

THE KRAFT-RUSSELL GENERIC EQUIVALENCE THEOREM AND ITS APPLICATION

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ABSTRACT. We find some extensions of the Kraft-Russell Generic Equivalence Theorem and using it we obtain a simple proof of a result of Dubouloz and Kishimoto.

1. INTRODUCTION

H. Kraft and P. Russell proved the following Generic Equivalence Theorem in [KrRu].

Theorem 1.1. *Let \mathbf{k} be a field and let $p : S \rightarrow Y$ and $q : T \rightarrow Y$ be two morphisms of \mathbf{k} -varieties. Suppose that*

- (a) \mathbf{k} is algebraically closed and of infinite transcendence degree over the prime field;
- (b) for all $y \in Y$ the two (schematic) fibers $S_y := p^{-1}(y)$ and $T_y := q^{-1}(y)$ are isomorphic; and
- (c) the morphisms p and q are affine.

Then there is a dominant morphism of finite degree $\varphi : X \rightarrow Y$ and an isomorphism $S \times_Y X = T \times_Y X$ over X .

The aim of this note is to establish the following facts:

- the assumption (c) is unnecessary;
- the conclusion of Theorem 1.1 remains valid if the assumption (c) is removed and (a) and (b) are replaced by the following assumptions (a1) \mathbf{k} is an uncountable (but not necessarily algebraically closed) field, (b1) there is a countable intersection W of Zariski open dense subsets of Y such that S_y and T_y are isomorphic for every $y \in W$;
- the conclusion of Theorem 1.1 remains valid if (a) and (c) are replaced by the assumptions (a2) \mathbf{k} is an algebraically closed field of finite transcendence degree over \mathbb{Q} and (c2) p and q are proper morphism.

Furthermore, using Minimal Model Program over non closed fields, Dubouloz and Kishimoto proved the following result [DuKi].

Theorem 1.2. *Let \mathbf{k} be an uncountable field of characteristic zero and let $f : X \rightarrow S$ be a dominant morphism between geometrically integral algebraic \mathbf{k} -varieties. Suppose that for general closed points $s \in S$, the fiber X_s contains an \mathbb{A}^1 -cylinder $U_s \simeq Z_s \times \mathbb{A}^1$ over a $\kappa(s)$ -variety Z_s . Then there exists an étale morphism $T \rightarrow S$ such that $X_T = X \times_S T$ contains an \mathbb{A}^1 -cylinder $U \simeq Z \times \mathbb{A}^1$ over a T -variety Z .*

We show by much simpler means that in the case, when \mathbf{k} is an algebraically closed field (of any characteristic) with an infinite transcendence degree over the prime field,

the Dubouloz-Kishimoto theorem is a simple consequence of the Kraft-Russell theorem.¹

2. ASSUMPTION (C)

The main result of this section (Theorem 2.5) is a straightforward adjustment of the argument in [KrRu] (known to Russell and Kraft) but we provide it for convenience of readers.

Notation 2.1. We suppose that $\rho : X \rightarrow Y$ is a dominant morphism of algebraic \mathbf{k} -varieties where \mathbf{k} is an algebraically closed field with an infinite transcendence degree over its prime field. Recall that there is a field $\mathbf{k}_0 \subset \mathbf{k}$ which is finitely generated over the prime field and a morphism $\rho_0 : X_0 \rightarrow Y_0$ of \mathbf{k}_0 -varieties such that the morphism $\rho : X \rightarrow Y$ is obtained from ρ_0 by the base extension $\text{Spec } \mathbf{k} \rightarrow \text{Spec } \mathbf{k}_0$. Denote by \mathbf{K}_0 the field of rational functions on Y_0 , i.e. $\text{Spec } \mathbf{K}_0$ is the generic point of Y_0 . Put $X_{0,\omega} = X_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{K}_0$.

The next fact was proven in [KrRu, Lemma 1]) (but unfortunately under the additional unnecessary assumption that X is affine).

Lemma 2.2. *Let Notation 2.1 hold. Then every \mathbf{k}_0 -embedding $\mathbf{K}_0 \hookrightarrow \mathbf{k}$ defines a closed point $y \in Y$ and an isomorphism*

$$X_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \rightarrow X_y = \rho^{-1}(y).$$

Proof. Without loss of generality we suppose that Y is affine. Let $\mathbf{k}_0[Y_0]$ be the algebra of regular functions on Y_0 . Following [KrRu, Lemma 1] we see that since $\mathbf{k}_0[Y_0] \subset \mathbf{K}_0$ any \mathbf{k}_0 -embedding $\mathbf{K}_0 \hookrightarrow \mathbf{k}$ yields a \mathbf{k}_0 -homomorphism $\mathbf{k}[Y] = \mathbf{k}_0[Y_0] \otimes_{\mathbf{k}_0} \mathbf{k} \rightarrow \mathbf{k}$ and, thus, a closed point y in Y . Let U_0 be a Zariski dense open affine subset of X_0 , $U_{0,\omega} = U_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{K}_0$, $U = U_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$ and U_y be the fiber over y of the restriction $U \rightarrow Y$ of ρ . Continuing the argument of Kraft and Russell we have

$$(1) \quad U_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \simeq U_0 \times_{Y_0} \text{Spec } \mathbf{k} \simeq U \times_Y \text{Spec } \mathbf{k} = U_y.$$

