Congruences on sums of super Catalan numbers

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Abstract. In this paper, we prove two congruences on the double sums of the super Catalan numbers (named by Gessel), which were recently conjectured by Apagodu.

Keywords: Congruences; Super Catalan numbers; Zeilberger's algorithm *MR Subject Classifications*: 11A07, 05A19, 05A10

1 Introduction

It is well-known that the Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

are integers and occur in various counting problems. We refer to [9] for many different combinatorial interpretations of the Catalan numbers. The closely related central binomial coefficients are given by $\binom{2n}{n}$ for $n \in \mathbb{N}$.

Both Catalan numbers and central binomial coefficients possess many interesting arithmetic properties. Sun and Tauraso [11] proved that for primes $p \ge 5$,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Recently, Mattarei and Tauraso [7] showed that

$$\sum_{k=0}^{q-1} \binom{2k}{k} x^k \equiv (1-4x)^{\frac{q-1}{2}} \pmod{p},$$
(1.1)

$$\sum_{k=0}^{q-1} C_k x^{k+1} \equiv \frac{1 - (1 - 4x)^{\frac{q+1}{2}}}{2} - x^q \pmod{p}, \tag{1.2}$$

where q is a power of an odd prime p. For more congruence properties on these numbers we refer to [6, 10, 12].

In 1874, E. Catalan observed that the numbers

$$S(m,n) = \frac{\binom{2m}{m}\binom{2n}{n}}{\binom{m+n}{m}}$$

are integers. Since S(1, n)/2 coincides with C_n , these numbers S(m, n) are named super Catalan numbers by Gessel [5]. These numbers should not confused with the Schröder– Hipparchus numbers, which are sometimes also called super Catalan numbers. Some interpretations of S(m, n) for some special values of m have been studied by several authors (see, e.g., [1, 4, 8]). It is still an open problem to find a general combinatorial interpretation for the super Catalan numbers.

Our interest concerns the following two conjectures by Apagodu [2, Conjecture 2].

Conjecture 1.1 (Apagodu) For any odd prime p, we have

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} S(i,j) \equiv \left(\frac{p}{3}\right) \pmod{p},$$
(1.3)

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (3i+3j+1)S(i,j) \equiv -7\left(\frac{p}{3}\right) \pmod{p}.$$
(1.4)

In Section 2, we provide a proof of (1.3) which makes use of a combinatorial identity.

Theorem 1.2 The congruence (1.3) is true.

We prove (1.4) by establishing the following congruence.

Theorem 1.3 For any prime $p \ge 5$, we have

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (i+j)S(i,j) \equiv -\frac{8}{3} \left(\frac{p}{3}\right) \pmod{p}.$$
(1.5)

From (1.3) and (1.5), we deduce (1.4) for $p \ge 5$. It is routine to check that (1.4) also holds for p = 3.

2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following identity.

Lemma 2.1 For any non-negative integer n, we have

$$\sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} \frac{\binom{n}{i}\binom{n}{j}}{\binom{i+j}{i}} = \frac{(-3)^n (2n-1)}{4(n+1)} + \frac{4^n}{\binom{2n}{n}} \left(\frac{1}{2} - \sum_{k=0}^{n} C_k \left(-\frac{3}{4}\right)^{k+1}\right), \quad (2.1)$$

where C_k denotes the kth Catalan number.

Proof. Applying the multi-Zeilberger algorithm [3], we find that the left-hand side of (2.1) satisfies the recurrence:

$$-18(n+1)s(n) + 3(2n-5)s(n+1) + 2(5n+6)s(n+2) + (5+2n)s(n+3) = 0.$$

It is routine to check that the right-hand side of (2.1) also satisfies this recurrence and both sides of (2.1) are equal for n = 0, 1, 2.

Proof of (1.3). Let $n = \frac{p-1}{2}$. We split the double sum on the left-hand side of (1.3) into four pieces:

$$S_1 = \sum_{i=0}^n \sum_{j=0}^n (\cdot), \quad S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} (\cdot), \quad S_3 = \sum_{i=n+1}^{2n} \sum_{j=0}^n (\cdot), \quad S_4 = \sum_{i=n+1}^{2n} \sum_{j=n+1}^{2n} (\cdot).$$

For $\binom{2i}{i} \equiv 0 \pmod{p}$ for $n+1 \leq i \leq 2n$, we have $S_4 \equiv 0 \pmod{p}$. By the symmetry $i \leftrightarrow j$, we get $S_2 = S_3$. It follows that

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}} \equiv S_1 + 2S_2 \pmod{p}.$$
(2.2)

