On an inverse boundary value problem for a nonlinear time harmonic Maxwell system

Cătălin I. Cârstea*

Abstract

This paper considers a class of nonlinear time harmonic Maxwell systems at fixed frequency, with nonlinear terms taking the form $\mathscr{X}(x, |\vec{E}(x)|^2)\vec{E}(x)$, $\mathscr{Y}(x, |\vec{H}(x)|^2)\vec{H}(x)$, such that $\mathscr{X}(x,s)$, $\mathscr{Y}(x,s)$ are both real analytic in s. Such nonlinear terms appear in nonlinear optics theoretical models. Under certain regularity conditions, it can be shown that boundary measurements of tangent components of the electric and magnetic fields determine the electric permittivity and magnetic permeability functions as well as the form of the nonlinear terms.

MSC(2000): 35R30, 35F60

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. The (macroscopic) Maxwell's equations for the electromagnetic field in a material filling the domain Ω, without (macroscopic) densities of charge or current, are

$$\begin{cases} \nabla\times\vec{\mathcal{E}} = -\partial_t\vec{\mathcal{B}}, \quad \nabla\times\vec{\mathcal{H}} = \partial_t\vec{\mathcal{D}}, \\ \nabla\cdot\vec{\mathcal{D}} = 0, \quad \nabla\cdot\vec{\mathcal{B}} = 0, \\ \vec{\mathcal{D}} = \underline{\epsilon}\vec{\mathcal{E}} + \vec{\mathscr{P}}_{NL}(\vec{\mathcal{E}}), \\ \vec{\mathcal{B}} = \underline{\mu}\vec{\mathcal{H}} + \vec{\mathscr{M}}_{NL}(\vec{\mathcal{H}}). \end{cases} \tag{1}$$

For a linear medium, $\vec{\mathscr{P}}_{NL}(\vec{\mathcal{E}}) = \vec{\mathscr{M}}_{NL}(\vec{\mathcal{H}}) = 0$, and the system takes the familiar form that has been studied extensively both from the point of view

*HKUST Jockey Club Institute for Advanced Study, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong; email: catalin.carstea@gmail.com

of the forward problem and also of the inverse problem. Nonlinear effects have been observed in practice, as (for example) the extensive literature on nonlinear optics indicates. As an example, see [12], [11], [15], where nonlinearities of the kind appearing in this paper are put forward.

We will consider time-harmonic fields of the form[1]

$$\vec{\mathcal{E}}(t,x) = \vec{E}(x)e^{-i\omega t} + \vec{E}^*(x)e^{i\omega t}, \quad \vec{\mathcal{H}}(t,x) = \vec{H}(x)e^{-i\omega t} + \vec{H}^*(x)e^{i\omega t}, \tag{2}$$

where $\omega > 0$ will be a given fixed frequency. At high frequency, the system (1) may be taken to reduce to

$$\begin{cases} \nabla\times\vec{E}(x) = i\omega\mu(\omega,x)\vec{H}(x) + \mathscr{Y}(\omega,x,|\vec{H}(x)|^2)\vec{H}(x), \\ \nabla\times\vec{H}(x) = -i\omega\epsilon(\omega,x)\vec{E}(x) - \mathscr{X}(\omega,x,|\vec{E}(x)|^2)\vec{E}(x). \end{cases} \tag{3}$$

A common model is that of a Kerr-type nonlinearity:

$$\mathscr{X}(x,|\vec{E}(x)|^2)\vec{E}(x) = a(x)|\vec{E}(x)|^2\vec{E}(x). \tag{4}$$

The inverse problem for a model in which both $\mathscr{X}$ and $\mathscr{Y}$ have this form has been investigated in [1]. However, more realistic models feature a saturation effect for $\mathscr{X}$ when the field intensity is high (see [12], [11]). One example, given in [15], is

$$\mathscr{X}(x,|\vec{E}(x)|^2)\vec{E}(x) = \frac{a(x)|\vec{E}(x)|^2}{1+b(x)|\vec{E}(x)|^2}\vec{E}(x). \tag{5}$$

A more complicated model is deduced in [11].

In this paper it will be assumed that $\mathscr{X}(x,s)$, $\mathscr{Y}(x,s)$ are analytic in s, having expansions at zero

$$\mathscr{X}(x,s) = \sum_{k=1}^{\infty} a_k(x)s^k, \quad \mathscr{Y}(x,s) = \sum_{k=1}^{\infty} b_k(x)s^k, \tag{6}$$

and

$$\epsilon,\mu \in C^5(\Omega), \quad a_k, b_k \in C^1(\Omega) \tag{7}$$

$$\mathfrak{Re}\,\epsilon, \mathfrak{Re}\,\mu > \lambda > 0, \tag{8}$$

$$||\epsilon||_{W^{5,\infty}(\Omega)}, ||\mu||_{W^{5,\infty}(\Omega)} < M < \infty, \tag{9}$$

$$\sum_{k=1}^{\infty} \left(||a_k||_{W^{1,\infty}(\Omega)} + ||b_k||_{W^{1,\infty}(\Omega)}\right) s^k < Ms, \quad \forall\, 0 < s < s_0, \tag{10}$$

[1]The $*$ denotes complex conjugation.

$$\sum_{k=1}^{\infty} k\left(||a_k||_{W^{1,\infty}(\Omega)} + ||b_k||_{W^{1,\infty}(\Omega)}\right) s^{k-1} < M, \quad \forall\, 0 < s < s_0, \tag{11}$$

$$\sum_{k=2}^{\infty} k(k-1)\left(||a_k||_{L^{\infty}(\Omega)} + ||b_k||_{L^{\infty}(\Omega)}\right) s^{k-2} < M, \quad \forall\, 0 < s < s_0, \tag{12}$$

where λ, M, s_0 are positive constants.

A note on notation: in order to make equations easier to read, the explicit dependence on x of various quantities will be suppressed. For example, $\mathscr{X}(|\vec{E}|^2)$ will stand for $\mathscr{X}(x, |\vec{E}(x)|^2)$ or $\mathscr{X}(\cdot, |\vec{E}(\cdot)|^2)$.

1.1 The forward problem

We will say that a vector field belongs to $L^p(\Omega)$, $W^{s,p}(\Omega)$, etc. if each component belongs to those respective spaces. Let

$$W^{s,p}_{div}(\Omega) = \left\{\vec{A} \in W^{s,p}(\Omega) : \nabla\cdot\vec{A} \in W^{s,p}(\Omega)\right\}, \tag{13}$$

with the natural choice of norms. If $\vec{n}$ is the outer unit normal to $\partial\Omega$, let

$$TW^{s,p}(\partial\Omega) = \left\{\vec{A} \in W^{s,p}(\partial\Omega) : \vec{n}\cdot\vec{A} = 0\right\}, \tag{14}$$

$$TW^{s,p}_{div}(\partial\Omega) = \left\{\vec{A} \in TW^{s,p}(\partial\Omega) : div(\vec{A}) \in W^{s,p}(\partial\Omega)\right\}, \tag{15}$$

where $div(\vec{A})$ is the divergence associated with the metric induced on the boundary by the Euclidean metric of $\mathbb{R}^3$. For a smooth vector field $\vec{A}$ on Ω, let

$$\mathfrak{t}(\vec{A}) = -\vec{n}\times(\vec{n}\times\vec{A}|_{\partial\Omega}), \tag{16}$$

i.e. the component tangential to the boundary of the restriction of $\vec{A}$. $\mathfrak{t}$ clearly extends to a bounded operator from $W^{s,p}(\Omega)$ to $TW^{s-1/p,p}(\partial\Omega)$. Let $W^{1,p}_b(\Omega) = \mathfrak{t}^{-1}\left(TW^{1-1/p,p}_{div}(\partial\Omega)\right)$, with the norm

$$||\vec{A}||_{W^{1,p}_b(\Omega)} = ||\vec{A}||_{W^{1,p}(\Omega)} + ||\mathfrak{t}(\vec{A})||_{TW^{1-1/p,p}_{div}(\partial\Omega)}. \tag{17}$$

