NONLINEAR EQUATIONS WITH GRADIENT NATURAL GROWTH AND DISTRIBUTIONAL DATA, WITH APPLICATIONS TO A SCHRÖDINGER TYPE EQUATION

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Abstract. We obtain necessary and sufficient conditions with sharp constants on the distribution σ for the existence of a globally finite energy solution to the quasilinear equation with a gradient source term of natural growth of the form $-\Delta_p u = |\nabla u|^p + \sigma$ in a bounded open set $\Omega \subset \mathbb{R}^n$. Here Δ_p, $p > 1$, is the standard p-Laplacian operator defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The class of solutions that we are interested in consists of functions $u \in W_0^{1,p}(\Omega)$ such that $e^{\mu u} \in W_0^{1,p}(\Omega)$ for some $\mu > 0$ and the inequality

$$\int_\Omega |\varphi|^p |\nabla u|^p dx \le A \int_\Omega |\nabla \varphi|^p dx$$

holds for all $\varphi \in C_c^\infty(\Omega)$ with some constant $A > 0$. This is a natural class of solutions at least when the distribution σ is nonnegative. The study of $-\Delta_p u = |\nabla u|^p + \sigma$ is applied to show the existence of globally finite energy solutions to the quasilinear equation of Schrödinger type $-\Delta_p v = \sigma\, v^{p-1}$, $v \ge 0$ in Ω, and $v = 1$ on $\partial\Omega$, via the exponential transformation $u \mapsto v = e^{\frac{u}{p-1}}$.

1. Introduction

The main goal of this paper is to address the solvability of quasilinear elliptic equations with gradient nonlinearity of natural growth of the form

$$\begin{cases} -\Delta_p u = |\nabla u|^p + \sigma & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded open set $\Omega \subset \mathbb{R}^n$. Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, is the p-Laplacian and the datum σ is a distribution in Ω. More generally, we also consider the equation

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) + \sigma & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

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where the principal operator $\operatorname{div} \mathcal{A}(x, u, \nabla u)$ is a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$ and $|\mathcal{B}(x, u, \nabla u)| \lesssim |\nabla u|^p$.

The precise assumptions on the nonlinearities $\mathcal{A}$, B and the the precise definition of solutions to (1.2) will be given in Section 2. Here we emphasize that in this paper we are interested only in *finite energy solutions* u with zero boundary condition in the sense that $u \in W_0^{1,p}(\Omega)$. The energy space $W_0^{1,p}(\Omega)$ is defined as the completion of $C_c^\infty(\Omega)$ under the semi-norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$.

As an application of the study of (1.1), we also obtain existence of finite energy solution to the quasilinear Schrödinger type equation

$$-\Delta_p v = (p-1)^{1-p}\, \sigma\, v^{p-1} \text{ in } \Omega, \qquad v \geq 0 \text{ in } \Omega, \qquad v = 1 \text{ on } \partial\Omega. \tag{1.3}$$

Equation (1.1) is a prototype for quasilinear equations with natural growth in the gradient that has attracted a lot of attention in the past years. It can be viewed as a quasilinear stationary version of a time-dependent viscous Hamilton-Jacobi equation, also known as the Kardar-Parisi-Zhang equation, which appears in the physical theory of surface growth [23, 24].

As far as existence is concerned, the nonlinearity $|\nabla u|^p$ in (1.1) is considered "to have the bad sign" and by now it is well-known that in order for (1.1) to have a solution the datum σ must be both *small and regular enough*. In particular, if σ is a nonnegative distribution in Ω (i.e., a nonnegative locally finite measure in Ω), then a necessary condition for the first equation in (1.1) to have a $W^{1,p}_{\rm loc}(\Omega)$ solution is that (see [20, 21, 22])

$$\int_\Omega |\varphi|^p d\sigma \leq \lambda \int_\Omega |\nabla \varphi|^p dx \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{1.4}$$

with $\lambda = (p-1)^{p-1}$. Moreover, when $\sigma \geq 0$ the nonlinear term itself also obeys a similar Poincaré-Sobolev inequality

$$\int_\Omega |\varphi|^p |\nabla u|^p dx \leq A \int_\Omega |\nabla \varphi|^p dx \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{1.5}$$

with $A = p^p$.

Thus a natural space of solutions associated to (1.1) is the space $\mathcal{S}$ of functions $u \in W_0^{1,p}(\Omega)$ such that (1.5) holds for some $A > 0$. The main question we wish to address here is to find an optimal (largest) space $\mathcal{D}$ of 'data' so that whenever $\sigma \in \mathcal{D}$ with sufficiently small norm $\|\sigma\|_{\mathcal{D}}$ then (1.1) admits a solution in $\mathcal{S}$. In the case $\sigma \geq 0$ we can completely characterize the existence of finite energy solutions to (1.1) in the following theorem. We remark again that in this case all $W_0^{1,p}(\Omega)$ solutions automatically belong to $\mathcal{S}$ and (1.5) holds with $A = p^p$.

Theorem 1.1. *Let σ be a nonnegative locally finite measure in Ω. If (1.1) has a solution in $u \in W_0^{1,p}(\Omega)$ then $\sigma \in (W_0^{1,p}(\Omega))^*$ and (1.4) holds with $\lambda = (p-1)^{p-1}$. Conversely, if $\sigma \geq 0$, $\sigma \in (W_0^{1,p}(\Omega))^*$, and (1.4) holds with $0 < \lambda < (p-1)^{p-1}$ then (1.1) has a* nonnegative *solution in $W_0^{1,p}(\Omega)$ such that $e^{\frac{\delta u}{p-1}} - 1 \in W_0^{1,p}(\Omega)$ for all $\delta \in [0, \delta_0)$ where $\delta_0 = (p-1)\lambda^{\frac{-1}{p-1}}$.*

In the linear case, $p = 2$, these necessary and sufficient conditions have been observed in [17]. See also [1] (for $p = 2$) and [19] for certain related results that were obtained by different methods. We remark that, under a mild restriction on the domain, by Hardy's inequality (see [3, 25]), Theorem 1.1 covers the case of unbounded measure such as $\sigma = \varepsilon\, \mathrm{dist}(x, \partial\Omega)^{-1}$ for some $\varepsilon > 0$. It is also worth mentioning that in the case $p = 2$ and σ is a nonnegative locally finite measure, other sharp existence results for (1.1) were obtained in [20] for $\Omega = \mathbb{R}^n$ and recently in [17] for bounded domains Ω with C^2 boundary under a very weak notion of solution and boundary conditions.

The first part of Theorem 1.1 follows from the known necessary condition (1.4), Hölder's inequality, and the assumption that $\nabla u \in L^p(\Omega)$, since we have

$$\sigma = -|\nabla u|^p - \mathrm{div}\,(|\nabla u|^{p-2}\nabla u) \leq -\mathrm{div}\,(|\nabla u|^{p-2}\nabla u).$$

On the other hand, the second part is a consequence of Theorem 1.2 below that treats even sign changing distribution datum σ. This in fact is the main result that will be obtained in this paper.

Theorem 1.2. (i) *Suppose that (1.1) has a solution in $u \in W_0^{1,p}(\Omega)$ such that (1.5) holds for some $A > 0$ then it necessarily holds that $\sigma = \mathrm{div}\,(F) - |F|^{\frac{p}{p-1}}$ for a vector field $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ such that*

$$\int_\Omega |F|^{\frac{p}{p-1}} |\varphi|^p dx \leq A \int_\Omega |\nabla\varphi|^p dx \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{1.6}$$

In particular, both σ and $|F|^{\frac{p}{p-1}}$ belong to the dual space $(W_0^{1,p}(\Omega))^*$.
(ii) *Conversely, suppose that $\sigma = \mathrm{div}\, F + f$ where $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f| \in (W_0^{1,p}(\Omega))^*$ such that*

$$p\int_\Omega |F||\varphi|^{p-1}|\nabla\varphi| dx + \int_\Omega |\varphi|^p d|f| \leq \lambda \int_\Omega |\nabla\varphi|^p dx \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{1.7}$$

for some $\lambda \in (0,(p-1)^{p-1})$*. Then equation* (1.1) *has a (possibly sign changing) solution* $u \in W_0^{1,p}(\Omega)$ *such that* $e^{\frac{\delta u}{p-1}} - 1 \in W_0^{1,p}(\Omega)$ *for all* $\delta \in [0,\delta_0)$ *where* $\delta_0 = (p-1)\lambda^{\frac{-1}{p-1}}$*. This solution satisfies the Poincaré-Sobolev inequality* (1.5) *for some* $A = A(p) > 0$*. Moreover, if* $\lambda \in (0,(p-1)^{\min\{1,p-1\}})$*, then both* $e^{\frac{u}{p-1}} - 1$ *and* $e^u - 1$ *belong to* $W_0^{1,p}(\Omega)$*.*

Several remarks regarding Theorem 1.2 are now in order.

Remark 1.3. *By approximation and Fatou's lemma, inequalities* (1.6) *and* (1.7) *actually hold for all* $\varphi \in W_0^{1,p}(\Omega)$*. The integral* $\int_\Omega |\varphi|^p d|f|$ *makes sense even for* $\varphi \in W_0^{1,p}(\Omega)$ *since* $|f|$ *is continuous with respect to the capacity* $\mathrm{cap}_p(\cdot,\Omega)$ *and* φ *has a* cap_p*-quasicontinuous representative, whose values are defined* cap_p*-quasieverywhere in* Ω*. Here* $\mathrm{cap}_p(\cdot,\Omega)$ *is the variational* p*-capacity associated to* Ω *defined for each compact set* $K \subset \Omega$ *by*

$$\mathrm{cap}_p(K,\Omega) := \inf\left\{\int_\Omega |\nabla\phi|^p dx : \phi \in C_c^\infty(\Omega) \text{ and } \phi \geq \chi_K\right\}.$$

Remark 1.4. *By Hölder's inequality we see that if* F *satisfies* (1.6) *for some* $A > 0$ *then*

$$p\int_\Omega |F||\varphi|^{p-1}|\nabla\varphi| dx \leq pA^{\frac{p-1}{p}}\int_\Omega |\nabla\varphi|^p dx \qquad \forall\varphi \in C_c^\infty(\Omega).$$

Thus by Theorem 1.2(ii) if $F \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$ *satisfies* (1.6) *for some* $0 < A < (p-1)^p p^{-\frac{p}{p-1}}$ *then the equation* $-\Delta_p u = |\nabla u|^p + \mathrm{div}\, F$ *has a solution in* $W_0^{1,p}(\Omega)$*.*

