# REGULAR S-ACTS WITH PRIMITIVE NORMAL AND ANTIADDITIVE THEORIES

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ABSTRACT. In this work we investigate the commutative monoids over which the axiomatizable class of regular S-acts is primitive normal and antiadditive. We prove that the primitive normality of an axiomatizable class of regular S-acts over the commutative monoid S is equivalent to the antiadditivity of this class and it is equivalent to a linearly order of semigroup R such that an S-act S is a maximum under the inclusion regular subact of S-act S.

## 1. Introduction

In [1] the primitive normal, primitive connected and additive theories of S-acts are studied. In particular it is proved that a class of all S-acts is primitive normal if and only if S is a linearly ordered monoid. In [2] on a language of a structure of primitive equivalences there are described S-acts with primitive normal, additive and antiadditive theories. It is shown that the class of all S-acts is antiadditive only for a linearly ordered monoid S, that is the class of all S-acts is antiadditive if and only if this class is primitive normal. In this work we investigate the commutative monoids over which the axiomatizable class of regular S-acts is primitive normal and antiadditive. We prove that the primitive normality of an axiomatizable class of regular S-acts over the commutative monoid S is equivalent to the antiadditivity of this class and it is equivalent to a linearly order of semigroup R such that an S-act S is a maximum under the inclusion regular subact of S-act S.

Let T be a complete first order theory of a language L. We fix some large much saturated model  $\mathcal{C}$  of T and we suppose that all considered models of the theory are its elementary submodels. All elements, tuples of elements and sets will be taken from  $\mathcal{C}$ . The tuples of elements  $\langle a_0, \ldots, a_{n-1} \rangle$  and variables  $\langle x_0, \ldots, x_{n-1} \rangle$  will be denoted by  $\bar{a}$  and  $\bar{x}$  accordingly. Let  $\bar{s} = \langle s_0, \ldots, s_{n-1} \rangle$  and  $\bar{t} = \langle t_0, \ldots, t_{k-1} \rangle$  be the tuples of variables or elements, A be a set. We will often write  $\bar{s} \in A$  instead  $s_0, \ldots, s_{n-1} \in A$ ,  $\bar{s}(i)$  instead  $s_i$ ,  $\exists \bar{s}$  instead  $\exists s_0 \ldots \exists s_{n-1}$ . The set  $\{s_0, \ldots, s_{n-1}, t_0, \ldots, t_{k-1}\}$  we will denote by  $\bar{s} \cup \bar{t}$ . We will denote the length of a tuple  $\bar{s}$  by  $|\bar{s}|$ , i.e.  $|\bar{s}| = n$ . If  $\Phi(\bar{x}, \bar{y})$  is a formula of language L, A is a model of the theory T,  $\bar{a}$  is a tuple of elements from A and  $|\bar{a}| = |\bar{y}|$ , then  $\Phi(A, \bar{a})$  will denote the set  $\{\bar{b} \mid A \models \Phi(\bar{b}, \bar{a})\}$ .

The formula of a form

$$\exists \bar{x} (\Phi_0 \wedge \cdots \wedge \Phi_k),$$

where  $\Phi_i$  are the atomic formulas  $(0 \le i \le k)$ , is called a primitive formula.

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Let  $\Phi(\bar{x}, \bar{y})$  be a primitive formula of language L,  $\bar{a}$ ,  $\bar{b}$  be the tuples of elements and  $|\bar{a}| = |\bar{b}| = |\bar{y}|$ . The set  $\Phi(\mathcal{C}, \bar{a})$  is called a primitive set. The sets  $\Phi(\mathcal{C}, \bar{a})$  and  $\Phi(\mathcal{C}, \bar{b})$  are called the primitive copies.

A theory T is called *primitive normal* if for each primitive copies X, Y we have X = Y or  $X \cap Y = \emptyset$ . An axiomatizable class of structures K of language L is called *primitive normal* if the theory of this class is primitive normal. It is known (see [3]) that the Cartesian closed stable class of structures is primitive normal.

An equivalence  $\alpha$  on some set X of n-tuples of elements from  $\mathcal{C}$ , which is defined in  $\mathcal{C}$  by some primitive formula  $\Phi(\bar{x}_1, \bar{x}_2)$ , is called a primitive equivalence. The domain X of such equivalence  $\alpha$  is defined in  $\mathcal{C}$  by primitive formula  $\Phi(\bar{x}, \bar{x})$  and is denoted by  $dom(\alpha)$ . If  $\bar{a} \in X$  then  $\alpha$ -class which contains  $\bar{a}$  will be denoted by  $\bar{a}/\alpha$ .

A set X is called  $\Delta$ -primitive if there exists a family S of primitive sets such that

$$X = \bigcap \{Y \mid Y \in S\}.$$

A set of form  $X = X^*/\alpha = \{\bar{a}/\alpha \mid \bar{a} \in X^*\}$ , where  $X^*$  is  $\Delta$ -primitive set,  $\alpha$  is primitive equivalence and  $X^* \subseteq dom(\alpha)$ , is called a generalized primitive set. A set  $X^*$  is called a basis and  $\alpha$  is called a generative equivalence of generalized primitive set X.

