ON NILPOTENT GENERATORS OF THE LIE ALGEBRA $\mathfrak{sl}_n$

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Abstract. Consider the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{K})$ over an infinite field of characteristic different from 2. We prove that for any nonzero nilpotent X there exists a nilpotent Y such that the matrices X and Y generate the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$.

1. Introduction

It is an important problem to find a minimal generating set of a given algebra. This problem was studied actively for semisimple Lie algebras. In 1951, Kuranishi [4] observed that any semisimple Lie algebra over a field of characteristic zero can be generated by two elements. Twenty-five years later, Ionescu [3] proved that for any nonzero element X of a complex or real simple Lie algebra $\mathcal{G}$ there exists an element Y such that the elements X and Y generate the Lie algebra $\mathcal{G}$. In the same year, Smith [5] proved that every traceless matrix of order $n \geqslant 3$ is the commutator of two nilpotent matrices. These results imply that the special linear Lie algebra $\mathfrak{sl}_n$ can be generated by three nilpotent matrices. In 2009, Bois [2] extended Kuranishi's result to algebraically closed fields of characteristic different from 2 and 3.

In this paper we obtain an analogue of Ionescu's result for nilpotent generators of the Lie algebra $\mathfrak{sl}_n$.

Theorem 1. *Let $\mathbb{K}$ be an infinite field of characteristic different from 2. For any nonzero nilpotent X there exists a nilpotent Y such that the matrices X and Y generate the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$.*

Our interest to this subject was motivated by the study of additive group actions on affine spaces, see [1, Theorem 5.17]. In a forthcoming publication we plan to extend Theorem 1 to arbitrary simple Lie algebras.

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2. Main results

Let $\mathbb{K}$ be an infinite field with $\operatorname{char}\mathbb{K} \neq 2$. A set of elements $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \in \mathbb{K}$) is called *consistent* if the following conditions hold:

(1) $\lambda_1 + \ldots + \lambda_n = 0$;
(2) $\lambda_i \neq 0$ for all i;
(3) $\lambda_i \neq \lambda_j$ for all $i \neq j$;
(4) $\lambda_i - \lambda_j = \lambda_k - \lambda_l$ only for $i = j,\ k = l$ or $i = k,\ j = l$.

Condition (1) defines $(n-1)$-dimensional subspace $W \subseteq \mathbb{K}^n$. Conditions (2)-(4) define linear inequalities on W whose set of solutions is nonempty. For example, if $\operatorname{char}\mathbb{K} = 0$, a set $\lambda_i = 2^{i-1}$ for $i = 1, \ldots, n-1$ and $\lambda_n = 1 - 2^{n-1}$ is consistent.

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A diagonal matrix $A = \operatorname{diag}(a_{11}, \dots, a_{nn})$ is called *consistent* if $a_{11}, \dots, a_{nn}$ is a consistent set.

Lemma 1. *Let T be a consistent matrix and A be a matrix with nonzero entries outside the principal diagonal. Then T and A generate the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$.*

Proof. Consider the following matrices:

$$A_1 = [T, A], \quad A_i = [T, A_{i-1}], \quad i = 2, \dots, n^2 - n.$$

Note that all matrices A_i have zeroes on the principal diagonal and are linearly independent. Indeed, if we consider the coordinates of these matrices in the basis of $n^2 - n$ matrix units, up to scalar multiplication of columns, we come to a Vandermonde matrix with nonzero determinant.

Hence, the matrices A_i form a basis of the space of $n \times n$-matrices with zeroes on the principal diagonal. This means that the subalgebra generated by T and A contains all matrix units E_{ij}, $i \neq j$. Since $E_{ii} - E_{jj} = [E_{ij}, E_{ji}]$, it follows that this subalgebra is $\mathfrak{sl}_n(\mathbb{K})$. □

Lemma 2. *Let C be a diagonal matrix with pairwise distinct nonzero entries on the principal diagonal. Then C can be represented as $C = A + B$ with A and B being nilpotent matrices, $\mathrm{rk}A = 1$, and all entries of A are nonzero.*

Proof. Let V be an n-dimensional vector space over $\mathbb{K}$. Let us fix a basis $e_1, \dots, e_n$ in V and consider linear operators given by matrices in this basis.

For a non-degenerate matrix $C = \operatorname{diag}(c_{11}, \dots, c_{nn})$, let us consider the following set of vectors:

$$v_1 = e_1 + \dots + e_n, \ v_2 = \frac{e_1}{c_{11}} + \dots + \frac{e_n}{c_{nn}}, \dots, v_n = \frac{e_1}{c_{11}^{n-1}} + \dots + \frac{e_n}{c_{nn}^{n-1}}.$$

Again using a Vandermonde matrix we obtain that $v_1, \dots, v_n$ form a basis in V. Moreover, we have $C(v_{i+1}) = v_i$ for all $i = 1, \dots, n-1$. Let us define the operator B as follows: $B(v_1) = 0$, $B(v_{i+1}) = v_i$ for $i = 1, \dots, n-1$, and let $A = C - B$.