Remark 1.5. *Let* μ *be a nonnegative locally finite measure in* Ω*. It is well-known that the inequality*

$$\int_\Omega |\varphi|^p d\mu \leq A_1\int_\Omega |\nabla\varphi|^p dx \qquad \forall\varphi \in C_c^\infty(\Omega)$$

is equivalent to the condition

$$\mu(K) \leq A_2\, \mathrm{cap}_p(K,\Omega) \tag{1.8}$$

for all compact sets $K \subset \Omega$ *(see* [28, Chapter 2]*).*

Thus in (*ii*) *of Theorem 1.2 , condition* (1.7) *can be replaced by* (1.8) *with* $\mu = |F|^{\frac{p}{p-1}} + |f|$ *for a sufficiently small constant* $A_2 > 0$*.*

Moreover, by (*ii*) *of Theorem 1.2, if* f *is a locally finite signed measure in* Ω *with* $|f| \in (W_0^{1,p}(\Omega))^*$ *such that* (1.8) *holds with* $d\mu = d|f|$*, the we have a decomposition*

$$f = \mathrm{div}\, F - g,$$

where $F \in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$ *and* $g \in L^1(\Omega)$*,* $g \geq 0$*, such that the* L^1 *function* $\mu := (|F|^{\frac{p}{p-1}} + g)$ *also satisfies* (1.8)*. See* [5, 18] *for a similar decomposition of measures that are continuous w.r.t the* p*-capacity.*

Remark 1.6. *Let $L^{s,\infty}(\Omega)$, $s \geq 1$, denote the weak L^s space on Ω with quasinorm*

$$\|g\|_{L^{s,\infty}(\Omega)} := \sup_{t>0} t|\{x \in \Omega : |g(x)| > t\}|^{1/s}.$$

For $g \in L^{\frac{n}{p},\infty}(\Omega)$ with $1 < p < n$, it is known that (see, e.g., [16, Eqn. (2.6)]*)*

$$\int_\Omega |\varphi|^p g dx \leq S_{n,p} \|g\|_{L^{\frac{n}{p},\infty}(\Omega)} \int_\Omega |\nabla \varphi|^p dx \qquad \forall \varphi \in C_c^\infty(\Omega),$$

where the constant $S_{n,p}$ is given by

$$S_{n,p} = \left[\frac{p}{\sqrt{\pi}(n-p)}\right]^p \Gamma(1+n/2)^{p/n}.$$

This shows that in Theorem 1.2(ii), condition (1.7) *can be replaced by $|F|^{\frac{p}{p-1}} + |f| \in L^{\frac{n}{p},\infty}(\Omega)$ with a sufficiently small norm. Existence results under this weak norm condition have been obtained in* [16]*. See also the earlier works* [14, 15] *where the strong norm condition involving $L^{\frac{n}{p}}(\Omega)$ was used instead. More general existence results in which $|F|^{\frac{p}{p-1}} + |f|$ is assumed to be small in the norm of certain Morrey spaces can be found in the recent paper* [29]*. Those Morrey space conditions are also stronger than condition* (1.7) *as they fall into the realm of Fefferman-Phong type conditions (see, e.g.,* [11, 12, 13, 31, 32]*).*

We now discuss the Schrödinger type equation with distributional potential (1.3). This equation is interesting in its own right and has a strong connection to equation (1.1) as being observed and exploited, e.g., in [1, 19, 21, 22].

By a solution to (1.3), we mean the following definition.

Definition 1.7. *Let $\sigma \in (W_0^{1,p}(\Omega))^*$. A function v defined in Ω is a solution of* (1.3) *if $v \geq 0$, $v - 1 \in W_0^{1,p}(\Omega)$, $v^{p-1} \in W_{\rm loc}^{1,p}(\Omega)$, and*

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi dx = (p-1)^{1-p} \langle \sigma, v^{p-1}\varphi \rangle \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{1.9}$$

Note that the right hand side of (1.9) makes sense since $v^{p-1}\varphi \in W_0^{1,p}(\Omega)$ and $\sigma \in (W_0^{1,p}(\Omega))^*$.

Formally, by making the change of unknowns $v = e^{\frac{u}{p-1}}$, equation (1.1) is transformed into the Schrödinger type equation (1.3). Indeed, it is possible to show rigorously that Theorem 1.2 implies the existence of finite energy solutions to (1.3):

Theorem 1.8. *Suppose that $\sigma = \operatorname{div} F + f$ where $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f| \in (W_0^{1,p}(\Omega))^*$ such that*

$$p \int_\Omega |F||\varphi|^{p-1}|\nabla\varphi| dx + \int_\Omega |\varphi|^p d|f| \leq \lambda \int_\Omega |\nabla\varphi|^p dx \qquad \forall \varphi \in C_c^\infty(\Omega),$$

for some $\lambda \in (0, (p-1)^{\min\{1,p-1\}})$. Then equation (1.3) *has a nonnegative solution v such that both $v-1$ and $v^{p-1}-1$ belong to $W_0^{1,p}(\Omega)$. Moreover, v satisfies the following Poincaré-Sobolev inequality*

$$\int_\Omega \left|\frac{\nabla v}{v}\right|^p |\varphi|^p dx \leq A \int_\Omega |\nabla\varphi|^p dx \qquad \forall \varphi \in C_c^\infty(\Omega), \tag{1.10}$$

with a constant $A = A(p) > 0$.

Remark 1.9. *If the factor $(p-1)^{1-p}$ on the right-hand side of* (1.3) *is dropped then the smallness condition on λ becomes $\lambda \in (0, p^\#)$, where $p^\# = (p-1)^{2-p}$ if $p > 2$ and $p^\# = 1$ if $p \leq 2$ as in* [22]. *The sharpness of $p^\#$ (and that of $(p-1)^{\min\{1,p-1\}}$ for* (1.3)*) was also justified in* [22].

One could also treat the Schrödinger type equation (1.3) in a more general fashion, where the standard p-Laplacian is replaced by a quasilinear elliptic operator with merely measurable 'coefficients'. See Remark 6.1 below and see also [22].

We mention that the existence of finite energy solutions to (1.3) in the case $\sigma \geq 0$ was obtained in [19] by a method that does not seem to work for sign changing σ (see also [1] for $p = 2$). On the other hand, the work [22] (see also [21]) obtains a locally finite energy solution $v \in W^{1,p}_{\rm loc}(\Omega)$ to the first two equations in (1.3) (*without any boundary conditions*) only under the mild restriction

$$-\Lambda \int_\Omega |\nabla\varphi|^p dx \leq \langle \sigma, |\varphi|^p \rangle \leq \lambda \int_\Omega |\nabla\varphi|^p dx \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

for some $\lambda \in (0, (p-1)^{\min\{1,p-1\}})$ and $\Lambda \in (0, +\infty)$. Moreover, v also satisfies (1.10) for some $A > 0$. Then, also under the restriction $\lambda \in (0, (p-1)^{\min\{1,p-1\}})$, by the logarithmic transformation $u = (p-1)\log(v)$ it was obtained in [22], a solution $u \in W^{1,p}_{\rm loc}(\Omega)$ to the first equation in (1.1) (but without any boundary condition) that also satisfies (1.5) for some $A > 0$.

In this paper, we follow an opposite route, i.e., we first treat equation (1.1) *directly and then deduce existence for the Schrödinger type equation* (1.3) *from it.* This way, we are able to treat equation (1.1) in its most general form, i.e., the nonlinear equation with general structure (1.2). Moreover, for equation (1.1) we obtain larger upper bound for λ in the existence condition (1.7) (i.e., $(p-1)^{p-1}$ versus $(p-1)^{\min\{1,p-1\}}$). Our approach to (1.2) is a refinement of the approach of V. Ferone and F. Murat in [15, 16]. The main difficulties to overcome here are the generality nature of σ and the sharpness of the smallness constants. In particular, in this scenario one does not gain any higher integrability on the nonlinear term $\mathcal{B}(x, u, \nabla u)$, which makes it impossible to follow a compactness argument as in [29]. Moreover, in order for us to apply the existence results of (1.1) to (1.3) we need to find a solution u of (1.1) with the additional property that both $e^{\frac{u}{p-1}} - 1$ and $e^u - 1$ belong to $W_0^{1,p}(\Omega)$ as stated in Theorem 1.2.

2. Equations with general nonlinear structure

As we have mentioned, existence results in the spirit of Theorem 1.2(ii) also hold for equations with a more general nonlinear structure (1.2). For that we need the following assumptions on the nonlinearities $\mathcal{A}$ and $\mathcal{B}$:

Assumption on $\mathcal{A}$. The nonlinearity $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory function, i.e., $\mathcal{A}(x, s, \xi)$ is measurable in x for every (s, ξ) and continuous in (s, ξ) for a.e. $x \in \Omega$. For some $p > 1$, it holds that

$$\langle \mathcal{A}(x, s, \xi) - \mathcal{A}(x, s, \eta), \xi - \eta \rangle > 0, \tag{2.1}$$

$$\langle \mathcal{A}(x, s, \xi), \xi \rangle \geq \alpha_0 |\xi|^p, \tag{2.2}$$

$$|\mathcal{A}(x, s, \xi)| \leq a_0 |\xi|^{p-1} + a_1 |s|^{p-1} \tag{2.3}$$

for every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$, $\xi \neq \eta$, and a.e. $x \in \Omega$. Here $\alpha_0 > 0$, and $a_0, a_1 \geq 0$.

Assumption on $\mathcal{B}$. The nonlinearity $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathédory function which satisfies, for a.e. $x \in \Omega$, every $s \in \mathbb{R}$, and every $\xi \in \mathbb{R}^n$,

$$|\mathcal{B}(x, s, \xi)| \leq b_0 |\xi|^p + b_1 |s|^m, \quad \mathcal{B}(x, s, \xi)\mathrm{sign}(s) \leq \alpha_0 \gamma_0 |\xi|^p, \tag{2.4}$$

where $m > 0$, and b_0, b_1, $\gamma_0 \geq 0$. Here α_0 is as given in (2.2).

By a solution of (1.2) we mean the following.

Definition 2.1. *Under* (2.1)-(2.4)*, a function* $u \in W_0^{1,p}(\Omega)$ *is a solution of* (1.2) *if* $\mathcal{B}(x, u, \nabla u) \in L^1_{\mathrm{loc}}(\Omega)$ *and*

$$\int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \mathcal{B}(x, u, \nabla u) \varphi \, dx + \langle \sigma, \varphi \rangle$$

holds for all test functions $\varphi \in C_c^\infty(\Omega)$.