Finally, let

$$W^{1,p}_D(\Omega) = \mathfrak{t}^{-1}(0), \quad ||\cdot||_{W^{1,p}_D(\Omega)} = ||\cdot||_{W^{1,p}(\Omega)}. \tag{18}$$

Before discussing the inverse problem a well-posedness result for the forward problem is necessary. In section 2 it will be proven that:

Theorem 1.1. *For $3 < p \leq 6$ there exists a discrete set $\Sigma \subset \mathbb{C}$ and a constant $\mathfrak{m} > 0$ such that if $\omega \notin \Sigma$ and $\vec{f} \in TW^{1-1/p,p}_{div}(\partial\Omega)$, $||\vec{f}||_{TW^{1-1/p,p}_{div}(\partial\Omega)} < \mathfrak{m}$ there exists a unique solution $\mathbf{U} = (\vec{E}, \vec{H}) \in W^{1,p}_b(\Omega) \times W^{1,p}_b(\Omega)$ of the system*

$$\begin{cases} \nabla \times \vec{E} = i\omega\mu\vec{H} + \mathscr{Y}(|\vec{H}|^2)\vec{H}, \\ \nabla \times \vec{H} = -i\omega\epsilon\vec{E} - \mathscr{X}(|\vec{E}|^2)\vec{E}, \end{cases} \tag{19}$$

such that $\mathfrak{t}(\vec{E}) = \vec{f}$ and

$$||\vec{E}||_{W^{1,p}_b(\Omega)} + ||\vec{H}||_{W^{1,p}_b(\Omega)} \leq C||\vec{f}||_{TW^{1-1/p,p}_{div}(\partial\Omega)}, \tag{20}$$

where $C > 0$ is a constant that does not depend on $\vec{f}$.

The proof of this result follows from estimates for the linear system obtained in [1] and a standard contraction principle argument.

1.2 The inverse problem

An inverse boundary value problem consists of the question of determining the interior physical properties of a possibly non-homogeneous object from measurements taken on the boundary of the object. A fundamental subproblem is the question of uniqueness: if two objects of the same shape give the same boundary measurement data, does it follow that their (relevant) physical properties are identical in the interior?

For time-harmonic electromagnetic fields in media in which (3) applies, in light of Theorem 1.1 we can then define the set of boundary measurements

$$\mathcal{B}_{\epsilon,\mu,\mathscr{F}} = \Big\{ (\mathfrak{t}(\vec{E}), \mathfrak{t}(\vec{H})) \in TW^{1-1/p,p}_{div}(\partial\Omega) \times TW^{1-1/p,p}_{div}(\partial\Omega) \\ : (\vec{E}, \vec{H}) \text{ is a solution of (3)} \Big\}. \tag{21}$$

In section 3 we prove that

Theorem 1.2. *Suppose $(\epsilon, \mu, \mathscr{F})$ and $(\epsilon, \mu, \mathscr{F})$ are as above, $\omega \notin \Sigma \cup \Sigma'$, and $\mathcal{B}_{\epsilon,\mu,\mathscr{F}} = \mathcal{B}_{\epsilon',\mu',\mathscr{F}'}$. Then $(\epsilon, \mu, \mathscr{F}) = (\epsilon', \mu', \mathscr{F}')$.*

The inverse boundary value problem has been studied extensively in the linear case. See for example [18], [4], [13], [14], [3], [2], [10], [21] etc. Uniqueness results similar to Theorem 1.2 for nonlinear equations have been obtained in [9], [8], [16], [19], [7], [6], [5], [17] using a linearization method. Here we will follow an idea from [1] and use the asymptotics in a small parameter t of solutions of (3) with boundary data $\mathfrak{t}(\vec{E}) = t\vec{f}$ in order to inductively prove uniqueness for the coefficients of the nonlinearity. We will also need to use certain special solutions, so called geometric optics (CGO) solutions, which we construct following the method in [2].

2 The forward problem

2.1 Preliminaries

The existence and uniqueness of $W^{1,p}$ solutions to the linear Maxwell system, for $p > 2$, has been investigated in [1]. First we quote an existence result for the boundary value problem for the homogeneous system:

Theorem 2.1 (see [1, Theorem 3.1]). *For $2 \le p \le 6$ there exists a discrete set $\Sigma \subset \mathbb{C}$ such that if $\omega \notin \Sigma$ and $\vec{f} \in TW^{1-1/p,p}_{div}(\partial\Omega)$ there exists a unique solution $(\vec{E}, \vec{H}) \in W^{1,p}_b(\Omega) \times W^{1,p}_b(\Omega)$ of the system*

$$\begin{cases} \nabla\times\vec{E} = i\omega\mu\vec{H}, \\ \nabla\times\vec{H} = -i\omega\epsilon\vec{E}, \end{cases} \tag{22}$$

such that $\mathfrak{t}(\vec{E}) = \vec{f}$ and

$$||\vec{E}||_{W^{1,p}_b(\Omega)} + ||\vec{H}||_{W^{1,p}_b(\Omega)} \le C||\vec{f}||_{TW^{1-1/p,p}_{div}(\partial\Omega)}, \tag{23}$$

where $C > 0$ is a constant that does not depend on $\vec{f}$.

We also need the following result for the inhomogeneous system:

Theorem 2.2 (see [1, Theorem 3.2]). *For $2 \le p \le 6$ exists a discrete set $\Sigma \subset \mathbb{C}$ such that if $\omega \notin \Sigma$ and $\vec{J}_e, \vec{J}_m \in W^{0,p}_{div}(\Omega)$, $\vec{n} \cdot \vec{J}_e|_{\partial\Omega}, \vec{n} \cdot \vec{J}_m|_{\partial\Omega} \in W^{1-1/p,p}(\partial\Omega)$, there exists a unique solution $(\vec{E}, \vec{H}) \in W^{1,p}_D(\Omega) \times W^{1,p}_b(\Omega)$ of the system*

$$\begin{cases} \nabla\times\vec{E} = i\omega\mu\vec{H} + \vec{J}_m, \\ \nabla\times\vec{H} = -i\omega\epsilon\vec{E} - \vec{J}_e, \end{cases} \tag{24}$$

such that

$$\begin{aligned} ||\vec{E}||_{W^{1,p}_b(\Omega)} + ||\vec{H}||_{W^{1,p}_b(\Omega)} \le C \Big(& ||\vec{J}_e||_{W^{0,p}_{div}(\Omega)} + ||\vec{J}_m||_{W^{0,p}_{div}(\Omega)} \\ & + ||\vec{n} \cdot \vec{J}_e|_{\partial\Omega}||_{W^{1-1/p,p}(\partial\Omega)} + ||\vec{n} \cdot \vec{J}_m|_{\partial\Omega}||_{W^{1-1/p,p}(\partial\Omega)} \Big), \end{aligned} \tag{25}$$

where $C > 0$ is a constant that does not depend on $\vec{J}_e, \vec{J}_m$.

Under the conditions of Theorem 2.2, we will write

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \mathcal{G}_{\epsilon,\mu}(\begin{pmatrix} \vec{J}_m \\ \vec{J}_e \end{pmatrix}). \tag{26}$$

It is a corollary of Theorem 2.2 that $\mathcal{G}_{\epsilon,\mu}$ is bounded from $W^{1,p}(\Omega)\times W^{1,p}(\Omega)$ to $W^{1,p}_D(\Omega)\times W^{1,p}_b(\Omega)$.