Let us check that A and B satisfy the conditions of Lemma 2. Indeed, B is nilpotent and $\mathrm{rk}A = 1$, because $A(v_i) = 0$ for $i = 1, \dots, n-1$. Since $\mathrm{tr}A = 0$, A is also nilpotent. It is only left to show that in the basis $e_1, \dots, e_n$ all entries of A are nonzero. All columns of A are proportional to the column $(c_{11}, \dots, c_{nn})$, because $\mathrm{rk}A = 1$ and $A(e_1 + \dots + e_n) = c_{11}e_1 + \dots + c_{nn}e_n$. Finally, we have $A(e_i) \neq 0$, because the vectors $v_2, v_3, \dots, v_n, e_i$ are linearly independent and $v_2, v_3, \dots, v_n$ is a basis of $\mathrm{Ker}A$. □

Lemma 3. *For each nilpotent matrix N with $\mathrm{rk}N = 1$ there exists a nilpotent matrix M such that N and M generate $\mathfrak{sl}_n(\mathbb{K})$.*

Proof. Let T be a consistent matrix, A and B be matrices from Lemma 2. It follows from Lemma 1 that A and B generate $\mathfrak{sl}_n(\mathbb{K})$. All nilpotent matrices of rank 1 are conjugate, i.e. for any nilpotent N with $\mathrm{rk}N = 1$ there exists $C \in \mathrm{GL}_n(\mathbb{K})$ such that $N = CAC^{-1}$. Moreover, if A and B generate $\mathfrak{sl}_n(\mathbb{K})$ and $C \in \mathrm{GL}_n(\mathbb{K})$, then N and CBC^{-1} also generate $\mathfrak{sl}_n(\mathbb{K})$. This completes the proof. □

Lemma 4. *Let V be a finite-dimensional vector space over an arbitrary field $\mathbb{K}$. Then a set of collections consisting of ordered n linearly independent vectors is open in the Zariski topology on V^n.*

Proof. We fix a basis of V. For every set $\{v_1, \dots, v_n \mid v_l \in V\}$ let us build a matrix consisting of n rows and $\dim V$ columns such that l-th row consists of the coordinates of v_l. It is only left to notice that $v_1, \dots, v_n$ are linearly independent if and only if there is at least one nonzero minor of order n. □

Lemma 5. *For any given matrix B a set of matrices X such that B and X generate $\mathfrak{sl}_n(\mathbb{K})$ is open in the Zariski topology on $\mathfrak{sl}_n(\mathbb{K})$.*

Proof. For any given matrix B and a variable X let us consider a set of all matrices that can be obtained from B and X by means of the Lie bracket $[\ ,\ ]$:

$$\mathrm{Com}(X, B) \coloneqq \{X,\ B,\ [X,\ B],\ [B,\ X],\ [X,\ [X,\ B]],\ [B,\ [X,\ B]],\ [[X,\ B],\ X], \dots\}$$

Let us enumerate all the matrices from $\mathrm{Com}(X, B)$ in some order independent of X. For example, we put $\mathrm{Com}_1(X) = X$, $\mathrm{Com}_2(X) = B$, $\mathrm{Com}_3(X) = [X,\ B]$, $\mathrm{Com}_4(X) = [X,\ [X,\ B]]$, $\mathrm{Com}_5(X) = [B,\ [X,\ B]], \dots$

Obviously, a subalgebra of $\mathfrak{sl}_n(\mathbb{K})$ generated by X and B is a linear span of elements of $\mathrm{Com}(X, B)$. Let I be an arbitrary set of indices $i_1, \dots, i_{n^2-1}$ and let M_I be a set of matrices X such that $\mathrm{Com}_{i_1}(X), \dots, \mathrm{Com}_{i_{n^2-1}}(X)$ are linearly independent. Let us construct a map

$$\varphi_I : \mathfrak{sl}_n(\mathbb{K}) \longrightarrow (\mathfrak{sl}_n(\mathbb{K}))^{n^2-1} \quad \text{by the rule} \quad X \to (\mathrm{Com}_{i_1}(X), \dots, \mathrm{Com}_{i_{n^2-1}}(X)).$$

It is defined by polynomials. Let us look at ordered collections consisting of $n^2 - 1$ linearly independent matrices. According to Lemma 4, a set compiled from all such collections is open in the Zariski topology on $(\mathfrak{sl}_n(\mathbb{K}))^{n^2-1}$. Then the preimage of this set under φ_I is open. Lemma 5 follows from the fact that a set of matrices X such that B and X generate $\mathfrak{sl}_n(\mathbb{K})$ is a union of all possible M_I. □