The theory T is called antiadditive if it is primitive normal and there is no infinite generalized primitive set which is an Abelian group under the defined by primitive formula operation. An axiomatizable class of structures K of language L is called antiadditive if the theory of this class is antiadditive.

Let us remind some concepts from the theory of S-acts. Throughout this paper S will denote a monoid with identity 1 and set of idempotents E. A structure  $\langle A; s \rangle_{s \in S}$  of the language  $L_S = \{s \mid s \in S\}$  is a (left) S-act if  $s_1(s_2a) = (s_1s_2)a$  and 1a = a for all  $s_1, s_2 \in S$  and  $a \in A$ . An S-act  $\langle A; s \rangle_{s \in S}$  we will denote by  $_SA$ . All S-acts, treated in the article, are left S-acts.

Let  ${}_SA$ ,  ${}_SB$  be S-acts. We call  $a \in A$  an act-regular element if there exists a homomorphism  $\varphi: {}_SSa \longrightarrow {}_SS$  such that  $\varphi(a)a = a$ . An S-act  ${}_SA$  is called regular if all its elements are act-regular. If for elements  $a \in A$  and  $b \in B$  there is an isomorphism  $f: {}_SSa \to {}_SSb$  such that f(a) = b then we will write  ${}_SSa \xrightarrow{}_SSb$ .

Note that the union of all regular subacts of an S-act is also a regular subact. The union of all regular subacts of an S-act  $_SS$  we will denote by  $_SR$ . Hereinafter, we assume that  $R \neq \emptyset$ .

A semigroup T is called *linearly ordered* if for all  $a, b \in T$  either  $Ta \subseteq Tb$  or  $Tb \subseteq Ta$ . A monoid S is called *regularly linearly ordered* if for all  $a \in R$  a semigroup Sa is linearly ordered.

We will distinguish the symbols of set-theoretic inclusion  $\subset$  and  $\subseteq$ .

# 2. Primitive normal Classes of Regular Acts

We will use the following remark without references to it.

**Remark 2.1.** For all  $a \in S$ ,  $e, f \in E$  we have

- 1)  $aS \subseteq eS \iff ea = a$ ;
- 2)  $Sa \subseteq Se \iff ae = a$ .

**Theorem 2.2.** An S-act <sub>S</sub>A is primitive normal if and only if for any pairwise disjoint finite sets of indexes I, J, K, any  $s_i, l_i, r_k^1, r_k^2 \in S$   $(i \in I, j \in J, k \in K), \bar{a}_1, \bar{a}_2, \bar{a}_3 \in A$ ,

$$|\bar{a}_1| = |\bar{a}_2| = |\bar{a}_3| = n, if$$
(1)

$$_{S}A \models \bigwedge_{i \in I} s_{i}\bar{a}_{1}(l_{i}) = s_{i}\bar{a}_{2}(l_{i}) \land \bigwedge_{j \in J} t_{j}\bar{a}_{2}(l_{j}) = t_{j}\bar{a}_{3}(l_{j}) \land \bigwedge_{k \in K} \bigwedge_{m \in \{1,2,3\}} r_{k}^{1}\bar{a}_{m}(l_{k}) = r_{k}^{2}\bar{a}_{m}(l_{k}),$$

where  $0 \le l_i, l_j, l_k \le n-1$ , then there exists  $b \in A$  such that |b| = n and

$$(2) \qquad {}_{S}A \models \bigwedge_{i \in I} s_{i}\bar{a}_{3}(l_{i}) = s_{i}\bar{b}(l_{i}) \land \bigwedge_{j \in J} t_{j}\bar{b}(l_{j}) = t_{j}\bar{a}_{1}(l_{j}) \land \bigwedge_{k \in K} r_{k}^{1}\bar{b}(l_{k}) = r_{k}^{2}\bar{b}(l_{k}).$$

*Proof. Necessity.* Let  ${}_SA$  be a primitive normal S-act and (1) hold for some pairwise disjoint finite sets of indexes I, J, K, some  $s_i, l_j, r_k^1, r_k^2 \in S$   $(i \in I, j \in J, k \in K)$  and  $\bar{a}_1, \bar{a}_2, \bar{a}_3 \in A, |\bar{a}_1| = |\bar{a}_2| = |\bar{a}_3| = n$ . We put

$$\Phi(\bar{x}, \bar{y}) \leftrightharpoons \exists \bar{u}(\bigwedge_{i \in I} s_i \bar{x}(l_i) = s_i \bar{u}(l_i) \land \bigwedge_{j \in J} t_j \bar{u}(l_j) = t_j \bar{y}(l_j) \land$$

