A method of induction the distances with Hilbert structure

Vesna Gotovac[a], Kateřina Helisova [b],
Lev B. Klebanov[c] and Irina V. Volchenkova[b d]

Abstract

A method of induction the distances with Hilbert structure is proposed. Some properties of the method are studied. Typical examples of corresponding metric spaces are discussed.

Key words: Hilbert spaces; metric spaces; isomtric embeddings into Hilbert spaces

1 Introduction

Let $\{\mathcal{X}, D\}$ be a metric space. We say D is a Hilbert-type distance if and only if there is an isometry from $\{\mathcal{X}, D\}$ on a subset of a Hilbert space. It is known (see [3]) this property is equivalent to negative definiteness of D^2. Namely, a real function $\mathcal{L}$ from $\mathcal{X}^2$ such that $\mathcal{L}(x_1, x_2) = \mathcal{L}(x_2, x_1)$ is called negative definite kernel if for arbitrary positive integer n and real numbers $c_1, \ldots, c_n$ satisfying to the condition $\sum_{j=1}^{n} c_j = 0$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{L}(x_i, x_j) c_i c_j \leq 0. \tag{1.1}$$

[a]Department of Mathematics, Faculty of Science, University of Split, 21000 Split, Croatia

[b]Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague 6, Czech Republic

[c]Department of Probability and Mathematical Statistics, Charles University, Sokolovska 83, 18675 Prague, Czech Republic, e-mail: levbkl@gmail.com

[d]The order of authors is alphabetical and has no other sense

That is, to proof that D is a Hilbert-type distance it is sufficient to verify $\mathcal{L} = D^2$ satisfies the relation (1.1). In the paper we provide a method of constructing Hilbert-type distance on a set $\mathcal{X}$ by using corresponding distance on image of $\mathcal{X}$ under a family of functions.

2 The method of defining Hilbert-type distances

Let $\mathcal{Z}$ be a metric space with a distance D on it. It is well-known that $\mathcal{Z}$ is isometric to a set of a Hilbert space if and only if $D^2(u, v)$ $(u, v \in Z)$ is negative definite kernel of $\mathcal{Z}^2$. Further on we suppose that D possesses this property. Let $\mathcal{X}$ be an abstract set, and let $f_y(.)$, $y \in \mathcal{Y}$ be a family of functions defined on $\mathcal{X}$ and taking values in $\mathcal{Z}$. Suppose that Ξ is a probability measure on $\mathcal{Y}$. Define

$$\rho(x_1, x_2) = \Big(\int_{\mathcal{Y}} D^2(f_y(x_1), f_y(x_2)) d\,\Xi(y) \Big)^{1/2}. \tag{2.1}$$

Our goal is to show that under some assumptions ρ is a distance on $\mathcal{X}$ such that ρ^2 is a negative definite kernel.

It is clear that for $x_1, x_2 \in \mathcal{X}$

A1. $\rho(x_1, x_2) \geq 0$;

A2. $\rho(x_1, x_2) = 0$ if and only if $f_y(x_1) = f_y(x_2)$ for Ξ-almost all $y \in \mathcal{Y}$;

A3. $\rho(x_1, x_2) = \rho(x_2, x_1)$.

Take arbitrary positive integer n and real numbers $c_1, \ldots, c_n$ satisfying to the condition $\sum_{j=1}^n c_j = 0$. For arbitrary $x_1, \ldots, x_n \in \mathcal{X}$ we have

$$\sum_{i=1}^n \sum_{j=1}^n \rho^2(x_i, x_j) c_i c_j = \int_{\mathcal{Y}} \Big(\sum_{i=1}^n \sum_{j=1}^n D^2(f_y(x_i), f_y(x_j)) c_i c_j \Big) d\,\Xi(y) \leq 0 \tag{2.2}$$

in view of negative definiteness of the kernel D^2. Therefore, ρ^2 is negative definite kernel on $\mathcal{X}$. From this fact it follows that ρ satisfies the triangle inequality (see, for example, [2]):

A4. $\rho(x_1, x_2) \leq \rho(x_1, x_3) + \rho(x_3, x_2)$ for all $x_1, x_2, x_3 \in \mathcal{X}$.