We remark that in the case $\mathcal{B}(x,u,\nabla u)\in L^1(\Omega)$ and $\sigma\in (W_0^{1,p}(\Omega))^*$, we can take any function $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ as a test function in the above definition. This follows from a result of Brézis and Browder [9] as we have $\mathcal{B}(x,u,\nabla u)\in (W_0^{1,p}(\Omega))^*\cap L^1(\Omega)$. It can also be seen by approximating φ in $W_0^{1,p}(\Omega)$ by a sequence $\varphi_j\in C_c^\infty(\Omega)$ such that $|\varphi_j|\leq|\varphi|\leq M$ a.e. (using Theorem 9.3.1 in [2] and suitable convolutions).

We mention that in the special case $|\mathcal{B}(x,u,\nabla u)|\in (W_0^{1,p}(\Omega))^*\cap L^1(\Omega)$, we can even drop the condition $\varphi\in L^\infty(\Omega)$. In fact, we have the following more general result.

Lemma 2.2. *Suppose that f is a locally finite signed measure in Ω with $|f|\in (W_0^{1,p}(\Omega))^*$. Then for any $\varphi\in W_0^{1,p}(\Omega)$ we have*

$$\langle f,\varphi\rangle=\int_\Omega \widetilde{\varphi}\,df,$$

where $\widetilde{\varphi}$ is any cap_p-quasicontinuous representative of φ.

In the case f is nonnegative, the proof of Lemma 2.2 can be found in [30, Lemma 2.5]. The general case also follows from that, since $f=f^+ + f^-$ and both f^+ and f^- belong to $(W_0^{1,p}(\Omega))^*$. In what follows, when dealing with pointwise behavior of functions in $W_0^{1,p}(\Omega)$ we will implicitly use their cap_p-quasicontinuous representatives. Lemma 2.2 will be used, e.g., in (3.11) below.

Under the above assumptions on $\mathcal{A}$ and $\mathcal{B}$, we obtain the following existence result.

Theorem 2.3. *Let $\sigma=\operatorname{div}F+f$ where $F\in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f|\in (W_0^{1,p}(\Omega))^*$ such that*

$$p\int_\Omega |F||\varphi|^{p-1}|\nabla\varphi|dx+\int_\Omega |\varphi|^p d|f|\leq\lambda\int_\Omega|\nabla\varphi|^p dx \tag{2.5}$$

holds for all $\varphi\in C_c^\infty(\Omega)$, with $\lambda\in(0,\gamma_0^{1-p}\alpha_0(p-1)^{p-1})$. Then there exists a solution $u\in W_0^{1,p}(\Omega)$ to the equation

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u)=\mathcal{B}(x,u,\nabla u)+\sigma \quad in \quad \Omega, \tag{2.6}$$

such that $e^{\frac{\delta|u|}{p-1}}-1\in W_0^{1,p}(\Omega)$ for all $\delta\in[\gamma_0,\delta_0)$, with $\delta_0=(p-1)\left(\frac{\alpha_0}{\lambda}\right)^{\frac{1}{p-1}}$.

Moreover, for any $\delta_1>\gamma_0$ such that (2.5) holds with

$$\lambda<\left(\frac{p-1}{\delta_1}\right)^p\alpha_0\left(\frac{\delta_1}{p-1}+\delta_1-\gamma_0\right), \tag{2.7}$$

we have $e^{\frac{\delta_1|u|}{p-1}}-1\in W_0^{1,p}(\Omega)$.

Remark 2.4. *It is easy to check that, for $\delta_1 > \gamma_0$ one has*

$$\left(\frac{p-1}{\delta_1}\right)^p \alpha_0 \left(\frac{\delta_1}{p-1} + \delta_1 - \gamma_0\right) < \gamma_0^{1-p}\alpha_0(p-1)^{p-1}.$$

Moreover, for example with $p > 2$ and $\alpha_0 = \gamma_0 = 1$, if (2.5) holds with $\lambda < p-1 \in (0, (p-1)^{p-1})$, then we see that (3.2) holds with $1 \le \delta < (p-1)(p-1)^{\frac{-1}{p-1}}$, but it does not allow us to take $\delta = p-1$! On the other hand, for $\lambda < p-1$ inequality (2.7) holds with $\delta_1 = p-1$ and thus $e^{|u|} - 1 \in W_0^{1,p}(\Omega)$.

Due to the general structures of $\mathcal{A}$ and $\mathcal{B}$, here we do not claim that the solution u obtained in Theorem 2.3 satisfies the Poincaré-Sobolev inequality (1.5).

The paper is organized as follows: In Section 5, we provide the proof of Theorem 2.3. This proof is based on the existence of solutions to an approximate equation along with certain uniform bounds given in Section 4. These important uniform bounds are in turn deduced from the a priori estimate of Section 3, though not directly. Finally, the proof of Theorems 1.2 and 1.8 will be given in Section 6.

3. An a priori estimate

In this section, we obtain certain exponential type a priori bounds for solutions of

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u) + \varepsilon\,|u|^{p-2}u = \mathcal{B}(x,u,\nabla u) + \sigma \quad \text{in} \quad \Omega, \tag{3.1}$$

where $\varepsilon \ge 0$. The case $\varepsilon > 0$ will be needed in the next section to absorb certain unfavorable terms in the approximating process; see (4.9) below. Earlier, this idea was implemented by V. Ferone and F. Murat in [16].

Theorem 3.1. *Let $\sigma = \operatorname{div} F + f$ where $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f| \in (W_0^{1,p}(\Omega))^*$ such that (2.5) holds for all $\varphi \in C_c^\infty(\Omega)$, with $\lambda \in (0, \gamma_0^{1-p}\alpha_0(p-1)^{p-1})$. Then for any $\varepsilon \ge 0$ and any $W_0^{1,p}(\Omega)$ solution u to equation (3.1) such that $e^{\frac{\delta|u|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$, we have*

$$\|u\|_{W_0^{1,p}(\Omega)} + \|e^{\frac{\delta|u|}{p-1}} - 1\|_{W_0^{1,p}(\Omega)} \le M_\delta. \tag{3.2}$$

provided $\delta \in [\gamma_0, \delta_0)$ where $\delta_0 = (p-1)(\alpha_0/\lambda)^{\frac{1}{p-1}}$. Here M_δ is independent of u and ε.

Moreover, for any $\delta_1 > \gamma_0$ such that $e^{\frac{\delta_1|u|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$, and (2.5) holds with λ satisfying (2.7), we have

$$\|e^{\frac{\delta_1|u|}{p-1}} - 1\|_{W_0^{1,p}(\Omega)} \le M_{\delta_1} + C_{\delta_1}\,\|\nabla u\|_{L^p(\Omega)}\,. \tag{3.3}$$

The constants M_{δ_1} and C_{δ_1} are independent of u and ε.

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (3.1) and define

$$w = \operatorname{sign}(u)[e^{\mu|u|} - 1]/\mu, \qquad \text{with } \mu = \delta/(p-1),$$

where $\operatorname{sign}(u) = 0$ if $u = 0$, $\operatorname{sign}(u) = 1$ if $u > 0$, and $\operatorname{sign}(u) = -1$ if $u < 0$. Then from the assumption $e^{\mu|u|} - 1 \in W_0^{1,p}(\Omega)$, we see that $w \in W_0^{1,p}(\Omega)$ with

$$\nabla w = e^{\mu|u|}\nabla u. \tag{3.4}$$

Indeed, for $\varepsilon > 0$ define $f_\varepsilon(x) = \frac{x}{\sqrt{x^2+\varepsilon^2}}$, $x \in \mathbb{R}$, and denote by T_s, $s > 0$, the two-sided truncation operator at level s, i.e.,

$$T_s(r) = r \ \text{ if } |r| \leq s \quad \text{and} \quad T_s(r) = \operatorname{sign}(r)s \ \text{ if } |r| > s,$$

then it follows that $T_s(u) \in W_0^{1,p}(\Omega)$ for any $s > 0$ and

$$\begin{aligned} \nabla\left[f_\varepsilon(T_s(u))(e^{\mu|T_s(u)|} - 1)/\mu\right] &= \frac{\nabla T_s(u)\varepsilon^2}{(T_s(u)^2 + \varepsilon^2)^{3/2}}(e^{\mu|T_s(u)|} - 1)/\mu \\ &\quad + f_\varepsilon(T_s(u))\nabla(e^{\mu|T_s(u)|} - 1)/\mu \end{aligned}$$

in the weak sense. Note that

$$\varepsilon^2(e^{\mu|T_s(u)|} - 1)/\mu \leq \frac{e^{\mu s}}{\mu s}\,\varepsilon^2|T_s(u)| \leq \frac{e^{\mu s}}{\mu s}\,(|T_s(u)|^2 + \varepsilon^2)^{3/2},$$

$$f_\varepsilon(T_s(u)) \to \operatorname{sign}(T_s(u)) = \operatorname{sign}(u) \text{ as } \varepsilon \to 0^+,$$

and thus by Dominated Convergence Theorem we find

$$\nabla\left[\operatorname{sign}(u)(e^{\mu|T_s(u)|} - 1)/\mu\right] = \operatorname{sign}(u)\nabla(e^{\mu|T_s(u)|} - 1)/\mu = e^{\mu|T_s(u)|}\nabla T_s(u).$$

Now using the assumption $e^{\mu|u|}|\nabla u| \in L^p(\Omega)$ and letting $s \to \infty$, we obtain (3.4).

For each $s > 0$, we will use the following test function for (3.1):

$$v_s = e^{\delta|u_s|}w_s,$$

where $u_s = T_s(u)$ and $w_s = \operatorname{sign}(u)[e^{\mu|u_s|} - 1]/\mu$ with $\mu = \delta/(p-1)$.