For the sake of simplifying notation let:

$$\mathbf{U}=\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix},\quad \mathcal{L}_{\epsilon,\mu}=\begin{pmatrix}\nabla\times & -i\omega\mu\\ i\omega\epsilon & \nabla\times\end{pmatrix},\quad \mathscr{F}(\mathbf{U})=\begin{pmatrix}\mathscr{Y}(|\vec{H}|^2)\vec{H}\\ -\mathscr{X}(|\vec{E}|^2)\vec{E}\end{pmatrix}. \tag{27}$$

Then equation (3) can be written

$$\mathcal{L}_{\epsilon,\mu}\mathbf{U}=\mathscr{F}(\mathbf{U}). \tag{28}$$

Given $\vec{f}\in TW^{1-1/p,p}_{div}(\partial\Omega)$, let

$$\mathbf{U}_0=\begin{pmatrix}\vec{E}_0 & \vec{H}_0\end{pmatrix}^t\in W^{1,p}_b(\Omega)\times W^{1,p}_b(\Omega) \tag{29}$$

be the solution given in Theorem 2.1. Then a solution of (3) with the boundary condition $\mathfrak{t}(\vec{E})=\vec{f}$ would be a fixed point of the operator

$$\mathcal{T}_{\vec{f},\epsilon,\mu}(\mathbf{U})=\mathbf{U}_0+\mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U})). \tag{30}$$

2.2 Existence of solutions

From now we will only consider $p>3$. Then $W^{1,p}(\Omega)\subset L^\infty(\Omega)$ and there exists a constant $c>0$ such that

$$||\mathbf{U}||_{L^\infty(\Omega)}\le c||\mathbf{U}||_{W^{1,p}(\Omega)},\quad \forall\ \mathbf{U}=\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix}\in W^{1,p}(\Omega). \tag{31}$$

Lemma 2.1. *Suppose* $\mathbf{U},\mathbf{U}'\in W^{1,p}(\Omega)$, *and* $||\mathbf{U}||_{W^{1,p}(\Omega)},||\mathbf{U}||_{W^{1,p}(\Omega)}\le\frac{s_0}{c}$, *then*

$$\|\mathscr{F}(\mathbf{U})-\mathscr{F}(\mathbf{U}')\|_{W^{1,p}(\Omega)} \le C\left(||\mathbf{U}||^2_{W^{1,p}(\Omega)}+||\mathbf{U}'||^2_{W^{1,p}(\Omega)}\right)||\mathbf{U}-\mathbf{U}'||_{W^{1,p}(\Omega)}, \tag{32}$$

where $C>0$ *does not depend on* $\mathbf{U}$ *and* $\mathbf{U}'$.

Proof. Suppose $\mathbf{U}=\begin{pmatrix}\vec{E} & \vec{H}\end{pmatrix}^t$, $\mathbf{U}'=\begin{pmatrix}\vec{E}' & \vec{H}'\end{pmatrix}^t$. Consider the difference

$$\mathscr{X}(|\vec{E}|^2)\vec{E}-\mathscr{X}(|\vec{E}'|^2)\vec{E}'=\mathscr{X}(|\vec{E}|^2)(\vec{E}-\vec{E}')+\left(\mathscr{X}(|\vec{E}|^2)-\mathscr{X}(|\vec{E}'|^2)\right)\vec{E}'. \tag{33}$$

Note that

$$|\mathscr{X}(x,|\vec{E}(x)|^2)| \leq \sum_{k=1}^{\infty} ||a_k||_{L^\infty(\Omega)}||\vec{E}||^{2k}_{L^\infty(\Omega)} \leq M||\vec{E}||^2_{W^{1,p}(\Omega)}, \tag{34}$$

$$|(D_x\mathscr{X})(x,|\vec{E}(x)|^2)| \leq M||\vec{E}||^2_{W^{1,p}(\Omega)}, \tag{35}$$

$$|(\partial_s\mathscr{X})(x,|\vec{E}(x)|^2)| \leq \sum_{k=1}^{\infty} k||a_k||_{L^\infty(\Omega)}||\vec{E}||^{2(k-1)}_{L^\infty} \leq M. \tag{36}$$

Therefore

$$\left\|\mathscr{X}(|\vec{E}|^2)(\vec{E}-\vec{E}')\right\|_{L^p(\Omega)} \leq C||\vec{E}||^2_{W^{1,p}(\Omega)}||\vec{E}-\vec{E}'||_{L^p(\Omega)}. \tag{37}$$

Also, since

$$\begin{aligned} D_x[\mathscr{X}(|\vec{E}|^2)(\vec{E}-\vec{E}')] &= \mathscr{X}(|\vec{E}|^2)D_x(\vec{E}-\vec{E}') \\ &+ (D_x\mathscr{X})(|\vec{E}|^2)(\vec{E}-\vec{E}') + 2\mathfrak{Re}\,(\vec{E}^*\cdot D_x\vec{E})(\partial_s\mathscr{X})(|\vec{E}|^2)(\vec{E}-\vec{E}') \end{aligned} \tag{38}$$

and

$$\left\|\mathscr{X}(|\vec{E}|^2)D_x(\vec{E}-\vec{E}')\right\|_{L^p(\Omega)} \leq M||\vec{E}||^2_{W^{1,p}(\Omega)}||D_x(\vec{E}-\vec{E}')||_{L^p(\Omega)}, \tag{39}$$

$$\left\|(D_x\mathscr{X})(|\vec{E}|^2)(\vec{E}-\vec{E}')\right\|_{L^p(\Omega)} \leq M||\vec{E}||^2_{W^{1,p}(\Omega)}||\vec{E}-\vec{E}'||_{L^p(\Omega)}, \tag{40}$$

$$\begin{aligned} &\left\|2\mathfrak{Re}\,(\vec{E}^*\cdot D_x\vec{E})(\partial_s\mathscr{X})(|\vec{E}|^2)(\vec{E}-\vec{E}')\right\|_{L^p(\Omega)} \\ &\qquad \leq 2M||\vec{E}||_{L^\infty(\Omega)}||D_x\vec{E}||_{L^p(\Omega)}||\vec{E}-\vec{E}'||_{L^\infty(\Omega)} \\ &\qquad\qquad \leq 2c^2M||\vec{E}||^2_{W^{1,p}(\Omega)}||\vec{E}-\vec{E}'||_{W^{1,p}(\Omega)}, \end{aligned} \tag{41}$$

it follows that

$$\left\|\mathscr{X}(|\vec{E}|^2)(\vec{E}-\vec{E}')\right\|_{W^{1,p}(\Omega)} \leq C||\vec{E}||^2_{W^{1,p}(\Omega)}||\vec{E}-\vec{E}'||_{W^{1,p}(\Omega)}. \tag{42}$$

In order to estimate the second term in (34), let $\vec{E}_t = \vec{E}' + t(\vec{E}-\vec{E}')$. Then

$$\mathscr{X}(|\vec{E}|^2) - \mathscr{X}(|\vec{E}'|^2) = \int_0^1 \partial_s\mathscr{X}(|\vec{E}_t|^2)2\mathfrak{Re}\,(\vec{E}_t^*\cdot(\vec{E}-\vec{E}'))\,\mathrm{d}t. \tag{43}$$

We have, for $q = p$ or $q = \infty$, that

$$\left\| \partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \right\|_{L^q(\Omega)} \leq C||\vec{E}_t||_{W^{1,p}(\Omega)} ||\vec{E} - \vec{E}'||_{W^{1,p}(\Omega)}$$
$$\leq C(||\vec{E}||_{W^{1,p}(\Omega)} + ||\vec{E}'||_{W^{1,p}(\Omega)}) ||\vec{E} - \vec{E}'||_{W^{1,p}(\Omega)} \quad (44)$$

