2, xu_3\}$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , again a contradiction. Hence  $N(s^+) \cap \{u_1, v_1, v_2, v_3\} = \emptyset$ .

Now,  $\{s^+, u_1, v_1, v_2, v_3\}$  is an independent set and  $G[\{s, s^+, u_1, v_1, v_2, v_3\}]$  is an induced  $K_{1,5}$  of  $G$ , giving a contradiction. So the assertion of the claim holds.  $\blacksquare$

By Claim 2.2,  $\{u_i\}$ ,  $N(u_i) \cap B_i$ ,  $(N(U - \{u_i\}))^- \cap B_i$  and  $(N_2(U) - N(u_i)) \cap B_i$  are pairwise disjoint subsets in  $B_i$  for each  $1 \leq i \leq 5$ , where  $(N(U - \{u_i\}))^- \cap B_i = \{x^- \mid x \in N(U - \{u_i\}) \cap B_i\}$ . Recall that  $N_5(U) = N_4(U) = (N_3(U) - N(u_i)) \cap B_i = \emptyset$  (for each  $1 \leq i \leq 5$ ) by Claims 2.8 and 2.9. Then for each  $1 \leq i \leq 3$ , we conclude that

$$\begin{aligned}
|B_i| &\geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}))^- \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
&= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
&= 1 + \sum_{j=1}^5 |N(u_j) \cap B_i| - |N_3(U) \cap N(u_i) \cap B_i| \\
&\geq \sum_{j=1}^5 |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}|, \tag{1}
\end{aligned}$$

where the last inequality follows from Claims 2.11 and 2.13. Similarly, for each  $4 \leq i \leq 5$ , we have

$$\begin{aligned}
|B_i| &\geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}))^- \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
&= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| + |(N_2(U) - N(u_i)) \cap B_i| \\
&= 1 + \sum_{j=1}^5 |N(u_j) \cap B_i| - |N_3(U) \cap N(u_i) \cap B_i| \\
&= 1 + \sum_{j=1}^5 |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}|, \tag{2}
\end{aligned}$$

where the last equality follows from Claims 2.6 and 2.10.

For each  $1 \leq i \leq 5$ , we define  $d_i = |N(u_i) \cap P|$ . Then  $d_4 = d_5 = 0$  by Claim 2.3. By applying Claim 2.4, we know that  $N(u_1) \cap P, N(u_2) \cap P$  and  $N(u_3) \cap P$  are pairwise disjoint. Therefore,

$$|P| \geq \sum_{i=1}^5 d_i = \sum_{i=1}^5 |N(u_i) \cap P|.$$

Moreover, if  $P \neq \emptyset$ , then by Claim 2.5, we see that  $N(U) \cap \{s^+\} = \emptyset$  and hence

$$|P| \geq 1 + \sum_{i=1}^5 d_i = 1 + \sum_{i=1}^5 |N(u_i) \cap P|.$$

It follows from Claim 2.8 that  $\sum_{i=1}^5 |N(u_i) \cap \{t\}| \leq 3$ . If  $\sum_{i=1}^5 |N(u_i) \cap \{t\}| \leq 2$ , then

$$|V(P_T[s, t])| = 2 + |P| \geq \sum_{i=1}^5 |N(u_i) \cap \{t\}| + \sum_{i=1}^5 |N(u_i) \cap P|.$$

If  $\sum_{i=1}^5 |N(u_i) \cap \{t\}| = 3$ , then by Claim 2.7, we deduce that  $P \neq \emptyset$ . This implies that

$$|V(P_T[s, t])| = 2 + |P| \geq 2 + \left(1 + \sum_{i=1}^5 |N(u_i) \cap P|\right) = \sum_{i=1}^5 |N(u_i) \cap \{t\}| + \sum_{i=1}^5 |N(u_i) \cap P|.$$