Furthermore, if V_0 is a Zariski open affine subset of U_0 then the way the isomorphism $U_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \simeq U_y$ was constructed in Formula (1) yields the commutative diagram

$$(2) \quad \begin{array}{ccc} V_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} & \simeq & V_y \\ \downarrow & & \downarrow \\ U_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} & \simeq & U_y \end{array}$$

where the vertical arrows are the natural embeddings (in other words, one has an isomorphism between the structure sheaves of $U_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k}$ and U_y). Consider a covering of X_0 (resp. $X_{0,\omega}$, resp. X_y) by affine charts $\{U_0^i\}_{i=1}^n$ (resp. $\{U_{0,\omega}^i\}_{i=1}^n$, resp. $\{U_y^i\}_{i=1}^n$). If $n = 2$ then applying Diagram (2) for the embeddings $U_{0,\omega}^1 \cap U_{0,\omega}^2 \hookrightarrow U_{0,\omega}^i$ and

¹ Dubouloz informed the author that he and Kishimoto knew that this version of their theorem can be extracted from the Kraft-Russell theorem.

$U_y^1 \cap U_y^2 \hookrightarrow U_y^i$ and gluing the affine charts we get an isomorphism between $X_{0,\omega} \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$ and $X_y = \rho^{-1}(y)$. Furthermore, we see that Diagram (2) remains true when U (resp. V) is not affine but only a union of two affine sets. Then the similar argument and the induction by n yields the desired isomorphism $X_{0,\omega} \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k} \rightarrow X_y$ for $n \geq 3$. \square

Notation 2.3. Let $\varphi : Z \rightarrow X$ be a morphism of algebraic \mathbf{k} -varieties and \mathbf{k}_0 be a subfield of \mathbf{k} such that for some \mathbf{k}_0 -varieties X_0 and Z_0 one has $X = X_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$ and $Z = Z_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$. Suppose that $\mathbf{k}_1 \subset \mathbf{k}$ is a finitely generated extension of \mathbf{k}_0 such that for \mathbf{k}_1 -varieties X_1 and Z_1 there exists a morphism $\varphi_1 : Z_1 \rightarrow X_1$ for which φ is obtained from φ_1 by the base extension $\text{Spec } \mathbf{k} \rightarrow \text{Spec } \mathbf{k}_1$. However, besides \mathbf{k}_0 the description of φ requires not the whole field \mathbf{k}_1 but only a finite number of elements of \mathbf{k}_1 (because φ is defined by the homomorphisms of rings of regular functions on affine charts and these rings are finitely generated). Thus for the \mathbf{k}_0 -algebra $C \subset \mathbf{k}_1$ generated by these elements we have the following observation used by Kraft and Russell in their proof for the affine case.

Lemma 2.4. *Let X and Z be algebraic varieties over a field \mathbf{k} and $\varphi : Z \rightarrow X$ be a morphism. Suppose that \mathbf{k}_0 , X_0 and Z_0 are as before. Then there exist a finitely generated \mathbf{k}_0 -algebra $C \subset \mathbf{k}$, ringed spaces \tilde{X} and \tilde{Z} with structure sheaves consisting of C -rings² and a C -morphism $\tilde{\varphi} : \tilde{Z} \rightarrow \tilde{X}$ such that $X = \tilde{X} \times_{\text{Spec } C} \text{Spec } \mathbf{k}$, $Z = \tilde{Z} \times_{\text{Spec } C} \text{Spec } \mathbf{k}$ and $\varphi = \tilde{\varphi} \times_{\text{Spec } C} \text{id}_{\text{Spec } \mathbf{k}}$.*

Theorem 2.5. *The Generic Equivalence Theorem is valid without the assumption (c).*

Proof. As before we can choose a field $\mathbf{k}_0 \subset \mathbf{k}$ which is finitely generated over the prime field such that for some morphisms $p_0 : S_0 \rightarrow Y_0$ and $q_0 : T_0 \rightarrow Y_0$ of \mathbf{k}_0 -varieties the morphisms $p : S \rightarrow Y$ and $q : T \rightarrow Y$ are obtained from these ones via the base extension $\text{Spec } \mathbf{k} \rightarrow \text{Spec } \mathbf{k}_0$. Suppose that \mathbf{K}_0 is the field of rational functions on Y_0 .