Now

$$D_x \left[\partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \right] = (D_x \partial_s \mathscr{X})(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}'))$$
$$+ \partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(D_x \vec{E}_t^* \cdot (\vec{E} - \vec{E}'))$$
$$+ \partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot D_x(\vec{E} - \vec{E}'))$$
$$+ \partial_s^2 \mathscr{X}(|\vec{E}_t|^2) 4\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \mathfrak{Re}\,(\vec{E}_t^* \cdot D_x \vec{E}_t). \quad (45)$$

Using the same type of estimates as above, we can obtain that

$$\left\| (D_x \partial_s \mathscr{X})(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \right\|_{L^p(\Omega)} \leq C||\vec{E}_t||_{L^\infty(\Omega)} ||\vec{E} - \vec{E}'||_{L^p(\Omega)}, \quad (46)$$

$$\left\| \partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(D_x \vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \right\|_{L^p(\Omega)} \leq C||\vec{E}_t||_{W^{1,p}(\Omega)} ||\vec{E} - \vec{E}'||_{L^\infty(\Omega)}, \quad (47)$$

$$\left\| \partial_s \mathscr{X}(|\vec{E}_t|^2) 2\mathfrak{Re}\,(\vec{E}_t^* \cdot D_x(\vec{E} - \vec{E}')) \right\|_{L^p(\Omega)} \leq C||\vec{E}_t||_{L^\infty(\Omega)} ||\vec{E} - \vec{E}'||_{W^{1,p}(\Omega)}, \quad (48)$$

$$\left\| \partial_s^2 \mathscr{X}(|\vec{E}_t|^2) 4\mathfrak{Re}\,(\vec{E}_t^* \cdot (\vec{E} - \vec{E}')) \mathfrak{Re}\,(\vec{E}_t^* \cdot D_x \vec{E}_t) \right\|_{L^p(\Omega)}$$
$$\leq C||\vec{E}_t||^2_{L^\infty(\Omega)} ||\vec{E}_t||_{W^{1,p}(\Omega)} ||\vec{E} - \vec{E}'||_{L^\infty(\Omega)}$$
$$\leq C||\vec{E}_t||_{W^{1,p}(\Omega)} ||\vec{E} - \vec{E}'||_{L^\infty(\Omega)}. \quad (49)$$

Putting these together with (43) it follows that

$$\left\| \left(\mathscr{X}(|\vec{E}|^2) - \mathscr{X}(|\vec{E}'|^2) \right) \vec{E}' \right\|_{W^{1,p}(\Omega)}$$
$$\leq C \left(||\vec{E}||^2_{W^{1,p}(\Omega)} + ||\vec{E}'||^2_{W^{1,p}(\Omega)} \right) ||\vec{E} - \vec{E}'||_{W^{1,p}(\Omega)}. \quad (50)$$

Equations (33), (37), (50) imply

$$\left\| \mathscr{X}(|\vec{E}|^2) \vec{E} - \mathscr{X}(|\vec{E}'|^2) \vec{E}' \right\|_{W^{1,p}(\Omega)}$$
$$\leq C \left(||\vec{E}||^2_{W^{1,p}(\Omega)} + ||\vec{E}'||^2_{W^{1,p}(\Omega)} \right) ||\vec{E} - \vec{E}'||_{W^{1,p}(\Omega)}. \quad (51)$$

A similar estimate holds for the $\mathscr{Y}$ component of $\mathscr{F}$. $\square$

Since $\mathscr{F}(0) = 0$, Lemma 2.1 has the corollary

Corollary 2.1. *If* $\mathbf{U} \in W^{1,p}(\Omega)$, *and* $||\mathbf{U}||_{W^{1,p}(\Omega)} \leq \frac{s_0}{c}$, *then*

$$\|\mathscr{F}(\mathbf{U})\|_{W^{1,p}(\Omega)} \leq C||\mathbf{U}||^3_{W^{1,p}(\Omega)}. \tag{52}$$

Proof of Theorem 1.1. Applying Lemma 2.1 and it's corollary together with Theorems 2.1 and 2.2 we can show that the operator $\mathcal{T}_{\vec{f},\epsilon,\mu}$ defined in (30) is a contraction on a sufficiently small ball in $W^{1,p}_D(\Omega) \times W^{1,p}_b(\Omega)$, of radius $\mathfrak{m}$, and therefore has a fixed point. □

In the following discussion we will assume that $\mathfrak{m}$ is chosen so that if $||\vec{f}||_{TW^{1-1/p,p}_{div}(\partial\Omega)} < \mathfrak{m}$, then

$$||\mathbf{U}_0||_{W^{1,p}(\Omega)} < \frac{\mathfrak{m}}{2}, \tag{53}$$

and if $||\mathbf{U}||_{W^{1,p}(\Omega)} < \mathfrak{m}$, then

$$||\mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U}))||_{W^{1,p}(\Omega)} < \frac{\mathfrak{m}}{2}. \tag{54}$$

2.3 Asymptotics

For $\mathbf{U} \in W^{1,p}(\Omega) = \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$, define

$$\mathscr{F}_k(\mathbf{U}) = \begin{pmatrix} b_k|\vec{H}|^{2k}\vec{H} \\ -a_k|\vec{E}|^{2k}\vec{E} \end{pmatrix}, \tag{55}$$

so $\mathscr{F}(\mathbf{U}) = \sum_{k=1}^{\infty} \mathscr{F}_k(\mathbf{U})$.

Let t be a small parameter. For $\vec{f} \in TW^{1-1/p,p}_{div}(\partial\Omega)$, let $\vec{f^t} = t\vec{f}$. Also, let $\mathbf{U}^t = \begin{pmatrix} \vec{E}^t \\ \vec{H}^t \end{pmatrix}$ be the solution of $\mathcal{L}_{\epsilon,\mu}\mathbf{U}^t = \mathscr{F}(\mathbf{U}^t)$ with boundary data $\mathfrak{t}(\vec{E}_t) = \vec{f}$, and let $\mathbf{U}^t_0 = \begin{pmatrix} \vec{E}^t_0 \\ \vec{H}^t_0 \end{pmatrix}$ be the solution of $\mathcal{L}_{\epsilon,\mu}\mathbf{U}^t_0 = 0$ with the same boundary data. Set $\mathbf{U}^t_k = \mathcal{T}^k_{\vec{f},\epsilon,\mu}(\mathbf{U}^t_0)$, $k = 1, 2, \ldots$. For $|t| < \mathfrak{m}/||\vec{f}||_{TW^{1-1/p,p}_{div}(\partial\Omega)}$, since $\mathcal{T}_{\vec{f},\epsilon,\mu}$ is a contraction,

$$||U^t_k||_{W^{1,p}(\Omega)} \leq \mathfrak{m} \text{ and } \mathbf{U}^t_k \to \mathbf{U}^t \text{ in } W^{1,p}(\Omega), \text{ as } k \to \infty. \tag{56}$$

Observe that $\mathbf{U}^t_0 = t\mathbf{U}_0$. Define

$$\mathbf{V}^t_1 = \begin{pmatrix} \vec{B}^t_1 \\ \vec{A}^t_1 \end{pmatrix} = \mathbf{U}^t_1 - \mathbf{U}^t_0 = \mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U}^t_0)), \tag{57}$$

$$\mathbf{V}_k^t = \begin{pmatrix} \vec{B}_k^t \\ \vec{A}_k^t \end{pmatrix} = \mathbf{U}_k^t - \mathbf{U}_{k-1}^t = \mathcal{G}_{\epsilon,\mu}\left(\mathscr{F}(\mathbf{U}_{k-2}^t + \mathbf{V}_{k-1}^t) - \mathscr{F}(\mathbf{U}_{k-2}^t)\right). \quad (58)$$