$$\wedge \bigwedge_{k \in K} r_k^1 \bar{x}(l_k) = r_k^2 \bar{x}(l_k) \wedge \bigwedge_{k \in K} r_k^1 \bar{u}(l_k) = r_k^2 \bar{u}(l_k) \wedge \bigwedge_{k \in K} r_k^1 \bar{y}(l_k) = r_k^2 \bar{y}(l_k)),$$

where  $|\bar{x}| = |\bar{u}| = |\bar{y}| = n$ . By a condition  $\bar{a}_1 \in \Phi(A, \bar{a}_1)$  and  $\bar{a}_1, \bar{a}_3 \in \Phi(A, \bar{a}_3)$ . Since the S-act  $_SA$  is primitive normal then  $\bar{a}_3 \in \Phi(A, \bar{a}_1)$  that is the condition (2) holds. Sufficiency. Let  $\Psi(\bar{x}, \bar{y})$  be a primitive formula,

$$\Psi(\bar{x}, \bar{y}) \leftrightharpoons \exists \bar{u} \Theta(\bar{x}, \bar{y}, \bar{u}),$$

where

$$\Theta(\bar{x}, \bar{y}, \bar{u}) \leftrightharpoons \bigwedge_{\langle i, j \rangle \in I_1} s_i^1 \bar{x}(l_i) = t_j^1 \bar{x}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_2} s_i^2 \bar{x}(l_i) = t_j^2 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_j) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) = t_j^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge_{\langle i, j \rangle \in I_3} s_i^3 \bar{u}(l_i) \land \bigwedge$$

$$\wedge \bigwedge_{\langle i,j\rangle \in I_4} s_i^4 \bar{u}(l_i) = t_j^4 \bar{y}(l_j) \wedge \bigwedge_{\langle i,j\rangle \in I_5} s_i^5 \bar{y}(l_i) = t_j^5 \bar{y}(l_j) \wedge \bigwedge_{\langle i,j\rangle \in I_6} s_i^6 \bar{y}(l_i) = t_j^6 \bar{x}(l_j),$$

 $s_i^k, t_j^k \in S \ (\langle i, j \rangle \in I_k, \ 1 \leqslant k \leqslant 6)$ . We will show that  ${}_SA$  is a primitive normal S-act. Suppose that  $\bar{a}_2 \in \Psi(A, \bar{a}_1), \ \bar{a}_2, \bar{a}_3 \in \Psi(A, \bar{a}_4)$  and  $\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4 \in A$ . Then

(3) 
$$SA \models \Theta(\bar{a}_2, \bar{a}_1, \bar{b}_{21}) \land \Theta(\bar{a}_2, \bar{a}_4, \bar{b}_{24}) \land \Theta(\bar{a}_3, \bar{a}_4, \bar{b}_{34})$$

for some  $\bar{b}_{21}, \bar{b}_{24}, \bar{b}_{34} \in A$ . It is enough to show that  $\bar{a}_3 \in \Psi(A, \bar{a}_1)$ . From (3) we have

(4) 
$$sA \models \bigwedge_{\langle i,j \rangle \in I_1} s_i^1 \bar{a}_3(l_i) = t_j^1 \bar{a}_3(l_j) \land \bigwedge_{\langle i,j \rangle \in I_5} s_i^5 \bar{a}_1(l_i) = t_j^5 \bar{a}_1(l_j);$$

(5) 
$$sA \models \bigwedge_{\langle i,j \rangle \in I_2} t_j^2 \bar{b}_{21}(l_j) = s_i^2 \bar{a}_2(l_i) = t_j^2 \bar{b}_{24}(l_j);$$

(6) 
$$_{S}A \models \bigwedge_{\langle i,j \rangle \in I_{4}} s_{i}^{4} \bar{b}_{24}(l_{i}) = t_{j}^{4} \bar{a}_{4}(l_{j}) = s_{i}^{4} \bar{b}_{34}(l_{i});$$

(7) 
$$sA \models \bigwedge_{\langle i,j\rangle \in I_3} s_i^3 \bar{b}_{km}(l_i) = t_j^3 \bar{b}_{km}(l_j);$$

(8) 
$$_{S}A \models \bigwedge_{\langle i,j \rangle \in I_{6}} s_{i}^{6} \bar{a}_{1}(l_{i}) = t_{j}^{6} \bar{a}_{2}(l_{j}) = s_{i}^{6} \bar{a}_{4}(l_{i}) = t_{j}^{6} \bar{a}_{3}(l_{j})$$

for all  $\langle k, m \rangle \in \{\langle 2, 1 \rangle, \langle 2, 4 \rangle, \langle 3, 4 \rangle\}$ . The conditions of Theorem and (5), (6), (7) imply