As in [KrRu] by Lemma 2.2 we get the following isomorphisms in self-evident notations

$$S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \simeq S_y \simeq T_y \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k}.$$

By Lemma 2.4, for the isomorphism $S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k}$ there exists a finitely generated \mathbf{K}_0 -algebra C in \mathbf{k} such that one has

$$S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } C \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } C.$$

Choosing a maximal ideal μ of C contained in the image of the morphism $S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } C \rightarrow \text{Spec } C$ and letting $\mathbf{L}_0 = C/\mu$ we get

$$(3) \quad S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{L}_0 \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{L}_0.$$

By construction the field \mathbf{L}_0 is a finite extension of \mathbf{K}_0 . It follows that there is a finite extension \mathbf{L} of the field \mathbf{K} of rational functions on Y such that

$$S_\omega \times_{\text{Spec } \mathbf{K}} \text{Spec } \mathbf{L} \simeq T_\omega \times_{\text{Spec } \mathbf{K}} \text{Spec } \mathbf{L}$$

² If C is a subring of a ring R we call R a C -ring and a homomorphism of two C -rings whose restriction to C is the identity map is called a C -homomorphism.

where S_ω and T_ω are generic fibers of p and q respectively. Since $S_\omega \times_{\text{Spec } \mathbf{k}} \text{Spec } \mathbf{L} \simeq S \times_Y \text{Spec } \mathbf{L}$ and $T_\omega \times_{\text{Spec } \mathbf{k}} \text{Spec } \mathbf{L} \simeq T \times_Y \text{Spec } \mathbf{L}$ there is a dominant morphism $X \rightarrow Y$ for which $S \times_Y X \simeq T \times_Y X$ and we are done. \square

Remark 2.6. It is interesting to discuss what happens to Theorem 2.5 if the field \mathbf{k} is not algebraically closed (but still of infinite transcendence degree over the prime field). Then there may be no embedding $\mathbf{K}_0 \hookrightarrow \mathbf{k}$ as in Lemma 2.2. However, for a finite extension \mathbf{k}_1 of \mathbf{k} one can find an embedding $\mathbf{K}_0 \hookrightarrow \mathbf{k}_1$. Consider the morphisms $p_1 : S_1 \rightarrow Y_1$ and $q_1 : T_1 \rightarrow Y_1$ of \mathbf{k}_1 -varieties obtained from $p : S \rightarrow Y$ and $q : T \rightarrow Y$ via the base extension $\text{Spec } \mathbf{k}_1 \rightarrow \text{Spec } \mathbf{k}$. Then until Formula (3) the argument remains valid with \mathbf{k} , p and q replaced by \mathbf{k}_1 , p_1 and q_1 . In Formula (3) the field \mathbf{L}_0 may contain a nontrivial finite extension \mathbf{k}_0^1 of \mathbf{k}_0 . Taking a bigger field \mathbf{k}_1 we can suppose that \mathbf{k}_0^1 is a subfield of \mathbf{k}_1 and proceed with the proof. Hence, though we cannot get the exact formulation of the Generic equivalence theorem in the case of non-closed fields, we can claim that for a finite extension \mathbf{k}_1 of \mathbf{k} and S_1, T_1 and Y_1 as before there is a dominant morphism of \mathbf{k}_1 -varieties of finite degree $X_1 \rightarrow Y_1$ and an isomorphism $S_1 \times_{Y_1} X_1 \simeq T_1 \times_{Y_1} X_1$ over X_1 .

3. VERY GENERAL FIBERS AND NON-CLOSED FIELDS

It is obvious that the assumption that an isomorphism $S_y \simeq T_y$ holds for every $y \in Y$ in the Kraft-Russell theorem can be replaced with the assumption that it is true for a general point of Y , i.e. for every point contained in some Zariski open dense subset of Y . However, the author does not know whether the proof of Kraft and Russell can be adjusted to the case when y is only a very general point of Y , i.e. it is in a complement of the countable union of proper closed subvarieties of Y . Hence we shall use a different approach. Namely, we shall use the technique which was communicated to the author by Vladimir Lin in 1980s and which was used in his unpublished work with Zaidenberg on a special case of The Generic Equivalence Theorem. The negative feature of this new proof is that we have to work over an **uncountable** field \mathbf{k} . However, we do not require that this field is algebraically closed.

Definition 3.1. We say that an uncountable subset W of an algebraic \mathbf{k} -variety X is Zariski locally dense if W is not contained in any countable union of proper closed subvarieties of X .

Example 3.2. Let W be the complement of a countable union $\bigcup_{i=1}^{\infty} Y_i$ of closed proper subvarieties of X . Then W is Zariski locally dense. Indeed, assume the contrary. That is, W is contained in a union $\bigcup_{i=1}^{\infty} Z_i$ of proper closed subvarieties of X and $X = \bigcup_{i=1}^{\infty} Y_i \cup \bigcup_{i=1}^{\infty} Z_i$. Without loss of generality we can suppose that X is affine and using a finite morphism of X onto some affine space $\mathbb{A}_{\mathbf{k}}^n$ we reduce the consideration to the case of $X \simeq \mathbb{A}_{\mathbf{k}}^n$. Note that equations of all Y_i 's and Z_i 's involve a countable number of coefficients. Let \mathbf{k}_0 be the smallest subfield of \mathbf{k} containing all these coefficients. Since \mathbf{k}_0 is countable there are points in $\mathbb{A}_{\mathbf{k}}^n$ whose coordinates are algebraically independent over \mathbf{k}_0 . Such a point cannot be contained in $\bigcup_{i=1}^{\infty} Y_i \cup \bigcup_{i=1}^{\infty} Z_i$. A contradiction.