Theorem 2.1. *Let $\mathcal{Z}$ be a metric space with a distance D on it. Suppose that $D^2(u,v)$ $(u,v \in Z)$ is negative definite kernel of $\mathcal{Z}^2$. Let $\mathcal{X}$ be an abstract set, and let $f_y(.)$, $y \in \mathcal{Y}$ be a family of functions defined on $\mathcal{X}$ and taking values in $\mathcal{Z}$. Suppose that Ξ is a probability measure on $\mathcal{Y}$ such that*

$$f_y(x_1) = f_y(x_2) \text{ for all } y \in \texttt{supp}(\Xi) \text{ implies } \texttt{x}_1 = \texttt{x}_2, \tag{2.3}$$

where, as usual, $\texttt{supp}(\Xi)$ is support of the measure Ξ. Then the function ρ defined by (2.1) is a distance on $\mathcal{X}$. Metric space $(\mathcal{X}, \rho)$ is isometric to a subset of a Hilbert space.

Proof. Properties $A1. - A4.$ together with (2.3) show that ρ is a distance on $\mathcal{X}$. Because ρ^2 is a negative definite kernel the conclusion of the Theorem follows from I.J. Schoenberg's Theorem ([3], see also [2]). □

Example 2.1. *Let $\mathcal{X}$ be a subset of a vector space $\mathcal{H}$ and $\mathcal{Y}$ is a subspace of algebraic conjugate $\mathcal{X}'$. Suppose that on $\mathcal{X}'$ with a σ-field of its subsets there exists a measure Ξ such that*

$$\langle x', x_1 \rangle = \langle x', x_2 \rangle \text{ for all } x' \text{ implies } x_1 = x_2.$$

Then there exists a Hilbert-type distance on $\mathcal{X}$.

Proof. It is sufficient to apply Theorem 2.1 to $\mathcal{Y} = \mathcal{X}'$ and $D(u,v) = |u - v|$ for $u, v \in \mathcal{Z} = \mathbb{R}^1$. □

Let us note that the distance

$$\rho(x_1, x_2) = \Big(\int_{\mathcal{Y}} (\langle x', x_1 \rangle - \langle x', x_2 \rangle)^2 d\,\Xi(x') \Big)^{1/2}$$

on $\mathcal{X}$ induces a norm on $\mathcal{X}$. Namely,

$$\|x\| = \rho(x, 0),\ x \in \mathcal{X}.$$

Corresponding inner product may be calculated as

$$(x_1, x_2) = \int_{\mathcal{Y}} \langle x', x_1 \rangle \cdot \langle x', x_2 \rangle d\,\Xi(x').$$

Basing on this, we can say that $\mathcal{X}$ is isometric to linear subspace of a Hilbert space. However, we cannot state that $\mathcal{X}$ is complete.

The conditions of Example 2.1 are close to necessary. Really, let $\mathcal{X}$ be a subset of a separable Hilbert space $\mathfrak{H}$. We may take as $\mathcal{Y}$ the dual space of all continuous linear functionals on $\mathfrak{H}$. Of course, in this situation there exists corresponding measure Ξ possessing desirable properties. Namely, from the proof of the result of [1] it follows that we can choose Ξ as a Gaussian measure with $\mathtt{supp}(\Xi) = \mathfrak{H}$.

The facts given by Example 2.1 and after it show that the condition of existence of inner product on a linear space may be changed by the condition of existence of suitable measure Ξ on reach enough subset of the dual space.

3 The method of defining L^m-type distances

Let $\mathcal{X}$ be an abstract set, and m be an even integer greater than 1. Assume that $\mathcal{L}(x_1, \dots, x_m)$ is a real continuous function on $\mathcal{X}^m$ symmetric with respect to permutations of its arguments. We say that $\mathcal{L}$ is an *m-negative definite kernel* (see [4]) if for any integer $n \geq 1$, any collection of points $x_1, \dots, x_n \in \mathcal{X}$, and any collection of real numbers $h_1, \dots, h_n$ satisfying the condition $\sum_{j=1}^{n} h_j = 0$, the following inequality holds:

$$(-1)^{m/2} \sum_{i_1=1}^{n} \dots \sum_{i_m=1}^{n} \mathcal{L}(x_{i_1}, \dots, x_{i_m}) h_{i_1} \cdots h_{i_m} \geq 0. \tag{3.1}$$

