

LIEB–THIRRING INEQUALITIES ON THE SPHERE

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ABSTRACT. We prove on the sphere \mathbb{S}^2 the Lieb–Thirring inequalities for orthonormal families of scalar and vector functions both on the whole sphere and on proper domains on \mathbb{S}^2 . By way of applications we obtain an explicit estimate for the dimension of the attractor of the Navier–Stokes system on a domain on the sphere with Dirichlet non-slip boundary conditions.

1. INTRODUCTION

The Schrödinger operator in $L_2(\mathbb{R}^d)$

$$-\Delta - V$$

with a real-valued potential V sufficiently fast decaying at infinity has a discrete negative spectrum satisfying the Lieb–Thirring spectral inequalities [27], [9], [25]

$$\sum_{\lambda_i \leq 0} |\lambda_i|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+(x)^{\gamma+d/2} dx, \quad (1.1)$$

where $V_+(x) = (|V(x)| + V(x))/2$ is the positive part of V .

For $\gamma \geq 3/2$ the sharp value of the Lieb–Thirring constants $L_{\gamma,d}$ was found in [24]:

$$L_{\gamma,d} = L_{\gamma,d}^{\text{cl}} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\gamma dx = \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2} \Gamma(d/2 + \gamma + 1)}. \quad (1.2)$$

For $1 \leq \gamma < 3/2$ the best known estimate of $L_{\gamma,d}$ was obtained in [6]:

$$L_{\gamma,d} \leq \frac{\pi}{\sqrt{3}} \cdot L_{\gamma,n}^{\text{cl}}, \quad \frac{\pi}{\sqrt{3}} = 1.8138\dots \quad (1.3)$$

The proof of (1.2) is essentially based on the following two ingredients, while the third one, in addition, is essential for the proof of (1.3):

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- 1). Sharp one-dimensional Lieb–Thirring estimates for the Schrödinger operators on \mathbb{R} with matrix-valued potentials for $\gamma \geq 3/2$ [24].
- 2). The Cartesian coordinates in \mathbb{R}^d , which make it possible to use the lifting argument with respect to dimensions [23], [24].
- 3). One-dimensional Lieb–Thirring estimates for the Schrödinger operators on \mathbb{R} with matrix-valued potentials for $\gamma = 1$, or, equivalently, one-dimensional generalized Sobolev inequality for traces of matrices [6], [7].

We are interested in this work in Lieb–Thirring inequalities on the sphere. Lieb–Thirring inequalities were generalized in [11], [33] to higher-order operators and Riemannian manifolds, however, the method involves extension operators for Sobolev spaces and therefore no information is available on the corresponding constants.

Explicit bounds for the Lieb–Thirring constants on the sphere were found in [14] by means of the Birman–Schwinger kernel, the constant for \mathbb{S}^2 and $\gamma = 1$ was improved in [18]. The same bound for the two-dimensional torus (with equal periods) was also obtained in [18].

The case of multi-dimensional anisotropic torus \mathbb{T}^d (a torus with different periods) was studied in [20]. The natural condition that the functions have zero mean value over the whole torus was replaced by a stronger condition that the functions have zero mean value over the shortest period uniformly for all remaining coordinates. Under this condition it was shown that independently of the ratios of the periods it holds

$$L_{\gamma,d}(\mathbb{T}^d) \leq \left(\frac{\pi}{\sqrt{3}}\right)^d \cdot L_{\gamma,n}^{\text{cl}}, \quad \gamma \geq 1. \quad (1.4)$$

Here we clearly have the Cartesian coordinates on \mathbb{T}^d , and, furthermore, one-dimensional Lieb–Thirring estimates for the Schrödinger operators with matrix-valued potentials in the periodic case [19], [20]. However, sharp semiclassical Lieb–Thirring inequalities for $\gamma \geq 3/2$ are not known to hold for the periodic boundary conditions even for scalar potentials, which leads, unlike (1.3), to the accumulation along with the dimension of the factor $\pi/\sqrt{3}$ in (1.4).

As far as the sphere is concerned, none of the three points mentioned above is available.

In this work we consider Lieb–Thirring inequalities in the dual formulation for orthonormal families on the two-dimensional sphere \mathbb{S}^2 and the torus \mathbb{T}^2 (with equal periods) by using the method of [30], [31] (see also [8], [9]). In the case of \mathbb{R}^d this approach does not give the best to date estimates

of the constants obtained in [6], however, as we shall see, this approach is very general and flexible, and works nicely in the case of the sphere producing rather good constants in the Lieb–Thirring inequalities. We restrict ourselves to the most important case of \mathbb{S}^2 and consider the Lieb–Thirring inequalities in the following three settings and prove in Section 2 that:

for an orthonormal family $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{S}^2)$ on the whole sphere with $\int_{\mathbb{S}^2} \psi_j dS = 0$ it holds

$$\int_{\mathbb{S}^2} \left(\sum_{j=1}^N \psi_j(s)^2 \right)^2 dS \leq \frac{9}{4\pi} \sum_{j=1}^N \|\nabla \psi_j\|^2; \quad (1.5)$$

for an orthonormal family of vector functions $\{u_j\}_{j=1}^N \in H_0^1(\Omega, T\mathbb{S}^2)$ with supports in a domain $\Omega \subseteq \mathbb{S}^2$

$$\int_{\Omega} \left(\sum_{j=1}^N |u_j(s)|^2 \right)^2 dS \leq \frac{9}{2\pi} \sum_{j=1}^N (\|\operatorname{div} u_j\|^2 + \|\operatorname{rot} u_j\|^2), \quad (1.6)$$

where the constant goes over to $9/(4\pi)$ if $\operatorname{div} u_j = 0$ (or $\operatorname{rot} u_j = 0$);

for an orthonormal family $\{\psi_j\}_{j=1}^N \in H_0^1(\Omega)$, $\Omega \subset \mathbb{S}^2$

$$\int_{\Omega} \left(\sum_{j=1}^N \psi_j(s)^2 \right)^2 dS \leq \frac{2}{\pi} \frac{4\pi + |\Omega|}{4\pi - |\Omega|} \sum_{j=1}^N \|\nabla \psi_j\|^2. \quad (1.7)$$

We observe that the constants in (1.5),(1.6) significantly improve the previously known estimate of the constants [18], which was $3/2$. Also, the constant in (1.7) blows up as $\Omega \rightarrow \mathbb{S}^2$, since on the whole sphere the Lieb–Thirring inequality cannot hold without the exclusion of the zero mode. Sharp estimates for the first eigenvalue of the Schrödinger operator on \mathbb{S}^{d-1} when the zero mode is important were obtained in [5].