The aim of this section is the following.

Theorem 3.3. *Let \mathbf{k} be an uncountable field of characteristic zero and $p : S \rightarrow Y$ and $q : T \rightarrow Y$ be morphisms of \mathbf{k} -varieties. Suppose that W is a Zariski locally dense subset of Y and for every $y \in W$ there is an isomorphism $p^{-1}(y) = S_y \simeq T_y = q^{-1}(y)$. Then there is a dominant morphism of finite degree $X \rightarrow Y$ such that $S \times_Y X$ and $T \times_Y X$ are isomorphic over X .*

The proof requires some preparations. We start with the following simple fact.

Proposition 3.4. *Let Y be an algebraic \mathbf{k} -variety, X and Z be subvarieties of $Y \times \mathbb{A}_{\mathbf{k}}^n$, $\rho : X \rightarrow Y$ and $\tau : Z \rightarrow Y$ be the natural projections, and \mathcal{P} be an algebraic family of rational maps $\mathbb{A}_{\mathbf{k}}^n \dashrightarrow \mathbb{A}_{\mathbf{k}}^n$. Suppose that \mathcal{P} is a subvariety of $Y \times \mathcal{P}$ such that for every $(y, f) \in \mathcal{P}$ the map f is regular on $X_y = \rho^{-1}(y)$. Let $\mathcal{P}_{X,Z}$ be the subset of \mathcal{P} that consists of all elements (y, f) such that $f(X_y) \subset Z_y$ for $Z_y := \tau^{-1}(y)$. Then $\mathcal{P}_{X,Z}$ is a constructible set.*

Proof. Consider the morphism $\kappa : X \times_Y \mathcal{P} \rightarrow Y \times \mathbb{A}_{\mathbf{k}}^n$ given by $(x, f) \rightarrow (\rho(x), f(x))$. Then $(Y \times \mathbb{A}_{\mathbf{k}}^n) \setminus Z$ and, therefore, $\kappa^{-1}((Y \times \mathbb{A}_{\mathbf{k}}^n) \setminus Z)$ are constructible sets. The image R of the latter under the natural projection $X \times_Y \mathcal{P} \rightarrow \mathcal{P}$ is a constructible set by the Chevalley's theorem (EGA IV, 1.8.4). Note that $\mathcal{P} \setminus R$ coincides with $\mathcal{P}_{X,Z}(N)$, i.e. it is also constructible and we are done. \square

Letting $Z = Y \times o$ where o is the origin in $\mathbb{A}_{\mathbf{k}}^n$ we get the following.

Corollary 3.5. *The subset $\mathcal{P}_X^0(N)$ of \mathcal{P} that consists of all elements (y, f) such that f vanishes on X_y is a constructible set.*

Notation 3.6. (1) Let $P(N)$ consist of $2n$ -tuples $\varphi = (f_1, g_1, f_2, g_2, \dots, f_n, g_n)$ of polynomials on $\mathbb{A}_{\mathbf{k}}^n$ of degree at most N such that g_1, \dots, g_n are not zero polynomials. We assign to φ the rational map $\check{\varphi} : \mathbb{A}_{\mathbf{k}}^n \dashrightarrow \mathbb{A}_{\mathbf{k}}^n$ given by $\check{\varphi} = (\frac{f_1}{g_1}, \dots, \frac{f_n}{g_n})$ and denote the variety of such rational maps by $R(N)$ with $\Theta : P(N) \rightarrow R(N)$ being the morphism given by $\Theta(\varphi) = \check{\varphi}$.

(2) Let the assumptions of Theorem 3.3 hold and Y be affine. Consider a cover of S (resp. T) by a collection $\mathcal{S} = \{S^i\}_{i \in J}$ of affine charts (resp. $\mathcal{T} = \{T^i\}_{i \in J}$) where J is finite set of indices. We can suppose that for some $n > 0$ every S^i (resp. T^i) is a closed subvariety of $Y \times \mathbb{A}_{\mathbf{k}}^n$ where the natural projection $S^i \rightarrow Y$ is the restriction of p (resp. $T^i \rightarrow Y$ is the restriction of q). By S_y^i (resp. T_y^i) we denote $S_y \cap S^i$ (resp. $T_y \cap T^i$). We treat each transition isomorphism α^{ij} of \mathcal{S} (resp. β^{ij} of \mathcal{T}) as the restriction of some rational map $\mathbb{A}_Y^n \dashrightarrow \mathbb{A}_Y^n$ and, choosing N large enough we assume that for every $y \in Y$ each rational map $\alpha^{ij}|_{S_y^i}$ (resp. $\beta^{ij}|_{T_y^i}$) is contained in $\Theta(P(N))$.