Then

$$\mathbf{V}_1^t = t^3 \mathcal{G}_{\epsilon,\mu}(\mathscr{F}_1(\mathbf{U}_0)) + t^5 \mathcal{G}_{\epsilon,\mu}(\mathscr{F}_2(\mathbf{U}_0)) + \cdots, \quad (59)$$

$$\begin{aligned} \mathbf{V}_2^t &= \mathcal{G}_{\epsilon,\mu}(\mathscr{F}_1(t\mathbf{U}_0 + \mathbf{V}_1^t) - \mathscr{F}_1(t\mathbf{U}_0)) + \cdots \\ &= \mathcal{G}_{\epsilon,\mu}(t^2 \begin{pmatrix} b_1|\vec{H}_0|^2\vec{B}_1^t + 2\vec{H}_0 \mathfrak{Re}\,(\vec{H}_0 \cdot \vec{B}_1^{t*}) \\ -a_1|\vec{E}_0|^2\vec{A}_1^t - 2\vec{E}_0 \mathfrak{Re}\,(\vec{E}_0 \cdot \vec{A}_1^{t*}) \end{pmatrix}) + \cdots = \mathscr{O}(t^5) \quad (60) \end{aligned}$$

and so on.

Lemma 2.2.

$$||\mathbf{V}_k^t||_{W^{1,p}(\Omega)} = \mathscr{O}(t^{2k+1}), \ \ as\ t \to 0. \quad (61)$$

Proof. Follows easily by induction. □

Lemma 2.3.

$$||\mathbf{U}^t - \mathbf{U}_k^t||_{W^{1,p}(\Omega)} = \mathscr{O}(t^{2k+3}), \ \ as\ t \to 0. \quad (62)$$

Proof. Let $\mathbf{u}_k^t = \mathbf{U}^t - \mathbf{U}_k^t$. Then

$$\begin{aligned} \mathbf{u}_k^t &= \mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U}_k^t + \mathbf{u}_k^t) - \mathscr{F}(\mathbf{U}_{k-1}^t)) \\ &= \mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U}_k^t + \mathbf{u}_k^t) - \mathscr{F}(\mathbf{U}_k^t)) + \mathcal{G}_{\epsilon,\mu}(\mathscr{F}(\mathbf{U}_k^t) - \mathscr{F}(\mathbf{U}_{k-1}^t)) \\ &= \left(\mathcal{T}_{\vec{f},\epsilon,\mu}(\mathbf{U}_k^t + \mathbf{u}_k^t) - \mathcal{T}_{\vec{f},\epsilon,\mu}(\mathbf{U}_k^t)\right) + \mathbf{V}_{k+1}^t. \quad (63) \end{aligned}$$

Since for small enough t, $\mathcal{T}_{\vec{f},\epsilon,\mu}$ is a contraction, the first term on the right hand side may be absorbed into the left hand side and applying Lemma 2.2, the result follows. □

Notice that the terms multiplying t^{2k+1} are the same for all $\mathbf{U}_{k'}^t$ with $k' \geq k$. Define then

$$\mathbf{W}_k = \frac{1}{(2k+1)!} \partial_t^{2k+1} \mathbf{U}_k^t \Big|_{t=0}. \quad (64)$$

A useful observation is that

$$\mathbf{W}_k = \mathcal{G}_{\epsilon,\mu}(\begin{pmatrix} b_k|\vec{H}_0|^{2k}H_0 \\ -a_k|\vec{E}_0|^{2k}E_0 \end{pmatrix}) + \left(\text{terms constructed from } \{a_l, b_l\}_{l=1}^{k-1} \text{ and } \mathbf{U}_0\right), \quad (65)$$

so

$$\mathcal{L}_{\epsilon,\mu}\mathbf{W}_k = \begin{pmatrix} b_k|\vec{H}_0|^{2k}H_0 \\ -a_k|\vec{E}_0|^{2k}E_0 \end{pmatrix} + \left(\text{terms constructed from } \{a_l, b_l\}_{l=1}^{k-1} \text{ and } \mathbf{U}_0\right). \quad (66)$$

3 The inverse problem

Suppose $\mathcal{B}_{\epsilon,\mu,\mathscr{F}} = \mathcal{B}_{\epsilon',\mu',\mathscr{F}'}$. For any $\vec{f} \in TW_{div}^{1-1/p,p}(\partial\Omega)$ let $\mathbf{U}_0 = \begin{pmatrix} \vec{E}_0 \\ \vec{H}_0 \end{pmatrix}$, $\mathbf{U}_0' = \begin{pmatrix} \vec{E}_0' \\ \vec{H}_0' \end{pmatrix}$, $\mathbf{W}_k = \begin{pmatrix} \vec{E}_k \\ \vec{H}_k \end{pmatrix}$, $\mathbf{W}_k' = \begin{pmatrix} \vec{E}_k' \\ \vec{H}_k' \end{pmatrix}$ be the constructed as in the previous sections using the two sets of coefficients respectively. Then we have

$$\mathfrak{t}(\vec{E}_0) = \mathfrak{t}(\vec{E}_0') = \vec{f}, \quad \mathfrak{t}(\vec{H}_0) = \mathfrak{t}(\vec{H}_0'), \tag{67}$$

$$\mathfrak{t}(\vec{E}_k) = \mathfrak{t}(\vec{E}_k') = 0, \quad \mathfrak{t}(\vec{H}_k) = \mathfrak{t}(\vec{H}_k'), \quad k = 1, 2, \ldots. \tag{68}$$

An immediate consequence is that

$$\begin{aligned} &\{(\mathfrak{t}(\vec{E}_0), \mathfrak{t}(\vec{H}_0)) \in TW_{div}^{1-1/p,p}(\partial\Omega) \times TW_{div}^{1-1/p,p}(\partial\Omega) : \mathcal{L}_{\epsilon,\mu} \begin{pmatrix} \vec{E}_0 \\ \vec{H}_0 \end{pmatrix} = 0\} \\ &= \{(\mathfrak{t}(\vec{E}_0'), \mathfrak{t}(\vec{H}_0')) \in TW_{div}^{1-1/p,p}(\partial\Omega) \times TW_{div}^{1-1/p,p}(\partial\Omega) : \mathcal{L}_{\epsilon',\mu'} \begin{pmatrix} \vec{E}_0' \\ \vec{H}_0' \end{pmatrix} = 0\}. \end{aligned} \tag{69}$$

It is a known result then (e.g. see [14], or [2]) that $\epsilon = \epsilon'$ and $\mu = \mu'$. It follows that $\mathbf{U}_0 = \mathbf{U}_0'$.

Suppose then that $\{a_l, b_l\}_{l=1}^{k-1} = \{a_l', b_l'\}_{l=1}^{k-1}$. We will show that then $a_k = a_k'$ and $b_k = b_k'$. Theorem 1.2 will follow by induction.