In both cases, we have

$$|V(P_T[s, t])| \geq \sum_{i=1}^5 |N(u_i) \cap \{t\}| + \sum_{i=1}^5 |N(u_i) \cap P|. \quad (3)$$

Note that  $N(U) \subseteq V(T)$ . By (1), (2) and (3), we conclude that

$$\begin{aligned} |V(T)| &= \sum_{i=1}^3 |B_i| + \sum_{i=4}^5 |B_i| + |V(P_T[s, t])| \\ &\geq \sum_{i=1}^3 \left( \sum_{j=1}^5 |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}| \right) + \sum_{i=4}^5 \left( 1 + \sum_{j=1}^5 |N(u_j) \cap B_i| + |N(u_i) \cap \{s\}| \right) \\ &\quad + \left( \sum_{i=1}^5 |N(u_i) \cap \{t\}| + \sum_{i=1}^5 |N(u_i) \cap P| \right) \\ &= 2 + \sum_{i=1}^5 \sum_{j=1}^5 |N(u_j) \cap B_i| + \sum_{i=1}^5 |N(u_i) \cap \{s, t\}| + \sum_{i=1}^5 |N(u_i) \cap P| \\ &= \sum_{j=1}^5 |N(u_j) \cap V(T)| + 2 \\ &= \sum_{j=1}^5 d(u_j) + 2 \\ &= d(U) + 2. \end{aligned}$$

Since  $U$  is an independent set in  $G$ , we have

$$n - 1 \leq \sigma_5(G) \leq d(U) \leq |V(T)| - 2 \leq n - 2,$$

a contradiction.

*Case 2.*  $T$  contains only one branch vertex.

Let  $r$  be the unique branch vertex in  $T$  with  $d_T(r) = 5$  and let  $N_T(r) = \{v_1, v_2, v_3, v_4, v_5\}$ . Since  $G$  is  $K_{1,5}$ -free, there exist two distinct  $i, j \in \{1, 2, 3, 4, 5\}$  such that  $v_i v_j \in E(G)$ . Let  $T' := T - rv_i + v_i v_j$ . If  $v_j$  is a leaf of  $T$ , then  $T'$  is a tree in  $G$  with 4 leaves and  $V(T') = V(T)$ , which contradicts Lemma 2.1. So we may assume that  $v_j$  has degree two in  $T$ . Then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $r$  and  $v_j$ ,  $d_{T'}(r) = 4$  and  $d_{T'}(v_j) = 3$ . By the same argument as in the proof of Case 1, we can also derive a contradiction.

*Case 3.*  $T$  contains three branch vertices.

Let  $s, w$  and  $t$  be the three branch vertices in  $T$  such that  $d_T(s) = d_T(w) = d_T(t) = 3$  and  $w \in V(P_T[s, t])$ . For each  $1 \leq i \leq 5$ , let  $B_i$  be the vertex set of the connected component of  $T - \{s, w, t\}$  containing  $u_i$  and let  $v_i$  be the unique vertex in  $B_i \cap N_T(\{s, w, t\})$ . Without loss of generality, we may assume that  $\{v_1, v_2\} \subseteq N_T(s)$ ,  $\{v_3, v_4\} \subseteq N_T(t)$  and  $v_5 \in N_T(w)$ . For each  $1 \leq i \leq 5$  and  $x \in B_i$ , we use  $x^-$  to denote the predecessor of  $x$  on  $P_T[s, u_i]$  or  $P_T[t, u_i]$  or  $P_T[w, u_i]$ . Define  $P := V(P_T[s, t]) - \{s, w, t\}$ .

For this case, we further choose  $T$  such that

(C3)  $d_T[s, t]$  is as small as possible.

It is easy to check that the following claim still holds in this case. (The proof is exactly the same as that of Claim 2.2.)

**Claim 2.14** *For all  $1 \leq i, j \leq 5$  and  $i \neq j$ , if  $x \in N(u_j) \cap B_i$ , then  $x \notin \{u_i, v_i\}$  and  $x^- \notin N(U - \{u_j\})$ .*

By applying Claim 2.14, we deduce that  $U$  is an independent set in  $G$ .