(3) Suppose that $\mathcal{Q}(N) = \prod_{i,j \in J} P(N)$, i.e. each element of $\mathcal{Q}(N)$ is $\Phi = \{\varphi^{ij} \in P(N) | i, j \in J\}$ where $\varphi^{ij} = (f_1^{ij}, g_1^{ij}, f_2^{ij}, g_2^{ij}, \dots, f_n^{ij}, g_n^{ij})$. Let $F(N)$ be the subset of $Y \times \mathcal{Q}(N)$ consisting of all elements (y, Φ) such that for all $i, j, i', j' \in J$ and $y \in Y$ one has the following

$$(4) \quad \check{\varphi}^{i'j'} \circ \alpha^{ii'}|_{S_y^i} = \beta^{jj'} \circ \check{\varphi}^{ij}|_{S_y^i};$$

$$(5) \quad \forall x \in S^i \exists j \text{ such that } \forall k = 1, \dots, n \ g_k^{ij}(x) \neq 0;$$

$$(6) \quad \check{\varphi}^{ij}(S_y^i) \subset T_y^j.$$

Lemma 3.7. *The set $F(N)$ is constructible.*

Proof. Every coordinate function of the rational map

$$(\check{\varphi}^{i'j'} \circ \alpha^{ii'} - \beta^{jj'} \circ \check{\varphi}^{ij}) : \mathbb{A}_{\mathbf{k}}^n \dashrightarrow \mathbb{A}_{\mathbf{k}}^n$$

can be presented as a quotient of two polynomials with the degrees of the numerator and the denominator bounded by some constant M depending on N only. That is, the ordered collection $\nu_{i,j,i',j'}$ of the numerators of this rational map is contained in $P(M)$. Consider the morphism $\tilde{\nu}_{i,j,i',j'} : Y \times \mathcal{Q}(N) \rightarrow Y \times P(M)$ where $\tilde{\nu}_{i,j,i',j'} = (\text{id}, \nu_{i,j,i',j'})$. Let \mathcal{Z}_i be the subvariety of $Y \times P(M)$ that consists of all elements (y, f_1, \dots, f_n) such that $f_k|_{S_y^i} \equiv 0$ for every k . By Corollary 3.5 \mathcal{Z}_i is a constructible set. Hence its preimages $\tilde{\mathcal{Z}}_{i,j,i',j'}$ in $Y \times \mathcal{Q}(N)$ under $\tilde{\nu}_{i,j,i',j'}$ is also constructible. Note that the variety $C = \bigcap_{i,j,i',j' \in J} \tilde{\mathcal{Z}}_{i,j,i',j'}$ consists of all elements satisfying Formula (4).

Consider the morphism $\theta_{ij} : S^i \times_Y C \rightarrow S^i \times \mathbb{A}_{\mathbf{k}}^n$ over S^i which sends each point (x, Φ) to $(x, g_1^{ij}(x), \dots, g_n^{ij}(x))$. Let L be the union of the coordinate hyperplanes in $\mathbb{A}_{\mathbf{k}}^n$ and $L_{ij} = \theta_{ij}^{-1}(S^i \times L)$. Then $K_i = \bigcap_{j \in J} L_{ij}$ is the subvariety of $S^i \times_Y C$ that consists of all points (x, Φ) such that for every $j \in J$ there exists $1 \leq k \leq n$ with $g_k^{ij}(x) = 0$. Let \mathcal{K}_i be the image of K_i in C under the natural projection $S^i \times_Y C \rightarrow C$ i.e. it is constructible by the Chevalley's theorem. Then its complement \mathcal{M}_i consists of elements $(y, \Phi) \in C$ such that for every $x \in S_y^i$ there exists $j \in J$ for which $g_k^{ij}(x) \neq 0$ for every $k = 1, \dots, n$. Hence the constructible set $D = \bigcap_{i \in J} \mathcal{M}_i$ satisfies Formula (5).

Let R^{ij} be the subvariety of $S^i \times_Y D$ that consists of points (x, Φ) for which $g_k^{ij}(x) \neq 0$ for every $k = 1, \dots, n$. That is, the map $\kappa_{ij} : R^{ij} \rightarrow Y \times \mathbb{A}_{\mathbf{k}}^n$ sending each point (x, Φ) to $(p(x), \check{\varphi}^{ij}(x))$ is regular. Let $\mathcal{R}^{ij} \subset R^{ij}$ be the preimage of $\mathbb{A}_{\mathbf{k}}^n \setminus T^j$ under this map. Then $\mathcal{R}_i = \bigcup_{j \in J} \mathcal{R}^{ij}$ is a constructible subset of $S^i \times_Y D$ and, therefore, its image R_i in D under the natural projection $S^i \times_Y D \rightarrow D$ is also constructible. Note that $F(N) := D \setminus \bigcup_{i \in J} R_i$ satisfies Formula (6) and we are done. \square

Remark 3.8. Formulas (4), (5) and (6) guarantee that any point $(y, \{\varphi^{ij}|i, j \in J\})$ in $F(N)$ defines a morphism $S_y \rightarrow T_y$. Hence we treat $F(N)$ further as collections of such morphisms.