For the case $m = 2$ we have the case of negative definite kernel. If the equality in (3.1) implies that $h_1 = \dots = h_n = 0$, then we call $\mathcal{L}$ *strictly m-negative definite kernel*. Equivalent form of (3.1) is

$$(-1)^{m/2} \int_{\mathcal{X}} \dots \int_{\mathcal{X}} \mathcal{L}(x_1, \dots, x_m) h(x_1) \cdots h(x_m) dQ(x_1) \dots dQ(x_m) \geq 0 \tag{3.2}$$

for any measure Q and any integrable function $h(x)$ such that

$$\int_{\mathcal{X}} h(x) dQ(x) = 0. \tag{3.3}$$

We call $\mathcal{L}$ *strongly m-negative definite kernel* if the equality in (3.3) holds for $h = 0$ Q-almost everywhere only.

Let $\mathcal{X}$ be an abstract set, and let $f_y(.)$, $y \in \mathcal{Y}$ be a family of functions defined on $\mathcal{X}$ and taking values in a set $\mathcal{Z}$. Suppose that $\mathcal{L}$ is m-negative definite kernel on $\mathcal{Z}^m$ and Ξ is a probability measure on $\mathcal{Y}$. Define

$$\mathfrak{R}_m(x_1, \ldots, x_m) = \int_{\mathcal{Y}} \mathcal{L}(f_y(x_1), \ldots, f_y(x_m)) d\,\Xi(y). \tag{3.4}$$

It is easy to see that $\mathfrak{R}_m$ is m-negative definite kernel on $\mathcal{X}^m$.

Assumption 1. *If*

$$(-1)^{m/2} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \mathcal{L}(f_y(x_{i_1}), \ldots, f_y(x_{i_m})) h_{i_1} \cdots h_{i_m} = 0$$

$$\textit{for } Q - \textit{ almost all } y \in \mathcal{Y}$$

implies that

$$(-1)^{m/2} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \mathcal{L}(x_{i_1}, \ldots, x_{i_m}) h_{i_1} \cdots h_{i_m} = 0$$

then strictly m-negativeness of $\mathcal{L}$ implies strictly m-negativeness of $\mathfrak{R}$.

Similarly statements with integrals instead of sums are true for strong negative definiteness. We omit precise formulation.

Suppose, as before, that m is an even integer greater than 1. Let D_m is a distance on $\mathcal{Z}$. From the results of [4] (see also [2]) it follows that $(\mathcal{Z}, D_m)$ is isometric to a subset of L^m space if and only if

$$D_m(u, v) = \Big((-1)^{m/2} \mathcal{L}(u - v, \ldots, u - v) \Big)^{1/m}, \quad u, v \in \mathcal{Z}, \tag{3.5}$$

where $\mathcal{L}(u_1, \ldots, u_m)$ is a strictly negative definite kernel on $\mathcal{Z}^m$. Therefore, under Assumption 1 and if $D_m(u, v)$ has the form (3.5) then

$$\rho_m(s, t) = \Big(\mathfrak{R}_m(s - t, \ldots, s - t) \Big)^{1/m}, \quad s, t \in \mathcal{X} \tag{3.6}$$

is a distance on $\mathcal{X}$, where L is strictly m-negative definite kernel used in (3.5). The set $\mathcal{X}$ with the distance ρ_m is isometric to a subset of L^m space.

4 Acknowledgment

The work was partially supported by Grant GACR 16-03708S.

References

[1] K. Ito (1970). *The Topological Support of Gauss Measure on Hilbert Space*, Nagoya Math. J. Vol. 38, 181–183.

[2] Lev B. Klebanov (2005). $\mathfrak{N}$-Distances and their Applications, *Charles University in Prague, The Karolinum Press*.

[3] I.J. Schoenberg (1938). *Metric spaces and positive definite functions*. Trans. Amer. Math. Soc., 44, 3, 552–563.

[4] A.A. Zinger, A.V. Kakosyan, L.B. Klebanov (1989). *Characterization of distributions by the mean values of statistics and some probability metrics*, Stability Problems of Stochastic Models, p.p. 47–55, Moscow: VNII Sistemnykh Isledovanii (in Russian).