**Claim 2.15**  $N(u_i) \cap P = \emptyset$  for each  $1 \leq i \leq 4$ .

*Proof.* Suppose to the contrary that there exists some vertex  $x \in P$  such that  $xu_i \in E(G)$  for some  $1 \leq i \leq 4$ . Without loss of generality, we may assume that  $x \in V(P_T[s, w]) - \{s, w\}$ . Let  $T' := T - v_i v_i^- + xu_i$ . If  $i \in \{1, 2\}$ , then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has three branch vertices  $x, w$  and  $t$ ,  $d_{T'}(x) = d_{T'}(w) = d_{T'}(t) = 3$ ,  $w \in V(P_{T'}[x, t])$  and  $d_{T'}[x, t] < d_T[s, t]$ , contradicting the condition (C3). Hence we have  $i \in \{3, 4\}$ . Now,  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has three branch vertices  $s, x$  and  $w$ ,  $d_{T'}(s) = d_{T'}(x) = d_{T'}(w) = 3$ ,  $x \in V(P_{T'}[s, w])$  and  $d_{T'}[s, w] < d_T[s, t]$ . But this also contradicts the condition (C3). So the claim holds. ■

**Claim 2.16**  $N(u_i) \cap \{w, t\} = \emptyset$  for each  $1 \leq i \leq 2$  and  $N(u_j) \cap \{s, w\} = \emptyset$  for each  $3 \leq j \leq 4$ .

*Proof.* Suppose there exists some  $i \in \{1, 2\}$  such that  $wu_i \in E(G)$  or  $tu_i \in E(G)$ . Then

$$T' := \begin{cases} T - sv_i + wu_i, & \text{if } wu_i \in E(G), \\ T - sv_i + tu_i, & \text{if } tu_i \in E(G), \end{cases}$$

is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$  and  $T'$  has two branch vertices  $w$  and  $t$ . By the same argument as in the proof of Case 1, we can obtain a contradiction. Therefore,  $N(u_i) \cap \{w, t\} = \emptyset$  for each  $1 \leq i \leq 2$ .

By a similar argument as above (by exchanging the roles of  $s$  and  $t$ ), we can also show that  $N(u_j) \cap \{s, w\} = \emptyset$  for each  $3 \leq j \leq 4$ . This completes the proof of Claim 2.16. ■

**Claim 2.17**  $N(u_5) \cap \{s, t\} = \emptyset$ .

*Proof.* Suppose for a contradiction that  $N(u_5) \cap \{s, t\} \neq \emptyset$ . Then we have  $su_5 \in E(G)$  or  $tu_5 \in E(G)$ . Define

$$T' := \begin{cases} T - wv_5 + su_5, & \text{if } su_5 \in E(G), \\ T - wv_5 + tu_5, & \text{if } tu_5 \in E(G). \end{cases}$$

Now,  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$  and  $T'$  has two branch vertices  $s$  and  $t$ . By the same argument as in the proof of Case 1, we can derive a contradiction. So the assertion of the claim holds. ■

**Claim 2.18**  $N_{\geq 2}(U - \{u_i\}) \cap B_i = \emptyset$  for each  $1 \leq i \leq 5$ .

*Proof.* Suppose the assertion of the claim is false. Then there exists some vertex  $x \in N_{\geq 2}(U - \{u_i\}) \cap B_i$  for some  $1 \leq i \leq 5$ . By Claim 2.14, we see that  $x \notin \{u_i, v_i\}$ . Since  $x \in N_{\geq 2}(U - \{u_i\})$ , there must exist two distinct  $j, k \in \{1, 2, 3, 4, 5\} - \{i\}$  such that  $xu_j, xu_k \in E(G)$ . By symmetry between  $j$  and  $k$ , we can always choose  $j$  such that  $v_i^- \neq v_j^-$ . Let  $T' := T - \{v_i v_i^-, v_j v_j^-\} + \{xu_j, xu_k\}$  and let  $y := \{s, w, t\} - \{v_i^-, v_j^-\}$ . Then  $T'$  is a tree in  $G$  with 5 leaves such that  $V(T') = V(T)$ ,  $T'$  has two branch vertices  $x$  and  $y$ ,  $d_{T'}(x) = 4$  and  $d_{T'}(y) = 3$ . By the same argument as in the proof of Case 1, we can deduce a contradiction. This proves Claim 2.18. ■