Notation 3.9. Exchanging the role of S and T we get a constructible set $G(N)$, i.e. each element of G defines a morphism $T_y \rightarrow S_y$. In particular, $F(N) \times_Y G(N)$ consists of elements $\{(y, f_y, g_y)\}$ where $f_y : S_y \rightarrow T_y$ and $g_y : T_y \rightarrow S_y$ are morphisms.

Lemma 3.10. *Let $H(N)$ be the subset of $F(N) \times_Y G(N)$ that consists of all elements (y, f_y, g_y) such that each f_y is an isomorphism and $g_y = f_y^{-1}$. Then $H(N)$ is a constructible set.*

Proof. Note that each element $h = (y, f_y, g_y)$ of $F(N) \times_Y G(N)$ defines the morphism $\kappa_h : S \rightarrow S \times_Y S$ which sends $s \in S_y$ to $(s, g_y \circ f_y(s))$. Let Δ_S be the diagonal in $S \times_Y S$

and let $H'(N) \subset F(N) \times_Y G(N)$ be the subset that consists of those elements h for which $\kappa_h(S) \subset \Delta_S$. By Proposition 3.4 $H'(N)$ is constructible. Exchanging the role of S and T we get the similar constructible set $H''(N)$. Letting $H(N) = H'(N) \cap H''(N)$ we get the desired conclusion. \square

Lemma 3.11. *Let the assumptions of Theorem 3.3 hold, $H(N)$ be as in Lemma 3.10 and $W(N) = \rho(H(N))$ where $\rho : H(N) \rightarrow Y$ is the natural projection. Then for some number N the set $W(N)$ contains a Zariski dense open subset of Y .*

Proof. Note that for any isomorphism $\varphi : S_y \rightarrow T_y$ we can find N for which (φ, φ^{-1}) is an element of $H(N)$. Hence the assumptions of Theorem 3.3 imply that $Y = \bigcup_{N=1}^{\infty} W(N)$. Therefore, one of $W(N)$'s is Zariski locally dense in Y . Furthermore, it is constructible by the Chevalley's theorem which implies that it contains a Zariski open dense subset U of Y . This is the desired conclusion. \square

Proof of Theorem 3.3. Without loss of generality we suppose that Y is affine. Let N be as in Lemma 3.11, $H = H(N)$ and $\rho : H \rightarrow Y$ be the natural morphism. It is dominant by Lemma 3.11. Taking a smaller H we can suppose that it is affine. Then we have the natural embedding $\rho^* : \mathbf{k}[Y] \hookrightarrow \mathbf{k}[H]$ of the rings of regular functions. For the field K of rational functions on Y consider the K -algebra $A = K \otimes_{\mathbf{k}[Y]} \mathbf{k}[H]$. By the Noether normalization lemma one can find algebraically independent elements $z_1, \dots, z_k \in \mathbf{k}[H]$ such that A is a finitely generated over the polynomial ring $K[z_1, \dots, z_n]$. Choose elements $b_1, \dots, b_k \in \mathbf{k}$ so that the subvariety X of H given by the system of equations $z_1 - b_1 = \dots = z_k - b_k = 0$ is not empty. Then the field of rational functions on X is a finite extension of K , i.e. we get a dominant morphism $X \rightarrow Y$ of finite degree.

Note that we can view $x \in X \subset H$ as an element (y, f_y, g_y) of $F(N) \times_Y G(M)$ as in Lemma 3.10 such that $y = \rho(x)$ and $f_y : S_y \rightarrow T_y$ is an isomorphism while $g_y : T_y \rightarrow S_y$ is its inverse. Hence the map $S \times_Y X \rightarrow T \times_Y X$ that sends every point $(s, x) \in S \times_Y X$ to $(f_y(s), x)$ is an isomorphism. This is the desired conclusion. \square

Remark 3.12. We do not know if the morphism $X \rightarrow Y$ in Theorem 3.3 can be made étale in the case of a positive characteristic. However, if \mathbf{k} has characteristic zero then over a Zariski dense open subset U of Y this morphism is smooth by the Generic Smoothness theorem [Har, Chapter III, Corollary 10.7] and replacing Y by U we can suppose that $X \rightarrow Y$ is étale.