In the same framework we consider in the end of Section 2 the two-dimensional square torus \mathbb{T}^2 and show that for an orthonormal family with orthogonality to constants it holds

$$\int_{\mathbb{T}^2} \left(\sum_{j=1}^N \psi_j(x)^2 \right)^2 dx \leq \frac{6}{\pi^2} \sum_{j=1}^N \|\nabla \psi_j\|^2. \quad (1.8)$$

In Section 3 we give applications of the obtained results (namely, inequality (1.6)) to the Navier–Stokes system in a domain Ω on \mathbb{S}^2 with Dirichlet

boundary conditions. The system reads:

$$\begin{cases} \partial_t u + \nabla_u u + \nabla p = \nu \Delta u + g, \\ u|_{t=0} = u^0, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0. \end{cases} \quad (1.9)$$

In the case $\Omega \subset \mathbb{R}^2$ we clearly have $\nabla_v u = \sum_{i=1}^2 v^i \partial_i u$, while on the sphere $\nabla_v u$ is the covariant derivative of u along v , and Δ is the vector Laplace–de Rham operator.

There is a vast literature devoted to the analysis of the long time behaviour of the solutions of this system described in terms of global attractors, see, for instance, [1], [33] and the references therein. A global attractor \mathcal{A} is a compact strictly invariant set which attracts bounded sets in the corresponding phase space as $t \rightarrow \infty$. An important geometric characterization of the attractor is its finite (Hausdorff and fractal) dimension. For more than thirty years the estimate in [33]

$$\dim_F \mathcal{A} \leq c(\Omega) \frac{\|f\| |\Omega|}{\nu^2}, \quad \Omega \subset \mathbb{R}^2 \quad (1.10)$$

remains the best available estimate in terms of the physical parameters: the viscosity coefficient ν , the size of the domain $|\Omega| = \operatorname{meas}(\Omega)$, and the magnitude of the forcing term $\|f\| = \|f\|_{L_2(\Omega)}$. We also point of the important paper [26] in this connection. Finally, this estimate was written in an explicit form in [17]:

$$\dim_F \mathcal{A} \leq \frac{1}{4\pi 3^{1/4}} \frac{\|f\| |\Omega|}{\nu^2} = 0.060 \dots \frac{\|f\| |\Omega|}{\nu^2}, \quad \Omega \subset \mathbb{R}^2. \quad (1.11)$$

We show in Section 3 that in the case of $\Omega \subset \mathbb{S}^2$ the estimate of the type (1.10) still holds, and, furthermore, can also be written in an explicit form

$$\dim_F \mathcal{A} \leq \frac{3}{2(2\pi)^{3/2}} \frac{\|f\| |\Omega|}{\nu^2} = 0.095 \dots \frac{\|f\| |\Omega|}{\nu^2}, \quad \Omega \subset \mathbb{S}^2. \quad (1.12)$$

To prove (1.12), in addition to (1.6) we use the following Li–Yau type bound

$$\sum_{k=1}^n \lambda_k \geq \frac{2\pi}{|\Omega|} n^2 \quad (1.13)$$

for the eigenvalues λ_k of the Stokes problem on $\Omega \subset \mathbb{S}^2$:

$$\begin{aligned} -\Delta u_j + \nabla p_j &= \lambda_j u_j, \\ \operatorname{div} u_j &= 0, \quad u_j|_{\partial\Omega} = 0, \end{aligned} \quad (1.14)$$

which was recently obtained in [21] and is exactly the same as in the case $\Omega \subset \mathbb{R}^2$ [17].

2. LIEB–THIRRING INEQUALITIES ON \mathbb{S}^2 AND \mathbb{T}^2

We first recall the basic facts concerning the Laplace operator on the sphere \mathbb{S}^2 (see, for instance, [32]). We have for the (scalar) Laplace–Beltrami operator $\Delta = \operatorname{div} \nabla$:

$$-\Delta Y_n^k = n(n+1)Y_n^k, \quad k = 1, \dots, 2n+1, \quad n = 0, 1, 2, \dots \quad (2.1)$$

Here the Y_n^k are the orthonormal real-valued spherical harmonics and each eigenvalue $\Lambda_n := n(n+1)$ has multiplicity $2n+1$.

The following identity is essential in what follows [28], [32]: for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} Y_n^k(s)^2 = \frac{2n+1}{4\pi}. \quad (2.2)$$

This identity, in turn, follows from the Laplace addition theorem for the spherical harmonics on \mathbb{S}^2 :

$$\sum_{k=1}^{2n+1} Y_n^k(s)Y_n^k(s_0) = \frac{2n+1}{4\pi} P_n(s \cdot s_0), \quad (2.3)$$

where $P_n(x)$, $x \in [-1, 1]$ are the classical Legendre polynomials.