By applying Claim 2.14, we conclude that  $\{u_i\}$ ,  $N(u_i) \cap B_i$  and  $(N(U - \{u_i\}))^- \cap B_i$  are pairwise disjoint subsets in  $B_i$  for each  $1 \leq i \leq 5$ , where  $(N(U - \{u_i\}))^- \cap B_i = \{x^- \mid x \in N(U - \{u_i\}) \cap B_i\}$ . It follows from Claims 2.15–2.18 that  $N_5(U) = N_4(U) = N_3(U) = (N_2(U) - N(u_i)) \cap B_i = \emptyset$  (for each  $1 \leq i \leq 5$ ). Therefore, for each  $1 \leq i \leq 5$ , we have

$$\begin{aligned} |B_i| &\geq 1 + |N(u_i) \cap B_i| + |(N(U - \{u_i\}))^- \cap B_i| \\ &= 1 + |N(u_i) \cap B_i| + |N(U - \{u_i\}) \cap B_i| \\ &= 1 + \sum_{j=1}^5 |N(u_j) \cap B_i|. \end{aligned} \quad (4)$$

By Claims 2.16 and 2.17, we know that

$$\begin{aligned} \sum_{i=1}^5 |N(u_i) \cap \{s, w, t\}| &= \sum_{i=1}^2 |N(u_i) \cap \{s, w, t\}| + \sum_{i=3}^4 |N(u_i) \cap \{s, w, t\}| + |N(u_5) \cap \{s, w, t\}| \\ &\leq 2 + 2 + 1 \\ &= 5. \end{aligned}$$

On the other hand, by Claim 2.15, we have

$$\sum_{i=1}^5 |N(u_i) \cap P| = |N(u_5) \cap P| \leq |P|.$$

Hence

$$|V(P_T[s, t])| = 3 + |P| = 5 + |P| - 2 \geq \sum_{i=1}^5 |N(u_i) \cap \{s, w, t\}| + \sum_{i=1}^5 |N(u_i) \cap P| - 2. \quad (5)$$

Since  $N(U) \subseteq V(T)$  and by (4) and (5), we deduce that

$$\begin{aligned} |V(T)| &= \sum_{i=1}^5 |B_i| + |V(P_T[s, t])| \\ &\geq \sum_{i=1}^5 \left( 1 + \sum_{j=1}^5 |N(u_j) \cap B_i| \right) + \left( \sum_{i=1}^5 |N(u_i) \cap \{s, w, t\}| + \sum_{i=1}^5 |N(u_i) \cap P| - 2 \right) \end{aligned}$$

$$\begin{aligned}
&= 5 + \sum_{i=1}^5 \sum_{j=1}^5 |N(u_j) \cap B_i| + \sum_{i=1}^5 |N(u_i) \cap \{s, w, t\}| + \sum_{i=1}^5 |N(u_i) \cap P| - 2 \\
&= \sum_{j=1}^5 |N(u_j) \cap V(T)| + 3 \\
&= \sum_{j=1}^5 d(u_j) + 3 \\
&= d(U) + 3.
\end{aligned}$$

This implies that

$$\sigma_5(G) \leq d(U) \leq |V(T)| - 3 \leq n - 3,$$

contradicting the assumption that  $\sigma_5(G) \geq n - 1$ . This completes the proof of Theorem 1.9. ■

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