Lemma 6. *For any matrix $A = \sum_{i=1}^{n-1} a_i E_{i\,i+1},\ a_1 = 1$, there exists a nilpotent matrix B such that A and B generate $\mathfrak{sl}_n(\mathbb{K})$.*

Proof. Lemma 3 implies that there exists a nilpotent matrix B_0 such that E_{12} and B_0 generate $\mathfrak{sl}_n(\mathbb{K})$. Consider all matrices of the form

$$X = \sum_{i=1}^{n-1} x_i a_i E_{i\,i+1}.$$

Obviously, matrices X and A are conjugate if $x_i \neq 0$ for all i. According to Lemma 5, there is a polynomial $F(x_1, \dots, x_{n-1})$ such that the following conditions hold:

(1) $F(1, 0, \dots, 0) \neq 0$;
(2) $F(x_1, \dots, x_{n-1}) \neq 0 \Longrightarrow B_0$ and X generate $\mathfrak{sl}_n(\mathbb{K})$.

Since F is a nonzero polynomial, there exists a set of nonzero numbers $\lambda_1, \dots, \lambda_{n-1}$ such that $F(\lambda_1, \dots, \lambda_{n-1}) \neq 0$. It implies that matrices B_0 and $X_0 = \sum_{i=1}^{n-1} \lambda_i a_i E_{i\,i+1}$ generate $\mathfrak{sl}_n(\mathbb{K})$. The fact that A and X_0 are conjugate completes the proof. □

Proof of Theorem 1. For any non-degenerate nilpotent linear operator X there exists a basis such that the matrix of X in this basis has the following form:

$$A = \sum_{i=1}^{n-1} a_i E_{i\,i+1}, \quad a_1 = 1.$$

In other words, if X is a nonzero nilpotent matrix, there exists $C \in \mathrm{GL}_n(\mathbb{K})$ such that $A = CXC^{-1}$. It follows from Lemma 6 that there exists a nilpotent matrix B such that A and B generate $\mathfrak{sl}_n(\mathbb{K})$. Thus, X and $C^{-1}BC$ generate $\mathfrak{sl}_n(\mathbb{K})$. □

3. EXAMPLES AND PROBLEMS

Let us give two examples with specific pairs of nilpotent matrices generating $\mathfrak{sl}_n(\mathbb{K})$. Matrices from the first example generate $\mathfrak{sl}_n(\mathbb{K})$ over an infinite field $\mathbb{K}$ of arbitrary characteristic except for $n = 4$ if $\operatorname{char}\mathbb{K} = 2$.

Example 1. Let $\mathbb{K}$ be an infinite field and $\lambda_1, \ldots, \lambda_n$ ($\lambda_i \in \mathbb{K}$) be a set such that following conditions hold:

(1) $\lambda_1 + \ldots + \lambda_n = 0$;
(2) $\lambda_{i+1} \neq \lambda_i$ for all i;
(3) $\lambda_{i+1} - \lambda_i = \lambda_{k+1} - \lambda_k$ only for $i = k$;
(4) $\lambda_1 + \cdots + \lambda_k \neq 0$ for all $k = 1, \ldots, n-1$.

Such sets exist except for $n = 4$ if $\operatorname{char}\mathbb{K} = 2$ (in this case condition (1) implies $\lambda_{44} - \lambda_{33} = \lambda_{22} - \lambda_{11}$). Let us denote by s_k the element $\lambda_1 + \ldots + \lambda_k$ and consider the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 \\ s_1 & \ldots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \ldots & s_{n-2} & 0 & 0 \\ 0 & \ldots & 0 & s_{n-1} & 0 \end{pmatrix}$$

Let us show that A and B generate $\mathfrak{sl}_n(\mathbb{K})$. Indeed, $T = [A, B]$ is a diagonal matrix with the entries $t_{ii} = \lambda_i$. Similarly to the proof of Lemma 1, we obtain that matrices T and A generate a subalgebra of $\mathfrak{sl}_n(K)$ containing $E_{i\,i+1}$ for $i = 1, \ldots, n-1$. Since $[E_{ik}, E_{kl}] = E_{il}$, this subalgebra contains all upper nil-triangular matrices. Similarly, T and B generate a subalgebra containing all lower nil-triangular matrices. Since $E_{ii} - E_{jj} = [E_{ij}, E_{ji}]$, A and B generate $\mathfrak{sl}_n(\mathbb{K})$.