From the definition of w_s we have $|w_s| \leq |w|$ and $\nabla w_s = e^{\mu|u_s|}\nabla u_s$. Thus both w_s and v_s belong to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and moreover,

$$\nabla v_s = \left[e^{\delta|u|}\nabla w + \delta|w|e^{\delta|u|}\nabla u\right]\chi_{\{|u|\leq s\}}.$$

Using v_s as a test function in (3.1), we get

$$\begin{aligned}\int_\Omega \mathcal{A}(x,u,\nabla u)\cdot\nabla w e^{\delta|u|}\chi_{\{|u|\le s\}}\,dx + \varepsilon\int_\Omega |u|^{p-2}u e^{\delta|u_s|}w_s\,dx \\ = -\int_\Omega \delta|w|e^{\delta|u|}\mathcal{A}(x,u,\nabla u)\cdot\nabla u\chi_{\{|u|\le s\}}\,dx + \\ + \int_\Omega \mathcal{B}(x,u,\nabla u)e^{\delta|u_s|}w_s\,dx + \int_\Omega F\cdot\nabla v_s\,dx + \int_\Omega v_s df.\end{aligned}$$

We now write this equality as

$$I_1 + I_2 = I_3 + I_4 + I_5 + I_6, \tag{3.5}$$

where I_i, $i\in\{1,\dots,6\}$, are the corresponding terms.

Estimate for I_1: Since $\nabla w_s = e^{\mu|u_s|}\nabla u_s$, using the coercivity condition (2.2), we see that

$$\begin{aligned} I_1 &= \int_\Omega \mathcal{A}(x,u_s,\nabla u_s)\cdot\nabla w_s e^{\delta|u_s|}\chi_{\{|u|\le s\}}\,dx \\ &= \int_\Omega \mathcal{A}(x,u_s,\nabla u_s)\cdot\nabla u_s e^{(\mu+\delta)|u_s|}\,dx \\ &\ge \alpha_0\int_\Omega |\nabla w_s|^p\,dx, \end{aligned}\tag{3.6}$$

where we used the fact $\mu+\delta = p\mu$.

Estimate for I_2: We have

$$I_2 = \varepsilon\int_\Omega |u|^{p-1}e^{\delta|u_s|}\frac{e^{\mu|u_s|}-1}{\mu}\,dx \ge \varepsilon\, s^{p-1}\int_\Omega e^{\delta s}\frac{e^{\mu s}-1}{\mu}\chi_{\{|u|>s\}} \ge 0. \tag{3.7}$$

Estimate for I_3+I_4: By (2.2) we have

$$\begin{aligned} I_3+I_4 &= -\int_\Omega \delta|w|e^{\delta|u|}\mathcal{A}(x,u,\nabla u)\cdot\nabla u\chi_{\{|u|\le s\}}dx + \int_\Omega \mathcal{B}(x,u,\nabla u)e^{\delta|u_s|}w_s dx \\ &\le -\int_\Omega \delta\alpha_0|w_s|e^{\delta|u_s|}|\nabla u_s|^p\,dx + \int_\Omega \mathcal{B}(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|dx \\ &= -\int_\Omega \delta\alpha_0|w_s|e^{\delta|u_s|}|\nabla u_s|^p\,dx + \\ &\quad + \int_\Omega \mathcal{B}(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|[\chi_{\{|u|\le s\}} + \chi_{\{|u|>s\}}]dx. \end{aligned}$$

Since we assume $\delta\ge\gamma_0$, this and the second condition in (2.4) imply that

$$\begin{aligned} I_3+I_4 &\le \int_\Omega (-\delta+\gamma_0)\alpha_0|w_s|e^{\delta|u_s|}|\nabla u_s|^p\,dx \\ &\quad + \int_\Omega \mathcal{B}(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|\chi_{\{|u|>s\}}dx \\ &\le \int_\Omega \mathcal{B}(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|\chi_{\{|u|>s\}}dx. \end{aligned}\tag{3.8}$$

Thus by the first inequality on (2.4) and the fact that

$$\begin{aligned} |v_s| &= e^{\delta|u_s|}|w_s| = (1+\mu|w_s|)^{\delta/\mu}|w_s| = (1+\mu|w_s|)^{p-1}|w_s| \\ &\leq \frac{1}{\mu}(1+\mu|w_s|)^p \leq \frac{1}{\mu}e^{p\mu|u_s|} \leq \frac{1}{\mu}e^{p\mu|u|}, \end{aligned} \tag{3.9}$$

we find

$$\begin{aligned} I_3 + I_4 &\leq \int_\Omega (b_0|\nabla u|^p + b_1|u|^m)\frac{1}{\mu}e^{p\mu|u|}\chi_{\{|u|>s\}}dx \\ &= \frac{1}{\mu}\int_\Omega b_0|\nabla w|^p\chi_{\{|u|>s\}}dx + \frac{1}{\mu}\int_\Omega b_1|u|^m e^{p\mu|u|}\chi_{\{|u|>s\}}dx. \end{aligned} \tag{3.10}$$

Estimate for $I_5 + I_6$: Using (3.9) again and Lemma 2.2 we have

$$\begin{aligned} I_5 + I_6 &= \int_\Omega F\cdot\nabla[(1+\mu|w_s|)^{p-1}w_s]\ dx + \int_\Omega v_s df \\ &= \int_\Omega F\cdot[(p-1)(1+\mu|w_s|)^{p-2}\nabla w_s\,\mathrm{sign}(w_s)\,\mu w_s]dx + \\ &\quad + \int_\Omega F\cdot[(1+\mu|w_s|)^{p-1}\nabla w_s]dx + \int_\Omega v_s df \\ &\leq p\int_\Omega |F|(1+\mu|w_s|)^{p-1}|\nabla w_s|dx + \int_\Omega (1+\mu|w_s|)^{p-1}|w_s|d|f|. \end{aligned} \tag{3.11}$$

Using the inequality

$$(1+\mu|w_s|)^{p-1} \leq (1+\tilde{\varepsilon})\mu^{p-1}|w_s|^{p-1} + C(\tilde{\varepsilon},p), \qquad \tilde{\varepsilon} > 0,$$

and Hölder's inequality we have

$$\begin{aligned} I_5 + I_6 &\leq (1+\tilde{\varepsilon})\mu^{p-1}\,p\int_\Omega |F||w_s|^{p-1}|\nabla w_s|dx + (1+\tilde{\varepsilon})\mu^{p-1}\int_\Omega |w_s|^p d|f| \\ &\quad + C(\tilde{\varepsilon},p)\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*}\Big)\,\|\nabla w_s\|_{L^p(\Omega)}\,. \end{aligned}$$

We recall that by approximation and Fatou's lemma (2.5) holds for all $\varphi \in W_0^{1,p}(\Omega)$. Then by (2.5) we get

$$\begin{aligned} I_5 + I_6 &\leq (1+\tilde{\varepsilon})\mu^{p-1}\lambda\,\|\nabla w_s\|^p_{L^p(\Omega)} + \\ &\quad + C(\tilde{\varepsilon},p)\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*}\Big)\,\|\nabla w_s\|_{L^p(\Omega)}\,. \end{aligned} \tag{3.12}$$

We now use estimates (3.6), (3.7), (3.10) and (3.12) in equality (3.5) to obtain the following bound

$$\begin{aligned}\kappa(\varepsilon)\,\|\nabla w_s\|^p_{L^p(\Omega)} \;\leq\; & \frac{1}{\mu}\int_\Omega b_0|\nabla w|^p\chi_{\{|u|>s\}}dx + \frac{1}{\mu}\int_\Omega b_1|u|^m e^{p\mu|u|}\chi_{\{|u|>s\}}dx + \\ & + C(\tilde{\varepsilon},p)\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)\,\|\nabla w_s\|_{L^p(\Omega)}\,,\end{aligned}$$

where $\kappa(\varepsilon) = \alpha_0 - (1+\varepsilon)\mu^{p-1}\lambda$. Observe that when $\delta < \delta_0 = (p-1)(\alpha_0/\lambda)^{\frac{1}{p-1}}$ we have

$$\mu^{p-1}\lambda = (\delta/(p-1))^{p-1}\lambda < (\delta_0/(p-1))^{p-1}\lambda = \alpha_0$$

and thus we can choose $\varepsilon > 0$ small enough so that $\kappa(\varepsilon) > 0$.

Since $(e^{\mu|u|}-1) \in W_0^{1,p}(\Omega)$, by Sobolev's embedding theorem it holds that $e^{p\mu|u|} \in L^{\frac{n}{n-p}}(\Omega)$ if $1<p<n$ and $e^{p\mu|u|} \in L^2(\Omega)$, say, if $p \geq n$. Thus we have $|u|^m e^{p\mu|u|} \in L^1(\Omega)$. Now letting $s\nearrow\infty$ in the last inequality, we find

$$\|\nabla w\|^p_{L^p(\Omega)} \leq C\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)\,\|\nabla w\|_{L^p(\Omega)}\,,$$

which yields

$$\Big\|e^{\delta|u|/(p-1)} - 1\Big\|_{W_0^{1,p}(\Omega)} \leq C(\delta,\lambda,p)\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)^{\frac{1}{p-1}}.$$

Finally, note that

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} \leq \frac{p-1}{\delta}\,\Big\|\nabla(e^{\delta|u|/(p-1)}-1)\Big\|_{L^p(\Omega)}$$

and hence, we also have

$$\|u\|_{W_0^{1,p}(\Omega)} \leq C(\delta,\lambda,p)\Big(\,\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)^{\frac{1}{p-1}}.$$

This proves inequality (3.2) for all $\delta \in [\gamma_0, \delta_0)$.

To prove inequality (3.3) for δ_1, we first define $\mu_1 = \frac{\delta_1}{p-1}$ and redefine

$$w = \mathrm{sign}(u)[e^{\mu_1|u|}-1]/\mu_1, \qquad w_s = \mathrm{sign}(u)[e^{\mu_1|u_s|}-1]/\mu_1. \tag{3.13}$$

Observe that

$$(e^{\mu_1|u_s|}-1)e^{\delta_1|u_s|} \geq (1-\varepsilon)e^{(\delta_1+\mu_1)|u_s|} - C(\varepsilon,\delta_1) \qquad \text{for all } \varepsilon\in(0,1),$$

and thus by the first inequality in (3.8), with (δ_1, μ_1) in place of (δ, μ), we have

$$\begin{aligned} I_3 + I_4 &\leq \int_\Omega (-\delta_1 + \gamma_0) \frac{\alpha_0}{\mu_1} (e^{\mu_1 |u_s|} - 1) e^{\delta_1 |u_s|} |\nabla u_s|^p \, dx \\ &\quad + \int_\Omega \mathcal{B}(x, u, \nabla u) \mathrm{sign}(u) e^{\delta_1 |u_s|} |w_s| \chi_{\{|u|>s\}} dx \\ &\leq \int_\Omega (1-\varepsilon)(-\delta_1 + \gamma_0) \frac{\alpha_0}{\mu_1} |\nabla w_s|^p dx + \int_\Omega C(\varepsilon, \delta_1)(\delta_1 - \gamma_0) \frac{\alpha_0}{\mu_1} |\nabla u_s|^p \, dx \\ &\quad + \int_\Omega \mathcal{B}(x, u, \nabla u) \mathrm{sign}(u) e^{\delta_1 |u_s|} |w_s| \chi_{\{|u|>s\}} dx. \end{aligned}$$

Here in the last inequality we used that $\delta_1 > \gamma_0$ and $|\nabla w_s|^p = e^{(\delta_1 + \mu_1)|u_s|} |\nabla u_s|^p$.