The scalar case. We first consider the scalar case. We exclude the zero mode and introduce the following notation labelling the eigenfunctions and the corresponding eigenvalues with a single subscript counting multiplicities: $-\Delta y_i = \lambda_i y_i$, where

$$\{y_i\}_{i=1}^\infty = \{Y_n^1, \dots, Y_n^{2n+1}\}_{n=1}^\infty, \quad \{\lambda_i\}_{i=1}^\infty = \{\Lambda_n, \dots, \Lambda_n\}_{n=1}^\infty. \quad (2.4)$$

2n+1 times

Theorem 2.1. *Let $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{S}^2)$ be an orthonormal family of scalar functions with zero average: $\int_{\mathbb{S}^2} \psi_j(s) dS = 0$. Then $\rho(s) := \sum_{j=1}^N \psi_j(s)^2$ satisfies the inequality*

$$\int_{\mathbb{S}^2} \rho(s)^2 dS \leq \frac{9}{4\pi} \sum_{j=1}^N \|\nabla \psi_j\|^2. \quad (2.5)$$

Proof. For $E \geq 0$ we define the spectral projections

$$P_E = \sum_{\lambda_j < E} (\cdot, y_j) y_j \quad \text{and} \quad P_E^\perp = \sum_{\lambda_j \geq E} (\cdot, y_j) y_j.$$

We denote by $\dot{\Delta}$ the Laplace operator restricted to the invariant subspace of functions orthogonal to constants, that is, $\dot{\Delta} = P_2^\perp \Delta P_2^\perp$. Then

$$-\dot{\Delta} = \sum_{j=1}^{\infty} \lambda_j(\cdot, y_j) y_j.$$

Next, since

$$\sum_{j=1}^{\infty} \lambda_j a_j = \lambda_1(a_1 + a_2 + \dots) + (\lambda_2 - \lambda_1)(a_2 + a_3 + \dots) + \dots = \int_0^\infty \sum_{\lambda_j \geq E} a_j dE,$$

it follows that

$$-\dot{\Delta} = \int_0^\infty \sum_{\lambda_j \geq E} (\cdot, y_j) y_j dE = \int_0^\infty P_E^\perp dE. \quad (2.6)$$

Given the orthonormal family $\{\psi_j\}_{j=1}^N$, let Γ be the finite rank orthogonal projection

$$\Gamma = \sum_{j=1}^N (\cdot, \psi_j) \psi_j.$$

Then $\text{Tr}(-\dot{\Delta})\Gamma = \text{Tr}(-\Delta)\Gamma = \sum_{j=1}^N \|\nabla \psi_j\|^2$, which is on the right-hand side in (2.5). On the other hand, by the cyclic property of the trace and (2.6) we have

$$\begin{aligned} \sum_{j=1}^N \|\nabla \psi_j\|^2 &= \text{Tr}(-\dot{\Delta})\Gamma = \text{Tr} \Gamma(-\dot{\Delta})\Gamma = \\ &= \int_0^\infty \text{Tr} \Gamma P_E^\perp \Gamma dE = \int_0^\infty \text{Tr} P_E^\perp \Gamma P_E^\perp dE = \\ &= \int_0^\infty \int_{\mathbb{S}^2} \rho_{P_E^\perp \Gamma P_E^\perp}(s) dS dE = \int_{\mathbb{S}^2} \int_0^\infty \rho_{P_E^\perp \Gamma P_E^\perp}(s) dE dS, \end{aligned} \quad (2.7)$$

where

$$\rho_{P_E^\perp \Gamma P_E^\perp}(s) = \sum_{j=1}^N \|P_E^\perp \psi_j\|^2 \psi_j(s)^2.$$

Now let B be a spherical cap around a point $a \in \mathbb{S}^2$ with area $|B|$ and let χ_B be the corresponding characteristic function. Also, let $E \geq 2 (= \lambda_1 = \Lambda_1)$.

Then denoting by $\|\cdot\|_{\text{HS}}$ the Hilbert–Schmidt norm we obtain

$$\begin{aligned} \left(\int_B \rho(s) dS\right)^{1/2} &= \|\Gamma \chi_B\|_{\text{HS}} \leq \|\Gamma P_E \chi_B\|_{\text{HS}} + \|\Gamma P_E^\perp \chi_B\|_{\text{HS}} = \\ &= \|\Gamma P_E \chi_B\|_{\text{HS}} + \left(\int_B \rho_{P_E^\perp \Gamma P_E^\perp}(s) dS\right)^{1/2}. \end{aligned} \quad (2.8)$$

Using that $\|\Gamma\| = 1$ and then the fact that both χ_B and P_E are projections, we find that

$$\begin{aligned} \|\Gamma P_E \chi_B\|_{\text{HS}}^2 &\leq \|P_E \chi_B\|_{\text{HS}}^2 = \sum_{\lambda_j < E} \int_{\mathbb{S}^2} y_j(s)^2 \chi_B(s) dS = \\ &= \sum_{n=1}^{n(n+1) < E} \int_{\mathbb{S}^2} \sum_{k=1}^{2n+1} Y_n^k(s)^2 \chi_B(s) dS = \frac{1}{4\pi} |B| \sum_{n=1}^{n(n+1) < E} (2n+1), \end{aligned} \quad (2.9)$$

where we used the key identity (2.2) at the last step.

Let $n(E) = (\sqrt{1+4E}-1)/2$ be the positive root of the quadratic equation $n(n+1) = E$ and let $[n(E)]$ be the integer part of it. We estimate the sum on the right-hand side

$$\begin{aligned} \sum_{n=1}^{n(n+1) < E} (2n+1) &= ([n(E)] + 1)^2 - 1 \leq (n(E) + 1)^2 - 1 = \\ &= \frac{4E + 2\sqrt{1+4E} - 2}{4} \leq E \max_{E \geq 2} \frac{4E + 2\sqrt{1+4E} - 2}{4E} = \frac{3}{2}E. \end{aligned}$$

Substituting this into (2.8) and letting $|B| \rightarrow 0$ we obtain

$$\rho(a)^{1/2} \leq C^{1/2} E^{1/2} + \rho_{P_E^\perp \Gamma P_E^\perp}(a)^{1/2}, \quad C = \frac{3}{8\pi}.$$

Since $\rho_{P_E^\perp \Gamma P_E^\perp} \geq 0$ we have actually shown that

$$\rho_{P_E^\perp \Gamma P_E^\perp}(a) \geq (\rho(a)^{1/2} - C^{1/2} E^{1/2})_+^2. \quad (2.10)$$

We derived this lower bound under the condition that $E \geq 2$. However, since $P_E^\perp = I$ for $0 \leq E < 2$ and $\rho_{P_E^\perp \Gamma P_E^\perp} = \rho$, lower bound (2.10) holds for all $E \geq 0$. Integrating with respect to E we obtain

$$\int_0^\infty \rho_{P_E^\perp \Gamma P_E^\perp}(a) dE \geq \int_0^\infty (\rho(a)^{1/2} - C^{1/2} E^{1/2})_+^2 dE = \frac{1}{6C} \rho(a)^2 = \frac{4\pi}{9} \rho(a)^2.$$