3.1 An integral identity

For two vector fields $\vec{A}$ and $\vec{B}$, we have

$$\int_{\partial\Omega} (\vec{n} \times \vec{A}) \cdot \vec{B} = \int_{\Omega} (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B}). \tag{70}$$

Let $\mathbf{u}_0 = \begin{pmatrix} \vec{e}_0 \\ \vec{h}_0 \end{pmatrix}$ be a solution of $\mathcal{L}_{\epsilon,\mu}\mathbf{u}_0 = 0$. Using (66) we get

$$\begin{aligned} \int_{\partial\Omega} (\vec{n} \times \vec{E}_k) \cdot \vec{h}_0 &= \int_{\Omega} i\omega\mu \vec{H}_k \cdot \vec{h}_0 + b_k |\vec{H}_0|^{2k} \vec{H}_0 \cdot \vec{h}_0 + i\omega\epsilon \vec{E}_k \cdot \vec{e}_0 \\ &\quad + \int_{\Omega} \left(\text{terms constructed from } \{a_l, b_l\}_{l=1}^{k-1}, \mathbf{U}_0, \text{ and } \mathbf{u}_0\right), \end{aligned} \tag{71}$$

$$\int_{\partial\Omega} (\vec{n} \times \vec{H}_k) \cdot \vec{e}_0 = -\int_{\Omega} i\omega\epsilon \vec{E}_k \cdot \vec{e}_0 + a_k |\vec{E}_0|^{2k} \vec{E}_0 \cdot \vec{e}_0 + i\omega\mu \vec{H}_k \cdot \vec{h}_0$$
$$+ \int_{\Omega} \left(\text{terms constructed from } \{a_l, b_l\}_{l=1}^{k-1}, \mathbf{U}_0, \text{ and } \mathbf{u}_0\right). \quad (72)$$

Then

$$\int_{\partial\Omega} (\vec{n} \times \vec{E}_k) \cdot \vec{h}_0 + (\vec{n} \times \vec{H}_k) \cdot \vec{e}_0 = \int_{\Omega} b_k |\vec{H}_0|^{2k} \vec{H}_0 \cdot \vec{h}_0 - a_k |\vec{E}_0|^{2k} \vec{E}_0 \cdot \vec{e}_0$$
$$+ \int_{\Omega} \left(\text{terms constructed from } \{a_l, b_l\}_{l=1}^{k-1}, \mathbf{U}_0, \text{ and } \mathbf{u}_0\right). \quad (73)$$

Subtracting the corresponding identities for the components of $\mathbf{W}'_k$ and using (68) we have

$$I_k(\mathbf{U}_0, u_0) = \int_{\Omega} (b_k - b'_k) |\vec{H}_0|^{2k} \vec{H}_0 \cdot \vec{h}_0 - (a_k - a'_k) |\vec{E}_0|^{2k} \vec{E}_0 \cdot \vec{e}_0 = 0, \quad (74)$$

which holds for any $\mathbf{u}_0$, $\mathbf{U}_0$ solutions of the homogeneous linear equation.

Let $\mathbf{u}_j = \begin{pmatrix} \vec{e}_j \\ \vec{h}_j \end{pmatrix} \in W_b^{1,p}(\Omega) \times W_b^{1,p}(\Omega)$ all satisfy $\mathcal{L}_{\epsilon,\mu}\mathbf{u}_j = 0$, $j = 1, 2, 3$, then

$$I_k(t_1\mathbf{u}_1 + t_2\mathbf{u}_2 + t_3\mathbf{u}_3, u_0) = 0, \quad \forall t_1, t_2, t_3 \in \mathbb{C}. \quad (75)$$

The left hand side of this identity is a polynomial in $t_1, t_2, t_3, t_1^*, t_2^*, t_3^*$, so the coefficient of each independent monomial must vanish. In particular, the coefficient of $t_1 t_2^k t_3^{*k}$ must be zero. The vanishing quantity is

$$\int_{\Omega} (b_k - b'_k) \left[\vec{h}_0 \cdot \vec{h}_1 (\vec{h}_2 \cdot \vec{h}_3^*)^k + k\vec{h}_0 \cdot \vec{h}_2 (\vec{h}_1 \cdot \vec{h}_3^*)(\vec{h}_2 \cdot \vec{h}_3^*)^{k-1}\right]$$
$$- \int_{\Omega} (a_k - a'_k) \left[\vec{e}_0 \cdot \vec{e}_1 (\vec{e}_2 \cdot \vec{e}_3^*)^k + k\vec{e}_0 \cdot \vec{e}_2 (\vec{e}_1 \cdot \vec{e}_3^*)(\vec{e}_2 \cdot \vec{e}_3^*)^{k-1}\right] = 0 \quad (76)$$

3.2 CGO solutions for the linear Maxwell system

CGO solutions for the linear Maxwell system have been constructed in many past works. The method given here is due to [13], [14]. We will mostly follow the construction as given in [2], summarizing the results when the argument proceeds identically and giving more detail when not. We show that

Proposition 3.1. *There exists a constant* $C(\rho, ||\epsilon||_{W^{5,\infty}(\Omega)}, ||\mu||_{W^{5,\infty}(\Omega)}) > 0$ *such that if* $\vec{\zeta} \in \mathbb{C}^3$, $\vec{\zeta} \cdot \vec{\zeta} = \omega^2$,

$$|\vec{\zeta}| > C(\rho, ||\epsilon||_{W^{5,\infty}(\Omega)}, ||\mu||_{W^{5,\infty}(\Omega)}), \tag{77}$$

then there exist solutions $\mathbf{U} = \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$ *of* $\mathcal{L}_{\epsilon,\mu}\mathbf{U} = 0$ *such that*

$$\vec{E} = e^{i\vec{\zeta}\cdot x}\left(\sigma_e \epsilon^{-1/2}\frac{\vec{\zeta}}{|\zeta|} + \vec{r}_e\right), \tag{78}$$

$$\vec{H} = e^{i\vec{\zeta}\cdot x}\left(\sigma_h \mu^{-1/2}\frac{\vec{\zeta}}{|\zeta|} + \vec{r}_h\right), \tag{79}$$

$$||\vec{r}_e||_{L^\infty(\Omega)}, ||\vec{r}_h||_{L^\infty(\Omega)} = \mathscr{O}(|\vec{\zeta}|^{-1}), \tag{80}$$

and $\sigma_e, \sigma_h \in \{0, 1\}$.

Let $\alpha = \log \epsilon$, $\beta = \log \mu$, and I_n be the identity matrix in dimension n. Suppose that

$$X = \left(\begin{array}{c} h \\ \vec{H} \\ \hline e \\ \vec{E} \end{array}\right) \tag{81}$$

satisfies the equation

$$(P + V)X = 0, \tag{82}$$

where

$$P = \frac{1}{i}\left(\begin{array}{cc|cc} & & & \nabla\cdot \\ & & \nabla & -\nabla\times \\ \hline & \nabla\cdot & & \\ \nabla & \nabla\times & & \end{array}\right), \tag{83}$$

$$V = \frac{1}{i}\left(\begin{array}{cc|cc} i\omega\mu & & & (\nabla\alpha)\cdot \\ & i\omega\mu I_3 & (\nabla\alpha) & \\ \hline & (\nabla\beta)\cdot & i\omega\epsilon & \\ (\nabla\beta) & & & i\omega\epsilon I_3 \end{array}\right). \tag{84}$$

Observe that if e and h vanish, then$(\vec{E}, \vec{H})$ is a solution of

$$\begin{cases} \nabla\times\vec{E} = i\omega\mu\vec{H}, \\ \nabla\times\vec{H} = -i\omega\epsilon\vec{E}. \end{cases} \tag{85}$$

Let

$$Y = \left(\begin{array}{c|c} \mu^{1/2} I_4 & \\ \hline & \epsilon^{1/2} I_4 \end{array}\right) X, \quad \kappa = \omega \mu^{1/2} \epsilon^{1/2}, \tag{86}$$

$$W = \kappa I_8 + \frac{1}{2i} \left(\begin{array}{cc|cc} & & & (\nabla\alpha)\cdot \\ & & (\nabla\alpha) & (\nabla\alpha)\times \\ \hline & (\nabla\beta)\cdot & & \\ (\nabla\beta) & -(\nabla\beta)\times & & \end{array}\right). \tag{87}$$