$$_{S}A \models \bigwedge_{\langle i,j \rangle \in I_{2}} t_{j}^{2} \bar{b}_{34}(l_{j}) = t_{j}^{2} \bar{b}(l_{j}) \land \bigwedge_{\langle i,j \rangle \in I_{4}} s_{i}^{4} \bar{b}(l_{i}) = s_{i}^{4} \bar{b}_{21}(l_{i}) \land \bigwedge_{\langle i,j \rangle \in I_{3}} s_{i}^{3} \bar{b}(l_{i}) = t_{j}^{3} \bar{b}(l_{j})$$

for some  $\bar{b} \in A$ . Using this and (3) we have

$$_{S}A \models \bigwedge_{\langle i,j \rangle \in I_{2}} s_{i}^{2} \bar{a}_{3}(l_{i}) = t_{j}^{2} \bar{b}(l_{j}) \wedge \bigwedge_{\langle i,j \rangle \in I_{4}} s_{i}^{4} \bar{b}(l_{i}) = t_{j}^{4} \bar{a}_{1}(l_{j}) \wedge \bigwedge_{\langle i,j \rangle \in I_{3}} s_{i}^{3} \bar{b}(l_{i}) = t_{i}^{3} \bar{b}(l_{i}).$$

This one, (4) and (8) imply  ${}_SA \models \Theta(\bar{a}_3, \bar{a}_1, b)$ , that is  $\bar{a}_3 \in \Psi(A, \bar{a}_1)$ . So we have proved that  ${}_SA$  is a primitive normal S-act.

**Proposition 2.3.** [4] Let  $_SA$  be an S-act,  $a \in A$ . An element a is act-regular if and only if there exists an idempotent  $e \in R$  such that  $_SSa \xrightarrow{\sim} _SSe$ .

**Proposition 2.4.** [5] If the class  $\Re$  of regular S-acts is axiomatizable then  $R = \bigcup \{e_i R \mid 1 \leqslant i \leqslant n\}$  for some  $n \geqslant 1$ ,  $e_i \in R$ ,  $e_i^2 = e_i$   $(1 \leqslant i \leqslant n)$ .

**Lemma 2.5.** Let the class  $\mathfrak{R}$  of regular S-acts is axiomatizable and primitive normal. Then R is a regularly linearly ordered monoid.

*Proof.* Assume that  $Sc \not\subseteq Sb$  and  $Sb \not\subseteq Sc$  for some  $b, c \in Sa$   $a \in R$ . There exists  $e \in E$  such that  ${}_{S}Sa \xrightarrow{\sim} {}_{S}Se$ . Let  ${}_{S}Se_{i}$   $(1 \leqslant i \leqslant 3)$  be the pairwise disjoint copies of S-act  ${}_{S}Se$ ,  $\Theta$  be a congruence of S-act  $\bigsqcup_{i=1}^{3} {}_{S}Se_{i}$  generated by  $\{\langle ce_{1}, ce_{2} \rangle, \langle be_{2}, be_{3} \rangle\}$ . Let

 $_{S}A$  denote an S-act  $\bigsqcup_{i=1}^{3} {_{S}Se_{i}}/\Theta$ ,  $d/\Theta$  denote a  $\Theta$ -class of  $d \in \bigsqcup_{i=1}^{3} {Se_{i}}$ . Let

$$\Phi(x,y) \rightleftharpoons \exists u(bu = x \land cu = y).$$

Hence

$$_{S}A\models\Phi(be_{1}/\Theta,ce_{1}/\Theta)\wedge\Phi(be_{2}/\Theta,ce_{1}/\Theta)\wedge\Phi(be_{2}/\Theta,ce_{3}/\Theta).$$

Since the class  $\mathfrak{R}$  is primitive normal then  ${}_{S}A \models \Phi(be_{1}/\Theta, ce_{3}/\Theta)$ . Let  $u^{0} \in \bigsqcup_{i=1}^{3} Se_{i}$  such that  ${}_{S}A \models bu^{0}/\Theta = be_{1}/\Theta \wedge cu^{0}/\Theta = ce_{3}/\Theta$ . Then  $be_{1}/\Theta, ce_{3}/\Theta \in Su^{0}/\Theta$ . But  $be_{1}/\Theta = \{be_{1}\}, ce_{3}/\Theta = \{ce_{3}\}$  and S-acts  ${}_{S}Se_{1}, {}_{S}Se_{3}$  do not intersect. A contradiction.

**Lemma 2.6.** Let S be a commutative monoid, the class  $\mathfrak{R}$  of regular S-acts is axiomatizable and primitive normal. Then for any idempotents  $e, f \in R$  either  $Se \subseteq Sf$  or  $Sf \subseteq Se$ .