Finally, integration over \mathbb{S}^2 in (2.7) completes the proof:

$$\sum_{j=1}^N \|\nabla \psi_j\|^2 \geq \frac{4\pi}{9} \int_{\mathbb{S}^2} \rho(s)^2 dS.$$

□

The vector case. In the vector case we have the identity for the gradients of spherical harmonics that is similar to (2.2) (see [13]): for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} |\nabla Y_n^k(s)|^2 = n(n+1) \frac{2n+1}{4\pi}. \quad (2.11)$$

This identity is essential for inequalities for vector functions on \mathbb{S}^2 . Substituting $\varphi(s) = Y_n^k(s)$ into the identity

$$\Delta \varphi^2 = 2\varphi \Delta \varphi + 2|\nabla \varphi|^2$$

we sum the results over $k = 1, \dots, 2n+1$. In view of (2.2) the left-hand side vanishes and we obtain (2.11) since the $Y_n^k(s)$'s are the eigenfunctions corresponding to Λ_n .

In the vector case we first define the Laplace operator acting on (tangent) vector fields on \mathbb{S}^2 as the Laplace–de Rham operator $-d\delta - \delta d$ identifying 1-forms and vectors. Then for a two-dimensional manifold (not necessarily \mathbb{S}^2) we have [12, 13]

$$\Delta u = \nabla \operatorname{div} u - \operatorname{rot} \operatorname{rot} u, \quad (2.12)$$

where the operators $\nabla = \operatorname{grad}$ and div have the conventional meaning. The operator rot of a vector u is a scalar and for a scalar ψ , $\operatorname{rot} \psi$ is a vector:

$$\operatorname{rot} u := -\operatorname{div}(n \times u), \quad \operatorname{rot} \psi := -n \times \nabla \psi,$$

where n is the unit outward normal vector, and in the local frame $n \times u = (-u_2, u_1)$.

Integrating by parts we obtain

$$(-\Delta u, u)_{L_2(T\mathbb{S}^2)} = \|\operatorname{rot} u\|^2 + \|\operatorname{div} u\|^2. \quad (2.13)$$

The vector Laplacian has a complete in $L_2(T\mathbb{S}^2)$ orthonormal basis of vector eigenfunctions. Using notation (2.4) we have

$$-\Delta w_j = \lambda_j w_j, \quad -\Delta v_j = \lambda_j v_j, \quad (2.14)$$

where

$$w_j = \lambda_j^{-1/2} n \times \nabla y_j, \quad \operatorname{div} w_j = 0, \quad v_j = \lambda_j^{-1/2} \nabla y_j, \quad \operatorname{rot} v_j = 0.$$

Hence, on \mathbb{S}^2 , corresponding to the eigenvalue $\Lambda_n = n(n+1)$, where $n = 1, 2, \dots$, there are two families of $2n+1$ orthonormal vector-valued eigenfunctions $w_n^k(s)$ and $v_n^k(s)$, where $k = 1, \dots, 2n+1$ and (2.11) gives the following important identities: for any $s \in \mathbb{S}^2$

$$\sum_{k=1}^{2n+1} |w_n^k(s)|^2 = \frac{2n+1}{4\pi}, \quad \sum_{k=1}^{2n+1} |v_n^k(s)|^2 = \frac{2n+1}{4\pi}. \quad (2.15)$$

We finally observe that since the sphere is simply connected, $-\Delta$ is strictly positive $-\Delta \geq \Lambda_1 I = 2I$.

Theorem 2.2. *Let $\{u_j\}_{j=1}^N \in H^1(T\mathbb{S}^2)$ be an orthonormal family of vector fields in $L^2(T\mathbb{S}^2)$. Then*

$$\int_{\mathbb{S}^2} \rho(s)^2 dS \leq \frac{9}{2\pi} \sum_{j=1}^N (\|\operatorname{rot} u_j\|^2 + \|\operatorname{div} u_j\|^2), \quad (2.16)$$

where $\rho(s) = \sum_{j=1}^N |u_j(s)|^2$. If, in addition, $\operatorname{div} u_j = 0$ (or $\operatorname{rot} u_j = 0$), then

$$\int_{\mathbb{S}^2} \rho(s)^2 dS \leq \frac{9}{4\pi} \cdot \begin{cases} \sum_{j=1}^N \|\operatorname{rot} u_j\|^2, & \operatorname{div} u_j = 0, \\ \sum_{j=1}^N \|\operatorname{div} u_j\|^2, & \operatorname{rot} u_j = 0. \end{cases} \quad (2.17)$$

Proof. We prove (2.16), the proof of (2.17) is similar. For $E \geq 0$ we set

$$P_E = \sum_{\lambda_j < E} ((\cdot, w_j)w_j + (\cdot, v_j)v_j), \quad P_E^\perp = \sum_{\lambda_j \geq E} ((\cdot, w_j)w_j + (\cdot, v_j)v_j).$$

Then as before

$$-\Delta = \int_0^\infty \sum_{\lambda_j \geq E} ((\cdot, w_j)w_j + (\cdot, v_j)v_j) dE = \int_0^\infty P_E^\perp dE.$$

The finite rank operator Γ is now $\Gamma = \sum_{j=1}^N (\cdot, u_j)u_j$, and in view of (2.13) equality (2.7) goes over to

$$\sum_{j=1}^N (\|\operatorname{div} u_j\|^2 + \|\operatorname{rot} u_j\|^2) = \operatorname{Tr}(-\Delta)\Gamma = \int_{\mathbb{S}^2} \int_0^\infty \rho_{P_E^\perp \Gamma P_E^\perp}(s) dE dS,$$

where

$$\rho_{P_E^\perp \Gamma P_E^\perp}(s) = \sum_{j=1}^N \|P_E^\perp u_j\|^2 |u_j(s)|^2.$$