Thus arguing as in (3.10) for the last term we find

$$\begin{aligned} I_3 + I_4 &\leq \int_\Omega (1-\varepsilon)(-\delta_1 + \gamma_0) \frac{\alpha_0}{\mu_1} |\nabla w_s|^p dx + C(\varepsilon) \int_\Omega |\nabla u_s|^p \, dx \\ &\quad + \frac{1}{\mu_1} \int_\Omega b_0 |\nabla w|^p \chi_{\{|u|>s\}} dx + \frac{1}{\mu_1} \int_\Omega b_1 |u|^m e^{p\mu_1 |u|} \chi_{\{|u|>s\}} dx. \end{aligned} \tag{3.14}$$

Using estimates (3.6), (3.7), (3.12) (with (δ_1, μ_1) in place of (δ, μ)) and (3.14) in equality (3.5) we then get

$$\begin{aligned} \kappa_1(\varepsilon) \|\nabla w_s\|^p_{L^p(\Omega)} &\leq \frac{1}{\mu_1} \int_\Omega b_0 |\nabla w|^p \chi_{\{|u|>s\}} dx + \frac{1}{\mu_1} \int_\Omega b_1 |u|^m e^{p\mu_1 |u|} \chi_{\{|u|>s\}} dx + \\ &\quad + C(\varepsilon) \Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*} \Big) \|\nabla w_s\|_{L^p(\Omega)} + \\ &\quad + C(\varepsilon) \int_\Omega |\nabla u_s|^p \, dx, \end{aligned}$$

where $\kappa_1(\varepsilon) = \alpha_0 + (1-\varepsilon)(\delta_1 - \gamma_0)\frac{\alpha_0}{\mu_1} - (1+\varepsilon)\mu_1^{p-1}\lambda$, with $\varepsilon \in (0,1)$. Thus when (2.7) holds we can find $\varepsilon \in (0,1)$ such that $\kappa_1(\varepsilon) > 0$. Then using Young's inequality and letting $s \to \infty$ we eventually obtain

$$\|\nabla w\|^p_{L^p(\Omega)} \leq C \Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*} \Big)^{\frac{1}{p-1}} + C \int_\Omega |\nabla u|^p \, dx.$$

This proves inequality (3.3) for all $\delta_1 > \gamma_0$ such (2.7) holds. □

4. Existence of solutions to an approximate equation

For $k > 0$, we now define a function $\mathcal{H}_k(x, s, \xi)$ by letting

$$\mathcal{H}_k(x, s, \xi) := \frac{\mathcal{B}(x, s, \xi)}{1 + \frac{1}{k} |\mathcal{B}(x, s, \xi)|}. \tag{4.1}$$

Note $|\mathcal{H}_k(x, s, \xi)| \leq k$, and (2.4) also holds with $\mathcal{H}_k(x, s, \xi)$ in place of $\mathcal{B}(x, s, \xi)$. Moreover,

$$\lim_{k \to \infty} \mathcal{H}_k(x, s, \xi) = \mathcal{B}(x, s, \xi).$$

The goal of this section is to obtain existence results for the approximate equation

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u) = \mathcal{H}_k(x,u,\nabla u) + \sigma \quad \text{in} \quad \Omega. \tag{4.2}$$

Proposition 4.1. *Let $\sigma = \operatorname{div} F + f$ where $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f| \in (W_0^{1,p}(\Omega))^*$ such that (2.5) holds for all $\varphi \in C_c^\infty(\Omega)$, with $\lambda \in (0, \gamma_0^{1-p}\alpha_0(p-1)^{p-1})$. Then for each $k > 0$, there exists a solution $u_k \in W_0^{1,p}(\Omega)$ to (4.2) such that $e^{\frac{\delta|u_k|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$ for all $\delta \in [\gamma_0, \delta_0)$, with $\delta_0 = (p-1)(\alpha_0/\lambda)^{\frac{1}{p-1}}$, and*

$$\|u_k\|_{W_0^{1,p}(\Omega)} + \|e^{\frac{\delta|u_k|}{p-1}} - 1\|_{W_0^{1,p}(\Omega)} \le M_\delta. \tag{4.3}$$

Moreover, for any $\delta_1 > \gamma_0$ such that (2.7) holds then we have

$$\|e^{\frac{\delta_1|u_k|}{p-1}} - 1\|_{W_0^{1,p}(\Omega)} \le M_{\delta_1}, \tag{4.4}$$

Here the constants M_δ and M_{δ_1} are independent of k.

Proof. Since $\sigma \in (W_0^{1,p}(\Omega))^*$ and $|\mathcal{H}_k(x,s,\xi)| \le k$, by the theory of pseudomonotone operators (see, e.g., [26], [27, Chapter 6], and [8]), for any $\varepsilon > 0$ there exists a solution $u_{k,\varepsilon} \in W_0^{1,p}(\Omega)$ to the equation

$$-\operatorname{div}\mathcal{A}(x,u,\nabla u) + \varepsilon|u|^{p-2}u = \mathcal{H}_k(x,u,\nabla u) + \sigma \quad \text{in} \quad \Omega. \tag{4.5}$$

The next step is to obtain uniform bounds of the form (4.3)-(4.4) for $\{u_{k,\varepsilon}\}$. However, we cannot directly apply Theorem 3.1 here since we do not know if $e^{\frac{\delta|u_{k,\varepsilon}|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$. The strategy here is to follow the proof of Theorem 3.1. For simplicity let us write $u = u_{k,\varepsilon}$, and for each $s > 0$, we set $v_s = e^{\delta|u_s|}w_s$, where $u_s = T_s(u)$ and $w_s = \operatorname{sign}(u)[e^{\mu|u_s|} - 1]/\mu$ with $\mu = \delta/(p-1)$. Then using v_s as a test function for (4.5) we obtain the following equality

$$I_1 + I_2 = I_3 + I_4' + I_5 + I_6, \tag{4.6}$$

where the expressions for I_1, I_2, I_3, I_5, I_6 are as in the proof of Theorem 3.1. The term I_4' is similar to I_4 given in the proof of Theorem 3.1 except that $\mathcal{B}(x,u,\nabla u)$ is now replaced by $\mathcal{H}_k(x,u,\nabla u)$. That is,

$$I_4' := \int_\Omega \mathcal{H}_k(x,u,\nabla u)e^{\delta|u_s|}w_s\, dx.$$

Thus lower estimates for I_1, I_2 and upper estimates for I_5+I_6 are unchanged; see (3.6), (3.7), and (3.12). As in (3.8) we have the following upper estimate for I_3+I_4':

$$\begin{aligned} I_3+I_4' &\leq \int_\Omega (-\delta+\gamma_0)\alpha_0|w_s|e^{\delta|u_s|}|\nabla u_s|^p\ dx \\ &\quad + \int_\Omega \mathcal{H}_k(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|\chi_{\{|u|>s\}}dx \\ &\leq \int_\Omega \mathcal{H}_k(x,u,\nabla u)\mathrm{sign}(u)e^{\delta|u_s|}|w_s|\chi_{\{|u|>s\}}dx. \end{aligned}$$

Thus, instead of (3.10), we now get

$$I_3+I_4' \leq k\int_\Omega e^{\delta s}\frac{e^{\mu s}-1}{\mu}\chi_{\{|u|>s\}}dx. \tag{4.7}$$

Similarly, instead of (3.14), we now obtain

$$\begin{aligned} I_3+I_4' &\leq \int_\Omega (1-\varepsilon)(-\delta_1+\gamma_0)\frac{\alpha_0}{\mu_1}|\nabla w_s|^p dx + C(\varepsilon)\int_\Omega |\nabla u_s|^p\ dx \\ &\quad + k\int_\Omega e^{\delta s}\frac{e^{\mu s}-1}{\mu}\chi_{\{|u|>s\}}dx. \end{aligned} \tag{4.8}$$

We recall that in (4.8), $\mu_1=\frac{\delta_1}{p-1}$ with w, w_s to be understood as in (3.13).

When $\varepsilon>0$ and s is such that $\varepsilon s^{p-1}\geq k$ by (3.7) and (4.7) we have

$$I_3+I_4'-I_2 \leq 0 \tag{4.9}$$

and thus it follows from (4.6) that

$$I_1 \leq I_5+I_6.$$

With this, employing (3.6) and (3.12) we find

$$\|\nabla w_s\|_{L^p(\Omega)} \leq C(\delta,\lambda,p)\Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)^{\frac{1}{p-1}}.$$

At this point we let $s\nearrow\infty$ to obtain that any solution $u=u_{k,\varepsilon}$ to (4.5) satisfies the bound

$$\begin{aligned} &\|u_{k,\varepsilon}\|_{W_0^{1,p}(\Omega)} + \left\|e^{\frac{\delta|u_{k,\varepsilon}|}{p-1}}-1\right\|_{W_0^{1,p}(\Omega)} \leq \\ &\qquad \leq C(\delta,\lambda,p)\Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + |||f|||_{(W_0^{1,p}(\Omega))^*}\Big)^{\frac{1}{p-1}} \end{aligned} \tag{4.10}$$

for every $\delta\in[\gamma_0,\delta_0)$.

For $\delta_1 > \gamma_0$ such that (2.7) holds, using (4.8) and arguing similarly we obtain

$$\begin{aligned} \left\| e^{\frac{\delta_1 |u_{k,\varepsilon}|}{p-1}} - 1 \right\|_{W_0^{1,p}(\Omega)} &\leq C\Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*} \Big)^{\frac{1}{p-1}} \\ &\quad + C \int_\Omega |\nabla u_{k,\varepsilon}|^p \, dx \\ &\leq C\Big(\|F\|_{L^{\frac{p}{p-1}}(\Omega)} + \||f|\|_{(W_0^{1,p}(\Omega))^*} \Big)^{\frac{1}{p-1}}. \end{aligned} \tag{4.11}$$

As the bound (4.10) is uniform in ε, we can extract a subsequence, still denoted by ε, such that

$$u_{k,\varepsilon} \to u_k \text{ weakly in } W_0^{1,p}(\Omega), \text{ strongly in } L^p(\Omega), \text{ and a.e. in } \Omega,$$

as $\varepsilon \searrow 0^+$ for a function $u_k \in W_0^{1,p}(\Omega)$. Due to the pointwise a.e. convergence, we see that u_k also satisfies (4.10)-(4.11) for every $\delta \in [\gamma_0, \delta_0)$ and every $\delta_1 > \gamma_0$ such that (2.7) holds.