4. CASE OF $\bar{\mathbb{Q}}$ -VARIETIES

Notation 4.1. In this section \mathbf{k}_0 is an algebraically closed field of finite transcendence degree over \mathbb{Q} (e.g., \mathbf{k}_0 is the field $\bar{\mathbb{Q}}$ of algebraic numbers) and $p_0 : S_0 \rightarrow Y_0$ and $q_0 : T_0 \rightarrow Y_0$ are morphisms of algebraic \mathbf{k}_0 -varieties. By the Lefschetz principle we treat \mathbf{k}_0 as a subfield of \mathbb{C} and we denote by $p : S \rightarrow Y$ and $q : T \rightarrow Y$ are complexifications of these morphisms p_0 and q_0 (i.e., say, S coincides with $S_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbb{C}$).

The analogue of the Kraft-Russell theorem for k_0 -varieties can be reduced to the complex case if the following is true.

Conjecture 4.2. *Let Notation 4.1 hold and the fibers $p_0^{-1}(y)$ and $q_0^{-1}(y_0)$ be isomorphic for general points $y_0 \in Y_0$. Then the fibers $S_y = p^{-1}(y)$ and $T_y = q^{-1}(y)$ are isomorphic for general points $y \in Y$.*

We can prove this conjecture only in the case of proper morphisms $p_0 : S_0 \rightarrow Y_0$ and $q_0 : T_0 \rightarrow Y_0$, and our proof is based on the theory of deformations of compact complex spaces.

Definition 4.3. A deformation of a compact complex space Z is a proper flat holomorphic map $\rho : \mathcal{Z} \rightarrow B$ of a complex spaces such that for a marked point $b_0 \in B$ one has $\rho^{-1}(b_0) = Z$. A deformation ρ is called versal if for any other deformation $\kappa : \mathcal{W} \rightarrow D$ of Z with $Z = \kappa^{-1}(d_0)$ there is a holomorphic map $g : (D, d_0) \rightarrow (B, b_0)$ of the germs such that $g^*(\rho) = \kappa|_{(D, d_0)}$.

We need the following crucial results of Palamodov [Pa76, Theorem 5.4] and [Pa73].

Theorem 4.4. (1) *Every compact complex space Z is a fiber $\rho^{-1}(b_0)$ of a proper flat map $\rho : \mathcal{Z} \rightarrow B$ of complex spaces which is a versal deformation of each of its fibers.*

(2) *The space \mathcal{M} of classes of isomorphic complex spaces admits a T_0 -topology such that every proper flat family $\rho : \mathcal{Z} \rightarrow B$ induces a continuous map $\theta : B \rightarrow \mathcal{M}$.*

(3) *Furthermore, if ρ in (2) is a versal deformation at every point of B then θ is an open map.*

Theorem 4.5. *Let Notation 4.1 hold and the morphisms p and q be proper. Then Conjecture 4.2 is true.*

Proof. Without loss of generality we suppose that Y_0 is affine, i.e. we view Y_0 as a closed subvariety of $\mathbb{A}_{\mathbf{k}_0}^n$. Hence Y is a closed subvariety of \mathbb{C}^n and we can treat the set of points in Y whose coordinates are in \mathbf{k}_0 as Y_0 . Let Y_1 be the closure of Y_0 in Y in the standard topology (i.e. Y_1 contains all points of $Y \subset \mathbb{C}^n$ with real coordinates). That is, Y_1 is Zariski locally dense in Y in the terminology of Definition 3.1. Hence by Theorem 3.3 it suffices to establish isomorphisms $S_y \simeq T_y$ for a general $y_0 \in Y_1$.

Let $\rho : \mathcal{Z} \rightarrow B$ be a versal deformation as in Theorem 4.4 (1) for $Z = S_{y_0} \simeq T_{y_0}$ and let $\theta : B \rightarrow \mathcal{M}$ be as in Theorem 4.4 (2). For some neighborhood Y' (in the standard topology) of y_0 in Y we have holomorphic maps $\hat{p} : (Y', y_0) \rightarrow (B, b_0)$ and $\hat{q} : (Y', y_0) \rightarrow (B, b_0)$ such that $(\hat{p})^*(\rho) = p|_{Y'}$ and $(\hat{q})^*(\rho) = q|_{Y'}$. To prove that S_y and T_y are isomorphic it suffices to prove that they are biholomorphic (by virtue of [SGA 1, XII, Theorem 4.4]). That is, it suffices for us to establish the equality $p' := \theta \circ \hat{p} = \theta \circ \hat{q} =: q'$ and, furthermore, as we mentioned before it is enough to establish equality $p'|_{Y'_1} = q'|_{Y'_1}$ where $Y'_1 = Y_1 \cap Y'$.