Next, (2.8) is formally unchanged and (2.9) in view of (2.15) goes over to

$$\|P_E \chi_B\|_{\text{HS}}^2 = \sum_{\lambda_j < E} \int_{\mathbb{S}^2} (|w_j(s)|^2 + |v_j(s)|^2) \chi_B(s) dS = \frac{2}{4\pi} |B| \sum_{n=1}^{n(n+1) < E} (2n+1),$$

and we complete the proof as in Theorem 2.1 with $C = 3/(4\pi)$. \square

Lieb–Thirring inequality on domains on \mathbb{S}^2 . In conclusion we consider Lieb–Thirring inequalities for scalar functions in a domain on \mathbb{S}^2 with Dirichlet boundary conditions. Let $\{\psi_j\}_{j=1}^N \in H_0^1(\Omega)$ be an orthonormal family in $L_2(\Omega)$, where $\Omega \Subset \mathbb{S}^2$ is a proper domain on \mathbb{S}^2 with $|\Omega| < 4\pi$. Of course, the Lieb–Thirring inequality holds for this system, and we find below an estimate for the constant.

We extend the ψ_j 's by zero to the whole sphere (preserving the same notation). The functions ψ_j , however, do not necessarily have zero average. Therefore we consider the operator

$$A = -\Delta + aI, \quad a > 0.$$

on the whole space $L_2(\mathbb{S}^2)$ (without the orthogonality condition). Then $\text{infspec} A = a$. We also include the eigenvalue $\lambda_0 = \Lambda_0 = 0$ into (2.4), so that it goes over to

$$\{y_i\}_{i=0}^\infty = \{Y_n^1, \dots, Y_n^{2n+1}\}_{n=0}^\infty, \quad \{\lambda_i\}_{i=0}^\infty = \{\Lambda_n, \dots, \Lambda_n\}_{n=0}^\infty. \quad (2.18)$$

2n+1 times

Following the proof of Theorem 2.1 for $E \geq 0$ the projections are now

$$P_E = \sum_{\lambda_j + a < E} (\cdot, y_j) y_j \quad \text{and} \quad P_E^\perp = \sum_{\lambda_j + a \geq E} (\cdot, y_j) y_j.$$

Then

$$A = \sum_{j=0}^\infty (\lambda_j + a) (\cdot, y_j) y_j,$$

and

$$A = \int_0^\infty \sum_{\lambda_j + a \geq E} (\cdot, y_j) y_j dE = \int_0^\infty P_E^\perp dE. \quad (2.19)$$

As in Theorem 2.1 the operator $\Gamma = \sum_{j=1}^N (\cdot, \psi_j) \psi_j$ and (2.7) goes over to

$$\sum_{j=1}^N (\|\nabla \psi_j\|^2 + a \|\psi_j\|^2) = \text{Tr} A \Gamma = \int_{\mathbb{S}^2} \int_0^\infty \rho_{P_E^\perp \Gamma P_E^\perp}(s) dE dS.$$

Inequality (2.8) is formally as before and (2.9) becomes

$$\begin{aligned} \|P_E \chi_B\|_{\text{HS}}^2 &= \sum_{\lambda_j + a < E} \int_{\mathbb{S}^2} y_j(s)^2 \chi_B(s) dS = \frac{1}{4\pi} |B| \sum_{n=0}^{n(n+1)+a < E} (2n+1) \leq \\ &\leq \frac{1}{4\pi} |B| (N(E, a) + 1)^2 = \frac{1}{4\pi} |B| \frac{(\sqrt{1+4(E-a)} + 1)^2}{4} \leq \frac{1}{4\pi} |B| E F(a), \end{aligned}$$

where $N(E, a)$ is the root of the equation $n(n+1) - (E-a) = 0$ and where

$$F(a) := \max_{E \geq a} \frac{(\sqrt{1+4(E-a)} + 1)^2}{4E} = \begin{cases} 1/a, & a \leq 1/2; \\ 4a/(4a-1), & a \geq 1/2. \end{cases} \quad (2.20)$$

We can now complete the argument as in Theorem 2.1 to obtain the inequality

$$\begin{aligned} \int_{\Omega} \rho(s)^2 dS &\leq \frac{3F(a)}{2\pi} \sum_{j=1}^N (\|\nabla \psi_j\|^2 + a\|\psi_j\|^2) \leq \\ &\leq \frac{3F(a)}{2\pi} (1 + a\lambda_1(\Omega)^{-1}) \sum_{j=1}^N \|\nabla \psi_j\|^2, \end{aligned} \quad (2.21)$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian in Ω , for which we have the following lower bound obtained in [21, Corollary 3.2]:

$$\lambda_1(\Omega) \geq \frac{2\pi}{|\Omega|} \left(1 - \frac{|\Omega|}{4\pi}\right).$$

Setting $a = 1$ (so that $F(a) = 4/3$) we have proved the following result.

Theorem 2.3. *Let Ω be a domain on \mathbb{S}^2 with $|\Omega| < |\mathbb{S}^2| = 4\pi$. Let $\{\psi_j\}_{j=1}^N \in H_0^1(\Omega)$ be an orthonormal family of scalar functions. Then $\rho(s) := \sum_{j=1}^N \psi_j(s)^2$ satisfies the inequality*

$$\int_{\Omega} \rho(s)^2 dS \leq k(\Omega) \sum_{j=1}^N \|\nabla \psi_j\|_{L^2(\Omega)}^2, \quad k(\Omega) \leq \frac{2}{\pi} \frac{4\pi + |\Omega|}{4\pi - |\Omega|}. \quad (2.22)$$

Remark 2.1. The constant in (2.22) blows up as $|\Omega| \rightarrow 4\pi$ as it should, since the Lieb–Thirring inequality for scalar functions cannot hold on the whole sphere without the exclusion on the zero mode. This does not happen in the vector case, since the vector Laplacian is positive definite, and for any $\Omega \subseteq \mathbb{S}^2$ and an orthonormal family $\{u_j\}_{j=1}^N \in H_0^1(\Omega, T\mathbb{S}^2)$ extension by zero shows that the corresponding Lieb–Thirring constants are uniformly bounded by the constant on the whole sphere.