*Proof.* Suppose that  $Se \not\subseteq Sf$  and  $Sf \not\subseteq Se$  for some idempotents  $e, f \in R$ . Note that  $ef \in E$ . If Sef = Sf then ef = fef = f, that is  $Sf \subseteq Se$ , a contradiction. Hence  $Sef \subset Sf$ . Similarly  $Sef = Sfe \subset Se$ . Let

$$\Phi(x,y) \rightleftharpoons \exists u(eu = ex \land fu = fy).$$

So

$$_{S}R \models \Phi(e,e) \land \Phi(f,f) \land \Phi(f,e).$$

Since the class  $\mathfrak{R}$  is primitive normal then  ${}_{S}R \models \Phi(e,f)$ . Let  $u^0 \in R$  such that  ${}_{S}R \models e = eu^0 \land f = fu^0$ . Then  $e, f \in Su^0$ . Hence by Lemma 2.5 either  $Se \subseteq Sf$  or  $Sf \subseteq Se$ , but that contradicts to assumption.

**Theorem 2.7.** Let S be a commutative monoid and the class  $\mathfrak{R}$  of regular S-acts is axiomatizable. The class  $\mathfrak{R}$  is primitive normal if and only if a semigroup R is linearly ordered.

Proof. Necessity. Let the class  $\mathfrak{R}$  is primitive normal. We will show that a semigroup R is linearly ordered. Since the class  $\mathfrak{R}$  is axiomatizable then by Proposition 2.4  $R = \bigcup \{e_i R \mid 1 \leqslant i \leqslant m\}$  for some  $m \geqslant 1$  and idempotents  $e_i \in R$   $(1 \leqslant i \leqslant m)$ . As  $sf = fsf \in fR$  for all  $s \in S$  then fR = Sf, where  $f \in R \cap E$ . So in view of commutativity of a monoid S and by Lemma 2.6 R = eR = Se for some idempotent  $e \in R$ . Thus by Lemma 2.5 R is a linearly ordered semigroup.

Sufficiency. Let  $_SA \in \mathfrak{R}, \ I, J, K$  be the pairwise disjoint finite sets of indexes,  $s_i, l_j, r_k^1, r_k^2 \in S$   $(i \in I, j \in J, k \in K), \ \bar{a}_1, \bar{a}_2, \bar{a}_3 \in A, \ |\bar{a}_1| = |\bar{a}_2| = |\bar{a}_3| = n$  and (1) holds, where  $0 \leq l_i, l_j, l_k \leq n-1$ . By Proposition 2.3 there exists the tuples of idempotents  $\bar{e}_1, \bar{e}_2, \bar{e}_3 \in R$  such that  $|\bar{e}_1| = |\bar{e}_2| = |\bar{e}_3| = n$  and  $_SS\bar{a}_i(j) \xrightarrow{\sim} _SS\bar{e}_i(j)$  for all  $i, j, 0 \leq i \leq 3, 0 \leq j \leq n-1$ . Then  $\bar{a}_i(j) = \bar{e}_i(j)\bar{a}_i(j)$  for all  $i, j, 0 \leq i \leq 3, 0 \leq j \leq n-1$ .

We will construct a tuple  $\bar{b}$  such that (2) holds.

Let us fix  $l \in \{0, 1, ..., n-1\}$ . We put  $I_l = \{i \in I \mid l_i = l\}$ ,  $J_l = \{j \in J \mid l_j = l\}$ . Let  $1 \leq k \leq 3$ . Since by the condition the set  $\{Sd \mid Sd \subseteq Se_k(l)\}$  is linearly ordered under the inclusion, then there exist  $s^k, t^k \in S$  such that  $Ss^k e_k(l) = \max\{Ss_i e_k(l) \mid i \in I_l\}$  and  $St^k e_k(l) = \max\{St_j e_k(l) \mid j \in J_l\}$ . Hence for all  $i \in I_l$  and  $j \in J_l$  there are  $r_i^k \in S$  and  $r_j^k \in S$  such that  $s_i \bar{e}_k(l) = r_i^k s^k \bar{e}_k(l)$  and  $t_j \bar{e}_k(l) = r_j^k t^k \bar{e}_k(l)$ , so  $s_i \bar{a}_k(l) = r_i^k s^k \bar{a}_k(l)$  and  $t_j \bar{a}_k(l) = r_i^k t^k \bar{a}_k(l)$ .

Assume that

(9) 
$$S\bar{e}_1(l) \subseteq S\bar{e}_2(l) \text{ and } S\bar{e}_3(l) \subseteq S\bar{e}_2(l),$$

that is  $\bar{e}_k(l) = \bar{e}_2(l)\bar{e}_k(l)$  and  $\bar{a}_k(l) = \bar{e}_2(l)\bar{a}_k(l)$  for all  $k, 1 \leqslant k \leqslant 3$ . Since the semigroup  $S\bar{e}_2(l)$  is linearly ordered, without loss of generality we can suppose that  $S\bar{e}_1(l) \subseteq Se_3(l)$ , i.e.  $\bar{e}_1(l) = \bar{e}_1(l)\bar{e}_3(l)$ . Assume that  $St^2\bar{e}_2(l) \subseteq Ss^2\bar{e}_2(l)$ , i.e.  $t^2\bar{e}_2(l) = r^2s^2\bar{e}_2(l)$  for some  $r^2 \in S$ . Let  $j \in J_l$ . Then  $t_j\bar{e}_2(l) = r_j^2t^2\bar{e}_2(l) = r_j^2r^2s^2\bar{e}_2(l)$  and  $t_j\bar{a}_2(l) = r_j^2r^2s^2\bar{a}_2(l)$ . So in view (1) we have