Note that for $0 \leq i \leq n$,

$$\binom{2i}{i} = (-4)^i \binom{-\frac{1}{2}}{i} \equiv (-4)^i \binom{n}{i} \pmod{p}.$$
(2.3)

Thus,

$$S_{1} \stackrel{(2.3)}{\equiv} \sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} \frac{\binom{n}{i}\binom{n}{j}}{\binom{i+j}{i}} \pmod{p}$$

$$\stackrel{(2.1)}{\equiv} \frac{(-3)^{n}(2n-1)}{4(n+1)} + \frac{4^{n}}{\binom{2n}{n}} \left(\frac{1}{2} - \sum_{k=0}^{n} C_{k} \left(-\frac{3}{4}\right)^{k+1}\right)$$

$$\equiv -\left(\frac{p}{3}\right) + \frac{(-1)^{n}}{2} - (-1)^{n} \sum_{k=0}^{n} C_{k} \left(-\frac{3}{4}\right)^{k+1} \pmod{p}, \qquad (2.4)$$

where we utilize $\binom{2n}{n} \equiv (-1)^n \pmod{p}$ in the last step. Since $C_k \equiv 0 \pmod{p}$ for $n+1 \leq k \leq 2n-1$, we have

$$\sum_{k=0}^{n} C_k \left(-\frac{3}{4}\right)^{k+1} \equiv \sum_{k=0}^{2n} C_k \left(-\frac{3}{4}\right)^{k+1} - C_{2n} \left(-\frac{3}{4}\right)^{2n+1}$$
$$\stackrel{(1.2)}{\equiv} \frac{1-4^{\frac{p+1}{2}}}{2} - \left(-\frac{3}{4}\right)^p - C_{p-1} \left(-\frac{3}{4}\right)^p \pmod{p}.$$

Using the Fermat's little theorem and

$$C_{p-1} = \frac{\binom{2p-2}{p-1}}{p} = \frac{\binom{2p-1}{p-1}}{2p-1} \equiv -1 \pmod{p},$$

we arrive at

$$\sum_{k=0}^{n} C_k \left(-\frac{3}{4}\right)^{k+1} \equiv -\frac{3}{2} \pmod{p}.$$
 (2.5)

Substituting (2.5) into (2.4) gives

$$S_1 \equiv 2(-1)^n - \left(\frac{p}{3}\right) \pmod{p}.$$
 (2.6)

Note that

$$S_2 = \sum_{i=0}^n \sum_{j=n+1}^{2n} \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}} = \sum_{i=0}^n \sum_{j=1}^n \frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}}.$$
 (2.7)

For $i + j \leq n$ and $1 \leq j \leq n$,

$$\frac{\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv 0 \pmod{p},$$

and so the summand on the right-hand side of (2.7) is congruent to 0 modulo p.

On the other hand, for $i + j \ge n + 1$ and $1 \le j \le n$,

$$\frac{\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} = \frac{i!}{(n+j)!} \cdot \frac{(2n+2)\cdots(2n+2j)}{(2n+2)\cdots(i+j+n)}$$
$$\equiv \frac{i!}{(n+j)!} \cdot \frac{(2j-1)!}{(i+j-n-1)!} \pmod{p}.$$
(2.8)

It follows from (2.3) and (2.8) that

$$\frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv \frac{(-4)^i\binom{j-1}{n-i}\binom{2j}{j}}{2\binom{n+j}{j}} \pmod{p}.$$

Since

$$\binom{n+j}{j} \equiv \binom{-\frac{1}{2}+j}{j} = \frac{\binom{2j}{j}}{4^j} \pmod{p},$$

we have

$$\frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}} \equiv \frac{(-1)^i \cdot 4^{i+j} \cdot \binom{j-1}{n-i}}{2} \pmod{p}.$$
 (2.9)

Substituting (2.9) into (2.7) gives

$$S_{2} \equiv \frac{1}{2} \sum_{j=1}^{n} 4^{j} \sum_{i=0}^{n} (-4)^{i} {\binom{j-1}{n-i}} \pmod{p}$$

$$= \frac{(-4)^{n}}{2} \sum_{j=1}^{n} 4^{j} \sum_{i=0}^{n} {\binom{j-1}{i}} \left(-\frac{1}{4}\right)^{i} \quad (i \to n-i)$$

$$= 2(-4)^{n} \sum_{j=1}^{n} 3^{j-1}$$

$$= (-12)^{n} - (-4)^{n}.$$

Thus,

$$S_2 \equiv \left(\frac{p}{3}\right) - (-1)^n \pmod{p}. \tag{2.10}$$

The proof of (1.3) follows from (2.2), (2.6) and (2.10).

