## Congruences on sums of super Catalan numbers

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Abstract. In this paper, we prove two congruences on the double sums of the super Catalan numbers (named by Gessel), which were recently conjectured by Apagodu.

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#### 1 Introduction

It is well-known that the Catalan numbers

$$
C_n = \frac{1}{n+1} \binom{2n}{n}
$$

are integers and occur in various counting problems. We refer to [\[9\]](#page-6-0) for many different combinatorial interpretations of the Catalan numbers. The closely related central binomial coefficients are given by  $\binom{2n}{n}$  $\binom{2n}{n}$  for  $n \in \mathbb{N}$ .

Both Catalan numbers and central binomial coefficients possess many interesting arith-metic properties. Sun and Tauraso [\[11\]](#page-7-0) proved that for primes  $p \geq 5$ ,

<span id="page-0-1"></span>
$$
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},
$$
  

$$
\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2},
$$

where  $\left(\frac{1}{r}\right)$  $\frac{1}{p}$  denotes the Legendre symbol. Recently, Mattarei and Tauraso [\[7\]](#page-6-1) showed that

$$
\sum_{k=0}^{q-1} \binom{2k}{k} x^k \equiv (1 - 4x)^{\frac{q-1}{2}} \pmod{p},\tag{1.1}
$$

<span id="page-0-0"></span>
$$
\sum_{k=0}^{q-1} C_k x^{k+1} \equiv \frac{1 - (1 - 4x)^{\frac{q+1}{2}}}{2} - x^q \pmod{p},\tag{1.2}
$$

where  $q$  is a power of an odd prime  $p$ . For more congruence properties on these numbers we refer to  $[6, 10, 12]$  $[6, 10, 12]$  $[6, 10, 12]$ .

In 1874, E. Catalan observed that the numbers

$$
S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m}}
$$

are integers. Since  $S(1, n)/2$  coincides with  $C_n$ , these numbers  $S(m, n)$  are named super  $\textit{Catalan numbers}$  by Gessel [\[5\]](#page-6-4). These numbers should not confused with the Schröder– Hipparchus numbers, which are sometimes also called super Catalan numbers. Some interpretations of  $S(m, n)$  for some special values of m have been studied by several authors (see, e.g., [\[1,](#page-6-5) [4,](#page-6-6) [8\]](#page-6-7)). It is still an open problem to find a general combinatorial interpretation for the super Catalan numbers.

Our interest concerns the following two conjectures by Apagodu [\[2,](#page-6-8) Conjecture 2].

Conjecture 1.1 (Apagodu) For any odd prime p, we have

<span id="page-1-5"></span><span id="page-1-3"></span><span id="page-1-1"></span><span id="page-1-0"></span>
$$
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} S(i,j) \equiv \left(\frac{p}{3}\right) \pmod{p},\tag{1.3}
$$

$$
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (3i+3j+1)S(i,j) \equiv -7\left(\frac{p}{3}\right) \pmod{p}.
$$
 (1.4)

In Section 2, we provide a proof of [\(1.3\)](#page-1-0) which makes use of a combinatorial identity.

**Theorem 1.2** The congruence  $(1.3)$  is true.

We prove [\(1.4\)](#page-1-1) by establishing the following congruence.

**Theorem 1.3** For any prime  $p \geq 5$ , we have

<span id="page-1-4"></span><span id="page-1-2"></span>
$$
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (i+j)S(i,j) \equiv -\frac{8}{3} \left(\frac{p}{3}\right) \pmod{p}.
$$
 (1.5)

From [\(1.3\)](#page-1-0) and [\(1.5\)](#page-1-2), we deduce [\(1.4\)](#page-1-1) for  $p \geq 5$ . It is routine to check that (1.4) also holds for  $p = 3$ .

#### 2 Proof of Theorem [1.2](#page-1-3)

In order to prove Theorem [1.2,](#page-1-3) we need the following identity.