Recall that we have

$$- \operatorname{div} \mathcal{A}(x, u_{k,\varepsilon}, \nabla u_{k,\varepsilon}) + \varepsilon |u_{k,\varepsilon}|^{p-2} u_{k,\varepsilon} = \mathcal{H}_k(x, u_{k,\varepsilon}, \nabla u_{k,\varepsilon}) + \sigma \quad \text{in } \mathcal{D}'(\Omega). \tag{4.12}$$

For each fixed $k > 0$, we know $\mathcal{H}_k(x, u_{k,\varepsilon}, \nabla u_{k,\varepsilon}) - \varepsilon |u_{k,\varepsilon}|^{p-2} u_{k,\varepsilon}$ is uniformly bounded in $\varepsilon \in (0,1)$ as finite measures in Ω. Thus by a convergence result shown in [6, Eqn (2.26)], we may further assume that

$$\nabla u_{k,\varepsilon} \to \nabla u_k \quad \text{a.e. in } \Omega, \text{ as } \varepsilon \searrow 0^+.$$

This allows us to pass to the limit in (4.12) as $\varepsilon \searrow 0^+$ to see that u_k solves (4.2) and satisfies the bounds (4.3)-(4.4). □

5. Proof of Theorem 2.3

This section is devoted to the proof of Theorem 2.3.

Proof. For each $k > 0$, let u_k be a solution of the approximate equation (4.2) as obtained in Proposition 4.1. Recall that $\mathcal{H}_k(x, s, \xi)$ is defined in (4.1). By (4.3)-(4.4) and Rellich's compactness theorem, there is a subsequence, still denoted by k, such that

$$u_k \xrightarrow{k} u \text{ weakly in } W_0^{1,p}(\Omega), \text{ strongly in } L^p(\Omega), \text{ and a.e. in } \Omega,$$

for some function $u \in W_0^{1,p}(\Omega)$ such that $e^{\frac{\delta |u|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$ for each $\delta \in [\gamma_0, \delta_0)$, and $e^{\frac{\delta_1 |u|}{p-1}} - 1 \in W_0^{1,p}(\Omega)$ for any $\delta_1 > \gamma_0$ such that (2.7) holds.

As u_k solves (4.2), we have

$$-\operatorname{div}\mathcal{A}(x,u_k,\nabla u_k)=\mathcal{H}_k(x,u_k,\nabla u_k)+\sigma \quad \text{in} \quad \Omega. \tag{5.1}$$

Thus to show that u is a solution of (2.6) it is enough to show that

$$u_k\to u \quad \text{strongly in } W_0^{1,p}(\Omega) \text{ as } k\nearrow\infty, \tag{5.2}$$

so that we can pass to the limit in (5.1) using (2.4), (4.3), and Vitali's Convergence Theorem.

For each $s>0$ we can write

$$\nabla u_k-\nabla u=\nabla T_s(u_k)-\nabla T_s(u)+\nabla G_s(u_k)-\nabla G_s(u),$$

where

$$G_s(r):=r-T_s(r), \qquad r\in\mathbb{R}.$$

In order to show (5.2) we shall show that the following limits hold:

$$\lim_{s\to\infty}\sup_{k>0}\|\nabla G_s(u_k)-\nabla G_s(u)\|_{L^p(\Omega)}=0 \tag{5.3}$$

$$\lim_{k\to\infty}\|\nabla T_s(u_k)-\nabla T_s(u)\|_{L^p(\Omega)}=0 \quad \text{for each } s>0. \tag{5.4}$$

The rest of the proof will be devoted to the verification of these limits.

Proof of (5.3)**.** Define $w_k=[e^{\frac{\delta}{p-1}|u_k|}-1]\frac{p-1}{\delta}$ and hence we get

$$\begin{aligned}\int_\Omega|\nabla G_s(u_k)|^p\,dx &= \int_{\{|u_k|>s\}}|\nabla u_k|^p\,dx\\ &= \int_{\{|u_k|>s\}}e^{-\frac{\delta p}{p-1}|u_k|}|\nabla w_k|^p\,dx\\ &\leq e^{-\frac{\delta p}{p-1}s}\int_{\{|u_k|>s\}}|\nabla w_k|^p\,dx.\end{aligned}$$

Using the estimate (4.3), we then find

$$\int_\Omega|\nabla G_s(u_k)|^p\,dx\leq C(\delta)e^{-\frac{\delta p}{p-1}s}, \tag{5.5}$$

which yields (5.3).

Proof of (5.4)**.** Following [16] (see also the earlier works [15, 4]), we shall make use of the following test function in (5.1):

$$v_k=e^{\delta|T_j(u_k)|}\psi(z_k), \quad \text{with } j\geq s,$$

where $z_k=T_s(u_k)-T_s(u)$ and ψ is a C^1 and increasing function from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$\psi(0)=0 \quad \text{and} \quad \psi'-H_0|\psi|\geq 1, \tag{5.6}$$

where $H_0 = \frac{b_0+(a_0+a_1)\delta}{\alpha_0}$. For example, the function $\psi(r) = 2re^{\frac{H_0^2 r^2}{4}}$ will do. We then have

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,u_k,\nabla u_k)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k\,dx \\
&= \int_\Omega \Big[\mathcal{H}_k(x,u_k,\nabla u_k) - \delta\mathcal{A}(x,\nabla u_k)\cdot\nabla T_j(u_k)\mathrm{sign}(u_k)\Big]e^{\delta|T_j(u_k)|}\psi(z_k)\,dx \\
&\quad + \langle\sigma, e^{\delta|T_j(u_k)|}\psi(z_k)\rangle.
\end{aligned}$$

Note that the term on the left-hand side in the above equality can be written as

$$\begin{aligned}
&\int_\Omega \mathcal{A}(x,u_k,\nabla u_k)\cdot(\nabla T_s(u_k)-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx \\
&= \int_{\{|u_k|\le s\}}(\mathcal{A}(x,T_s(u_k),\nabla T_s(u_k)) - \mathcal{A}(x,T_s(u_k),\nabla T_s(u)))\cdot \\
&\qquad\qquad \cdot(\nabla T_s(u_k)-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx \\
&\quad + \int_{\{|u_k|\le s\}} \mathcal{A}(x,T_s(u_k),\nabla T_s(u))\cdot(\nabla T_s(u_k)-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx \\
&\quad + \int_{\{|u_k|>s\}} \mathcal{A}(x,u_k,\nabla u_k)\cdot(-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx.
\end{aligned}$$

Thus combining the last two equalities we obtain

$$I_1 = -I_2 - I_3 + I_4 + I_5, \tag{5.7}$$

where we have defined

$$\begin{aligned}
I_1 &= \int_{\{|u_k|\le s\}}(\mathcal{A}(x,T_s(u_k),\nabla T_s(u_k)) - \mathcal{A}(x,T_s(u_k),\nabla T_s(u)))\cdot \\
&\qquad\qquad \cdot(\nabla T_s(u_k)-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx,
\end{aligned}$$

$$I_2 = \int_{\{|u_k|\le s\}} \mathcal{A}(x,T_s(u_k),\nabla T_s(u))\cdot(\nabla T_s(u_k)-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx,$$

$$I_3 = \int_{\{|u_k|>s\}} \mathcal{A}(x,u_k,\nabla u_k)\cdot(-\nabla T_s(u))e^{\delta|T_j(u_k)|}\psi'(z_k)\,dx,$$

$$I_4 = \int_\Omega \Big[\mathcal{H}_k(x,u_k,\nabla u_k) - \delta\mathcal{A}(x,u_k,\nabla u_k)\cdot\nabla T_j(u_k)\mathrm{sign}(u_k)\Big]e^{\delta|T_j(u_k)|}\psi(z_k)\,dx,$$

and

$$I_5 = \langle\sigma, e^{\delta|T_j(u_k)|}\psi(z_k)\rangle.$$

We now write I_4 as

$$I_4 = I_4' + I_4'', \tag{5.8}$$

where

$$I_4' := \int_{\{|u_k|>s\}} H_{k,j}(x)\, e^{\delta|T_j(u_k)|}\psi(z_k)\, dx,$$

$$I_4'' := \int_{\{|u_k|\le s\}} H_{k,j}(x)\, e^{\delta|T_j(u_k)|}\psi(z_k)\, dx,$$

with

$$H_{k,j}(x) := \mathcal{H}_k(x,u_k,\nabla u_k) - \delta\mathcal{A}(x,u_k,\nabla u_k)\cdot\nabla T_j(u_k)\mathrm{sign}(u_k).$$