Then

$$(P + W)Y = 0. \tag{88}$$

Note that

$$(P + W)(P - W^t) = -\triangle + Q, \tag{89}$$

where

$$Q = \frac{1}{2} \left(\begin{array}{cc|cc} (\triangle\alpha) & & & \\ & 2(\partial_i\partial_j\alpha)_{ij} - (\triangle\alpha) I_3 & & \\ \hline & & (\triangle\beta) & \\ & & & 2(\partial_i\partial_j\beta)_{ij} - (\triangle\beta) I_3 \end{array}\right)$$

$$- \left(\begin{array}{c|cc} (\kappa^2 - \frac{1}{4}(\nabla\alpha\cdot\nabla\alpha)) I_4 & & -2i(\nabla\kappa)\cdot \\ & -2i(\nabla\kappa) & \\ \hline & -2i(\nabla\kappa)\cdot & (\kappa^2 - \frac{1}{4}(\nabla\beta\cdot\nabla\beta)) I_4 \\ -2i(\nabla\kappa) & & \end{array}\right). \tag{90}$$

If Z is a solution of

$$(-\triangle + Q)Z = 0, \tag{91}$$

then $Y = (P - W^t)Z$ is a solution to (88). We would like to construct solutions of (91) that are of the form

$$Z(\vec{\zeta}, x) = e^{i\vec{\zeta}\cdot x}(L(\vec{\zeta}) + R(\vec{\zeta}, x)), \quad \vec{\zeta} \in \mathbb{C}^3. \tag{92}$$

To do so, first extend the coefficients ϵ, μ to $\mathbb{R}^3$ so that $\epsilon - 1, \mu - 1 \in C_0^5(\mathbb{R}^3)$. Then $\omega^2 I_8 + Q \in C_0^3(\mathbb{R}^3)$. Let $\rho > 0$ be such that supp $(\omega^2 I_8 + Q)$ is contained in the ball of radius ρ. We can prove the following

Lemma 3.1 (compare to [2, Lemma 8])**.** *There exist a $C(\rho) > 0$ such that for any $L \in \mathbb{C}^8$, $\vec{\zeta} \in \mathbb{C}^3$ with $\vec{\zeta}\cdot\vec{\zeta} = \omega^2$ and*

$$|\vec{\zeta}| > C(\rho) ||\omega^2 I_8 + Q||_{L^\infty(\mathbb{R}^3)}, \tag{93}$$

there exists

$$Z = e^{i\vec{\zeta}\cdot x}(L + R) \tag{94}$$

a solution of (91) *in* $\mathbb{R}^3$, $Z \in W^{3,2}(\Omega)$ *and with*

$$||R||_{W^{3,2}(\Omega)} \leq \frac{1}{|\vec{\zeta}|} C(\rho)|L| \, ||\omega^2 + Q||_{W^{3,\infty}(\mathbb{R}^3)}. \tag{95}$$

Proof. We only need to show that such an R exists. The equation it need to satisfy is

$$(-\triangle - 2i\vec{\zeta} \cdot \nabla)R + (\omega^2 I_8 + Q)R = -(\omega^2 I_8 + Q)L. \tag{96}$$

We would like to, in a certain sense, invert $(-\triangle - 2i\vec{\zeta} \cdot \nabla)$. For some $-1 < \delta < 0$, define the spaces

$$L^2_\delta(\mathbb{R}^3) = \left\{ f : ||f||_{L^2_\delta} = ||(1 + |x|^2)^{\delta/2} f||_{L^2(\mathbb{R}^3)} < \infty \right\}, \tag{97}$$

$$W^{s,2}_\delta(\mathbb{R}^3) = \left\{ f : ||f||_{W^{s,2}_\delta} = ||(1 + |x|^2)^{\delta/2} f||_{W^{s,2}(\mathbb{R}^3)} < \infty \right\}. \tag{98}$$

There exists (see, for example, [20, Corollary 2.2]) $G_{\vec{\zeta}} : W^{s,2}_{\delta+1}(\mathbb{R}^3) \to W^{s,2}_\delta(\mathbb{R}^3)$ such that $(-\triangle - 2i\vec{\zeta} \cdot \nabla)G_{\vec{\zeta}}\phi = \phi$ and

$$||G_{\vec{\zeta}}\phi||_{W^{s,2}_\delta(\mathbb{R}^3)} \leq \frac{1}{|\vec{\zeta}|} C(\delta)||f||_{W^{s,2}_{\delta+1}(\mathbb{R}^3)} \tag{99}$$

The equation R should satisfy can then be written as

$$(I_8 + G_{\vec{\zeta}}(\omega^2 I_8 + Q))R = -G_{\vec{\zeta}}(\omega^2 I_8 + Q)L. \tag{100}$$

We can choose the constant $C(\rho)$ in (93) so that

$$||G_{\vec{\zeta}}(\omega^2 I_8 + Q)R||_{W^{3,2}_\delta(\mathbb{R}^3)} \leq \frac{1}{2}||R||_{W^{3,2}_\delta(\mathbb{R}^3)}, \tag{101}$$

in which case there exists a solution

$$R = -\left(I_8 + G_{\vec{\zeta}}(\omega^2 I_8 + Q) \right)^{-1} G_{\vec{\zeta}}(\omega^2 I_8 + Q)L, \tag{102}$$

and it satisfies the estimate (106). □

The following lemma is a restatement of a result in [2]:

Lemma 3.2 (see [2, Proposition 9])**.** *There exists a constant* $C(\rho, ||\epsilon - 1||_{W^{5,\infty}(\mathbb{R}^3)}, ||\mu - 1||_{W^{5,\infty}(\mathbb{R}^3)}) > 0$ *such that if* $\vec{\zeta} \in \mathbb{C}^3$, $\vec{\zeta} \cdot \vec{\zeta} = \omega^2$,

$$|\vec{\zeta}| > C(\rho, ||\epsilon - 1||_{W^{5,\infty}(\mathbb{R}^3)}, ||\mu - 1||_{W^{5,\infty}(\mathbb{R}^3)}), \tag{103}$$

$$L = \frac{1}{|\vec{\zeta}|}\begin{pmatrix} \vec{\zeta}\cdot\vec{a} \\ \omega\vec{b} \\ \hline \vec{\zeta}\cdot\vec{b} \\ \omega\vec{a} \end{pmatrix}, \quad \vec{a},\vec{b}\in C^3, \tag{104}$$

then there exists

$$Z = e^{i\vec{\zeta}\cdot x}(L+R) \tag{105}$$

a solution of (91) *in* $\mathbb{R}^3$, $Z \in W^{3,2}(\Omega)$ *and with*

$$||R||_{W^{3,2}(\Omega)} \le \frac{1}{|\vec{\zeta}|}C(\rho)|L|\,||\omega^2+Q||_{W^{3,\infty}(\mathbb{R}^3)}. \tag{106}$$

Additionally, $Y = (P - W^t)Z$ *solves* $(P+W)Y = 0$ *and is of the form*

$$Y = \begin{pmatrix} 0 \\ \mu^{1/2}\vec{H} \\ \hline 0 \\ \epsilon^{1/2}\vec{E} \end{pmatrix}. \tag{107}$$

Under the conditions of the previous lemma, we get

$$\vec{E} = e^{i\vec{\zeta}\cdot x}\left(\epsilon^{-1/2}\frac{\vec{\zeta}\cdot\vec{a}}{|\vec{\zeta}|}\vec{\zeta} + \vec{r}_e\right), \tag{108}$$

$$\vec{H} = e^{i\vec{\zeta}\cdot x}\left(\mu^{-1/2}\frac{\vec{\zeta}\cdot\vec{b}}{|\vec{\zeta}|}\vec{\zeta} + \vec{r}_h\right). \tag{109}$$

For $\sigma_e, \sigma_h \in \{0,1\}$, choose

$$\vec{a} = \sigma_e\frac{\vec{\zeta}^*}{|\vec{\zeta}|^2}, \quad \vec{b} = \sigma_h\frac{\vec{\zeta}^*}{|\vec{\zeta}|^2}. \tag{110}$$