3 Proof of Theorem 1.3

Lemma 3.1 For any non-negative integer n, we have

$$\sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} (i+j) \frac{\binom{n}{i}\binom{n}{j}}{\binom{i+j}{i}} = 16n(-3)^{n-1} + \frac{8n4^n}{\binom{2n}{n}} \sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^k.$$
(3.1)

Proof. By the multi-Zeilberger algorithm [3], we obtain the recurrence for the left-hand side of (3.1):

$$-6(n+1)(581n+793)s(n) + (818n^2 - 6653n - 9936)s(n+1) + (2166n^2 + 3474n + 2898)s(n+2) + (2n+5)(251n+92)s(n+3) = 0.$$

It is easy to verify that the right-hand side of (3.1) also satisfies the above recurrence and both sides of (3.1) are equal for n = 0, 1, 2.

Proof of (1.5). Let $n = \frac{p-1}{2}$. In a similar way,

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (i+j) \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}} \equiv S_1 + 2S_2 \pmod{p}, \tag{3.2}$$

where

$$S_1 = \sum_{i=0}^n \sum_{j=0}^n (i+j) \frac{\binom{2i}{i} \binom{2j}{j}}{\binom{i+j}{i}},$$

and

$$S_2 = \sum_{i=0}^{n} \sum_{j=n+1}^{2n} (i+j) \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}}.$$

By (2.3) and (3.1), we have

$$S_{1} \stackrel{(2.3)}{\equiv} \sum_{i=0}^{n} \sum_{j=0}^{n} (-4)^{i+j} (i+j) \frac{\binom{n}{i}\binom{n}{j}}{\binom{i+j}{i}} \pmod{p}$$

$$\stackrel{(3.1)}{\equiv} 16n(-3)^{n-1} + \frac{8n4^{n}}{\binom{2n}{n}} \sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^{k}$$

$$\equiv \frac{8}{3} \left(\frac{p}{3}\right) - 4(-1)^{n} \sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^{k} \pmod{p}, \qquad (3.3)$$

where we make use of $\binom{2n}{n} \equiv (-1)^n \pmod{p}$ in the last step. Since $\binom{2k}{k} \equiv 0 \pmod{p}$ for $n+1 \leq k \leq 2n$, we have

$$\sum_{k=0}^{n} \binom{2k}{k} \left(-\frac{3}{4}\right)^{k} \equiv \sum_{k=0}^{2n} \binom{2k}{k} \left(-\frac{3}{4}\right)^{k} \stackrel{(1.1)}{\equiv} 4^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$
(3.4)

Substituting (3.4) into (3.3) gives

$$S_1 \equiv \frac{8}{3} \left(\frac{p}{3}\right) - 4(-1)^n \pmod{p}.$$
 (3.5)

On the other hand, by (2.9) we have

$$S_{2} = \sum_{i=0}^{n} \sum_{j=n+1}^{2n} (i+j) \frac{\binom{2i}{i}\binom{2j}{j}}{\binom{i+j}{i}}$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{n} (i+j+n) \frac{\binom{2i}{i}\binom{2j+2n}{j+n}}{\binom{i+j+n}{i}}$$

$$\stackrel{(2.9)}{\equiv} \frac{1}{2} \sum_{j=1}^{n} 4^{j} \sum_{i=0}^{n} (-4)^{i} (i+j+n) \binom{j-1}{n-i} \pmod{p}$$

$$= \frac{(-4)^{n}}{2} \sum_{j=1}^{n} 4^{j} \sum_{i=0}^{n} (2n+j-i) \binom{j-1}{i} \left(-\frac{1}{4}\right)^{i},$$
(3.6)

where we set $i \to n - i$ in the last step. Note that

$$\sum_{i=0}^{n} (2n+j-i) {\binom{j-1}{i}} {\binom{-1}{4}}^{i}$$

= $(2n+j) {\binom{3}{4}}^{j-1} - (j-1) \sum_{i=1}^{n} {\binom{j-2}{i-1}} {\binom{-1}{4}}^{i}$
= $(2n+j) {\binom{3}{4}}^{j-1} + \frac{j-1}{4} {\binom{3}{4}}^{j-2}.$ (3.7)

Substituting (3.7) into (3.6) and making elementary calculation gives

$$S_2 \equiv \frac{(-12)^n (10n-3) + (-4)^n (3-6n)}{3} \pmod{p}.$$

It follows that

$$S_2 \equiv 2(-1)^n - \frac{8}{3}\left(\frac{p}{3}\right) \pmod{p}.$$
 (3.8)

Combining (3.2), (3.5) and (3.8), we complete the proof of (1.5).

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