Note that $|\nabla T_j(u_k)| \le |\nabla u_k|$ and hence using the growth conditions in (2.3) and (2.4) we get

$$\begin{aligned}
|I_4''| \;&\le\; \int_{\{|u_k|\le s\}} \Big(b_0|\nabla u_k|^p + b_1|u_k|^m + \delta a_0|\nabla u_k|^p + \delta a_1|u_k|^{p-1}|\nabla u_k|\Big)\times \\
&\qquad\qquad \times e^{\delta|T_j(u_k)|}\; |\psi(z_k)|\;\; dx \\
&\le\; \int_{\{|u_k|\le s\}} (b_0+\delta(a_0+a_1))|\nabla u_k|^p e^{\delta|T_j(u_k)|}\; |\psi(z_k)|\;\; dx \\
&\quad + \int_{\{|u_k|\le s\}} \Big(b_1|u_k|^m + c(p)\delta a_1|u_k|^p\Big) e^{\delta|T_j(u_k)|}\; |\psi(z_k)|\;\; dx \\
&\le\; \frac{b_0+\delta(a_0+a_1)}{\alpha_0}\int_{\{|u_k|\le s\}} \mathcal{A}(x,T_s(u_k),\nabla T_s(u_k))\cdot\nabla T_s(u_k) e^{\delta|T_j(u_k)|}\; |\psi(z_k)|\;\; dx \\
&\quad + \int_{\{|u_k|\le s\}} \Big(b_1|u_k|^m + c(p)\delta a_1|u_k|^p\Big) e^{\delta|T_j(u_k)|}\; |\psi(z_k)|\;\; dx,
\end{aligned}$$

where we used Young's inequality in the second inequality and the coercivity condition (2.2) in the last inequality. Thus, recalling that $H_0 = \frac{b_0+\delta(a_0+a_1)}{\alpha_0}$, we find

$$\begin{aligned}
|I_4''| \;&\le\; H_0\int_{\{|u_k|\le s\}} [\mathcal{A}(x,T_s(u_k),\nabla T_s(u_k)) - \mathcal{A}(x,T_s(u_k),\nabla T_s(u))]\,\cdot \\
&\qquad\qquad \cdot\, [\nabla T_s(u_k) - \nabla T_s(u)]\, e^{\delta|T_j(u_k)|}\; |\psi(z_k)| dx \\
&\quad + H_0\int_{\{|u_k|\le s\}} \mathcal{A}(x,T_s(u_k),\nabla T_s(u))\cdot[\nabla T_s(u_k) - \nabla T_s(u)]\, e^{\delta|T_j(u_k)|}\; |\psi(z_k)| dx \\
&\quad + H_0\int_{\{|u_k|\le s\}} \mathcal{A}(x,T_s(u_k),\nabla T_s(u_k))\cdot\nabla T_s(u) e^{\delta|T_j(u_k)|}\; |\psi(z_k)| dx \\
&\quad + \int_{\{|u_k|\le s\}} \Big(b_1|u_k|^m + c(p)\delta a_1|u_k|^{p-1}|\nabla T_s(u_k)|\Big) e^{\delta|T_j(u_k)|}\; |\psi(z_k)| dx.
\end{aligned}$$

Using this bound, equalities (5.7)-(5.8), and the inequality in (5.6), we now obtain

$$I_1' \le -I_2 - I_3 + I_4' + I_5 + I_6 + I_7 + I_8, \tag{5.9}$$

where

$$I_1' = \int_{\{|u_k|\le s\}} (\mathcal{A}(x,T_s(u_k),\nabla T_s(u_k)) - \mathcal{A}(x,T_s(u),\nabla T_s(u)))\cdot(\nabla T_s(u_k) - \nabla T_s(u))\; dx,$$

$$I_6 = H_0 \int_{\{|u_k|\le s\}} \mathcal{A}(x, T_s(u_k), \nabla T_s(u)) \cdot [\nabla T_s(u_k) - \nabla T_s(u)]\, e^{\delta |T_j(u_k)|}\ |\psi(z_k)|\ dx,$$

$$I_7 = H_0 \int_{\{|u_k|\le s\}} \mathcal{A}(x, T_s(u_k), \nabla T_s(u_k)) \cdot \nabla T_s(u) e^{\delta |T_j(u_k)|}\ |\psi(z_k)|\ dx,$$

and

$$I_8 = \int_{\{|u_k|\le s\}} \Big(b_1 |u_k|^m + c(p)\delta a_1 |u_k|^{p-1} |\nabla T_s(u_k)| \Big) e^{\delta |T_j(u_k)|}\ |\psi(z_k)|\ dx.$$

We shall next treat each term on the right-hand side of (5.9).

The term I_2: We know that $u_k \xrightarrow{k} u$ a.e., from which we see that $z_k \xrightarrow{k} 0$ a.e. and hence

$$\mathcal{A}(x, T_s(u_k), \nabla T_s(u)) e^{\delta |T_j(u_k)|} \psi'(z_k) \xrightarrow{k} \mathcal{A}(x, T_s(u), \nabla T_s(u)) e^{\delta |T_j(u)|} \psi'(0) \quad \text{a.e.}$$

Thus using the pointwise estimate, which follows from (2.3),

$$|\mathcal{A}(x, T_s(u_k), \nabla T_s(u)) e^{\delta |T_j(u_k)|} \psi'(z_k)| \le e^{\delta j} \max_{r\in[-2s,2s]} |\psi'(r)| \Big[a_0 |\nabla T_s(u)|^{p-1} + a_1 s^{p-1} \Big]$$

and the fact that $|\nabla T_s(u)|^{p-1} \in L^{\frac{p}{p-1}}(\Omega)$, it follows from Lebesgue's Dominated Convergence Theorem that

$$\mathcal{A}(x, T_s(u_k), \nabla T_s(u)) e^{\delta |T_j(u_k)|} \psi'(z_k) \xrightarrow{k} \mathcal{A}(x, T_s(u), \nabla T_s(u)) e^{\delta |T_j(u)|} \psi'(0)$$

strongly in $L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$.

Since $\|T_s(u_k)\|_{W_0^{1,p}(\Omega)}$ is uniformly bounded in k and $T_s(u_k) \xrightarrow{k} T_s(u)$ a.e. we get that $\nabla T_s(u_k) \stackrel{k}{\rightharpoonup} \nabla T_s(u)$ weakly in $L^p(\Omega, \mathbb{R}^n)$. Also, since

$$\chi_{\{|u_k|\le s\}} \xrightarrow{k} \chi_{\{|u|\le s\}} \text{ a.e. in } \Omega \setminus \{|u| = s\} \text{ while } |\nabla T_s(u)| = 0 \text{ a.e. on } \{|u| = s\}, \tag{5.10}$$

we have from Lebesgue's Dominated Convergence Theorem that

$$\nabla T_s(u) \chi_{\{|u_k|\le s\}} \xrightarrow{k} \nabla T_s(u) \chi_{\{|u|\le s\}} = \nabla T_s(u) \quad \text{strongly in} \quad L^p(\Omega, \mathbb{R}^n).$$

Thus with the observation $\chi_{\{|u_k|\le s\}}(\nabla T_s(u_k) - \nabla T_s(u)) = \nabla T_s(u_k) - \nabla T_s(u) \chi_{\{|u_k|\le s\}}$, we see that

$$\chi_{\{|u_k|\le s\}}(\nabla T_s(u_k) - \nabla T_s(u)) \stackrel{k}{\rightharpoonup} 0 \quad \text{weakly in} \quad L^p(\Omega, \mathbb{R}^n). \tag{5.11}$$

The above calculations imply that $\lim_{k\to\infty} I_2 = 0$.

The term I_3: By (2.3), $|\mathcal{A}(x, u_k, \nabla u_k)|$ is uniformly bounded in $L^{\frac{p}{p-1}}(\Omega)$. On the other hand, again by (5.10) and Lebesgue's Dominated Convergence Theorem we have

$$|\chi_{\{|u_k|>s\}}(-\nabla T_s(u)) e^{\delta |T_j(u_k)|} \psi'(z_k)| \xrightarrow{k} 0 \quad \text{strongly in} \quad L^p(\Omega).$$

Thus we see that $\lim_{k\to\infty} I_3 = 0$.

The term I_4': We have the inequalities $\mathcal{A}(x,u_k,\nabla u_k)\cdot\nabla T_j(u_k)\ge \alpha_0|\nabla u_k|^p\chi_{\{|u_k|\le j\}}$ and $\chi_{\{|u_k|>s\}}\mathrm{sign}(u_k)\psi(z_k)\ge 0$. Thus using the second inequality in (2.4) we see that

$$\begin{aligned}
I_4' &= \int_{\{|u_k|>s\}}[\mathrm{sign}(u_k)\mathcal{H}_k(x,u_k,\nabla u_k)-\delta\mathcal{A}(x,u_k,\nabla u_k)\cdot\nabla T_j(u_k)]\times\\
&\qquad\qquad \times\ \mathrm{sign}(u_k)e^{\delta|T_j(u_k)|}\psi(z_k)\,dx\\
&\le \int_{\{|u_k|>s\}}\left[\gamma_0\alpha_0|\nabla u_k|^p-\delta\alpha_0|\nabla u_k|^p\chi_{\{|u_k|\le j\}}\right]\mathrm{sign}(u_k)e^{\delta|T_j(u_k)|}\psi(z_k)\,dx\\
&\le \int_{\{|u_k|>j\}}\gamma_0\alpha_0|\nabla u_k|^p\mathrm{sign}(u_k)e^{\delta|T_j(u_k)|}\psi(z_k)\,dx,
\end{aligned}$$

where we used that $\delta\ge\gamma_0$ and $j\ge s$ in the last inequality. At this point, using (5.5) with j in place of s, we get

$$\begin{aligned}
I_4' &\le \gamma_0\alpha_0\max_{r\in[-2s,2s]}|\psi(r)|\ e^{\delta j}\int_{\{|u_k|>j\}}|\nabla u_k|^p\,dx\\
&\le C(\delta)\gamma_0\alpha_0\max_{r\in[-2s,2s]}|\psi(r)|\ e^{\delta j}\ e^{-\frac{\delta p}{p-1}j}.
\end{aligned}$$

This yields that $\limsup_{j\to\infty}\sup_{k>0}I_4'=0$.

The term I_5: Since $f\in(W_0^{1,p}(\Omega))^*$, there is a vector field $F_1\in L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$ such that $\mathrm{div}\,F_1=f$ in $\mathcal{D}'(\Omega)$. Thus $\sigma=\mathrm{div}\,(F+F_1)$ which yields

$$\begin{aligned}
I_5 &= \delta\int_\Omega(F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi(z_k)\nabla T_j(u_k)\mathrm{sign}(u_k)\ dx\\
&\quad+\int_\Omega(F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k\ dx.
\end{aligned}\tag{5.12}$$

As $\psi(0)=0$ we have $(F+F_1)e^{\delta|T_j(u_k)|}\psi(z_k)\xrightarrow{k}(0,\dots,0)$ a.e. in Ω. Thus by Lebesgue's Dominated Convergence Theorem we find

$$(F+F_1)e^{\delta|T_j(u_k)|}\psi(z_k)\xrightarrow{k}(0,\dots,0)\quad\text{strongly in}\quad L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n).$$

Since $\nabla T_j(u_k)\mathrm{sign}(u_k)$ is uniformly bounded in $L^p(\Omega,\mathbb{R}^n)$, we then conclude that

$$\delta\int_\Omega(F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi(z_k)\nabla T_j(u_k)\mathrm{sign}(u_k)\ dx\xrightarrow{k}0.\tag{5.13}$$

We now write

$$\int_\Omega(F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k\ dx=R_1+R_2,\tag{5.14}$$

where

$$R_1 := \int_{\{|u_k|\le s\}} (F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k \, dx$$
$$R_2 := \int_{\{|u_k|> s\}} (F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k \, dx.$$

Again by Lebesgue's Dominated Convergence Theorem we have

$$(F+F_1)e^{\delta|T_j(u_k)|}\psi'(z_k) \xrightarrow{k} (F+F_1)e^{\delta|T_j(u)|}\psi'(0) \quad \text{strongly in} \quad L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n).$$

Thus using (5.11) (recall that $\nabla z_k = \nabla T_s(u_k) - \nabla T_s(u)$) we obtain that

$$R_1 \xrightarrow{k} 0.$$

On the other hand, from the definition of z_k we have

$$R_2 = \int_\Omega (F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)(-\nabla T_s(u))\chi_{\{|u_k|>s\}} \, dx.$$

Then by (5.10), Hölder's inequality, and Lebesgue's Dominated Convergence Theorem, it follows that

$$R_2 \xrightarrow{k} 0.$$

Now recalling (5.14) we get

$$\int_\Omega (F+F_1)\cdot e^{\delta|T_j(u_k)|}\psi'(z_k)\nabla z_k \, dx \xrightarrow{k} 0. \tag{5.15}$$

Hence using (5.13) and (5.15) in (5.12) we conclude that $\lim_{k\to\infty} I_5 = 0$.