Then

$$\vec{E} = e^{i\vec{\zeta}\cdot x}\left(\sigma_e\epsilon^{-1/2}\frac{\vec{\zeta}}{|\zeta|} + \vec{r}_e\right), \tag{111}$$

$$\vec{H} = e^{i\vec{\zeta}\cdot x}\left(\sigma_h\mu^{-1/2}\frac{\vec{\zeta}}{|\zeta|} + \vec{r}_h\right), \tag{112}$$

and, applying the previous lemma and Sobolev embedding

$$||\vec{r}_e||_{L^\infty(\Omega)}, ||\vec{r}_h||_{L^\infty(\Omega)} = \mathscr{O}(|\vec{\zeta}|^{-1}). \tag{113}$$

3.3 Proof of the main theorem

Let $\xi \in \mathbb{R}^3$. WLOG, $\xi = \xi_1 e_1$. Let $\vec{\zeta}_j = \alpha_j + i\beta_j \in \mathbb{C}^n$, $j = 0, 1, 2, 3$,

$$\beta_j = (-1)^j \left(\tau^2 + \xi_1^2/4\right)^{1/2} e_3, \tag{114}$$

$$\alpha_0 = \frac{\xi_1}{2} e_1 - (\omega^2 + \tau^2)^{1/2} e_2, \tag{115}$$

$$\alpha_1 = \frac{\xi_1}{2} e_1 + (\omega^2 + \tau^2)^{1/2} e_2, \tag{116}$$

$$\alpha_2 = \alpha_3 = -\frac{\xi_1}{2} e_1 - (\omega^2 + \tau^2)^{1/2} e_2. \tag{117}$$

Then

$$\vec{\zeta}_j \cdot \vec{\zeta}_j = \omega^2, \quad |\vec{\zeta}_j|^2 = 2\tau^2 + \omega^2 + \xi_1^2/4. \tag{118}$$

For sufficiently large $\tau > 0$, let

$$\vec{e}_j = e^{i\vec{\zeta}_j \cdot x}(\sigma_e \epsilon^{-1/2} |\vec{\zeta}_j|^{-1} \vec{\zeta}_j + r_{ej}) \tag{119}$$

$$\vec{h}_j = e^{i\vec{\zeta}_j \cdot x}(\sigma_h \mu^{-1/2} |\vec{\zeta}_j|^{-1} \vec{\zeta}_j + r_{hj}) \tag{120}$$

be the special solutions given by Proposition 3.1.

Note that

$$\vec{\zeta}_0 \cdot \vec{\zeta}_1 = -\omega^2 - 2\tau^2, \quad \vec{\zeta}_2 \cdot \vec{\zeta}_3^* = 2\tau^2 + \omega^2 + \xi_1^2/4, \tag{121}$$

$$\vec{\zeta}_0 \cdot \vec{\zeta}_2 = 2\tau^2 + \omega^2, \quad \vec{\zeta}_1 \cdot \vec{\zeta}_3 = -\omega^2. \tag{122}$$

Then

$$\vec{e}_0 \cdot \vec{e}_1 (\vec{e}_2 \cdot \vec{e}_3^*)^k = -\sigma_e |\epsilon|^{-k} \epsilon^{-1} \exp(i\xi \cdot x) + \mathscr{O}(\tau^{-1}), \tag{123}$$

$$\vec{e}_0 \cdot \vec{e}_2 (\vec{e}_1 \cdot \vec{e}_3)(\vec{e}_2 \cdot \vec{e}_3^*)^{k-1} = \mathscr{O}(\tau^{-1}), \tag{124}$$

where $\mathscr{O}(\tau^{-1})$ is to be understood in the sense of $L^\infty(\Omega)$ norms. Choosing $\sigma_e = 1$, $\sigma_h = 0$ and taking the limit $\tau \to \infty$ in (76), we get

$$\left(\frac{a_k - a_k'}{|\epsilon|^k \epsilon} \chi_\Omega\right)^\wedge (\xi) = 0, \quad \forall \xi \in \mathbb{R}^3. \tag{125}$$

This implies $a_k = a_k'$. By an identical argument it follows that $b_k = b_k'$. This concludes the induction step.

Acknowledgement The author is grateful to Prof. Gunther Uhlmann for proposing this problem and for suggesting improvements to the manuscript.

References

[1] Y. M. Assylbekov and T. Zhou. Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media. *arXiv preprint arXiv:1709.07767*, 2017.

[2] P. Caro. Stable determination of the electromagnetic coefficients by boundary measurements. *Inverse Problems*, 26(10):105014, 25, 2010.

[3] P. Caro, P. Ola, and M. Salo. Inverse boundary value problem for Maxwell equations with local data. *Comm. Partial Differential Equations*, 34(10-12):1425–1464, 2009.

[4] D. Colton and L. Päivärinta. The uniqueness of a solution to an inverse scattering problem for electromagnetic waves. *Arch. Rational Mech. Anal.*, 119(1):59–70, 1992.

[5] D. Hervas and Z. Sun. An inverse boundary value problem for quasilinear elliptic equations. *Comm. Partial Differential Equations*, 27(11-12):2449–2490, 2002.

[6] V. Isakov. Uniqueness of recovery of some quasilinear partial differential equations. *Comm. Partial Differential Equations*, 26(11-12):1947–1973, 2001.

[7] V. Isakov. Uniqueness of recovery of some systems of semilinear partial differential equations. *Inverse Problems*, 17(4):607–618, 2001. Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000).

[8] V. Isakov and A. Nachman. Global uniqueness for a two-dimensional semilinear elliptic inverse problem. *Trans. Amer. Math. Soc.*, 347(9):3375–3390, 1995.

[9] V. Isakov and J. Sylvester. Global uniqueness for a semilinear elliptic inverse problem. *Comm. Pure Appl. Math.*, 47(10):1403–1410, 1994.

[10] C. E. Kenig, M. Salo, and G. Uhlmann. Inverse problems for the anisotropic Maxwell equations. *Duke Math. J.*, 157(2):369–419, 2011.

[11] J. Marburger, L. Huff, J. D. Reichert, and W. G. Wagner. Stationary self-trapping of optical beams in dense media with Lorentz local-field corrections. *Phys. Rev.*, 184:255–259, Aug 1969.

[12] J. H. Marburger. Self-focusing: theory. *Progress in quantum electronics*, 4:35–110, 1975.

[13] P. Ola, L. Päivärinta, and E. Somersalo. An inverse boundary value problem in electrodynamics. *Duke Math. J.*, 70(3):617–653, 1993.

[14] P. Ola and E. Somersalo. Electromagnetic inverse problems and generalized Sommerfeld potentials. *SIAM J. Appl. Math.*, 56(4):1129–1145, 1996.

[15] C. A. Stuart. Self-trapping of an electromagnetic field and bifurcation from the essential spectrum. *Arch. Rational Mech. Anal.*, 113(1):65–96, 1990.

[16] Z. Sun. On a quasilinear inverse boundary value problem. *Math. Z.*, 221(2):293–305, 1996.

[17] Z. Sun. Inverse boundary value problems for a class of semilinear elliptic equations. *Adv. in Appl. Math.*, 32(4):791–800, 2004.

[18] Z. Sun and G. Uhlmann. An inverse boundary value problem for Maxwell's equations. *Arch. Rational Mech. Anal.*, 119(1):71–93, 1992.

[19] Z. Sun and G. Uhlmann. Inverse problems in quasilinear anisotropic media. *Amer. J. Math.*, 119(4):771–797, 1997.

[20] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)*, 125(1):153–169, 1987.

[21] T. Zhou. Reconstructing electromagnetic obstacles by the enclosure method. *Inverse Probl. Imaging*, 4(3):547–569, 2010.