Remark 2.2. In (2.22) we have just set $a = 1$, while of course

$$k(\Omega) \leq \frac{3}{2\pi} \min_{a \geq 1/2} F(a)(1 + a\lambda_1(\Omega)^{-1}).$$

Given (2.20) we can find the minimum explicitly. Avoiding writing a long expression we just write the asymptotic behaviour of our estimate of the constant in the two regimes:

$$k(\Omega) \rightarrow \frac{3}{2\pi} \text{ as } |\Omega| \rightarrow 0; \quad k(\Omega) \sim \frac{12}{4\pi - |\Omega|} \text{ as } |\Omega| \rightarrow 4\pi.$$

However, setting $a = 1/2$ in (2.21) we see that $k(\Omega) \leq 12/(4\pi - |\Omega|)$ for all $|\Omega| < 4\pi$. It is also worth pointing out that $3/(2\pi)$ is the estimate of the Lieb–Thirring constant in \mathbb{R}^2 by the approach of [30], [31], (see also [9]).

Lieb–Thirring inequality on the torus. The case of the square 2D torus $\mathbb{T}^2 = [0, 2\pi]^2$ is treated in exactly the same way as in Theorem 2.1. The orthonormal eigenfunctions of $-\Delta$ in the space of functions with zero average are

$$\frac{1}{2\pi} e^{ik \cdot x}, \quad k \in \mathbb{Z}_0^2 = \{k = (k_1, k_2) \in \mathbb{Z}^2, |k|^2 > 0\}$$

with eigenvalues $k_1^2 + k_2^2$. The first eigenvalue is 1 and for $E \geq 1$ the number $N(E)$ of eigenvalues less than or equal to E satisfies

$$N(E) \leq 4E. \tag{2.23}$$

In fact, it is well known that $\lim_{E \rightarrow \infty} N(E)/E = \pi$. Therefore it suffices to verify (2.23) for finitely many eigenvalues. We have $N(1) = 4$, $N(5) = 20$ and for other E we have the strict inequality in (2.23).

Theorem 2.4. *Let $\{\psi_j\}_{j=1}^N \in H^1(\mathbb{T}^2)$ be an orthonormal family of scalar functions with zero average: $\int_{\mathbb{T}^2} \psi_j(x) dx = 0$. Then $\rho(x) := \sum_{j=1}^N \psi_j(x)^2$ satisfies the inequality*

$$\int_{\mathbb{T}^2} \rho(x)^2 dx \leq \frac{6}{\pi^2} \sum_{j=1}^N \|\nabla \psi_j\|^2. \tag{2.24}$$

Proof. Up to (2.8) the proof is the same as in Theorem 2.1. Setting now $E \geq 1$, in view of (2.23) we have

$$\|P_E \chi_B\|_{\text{HS}}^2 = \frac{1}{4\pi^2} \sum_{k \in \mathbb{Z}_0^2, |k|^2 < E} \int_{\mathbb{T}^2} \chi_B(x) dx \leq \frac{1}{4\pi^2} N(E) |B| \leq \frac{1}{\pi^2} E |B|,$$

and we obtain the analog of (2.10) with $C = 1/\pi^2$. The completion of the proof is again exactly the same. \square

Remark 2.3. The case of multidimensional anisotropic torus with arbitrary periods was studied in [20] under a stronger orthogonality condition that the functions have zero average with respect to the shortest coordinate. Then, for example, in the 2D case the constant is $\pi/6 < 6/\pi^2$ and is independent of the aspect ratio.

Remark 2.4. It is also worth pointing out that the 2D Lieb–Thirring inequalities for one function, that is, for $N = 1$ turn into multiplicative inequality of the form

$$\|\psi\|_{L^4}^4 \leq k \|\psi\|^2 \|\nabla\psi\|^2,$$

where k is the corresponding Lieb–Thirring constant. In the context of the Navier–Stokes equations this inequality is called the Ladyzhenskaya inequality and is important both in the theory, and applications. For example, the estimate for k specifically for this inequality on the torus \mathbb{T}^2 in [10] is $1/(2\pi^2) + 1/(\sqrt{2}\pi) + 2 = 2.27\dots$, while (1.8) gives $6/\pi^2 = 0.60\dots$

3. NAVIER–STOKES SYSTEM IN A DOMAIN ON \mathbb{S}^2

We now consider the Navier–Stokes system in a domain $\Omega \subset \mathbb{S}^2$, see (1.9) and write it as an evolution equation

$$\partial_t u + \nu Au + B(u, u) = f, \quad u(0) = u_0, \quad (3.1)$$

in the phase space H , which is the completion in $L_2(\Omega, T\mathbb{S}^2)$ of the linear set of smooth divergence free tangent vector functions compactly supported in Ω . Let P be the corresponding orthogonal projection. Then $A = -P\Delta$ is the Stokes operator and $B(u, v) = P(\nabla_u v)$ is the bilinear term defined by duality as follows: for all $u, v, w \in H_0^1(\Omega, T\mathbb{S}^2) \cap H$

$$\langle Au, v \rangle = (\text{rot } u, \text{rot } v),$$

$$\langle B(u, v), w \rangle = \int_{\Omega} \nabla_{u(s)} v(s) \cdot w(s) dS =: b(u, v, w).$$

Equation (3.1) generates the semigroup of solution operators $S_t: H \rightarrow H$, $S_t u_0 = u(t)$ which has a compact global attractor $\mathcal{A} \Subset H$. The case of whole sphere (or a two-dimensional manifold without boundary) was studied in [12], [13]. The fact that Ω is not the whole sphere does not play a role here (see, however, Remark 3.2). The existence of the attractor is shown similarly to the case of a bounded domain in \mathbb{R}^2 with Dirichlet boundary conditions (see, for instance, [1], [33] for the case of a domain with smooth boundary and [22], [29] for a nonsmooth domain). The attractor \mathcal{A} is the maximal strictly invariant compact set.