The terms I_6, I_7, and I_8: Since $\psi(0)=0$, by Lebesgue's Dominated Convergence Theorem we find

$$\chi_{\{|u_k|\le s\}}\mathcal{A}(x,T_s(u_k),\nabla T_s(u))e^{\delta|T_j(u_k)|}|\psi(z_k)| \xrightarrow{k} 0 \quad \text{strongly in} \quad L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$$

and

$$\chi_{\{|u_k|\le s\}}\nabla T_s(u)e^{\delta|T_j(u_k)|}|\psi(z_k)| \xrightarrow{k} 0 \quad \text{strongly in} \quad L^p(\Omega,\mathbb{R}^n).$$

On the other hand, $\nabla T_s(u_k) - \nabla T_s(u)$ and $\mathcal{A}(x,T_s(u_k),\nabla T_s(u_k))$ are uniformly bounded in $L^p(\Omega,\mathbb{R}^n)$ and in $L^{\frac{p}{p-1}}(\Omega,\mathbb{R}^n)$, respectively. Thus we obtain that

$$\lim_{k\to\infty} I_6 = \lim_{k\to\infty} I_7 = 0.$$

As for the term I_8, we estimate

$$I_8 \leq \int_{\{|u_k|\leq s\}} (b_1|s|^m + c(p)\delta a_1 s^{p-1})e^{\delta s}|\psi(z_k)|dx,$$

which also converges to zero, as $k\nearrow\infty$, by Lebesgue's Dominated Convergence Theorem.

We have shown that $\lim_{k\to\infty}(-I_2-I_3+I_5+I_6+I_7+I_8) = 0$ and $\limsup_{j\to\infty}\sup_{k>0} I_4' = 0$. For each fixed $s > 0$, we now let

$$D_k = (\mathcal{A}(x, T_s(u_k), \nabla T_s(u_k)) - \mathcal{A}(x, T_s(u_k), \nabla T_s(u))) \cdot (\nabla T_s(u_k) - \nabla T_s(u)).$$

As $D_k \geq 0$ (by (2.1)), in view of (5.9) we find that

$$\int_{\{|u_k|\leq s\}} D_k \, dx \overset{k}{\to} 0. \tag{5.16}$$

On the other hand, by (5.10),

$$\begin{aligned} \chi_{\{|u_k|>s\}} D_k &= \chi_{\{|u_k|>s\}}[\mathcal{A}(x, T_s(u_k), 0) - \mathcal{A}(x, T_s(u_k), \nabla T_s(u))] \cdot (-\nabla T_s(u)) \\ &\to 0 \quad \text{a.e. as } k\nearrow\infty. \end{aligned}$$

It then follows from Lebesgue's Dominated Convergence Theorem that

$$\int_{\{|u_k|>s\}} D_k \, dx \overset{k}{\to} 0. \tag{5.17}$$

Combining (5.16)-(5.17) we obtain

$$\int_{\Omega} D_k \, dx \overset{k}{\to} 0.$$

At this point we use the conditions (2.1)-(2.3) and a result of F. E. Browder (see [10] or [7, Lemma 5]) to complete the proof of (5.4). □

6. Proof of Theorems 1.2 and 1.8

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. (i) Suppose that (1.1) has a solution in $u \in W_0^{1,p}(\Omega)$ such that (1.5) holds for some $A > 0$. Then letting $F = |\nabla u|^{p-2}\nabla u$, we immediately have the desired representation for σ.

(ii) Suppose that $\sigma = \operatorname{div} F + f$ where $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$ and f is a locally finite signed measure in Ω with $|f| \in (W_0^{1,p}(\Omega))^*$ such that (1.7) holds for some $\lambda \in (0, (p-1)^{p-1})$. Applying Theorem 2.3 we obtain a solution to (1.1) that satisfies all of the properties stated in Theorem 1.2(ii) except the Poincaré-Sobolev inequality (1.5). To verify it, we use $|\varphi|^p$, $\varphi \in C_c^\infty(\Omega)$, as a test function in (1.1) to get

$$\int_\Omega |\varphi|^p |\nabla u|^p dx = p \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla |\varphi| |\varphi|^{p-1} dx + \langle \sigma, |\varphi|^p \rangle.$$

Thus by Hölder's inequality and condition (1.7) we find

$$\int_\Omega |\varphi|^p |\nabla u|^p dx \le p \left(\int_\Omega |\nabla u|^p |\varphi|^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla \varphi|^p dx \right)^{\frac{1}{p}} + (p-1)^{p-1} \int_\Omega |\nabla \varphi|^p dx.$$

At this point applying Young's inequality we obtain the Poincaré-Sobolev inequality (1.5) with some $A = A(p) > 0$. □

Finally, we prove Theorem 1.8.

Proof of Theorem 1.8. By Theorem 1.2(ii) we can find a solution $u \in W_0^{1,p}(\Omega)$ to (1.1) such that both $e^u - 1$ and $e^{\frac{u}{p-1}} - 1 \in W_0^{1,p}(\Omega)$. Thus if we define $v = e^{\frac{u}{p-1}}$ then it holds that $v - 1 \in W_0^{1,p}(\Omega)$ and $v^{p-1} = e^u \in W^{1,p}(\Omega)$. We will show that v is indeed a solution of (1.3).

We first observe that the function $e^u |\nabla u|^p$ belongs to $L^1(\Omega)$. Indeed,

$$\begin{aligned} \int_\Omega e^u |\nabla u|^p dx &= \int_{\{u \ge 0\} \cap \Omega} e^u |\nabla u|^p dx + \int_{\{u<0\} \cap \Omega} e^u |\nabla u|^p dx \\ &\le \int_{\{u \ge 0\} \cap \Omega} e^{pu} |\nabla u|^p dx + \int_{\{u<0\} \cap \Omega} |\nabla u|^p dx \\ &\le \int_\Omega |\nabla (e^u)|^p dx + \int_\Omega |\nabla u|^p dx < +\infty. \end{aligned}$$

Let $\varphi \in C_c^\infty(\Omega)$. Using $\phi_j := \varphi \min\{e^u, j\}$, $j > 0$, as a test function for (1.1) we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi_j dx = \int_\Omega |\nabla u|^p \phi_j dx + \langle \sigma, \phi_j \rangle. \tag{6.1}$$

We now send $j \nearrow \infty$ in (6.1) to obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla (\varphi e^u) dx = \int_\Omega |\nabla u|^p \varphi e^u dx + \langle \sigma, \varphi e^u \rangle.$$

Here we use $e^u |\nabla u|^p \in L^1(\Omega)$ and Lebesgue's Dominated Convergence Theorem. We note that actually by Lemma 2.2 we can immediately use φe^u as a test function. Thus after expanding and simplifying we get

$$\int_\Omega [|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi] e^u dx = \langle \sigma, \varphi e^u \rangle = \langle \sigma, \varphi v^{p-1} \rangle.$$

Note that $\nabla v = (p-1)^{-1} e^{\frac{u}{p-1}} \nabla u$ and thus $\nabla u = (p-1) e^{-\frac{u}{p-1}} \nabla v$. This yields that

$$(|\nabla u|^{p-2} \nabla u) e^u = (p-1)^{p-1} |\nabla v|^{p-2} \nabla v,$$

and hence

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi dx = (p-1)^{1-p} \langle \sigma, \varphi v^{p-1} \rangle$$

for all $\varphi \in C_c^{\infty}(\Omega)$. This shows that v is a solution of (1.3) as claimed.

Finally, inequality (1.10) follows from (1.5) and the equality $|\frac{\nabla v}{v}|^p = (p-1)^{-p} |\nabla u|^p$. □

Remark 6.1. *The above argument also works for the more general equation*

$$-\mathrm{div}\, \mathcal{A}(x, \nabla v) = (p-1)^{1-p}\, \sigma\, v^{p-1} \text{ in } \Omega, \qquad v \geq 0 \text{ in } \Omega, \qquad v = 1 \text{ on } \partial\Omega,$$

where $\mathcal{A}(x, \xi)$ *satisfies* (2.1)-(2.3) *with* $0 < \alpha_0 \leq a_0$ *and the homogeneity condition*

$$\mathcal{A}(x, t\xi) = t^{p-1} \mathcal{A}(x, \xi) \qquad \text{for all } t > 0.$$

In this case $v = e^{\frac{u}{p-1}}$, *where* $u \in W_0^{1,p}(\Omega)$ *solves the equation*

$$-\mathrm{div}\, \mathcal{A}(x, \nabla u) = \mathcal{A}(x, \nabla u) \cdot \nabla u + \sigma.$$

By Theorem 2.3, to guarantee that both $e^u - 1$ *and* $e^{\frac{u}{p-1}} - 1 \in W_0^{1,p}(\Omega)$, *we also need to assume*

$$\lambda \in \left(0,\ a_0^{1-p} \alpha_0^p\, (p-1)^{p-1}\right) \quad \text{if} \quad \frac{a_0}{\alpha_0} \geq p-1$$

and

$$\lambda \in (0,\ \alpha_0 p - a_0) \quad \text{if} \quad \frac{a_0}{\alpha_0} < p-1.$$

However, note that no regularity assumption in the x*-variable of* $\mathcal{A}(x, \xi)$ *is needed here.*

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