Example 2. If $\operatorname{char}\mathbb{K} = 0$ and n is odd, then the matrices

$$M = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

generate $\mathfrak{sl}_n(\mathbb{K})$. Firstly let us consider the case $\mathbb{K} = \mathbb{C}$. It is possible to make a consistent set from different complex n-th roots of unity, since if n is odd, a regular n-gon does not have any parallel and equal sides/diagonals. Let T be a corresponding consistent matrix. It can be represented as $T = A + B$, where A, B are the nilpotent matrices from Lemma 2. Using notations of Lemma 2, in the basis $v_1, \ldots, v_n$ the operators A and B have matrices $N = E_{1n}$ and $M = \sum_{i=1}^{n-1} E_{i\,i+1}$, respectively. Thus, M and N generate $\mathfrak{sl}_n(\mathbb{K})$. We conclude that the set $\operatorname{Com}(M, N)$ from Lemma 5 contains $n^2 - 1$ linearly independent matrices. Since linear independence of matrices does not depend on the ground field, the matrices M and N generate $\mathfrak{sl}_n(\mathbb{Q})$ and $\mathfrak{sl}_n(\mathbb{K})$, where $\mathbb{K}$ is an extension of the field $\mathbb{Q}$.

Remark 1. If n is even, M and N do not generate $\mathfrak{sl}_n(\mathbb{K})$.

Let us look at the set of matrices

$$\Lambda = \{A \in \mathfrak{sl}_n(\mathbb{K}) \mid AC^{-1} + C^{-1}A^T = 0\}, \quad \text{where} \quad C = \begin{pmatrix} 0 & 0 & \dots & (-1)^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 0 \\ -1 & 0 & \dots & 0 \end{pmatrix}.$$

If $n \geqslant 3$ then Λ is a proper subalgebra of $\mathfrak{sl}_n(\mathbb{K})$, and if n is even, we have $M,\, N \in \Lambda$.

The proof of Theorem 1 implies that for any $n > 1$ there exists a number N such that Theorem 1 holds for $\mathfrak{sl}_n(\mathbb{K})$, $|\mathbb{K}| \geqslant N$. It may be interesting to extend Theorem 1 to finite fields and fields of characteristic 2.

Problem 1. *What is the minimal generating set consisting of nilpotent matrices of the Lie algebra $\mathfrak{sl}_n$ over a finite field?*

Problem 2. *Does Theorem 1 hold for infinite field of characteristic 2?*

The following example shows that at least some conditions of Theorem 1 are necessary.

Example 3. Let $\mathbb{F}_2$ be the field $\mathbb{Z}/2\mathbb{Z}$. Let us show that for $\mathbb{K} = \mathbb{F}_2$, Theorem 1 does not hold. We claim that for the matrix $E_{12} \in \mathfrak{sl}_3(\mathbb{F}_2)$ there does not exist a nilpotent matrix Y such that E_{12} and Y generate $\mathfrak{sl}_3(\mathbb{F}_2)$.

Consider linear operators given by the matrices E_{12} and Y. If $\mathrm{rk}Y = 1$ then $\mathrm{Ker}Y$ and $\mathrm{Ker}E_{12}$ have nonempty intersection, hence the subalgebra generated by E_{12} and Y is not $\mathfrak{sl}_3(\mathbb{F}_2)$. Thus $\mathrm{rk}Y = 2$. Since all nilpotent matrices of rank 2 are conjugate in $\mathfrak{sl}_3(\mathbb{F}_2)$, we only have to check that there is no A of rank 1 such that A and $B = E_{12} + E_{23}$ generate $\mathfrak{sl}_3(\mathbb{F}_2)$.

Since the first column and the last row of the matrix B are zero, the first column and the last row of the matrix A are nonzero. It implies that $a_{31} = 1$. There are only 8 such matrices. Two matrices of these eight are persymmetric and we can split other six matrices into pairs symmetric with respect to the antidiagonal matrices. Let X' be a matrix such that X and X' are symmetric with respect to the antidiagonal. Since symmetry and antisymmetry are the same in $\mathbb{F}_2$ and $B = B'$, we have $[X, B]' = [X', B]$. It implies that if $X = X'$ the subalgebra $\mathfrak{sl}_3(\mathbb{F}_2)$ generated by X and B consists of matrices symmetric with respect to the antidiagonal. Moreover, if X and B generate $\mathfrak{sl}_3(\mathbb{F}_2)$, then X' and B' generate $\mathfrak{sl}_3(\mathbb{F}_2)$. So it is only left to show that A and B do not generate $\mathfrak{sl}_3(\mathbb{F}_2)$, where A is one of the following three matrices:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let us denote by Λ_1 the linear span of the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and by Λ_2 the linear span of the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We have A_1, $B \in \Lambda_1$ and A_2, A_3, $B \in \Lambda_2$, and it is easy to verify directly that Λ_1 and Λ_2 are subalgebras of $\mathfrak{sl}_n(\mathbb{K})$.

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