**Lemma 2.1** For any non-negative integer n, we have

$$
\sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} \frac{\binom{n}{i} \binom{n}{j}}{\binom{i+j}{i}} = \frac{(-3)^n (2n-1)}{4(n+1)} + \frac{4^n}{\binom{2n}{n}} \left( \frac{1}{2} - \sum_{k=0}^{n} C_k \left( -\frac{3}{4} \right)^{k+1} \right), \tag{2.1}
$$

where  $C_k$  denotes the kth Catalan number.

Proof. Applying the multi-Zeilberger algorithm [\[3\]](#page-6-9), we find that the left-hand side of [\(2.1\)](#page-1-4) satisfies the recurrence:

$$
-18(n + 1)s(n) + 3(2n - 5)s(n + 1) + 2(5n + 6)s(n + 2) + (5 + 2n)s(n + 3) = 0.
$$

It is routine to check that the right-hand side of [\(2.1\)](#page-1-4) also satisfies this recurrence and both sides of  $(2.1)$  are equal for  $n = 0, 1, 2$ .

*Proof of* [\(1.3\)](#page-1-0). Let  $n = \frac{p-1}{2}$  $\frac{-1}{2}$ . We split the double sum on the left-hand side of  $(1.3)$  into four pieces:

$$
S_1 = \sum_{i=0}^n \sum_{j=0}^n (\cdot), \quad S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} (\cdot), \quad S_3 = \sum_{i=n+1}^{2n} \sum_{j=0}^n (\cdot), \quad S_4 = \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} (\cdot).
$$

For  $\binom{2i}{i}$  $\binom{2i}{i} \equiv 0 \pmod{p}$  for  $n+1 \leq i \leq 2n$ , we have  $S_4 \equiv 0 \pmod{p}$ . By the symmetry  $i \leftrightarrow j$ , we get  $S_2 = S_3$ . It follows that

<span id="page-2-2"></span><span id="page-2-0"></span>
$$
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}} \equiv S_1 + 2S_2 \pmod{p}.
$$
 (2.2)

Note that for  $0 \leq i \leq n$ ,

<span id="page-2-1"></span>
$$
\binom{2i}{i} = (-4)^i \binom{-\frac{1}{2}}{i} \equiv (-4)^i \binom{n}{i} \pmod{p}.
$$
 (2.3)

Thus,

$$
S_1 \stackrel{(2.3)}{\equiv} \sum_{i=0}^n \sum_{j=0}^n (-4)^{i+j} \frac{\binom{n}{i} \binom{n}{j}}{\binom{i+j}{i}} \pmod{p}
$$
  
\n
$$
\stackrel{(2.1)}{=} \frac{(-3)^n (2n-1)}{4(n+1)} + \frac{4^n}{\binom{2n}{n}} \left(\frac{1}{2} - \sum_{k=0}^n C_k \left(-\frac{3}{4}\right)^{k+1}\right)
$$
  
\n
$$
\equiv -\left(\frac{p}{3}\right) + \frac{(-1)^n}{2} - (-1)^n \sum_{k=0}^n C_k \left(-\frac{3}{4}\right)^{k+1} \pmod{p}, \qquad (2.4)
$$

where we utilize  $\binom{2n}{n}$  ${n \choose n} \equiv (-1)^n \pmod{p}$  in the last step. Since  $C_k \equiv 0 \pmod{p}$  for  $n + 1 \le k \le 2n - 1$ , we have

$$
\sum_{k=0}^{n} C_k \left( -\frac{3}{4} \right)^{k+1} \equiv \sum_{k=0}^{2n} C_k \left( -\frac{3}{4} \right)^{k+1} - C_{2n} \left( -\frac{3}{4} \right)^{2n+1}
$$

$$
\stackrel{(1.2)}{\equiv} \frac{1 - 4^{\frac{p+1}{2}}}{2} - \left( -\frac{3}{4} \right)^p - C_{p-1} \left( -\frac{3}{4} \right)^p \pmod{p}.
$$

Using the Fermat's little theorem and

$$
C_{p-1}=\frac{\binom{2p-2}{p-1}}{p}=\frac{\binom{2p-1}{p-1}}{2p-1}\equiv -1\pmod{p},
$$

we arrive at

$$
\sum_{k=0}^{n} C_k \left( -\frac{3}{4} \right)^{k+1} \equiv -\frac{3}{2} \pmod{p}.
$$
 (2.5)