Theorem 3.1. *The fractal dimension of \mathcal{A} satisfies the estimate*

$$\dim_F \mathcal{A} \leq \frac{3}{2(2\pi)^{3/2}} \frac{\|f\| |\Omega|}{\nu^2}. \quad (3.2)$$

Proof. Following the general scheme we have to estimate the m -trace of the semigroup linearized on the solution lying on the attractor [33]. The semigroup S_t is uniformly differentiable in H with differential $L(t, u_0): \xi \rightarrow U(t) \in H$, where $U(t)$ is the solution of the variational equation

$$\partial_t U = -\nu AU - B(U, u(t)) - B(u(t), U) =: \mathcal{L}(t, u_0)U, \quad U(0) = \xi. \quad (3.3)$$

We estimate the numbers $q(m)$ (the sums of the first m global Lyapunov exponents):

$$q(m) := \limsup_{t \rightarrow \infty} \sup_{u_0 \in \mathcal{A}} \sup_{\{v_j\}_{j=1}^m} \frac{1}{t} \int_0^t \sum_{j=1}^m (\mathcal{L}(\tau, u_0)v_j, v_j) d\tau, \quad (3.4)$$

where $\{v_j\}_{j=1}^m \in H_0^1(\Omega, T\mathbb{S}^2) \cap H$ is an arbitrary orthonormal system (see [1], [4], [33]). Using below the well-known identity $b(u, v, v) = 0$ and the estimate

$$\sum_{j=1}^m b(v_j, u, v_j) \leq 2^{-1/2} \|\rho\| \|\operatorname{rot} u\|,$$

where $\rho(s) = \sum_{j=1}^m |v_j(s)|^2$, see [16, Lemma 3.2], we obtain also using (2.13)

$$\begin{aligned} \sum_{j=1}^m (\mathcal{L}(t, u_0)v_j, v_j) &= -\nu \sum_{j=1}^m \|\operatorname{rot} v_j\|^2 - \sum_{j=1}^m b(v_j, u(t), v_j) \leq \\ &\leq -\nu \sum_{j=1}^m \|\operatorname{rot} v_j\|^2 + 2^{-1/2} \|\rho\| \|\operatorname{rot} u(t)\| \leq \\ &\leq -\nu \sum_{j=1}^m \|\operatorname{rot} v_j\|^2 + 2^{-1/2} \left(c_{\text{LT}} \sum_{j=1}^m \|\operatorname{rot} v_j\|^2 \right)^{1/2} \|\operatorname{rot} u(t)\| \leq \\ &\leq -\frac{\nu}{2} \sum_{j=1}^m \|\operatorname{rot} v_j\|^2 + \frac{c_{\text{LT}}}{4\nu} \|\operatorname{rot} u(t)\|^2 \leq -\frac{\nu c_{\text{LY}} m^2}{2|\Omega|} + \frac{c_{\text{LT}}}{4\nu} \|\operatorname{rot} u(t)\|^2. \end{aligned}$$

Here we used the Lieb–Thirring inequality (2.17) with $c_{\text{LT}} \leq 9/(4\pi)$ and inequality (1.13), which gives by the variational principle that for the orthonormal family $\{v_j\}_{j=1}^m \in H_0^1(\Omega, T\mathbb{S}^2) \cap H$ it holds

$$\sum_{k=1}^m \|\operatorname{rot} v_k\|^2 \geq \sum_{k=1}^m \lambda_k \geq \frac{c_{\text{LY}} m^2}{|\Omega|}, \quad c_{\text{LY}} = 2\pi, \quad (3.5)$$

where λ_k are the eigenvalues of the Stokes operator (1.14). Using the well-known estimate [1], [33]

$$\limsup_{t \rightarrow \infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t \|\operatorname{rot} u(\tau)\|^2 d\tau \leq \frac{\|f\|^2}{\lambda_1 \nu^2},$$

for the solutions lying on the attractor we obtain for the numbers $q(m)$:

$$q(m) \leq -\frac{\nu c_{LY} m^2}{2|\Omega|} + \frac{c_{LT} \|f\|^2}{4\lambda_1 \nu^3} \leq -\frac{\nu c_{LY} m^2}{2|\Omega|} + \frac{c_{LT} \|f\|^2 |\Omega|}{4c_{LY} \nu^3}.$$

It was shown in [4], [33] and in [2], [3], respectively, that both the Hausdorff and fractal dimensions of \mathcal{A} are bounded by the number m_* for which $q(m_*) = 0$. This gives that

$$\dim_F \mathcal{A} \leq \frac{1}{c_{LY}} \left(\frac{c_{LT}}{2} \right)^{1/2} \frac{\|f\| |\Omega|}{\nu^2}$$

and completes the proof since $c_{LY} = 2\pi$ and $c_{LT} \leq 9/(4\pi)$. \square

Remark 3.1. In terms of the physical parameters estimate (3.2) is the same as (1.11) in the case $\Omega \subset \mathbb{R}^2$, so that the (constant) curvature does not seem to play a role. Furthermore, the numerical coefficients are different only due to the fact that the current estimate for the Lieb–Thirring constant for orthonormal families of two-dimensional divergence free vector functions in $\Omega \subset \mathbb{R}^2$ is $1/(2\sqrt{3})$ (see [6], [17]), which is better than our estimate for the sphere $9/(4\pi)$, while, as mentioned above, the constants in the Li–Yau lower bounds for the Stokes operator in $\Omega \subset \mathbb{R}^2$ and $\Omega \subset \mathbb{S}^2$ are the same.