$$t_i \bar{a}_3(l) = t_i \bar{a}_2(l) = r_i^2 r^2 s^2 \bar{a}_2(l) = r_i^2 r^2 s^2 \bar{a}_1(l) = r_i^2 r^2 s^2 \bar{e}_2(l) \bar{a}_1(l) = t_i \bar{e}_2(l) \bar{a}_1(l) = t_i \bar{a}_1(l),$$

that is  $t_j \bar{a}_3(l) = t_j \bar{a}_1(l)$ . We put  $\bar{b}(l) = \bar{a}_3(l)$ . If  $Ss^2 \bar{e}_2(l) \subseteq St^2 \bar{e}_2(l)$  then in the same way we have  $\bar{b}(l) = \bar{a}_1(l)$ .

Let (9) be wrong. Then without loss of generality we can suppose that

$$S\bar{e}_3(l) \subseteq S\bar{e}_1(l)$$
 and  $S\bar{e}_2(l) \subseteq S\bar{e}_1(l)$ ,

that is  $\bar{e}_k(l) = \bar{e}_1(l)\bar{e}_k(l)$  and  $\bar{a}_k(l) = \bar{e}_1(l)\bar{a}_k(l)$  for all  $k, 1 \leq k \leq 3$ .

Assume that  $St^1\bar{e}_1(l) \subseteq Ss^1\bar{e}_1(l)$ , that is  $t^1\bar{e}_1(l) = r^1s^1\bar{e}_1(l)$  for some  $r^1 \in S$ . Let  $j \in J_l$ . Then  $t_j\bar{e}_1(l) = r_j^1t^1\bar{e}_1(l) = r_j^1r^1s^1\bar{e}_1(l)$  and  $t_j\bar{a}_1(l) = r_j^1r^1s^1\bar{a}_1(l)$ . Hence using (1) we get

$$t_j \bar{a}_3(l) = t_j \bar{a}_2(l) = t_j \bar{e}_1(l) \bar{a}_2(l) = r_j^1 r^1 s^1 \bar{e}_1(l) \bar{a}_2(l) = r_j^1 r^1 s^1 \bar{a}_2(l) = r_j^1 r^1 s^1 \bar{a}_1(l) = t_j \bar{a}_1(l),$$

that is  $t_j \bar{a}_3(l) = t_j \bar{a}_1(l)$ . We set  $\bar{b}(l) = \bar{a}_3(l)$ .

Suppose that  $\check{S}s^1\bar{e}_1(l)\subseteq St^1\bar{e}_1(l)$ , that is  $s^1\bar{e}_1(l)=r_1t^1\bar{e}_1(l)$  for some  $r_1\in S$ . Let  $i\in I_l$ . Then  $s_i\bar{e}_1(l)=r_i^1s^1\bar{e}_1(l)=r_i^1r_1t^1\bar{e}_1(l)$ . So using (1) we get

$$s_i\bar{a}_3(l) = s_i\bar{e}_1(l)\bar{a}_3(l) = r_i^1r_1t^1\bar{e}_1(l)\bar{a}_3(l) = r_i^1r_1t^1\bar{e}_1(l)\bar{a}_2(l) = r_i^2r_1t^1\bar{e}_1(l)\bar{a}_2(l) = r_i^2r_1t^1\bar{e}_1(l)\bar{e}_1(l)\bar{e}_1(l)$$

$$= s_i \bar{e}_1(l) \bar{a}_2(l) = s_i \bar{a}_2(l) = s_i \bar{a}_1(l),$$

that is  $s_i \bar{a}_3(l) = s_i \bar{a}_1(l)$ . We set  $\bar{b}(l) = \bar{a}_1(l)$ .

Therefore the tuple  $\bar{b}$  such that (2) holds is construct.

The following example shows that the condition of commutativity of monoid S in Theorem 2.7 is essentially.

Let  $S = \{e_1, e_2\} \cup T \cup \{1\}$ , where T is a semigroup with  $\{a, b\}$  generators and  $ab^2 = ab$ ,  $ba^2 = ba$  defining relationships. Binary operation on S is defined in the following way:  $se_i = e_i$ ,  $e_it = e_i$  for all  $i \in \{1, 2\}$ ,  $s \in S$ ,  $t \in T$ ; 1 is an unit element. It is easy to check that S is a monoid under the operation and  $E = \{e_1, e_2, 1\}$ . Note that  $Se_i = \{e_i\}$  for all  $i \in \{1, 2\}$ . Since  $ba^2 = ba$  and  $ba \neq b$  then the assertion  ${}_SSa \xrightarrow{\sim} {}_S S \cdot 1$  is false. Since  $a^2 \neq a$  and  $ae_i = e_i$  then the assertion  ${}_SSa \xrightarrow{\sim} {}_S Se_i$  is false for all i,  $i \in \{1, 2\}$ . In the same way it is shown the falsity of the assertion  ${}_SSb \xrightarrow{\sim} {}_S Se_i$  ( $i \in \{1, 2\}$ ).