Substituting [\(2.5\)](#page-3-0) into [\(2.4\)](#page-2-1) gives

<span id="page-3-4"></span><span id="page-3-0"></span>
$$
S_1 \equiv 2(-1)^n - \left(\frac{p}{3}\right) \pmod{p}.\tag{2.6}
$$

Note that

$$
S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}} = \sum_{i=0}^n \sum_{j=1}^n \frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}}.
$$
 (2.7)

For  $i + j \leq n$  and  $1 \leq j \leq n$ ,

<span id="page-3-2"></span><span id="page-3-1"></span>
$$
\frac{\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv 0 \pmod{p},
$$

and so the summand on the right-hand side of  $(2.7)$  is congruent to 0 modulo p.

On the other hand, for  $i + j \ge n + 1$  and  $1 \le j \le n$ ,

$$
\frac{\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} = \frac{i!}{(n+j)!} \cdot \frac{(2n+2)\cdots(2n+2j)}{(2n+2)\cdots(i+j+n)} \n\equiv \frac{i!}{(n+j)!} \cdot \frac{(2j-1)!}{(i+j-n-1)!} \pmod{p}.
$$
\n(2.8)

It follows from  $(2.3)$  and  $(2.8)$  that

$$
\frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv \frac{(-4)^i \binom{j-1}{n-i} \binom{2j}{j}}{2\binom{n+j}{j}} \pmod{p}.
$$

Since

$$
\binom{n+j}{j} \equiv \binom{-\frac{1}{2} + j}{j} = \frac{\binom{2j}{j}}{4^j} \pmod{p},
$$

we have

<span id="page-3-3"></span>
$$
\frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv \frac{(-1)^i \cdot 4^{i+j} \cdot \binom{j-1}{n-i}}{2} \pmod{p}.
$$
\n(2.9)

Substituting [\(2.9\)](#page-3-3) into [\(2.7\)](#page-3-1) gives

$$
S_2 \equiv \frac{1}{2} \sum_{j=1}^n 4^j \sum_{i=0}^n (-4)^i {j-1 \choose n-i} \pmod{p}
$$
  
= 
$$
\frac{(-4)^n}{2} \sum_{j=1}^n 4^j \sum_{i=0}^n {j-1 \choose i} \left(-\frac{1}{4}\right)^i \quad (i \to n-i)
$$
  
= 
$$
2(-4)^n \sum_{j=1}^n 3^{j-1}
$$
  
= 
$$
(-12)^n - (-4)^n.
$$

Thus,

<span id="page-4-1"></span><span id="page-4-0"></span>
$$
S_2 \equiv \left(\frac{p}{3}\right) - (-1)^n \pmod{p}.\tag{2.10}
$$

The proof of  $(1.3)$  follows from  $(2.2)$ ,  $(2.6)$  and  $(2.10)$ .

# 3 Proof of Theorem [1.3](#page-1-5)

**Lemma 3.1** For any non-negative integer  $n$ , we have

$$
\sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} (i+j) \frac{\binom{n}{i} \binom{n}{j}}{\binom{i+j}{i}} = 16n(-3)^{n-1} + \frac{8n4^n}{\binom{2n}{n}} \sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^k.
$$
 (3.1)

Proof. By the multi-Zeilberger algorithm [\[3\]](#page-6-9), we obtain the recurrence for the left-hand side of [\(3.1\)](#page-4-1):

$$
-6(n+1)(581n+793)s(n) + (818n^2 - 6653n - 9936)s(n+1)
$$
  
+ (2166n<sup>2</sup> + 3474n + 2898)s(n+2) + (2n + 5)(251n + 92)s(n+3) = 0.

It is easy to verify that the right-hand side of [\(3.1\)](#page-4-1) also satisfies the above recurrence and both sides of [\(3.1\)](#page-4-1) are equal for  $n = 0, 1, 2$ .