Remark 3.2. In the case when $\Omega = \mathbb{S}^2$ (or, more generally, when Ω is a 2D manifold without boundary) we have an additional important orthogonality relation

$$b(u, u, Au) = 0,$$

which is the same as in the 2D space periodic case. This makes it possible to improve the estimate of the dimension in terms of the physical parameters and obtain that as in the case of the torus \mathbb{T}^2 [33] it holds (see [13, 15])

$$\dim_F \mathcal{A} \leq c G^{2/3} (1 + \log G)^{1/3}, \quad G = \frac{\|f\| |\Omega|}{\nu^2}.$$

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REFERENCES

- [1] A. Babin and M. Vishik, *Attractors of evolution equations*. Studies in Mathematics and its Applications, 25. North-Holland Publishing Co., Amsterdam, 1992.
- [2] V.V.Chepyzhov and A.A.Ilyin, A note on the fractal dimension of attractors of dissipative dynamical systems, *Nonlinear Anal.* **44** (2001), 811–819.
- [3] V.V.Chepyzhov and A.A.Ilyin, On the fractal dimension of invariant sets; applications to Navier–Stokes equations. *Discrete Contin. Dyn. Syst.* **10**:1, 2 (2004), 117–135.
- [4] P.Constantin and C.Foias, Global Lyapunov exponents, Kaplan–Yorke formulas and the dimension of the attractors for the 2D Navier–Stokes equations. *Comm. Pure Appl. Math.* **38**:1 (1985), 1–27.
- [5] J.Dolbeault, M.J.Esteban, and A.Laptev, Spectral estimates on the sphere. *Anal. PDE* **7**:2 (2014), 435–460.
- [6] J.Dolbeault, A.Laptev, and M.Loss, Lieb–Thirring inequalities with improved constants. *J. European Math. Soc.* **10**:4 (2008), 1121–1126.
- [7] A.Eden and C.Foias, A simple proof of the generalized Lieb–Thirring inequalities in one space dimension. *J. Math. Anal. Appl.* **162** (1991), 250–254.
- [8] R.L.Frank, Cwikel’s theorem and the CLR inequality. *J. Spectr. Theory* **4** (2014), 1–21.
- [9] R.L.Frank, A.Laptev, and T.Weidl, *Lieb–Thirring inequalities*, book in preparation.
- [10] C.Foias, M.S.Jolly, and M.Yang, On single mode forcing of the 2D-NSE. *J. Dyn. Diff. Equat.* **25** (2013), 393–433.
- [11] J. M. Ghidaglia, M. Marion, and R. Temam, Generalization of the Sobolev–Lieb–Thirring inequalities and applications to the dimension of attractors. *Differential and Integral Equations* **1**:1 (1988), 1–21.
- [12] A.A.Ilyin, Navier–Stokes and Euler equations on two-dimensional closed manifolds. *Mat. Sb.* **181**:4 (1990), 521–539; English transl. in *Math. USSR-Sb.* **69**:2 (1991).
- [13] A.A.Ilyin, Partly dissipative semigroups generated by the Navier–Stokes system on two-dimensional manifolds and their attractors. *Mat. Sbornik* **184**, no. 1, 55–88 (1993) English transl. in *Russ. Acad. Sci. Sb. Math.* **78**, no. 1, 47–76 (1993).
- [14] A.A.Ilyin, Lieb–Thirring inequalities on the N -sphere and in the plane, and some applications. *Proc. London Math. Soc.* **67**, 159–182 (1993).
- [15] A.A.Ilyin, Navier–Stokes equations on the rotating sphere. A simple proof of the attractor dimension estimate. *Nonlinearity* **7** (1994), 31–39.
- [16] A.A.Ilyin, A.Miranville and E.S.Titi, Small viscosity sharp estimates for the global attractor of the 2-D damped-driven Navier–Stokes equations. *Commun. Math. Sci.* **2**:3 (2004), 403–426.
- [17] A.A.Ilyin, On the spectrum of the Stokes operator. *Funktsional. Anal. i Prilozhen.* **43**:4 (2009), 14–25; English transl. in *Funct. Anal. Appl.* **43**:4 (2009).
- [18] A.A.Ilyin, Lieb–Thirring inequalities on some manifolds. *J. Spectr. Theory* **2** (2012), 57–78.
- [19] A.Ilyin, A.Laptev, M.Loss, and S.Zelik, One-dimensional interpolation inequalities, Carlson–Landau inequalities and magnetic Schrödinger operators. *Int. Math. Res. Not.* **2016**:4 (2016), 1190–1222.

- [20] A.A. Ilyin and A.A. Laptev, Lieb–Thirring inequalities on the torus. *Mat. Sb.* **207**:10 (2016), 56–79; English transl. in *Sb. Math.* **207**:9–10 (2016).
- [21] A.A. Ilyin and A.A. Laptev, Berezin–Li–Yau inequalities on domains on the sphere. Submitted, arXiv:1712.10078.
- [22] O.A. Ladyzhenskaya, First boundary value problem for Navier–Stokes equations in domain with nonsmooth boundaries. *C. R. Acad. Sci. Paris, Ser. 1* **314**:4 (1992), 253–258.
- [23] A. Laptev, Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. *J. Funct. Anal.* **151** (1997), 531–545.
- [24] A. Laptev and T. Weidl, Sharp Lieb–Thirring inequalities in high dimensions. *Acta Math.* **184** (2000), 87–111.
- [25] A. Laptev and T. Weidl, Recent results on Lieb–Thirring inequalities. In *Journées Équations aux Dérivées Partielles (La Chapelle sur Erdre, 2000)*, Exp. No. XX, 14 pp. Univ. Nantes, Nantes, 2000.
- [26] E.H. Lieb, On characteristic exponents in turbulence. *Comm. Math. Phys.* **92** (1984), 473–480.
- [27] E. Lieb and W. Thirring, Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities, *Studies in Mathematical Physics. Essays in honor of Valentine Bargmann*, Princeton University Press, Princeton NJ, 269–303 (1976).
- [28] S.G. Mikhailin, *Linear partial differential equations*. Vyssh. Shkola, Moscow, 1977; German transl. *Partielle Differentialgleichungen in der mathematischen Physik*. Akademie-Verlag, Berlin, 1978.
- [29] R. Rosa, The global attractor for the 2D Navier–Stokes flow on some unbounded domains. *Nonlinear Anal.* **32**:1 (1998), 71–85.
- [30] M. Rumin, Spectral density and Sobolev inequalities for pure and mixed states. *Geom. Funct. Anal.* **20** (2010), 817–844.
- [31] M. Rumin, Balanced distribution-energy inequalities and related entropy bounds. *Duke Math. J.* **160** (2011), 567–597.
- [32] E.M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton NJ, 1972.
- [33] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer-Verlag, New York, 1997.

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