So  $R = \{e_1, e_2\}$ . It is clear that the semigroup R is not linearly order. For all S-act  $sA \in \mathfrak{R}$ ,  $a \in A$  and  $s \in S$  we have sa = a, that is any regular S-act is represent as a coproduct of one-element S-acts. Hence the class  $\mathfrak{R}$  is axiomatizable and primitive normal.

### 3. Antiadditive Classes of Regular Acts

The axiomatizable classes of regular S-acts were investigated in [6]. Particularly in that work there was proved the following proposition.

**Proposition 3.1.** If the class  $\mathfrak{R}$  of regular S-acts is axiomatizable then  $R = \bigcup \{e_i R \mid 1 \leqslant i \leqslant n\}$  for some  $n \geqslant 1$ ,  $e_i \in R$ ,  $e_i^2 = e_i$   $(1 \leqslant i \leqslant n)$ .

This statement implies

Corollary 3.2. If the class  $\mathfrak{R}$  of regular S-acts is axiomatizable, monoid S is commutative and R will be a linearly order semigroup then R = eR for some idempotent  $e \in R$ .

Throughout T will denote a theory of the axiomatizable primitive normal class  $\mathfrak{R}$  of regular S-acts, S will be a commutative monoid and R is a linearly order semigroup.

A proof of following Lemma is a modification of a proof of Lemma 2 in [2].

**Lemma 3.3.** Let  $\Phi(x_0, \bar{x})$  be a conjunction of atomic formulas,  $\bar{x} = \langle x_1, \dots, x_n \rangle$ . Then there is a formula  $\Psi(\bar{x})$ , which is a conjunction of atomic formulas,  $s, t \in S$  and  $i, 0 \le i \le n$ , such that in theory T

$$\Phi(x_0, \bar{x}) \equiv \Psi(\bar{x}) \wedge tx_i = sx_0.$$

*Proof.* By Corollary 3.2 there exists an idempotent  $e \in R$  such that R = eR. Then

$$(10) T \vdash \forall x(x = ex).$$

Let  $\Phi(x_0, \bar{x})$  be a conjunction of atomic formulas,  $\bar{x} = \langle x_1, \dots, x_n \rangle$ . We will prove Lemma by the induction on a number k of atomic subformulas of formula  $\Phi(x_0, \bar{x})$  containing a variable  $x_0$ . Suppose that

$$\Phi(x_0, \bar{x}) \rightleftharpoons \Psi_1(x_0, \bar{x}) \land t_1 x_i = s_1 x_0,$$

where  $\Psi_1(x_0, \bar{x})$  is a conjunction of atomic formulas,  $s_1, t_1 \in S$ ,  $0 \leq i \leq n$ . On the suggestion of the induction

$$\Psi_1(x_0, \bar{x}) \equiv \Psi_2(\bar{x}) \wedge t_2 x_j = s_2 x_0$$

for some formula  $\Psi_2(\bar{x})$ , which is a conjunction of atomic formulas, some  $s_2, t_2 \in S$  and  $j, 0 \leq j \leq n$ . In view of linearly order of a semigroup R we have either  $Ss_1e \subseteq Ss_2e$  or  $Ss_2e \subseteq Ss_1e$ . Let for example  $Ss_1e \subseteq Ss_2e$ . Then there exists  $r \in S$  such that  $s_1e = rs_2e$ . Hence in view of (10) we have

$$\Phi(x_0, \bar{x}) \equiv \Psi_2(\bar{x}) \wedge t_2 x_j = s_2 x_0 \wedge t_1 x_i = s_1 x_0 \equiv$$
$$\equiv \Psi_2(\bar{x}) \wedge r t_2 x_j = t_1 x_i \wedge t_2 x_j = s_2 x_0.$$

Lemma is proved.

The proof of following Lemma coincides exactly with the proof of Lemma 3 in [2].

**Lemma 3.4.** Let  $\Phi(\bar{x})$  be not always-false primitive formula,  $\bar{x} = \langle x_1, \dots, x_n \rangle$ . Then there exists the formula  $\Phi_0(\bar{x})$ , which is a conjunction of atomic formulas, and the primitive formulas  $\Phi_i(x_i)$ ,  $1 \leq i \leq n$ , such that in theory T

$$\Phi(\bar{x}) \equiv \Phi_0(\bar{x}) \wedge \bigwedge_{1 \leqslant i \leqslant n} \Phi_i(x_i).$$

**Lemma 3.5.** Let  $\bar{a} \in C$ ,  $\Phi(\bar{x}, \bar{y}, \bar{z}, \bar{a})$  be a primitive formula which defines on the infinite generalized primitive set X a binary operation  $+: \bar{x} + \bar{y} = \bar{z}$ . Then the set X is not a group under this operation.