*Proof of* [\(1.5\)](#page-1-2). Let  $n = \frac{p-1}{2}$  $\frac{-1}{2}$ . In a similar way,

$$
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (i+j) \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}} \equiv S_1 + 2S_2 \pmod{p},\tag{3.2}
$$

where

<span id="page-4-2"></span>
$$
S_1 = \sum_{i=0}^{n} \sum_{j=0}^{n} (i+j) \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}},
$$

and

<span id="page-5-1"></span>
$$
S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} (i+j) \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}}.
$$

By  $(2.3)$  and  $(3.1)$ , we have

$$
S_1 \stackrel{(2.3)}{\equiv} \sum_{i=0}^n \sum_{j=0}^n (-4)^{i+j} (i+j) \frac{\binom{n}{i} \binom{n}{j}}{\binom{i+j}{i}} \pmod{p}
$$
  
\n
$$
\stackrel{(3.1)}{=} 16n(-3)^{n-1} + \frac{8n4^n}{\binom{2n}{n}} \sum_{k=0}^n \binom{2k}{k} \left(-\frac{3}{4}\right)^k
$$
  
\n
$$
\equiv \frac{8}{3} \left(\frac{p}{3}\right) - 4(-1)^n \sum_{k=0}^n \binom{2k}{k} \left(-\frac{3}{4}\right)^k \pmod{p},
$$
\n(3.3)

where we make use of  $\binom{2n}{n}$  ${n \choose n} \equiv (-1)^n \pmod{p}$  in the last step.

Since  $\binom{2k}{k}$  $\binom{2k}{k} \equiv 0 \pmod{p}$  for  $n+1 \leq k \leq 2n$ , we have

$$
\sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^k \equiv \sum_{k=0}^{2n} \binom{2k}{k} \left(-\frac{3}{4}\right)^k \stackrel{(1.1)}{\equiv} 4^{\frac{p-1}{2}} \equiv 1 \pmod{p}.\tag{3.4}
$$

Substituting [\(3.4\)](#page-5-0) into [\(3.3\)](#page-5-1) gives

<span id="page-5-3"></span><span id="page-5-2"></span><span id="page-5-0"></span>
$$
S_1 \equiv \frac{8}{3} \left(\frac{p}{3}\right) - 4(-1)^n \pmod{p}.
$$
 (3.5)

On the other hand, by [\(2.9\)](#page-3-3) we have

$$
S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} (i+j) \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}}
$$
  
\n
$$
= \sum_{i=0}^n \sum_{j=1}^n (i+j+n) \frac{\binom{2i}{i} \binom{2j+2n}{j+n}}{\binom{i+j+n}{i}}
$$
  
\n
$$
\stackrel{(2.9)}{=} \frac{1}{2} \sum_{j=1}^n 4^j \sum_{i=0}^n (-4)^i (i+j+n) \binom{j-1}{n-i} \pmod{p}
$$
  
\n
$$
= \frac{(-4)^n}{2} \sum_{j=1}^n 4^j \sum_{i=0}^n (2n+j-i) \binom{j-1}{i} \left(-\frac{1}{4}\right)^i,
$$
\n(3.6)

where we set  $i \rightarrow n - i$  in the last step. Note that

$$
\sum_{i=0}^{n} (2n+j-i) \binom{j-1}{i} \left(-\frac{1}{4}\right)^{i}
$$
  
=  $(2n+j) \left(\frac{3}{4}\right)^{j-1} - (j-1) \sum_{i=1}^{n} \binom{j-2}{i-1} \left(-\frac{1}{4}\right)^{i}$   
=  $(2n+j) \left(\frac{3}{4}\right)^{j-1} + \frac{j-1}{4} \left(\frac{3}{4}\right)^{j-2}$ . (3.7)

Substituting [\(3.7\)](#page-6-10) into [\(3.6\)](#page-5-2) and making elementary calculation gives

$$
S_2 \equiv \frac{(-12)^n (10n - 3) + (-4)^n (3 - 6n)}{3} \pmod{p}.
$$

It follows that

$$
S_2 \equiv 2(-1)^n - \frac{8}{3} \left(\frac{p}{3}\right) \pmod{p}.
$$
 (3.8)

Combining  $(3.2)$ ,  $(3.5)$  and  $(3.8)$ , we complete the proof of  $(1.5)$ .

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<span id="page-6-11"></span><span id="page-6-10"></span>

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