*Proof.* Let  $\bar{a} \in \mathcal{C}$ ,  $\Phi(\bar{x}, \bar{y}, \bar{z}, \bar{a})$  be a primitive formula defining a structure of group relative to a binary operation +,  $X^*$  be a basis,  $\alpha$  be a generative equivalence of the generalized primitive set X,  $|\bar{a}| = |\bar{u}|$ . By Lemma 3.4 in the theory T

$$\Phi(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \equiv \Phi_0(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \wedge \Phi_1(\bar{x}) \wedge \Phi_2(\bar{y}) \wedge \Phi_3(\bar{z}) \wedge \Phi_4(\bar{u})$$

for some formulas  $\Phi_0(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ ,  $\Phi_1(\bar{x})$ ,  $\Phi_2(\bar{y})$ ,  $\Phi_3(\bar{z})$ ,  $\Phi_4(\bar{u})$ , where  $\Phi_0(\bar{x}, \bar{y}, \bar{z}, \bar{u})$  is a conjunction of atomic formulas,  $\Phi_1(\bar{x})$ ,  $\Phi_2(\bar{y})$ ,  $\Phi_3(\bar{z})$ ,  $\Phi_4(\bar{u})$  are the primitive formulas. By Lemma 3.3 there exist the formula  $\Psi(\bar{x}, \bar{y}, \bar{u})$ ,  $t_i, s_i \in S$  and  $\bar{w} = \langle w_1, \ldots, w_n \rangle$ , where  $w_i \in \bar{x} \cup \bar{y} \cup \bar{z} \cup \bar{u}$ ,  $1 \leq i \leq n$ , such that  $\Psi(\bar{x}, \bar{y}, \bar{u})$  is a conjunction of atomic formulas and in the theory T

$$\Phi_0(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \equiv \Psi(\bar{x}, \bar{y}, \bar{u}) \wedge \Theta(\bar{x}, \bar{y}, \bar{z}, \bar{u}),$$

where

$$\Theta(\bar{x}, \bar{y}, \bar{z}, \bar{u}) \rightleftharpoons \bigwedge_{1 \le i \le n} t_i z_i = s_i w_i.$$

Let  $\bar{b}, \bar{c} \in X^*$ ,  $\bar{0}/\alpha$  be a null element of the group X. Suppose that  $t_i z_i = s_i x_j$  is an atomic subformula of the formula  $\Theta(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ . Since  $\bar{0}/\alpha + \bar{b}/\alpha = \bar{b}/\alpha$  then  $t_i \bar{b}(i) = s_i \bar{0}(j)$ . Since

(11) 
$$\bar{0}/\alpha + \bar{c}/\alpha = \bar{c}/\alpha + \bar{0}/\alpha = \bar{c}/\alpha.$$

then  $t_i \bar{c}(i) = s_i \bar{0}(j) = s_i \bar{c}(j)$ . So  $t_i \bar{b}(i) = s_i \bar{c}(j)$ . Suppose  $t_i z_i = s_i y_j$  is an atomic subformula of the formula  $\Theta(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ . Since

$$(12) \bar{b}/\alpha + \bar{0}/\alpha = \bar{b}/\alpha,$$

then  $t_i\bar{b}(i) = s_i\bar{0}(j)$ . If  $t_iz_i = s_iz_j$  is an atomic subformula of the formula  $\Theta(\bar{x}, \bar{y}, \bar{z}, \bar{u})$ , then (11) implies the equality  $t_i\bar{b}(i) = s_i\bar{a}(j)$ . Moreover from (11) and (12) we have

$$\mathcal{C} \models \Psi(\bar{c}, \bar{0}, \bar{a}) \land \Phi_1(\bar{c}) \land \Phi_2(\bar{0}) \land \Phi_3(\bar{b}) \land \Phi_4(\bar{a}).$$

Hence  $\bar{c}/\alpha + \bar{0}/\alpha = \bar{b}/\alpha$ , that is  $\bar{c}/\alpha = \bar{b}/\alpha$  and |X| = 1. Contradiction.

Lemma 3.5 implies

**Theorem 3.6.** If S is a commutative monoid, the class  $\Re$  of regular S-acts is axiomatizable and primitive normal then the class  $\Re$  is antiadditive.

By Theorems 2.7, 3.6 and definition of antiadditive class we have

Corollary 3.7. Let S be a commutative monoid and the class  $\mathfrak{R}$  of regular S-acts is axiomatizable. Then the following conditions are equivalent:

- 1) the class  $\Re$  is primitive normal;
- 2) the class  $\Re$  is antiadditive;
- 3) the semigroup R is linearly order.

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