Assume the contrary. Then by Theorem 3.3 the set $R_0 \subset Y'_1$ of points y for which $p'(y) = q'(y)$ cannot be Zariski locally dense, i.e. it is contained in a countable union of subsets of Y'_1 which are nowhere dense in Y'_1 . Let $R_1 \subset Y'_1$ (resp. $R_2 \subset Y'_1$) be the set of points y such that there is a neighborhood $U_y \subset \mathcal{M}$ of $p'(y)$ that does not contain $q'(y)$ (resp. a neighborhood $V_y \subset \mathcal{M}$ of $q'(y)$ that does not contain $p'(y)$). Since \mathcal{M} is a T_0 -space we see that $R_0 \cup R_1 \cup R_2 = Y'_1$. Furthermore, since the map θ is open

(by Theorem 4.4(3)) we can suppose that $U_y = \theta(\tilde{U}_y)$ (resp. $V_y = \theta(\tilde{V}_y)$) where \tilde{U}_y is a neighborhood of $\hat{p}(y)$ in B (resp. \tilde{V}_y is a neighborhood of $\hat{q}(y)$ in B). Since B is a germ of a complex space we can consider a metric on it which induces the standard topology. Let R_1^n be the set of points $y \in R_1$ such that \tilde{U}_y contains the ball $D(y, \frac{1}{n}) \subset B$ of radius $\frac{1}{n}$ (in this metric) with center at y and let R_2^n be the similar subset of R_2 . Then we have

$$Y'_1 = R_0 \cup \bigcup_{i=1}^{\infty} R_1^n \cup \bigcup_{i=1}^{\infty} R_2^n.$$

By the Baire category theorem there is a nonempty open subset $W \subset Y'_1$ and n such that, say, R_1^n is everywhere dense in W . In particular, for every point $y_1 \in W \cap R_1^n$ the ball $D(\hat{p}(y_1), \frac{1}{n})$ does not meet $\theta^{-1}(q'(y_1))$. Without loss of generality we can suppose that $\hat{p}(y_1)$ is a smooth point of $\hat{p}(Y')$ and taking a larger n we can also assume that $\hat{p}(W)$ coincides with the intersection of $\hat{p}(Y'_1)$ with $D(\hat{p}(y_1), \frac{1}{2n})$. Hence we can choose a point $y_2 \in W \cap Y_0$ near y_1 such that for $b_2 := \hat{p}(y_2)$ the ball $D(b_2, \frac{1}{3n})$ does not meet $\theta^{-1}(q'(R_1^n \cap W))$.

On the other hand by the assumption of Conjecture 4.2 we have $S_{y_2} \simeq T_{y_2}$ and since $\rho : \mathcal{Z} \rightarrow B$ is a versal deformation at every point of B (by Theorem 4.4 (1)) there exists a map $\check{q} : (Y', y_2) \rightarrow (B, b_2)$ such that $(\check{q})^*(\rho) = q|_{(Y', y_2)}$. By continuity $D(b_2, \frac{1}{3n})$ contains points from $\check{q}(R_1^n \cap W)$. Thus it must contain points from $\theta^{-1}(q'(R_1^n \cap W))$ since $q' = \theta \circ \hat{q}$ and $\theta \circ \hat{q}(R_1^n \cap W) = \theta \circ \check{q}(R_1^n \cap W)$. This contradiction shows that R_0 is Zariski locally dense in Y . Now the desired conclusion follows from Theorem 3.3. \square

Remark 4.6. The assumption that \mathbf{k}_0 is algebraically closed can be dropped from the formulation of Theorem 4.5 since it is not used in the proof.

Theorem 4.7. *Let Notation 4.1 hold and let $p_0 : S_0 \rightarrow Y_0$ and $q_0 : T_0 \rightarrow Y_0$ be proper morphisms. Suppose that for all $y_0 \in Y_0$ the two (schematic) fibers $p_0^{-1}(y_0)$ and $q_0^{-1}(y_0)$ are isomorphic. Then there is a dominant morphism of finite degree $X_0 \rightarrow Y_0$ and an isomorphism $S_0 \times_{Y_0} X_0 = T_0 \times_{Y_0} X_0$ over X_0 .*

Proof. Let \mathbf{K}_0 be the field of rational functions on Y_0 and ω be the generic point of Y_0 . Since \mathbb{C} is algebraically closed and of infinite transcendence degree over \mathbf{k}_0 we can always find an injective homomorphism $\mathbf{K}_0 \hookrightarrow \mathbb{C}$. Then by Lemma 2.2 this homomorphism defines a closed point $y \in Y$ and

$$S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbb{C} \rightarrow S_y$$

where $S_{0,\omega}$ is the generic fiber of p_0 . Since the similar fact holds for $q_0 : T_0 \rightarrow Y_0$ and since $S_y \simeq T_y$ by Theorem 4.5 we have

$$S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbb{C} \simeq S_y \simeq T_y \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbb{C}.$$

Then repeating the argument from the proof of Theorem 2.5 we construct a finite extension \mathbf{L}_0 of \mathbf{K}_0 for which

$$S_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{L}_0 \simeq T_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{L}_0.$$

Since $S_{0,\omega} \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{L}_0 \simeq S_0 \times_{Y_0} \text{Spec } \mathbf{L}_0$ and $T_{0,\omega} \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{L}_0 \simeq T_0 \times_{Y_0} \text{Spec } \mathbf{L}_0$ there is a dominant morphism of finite degree $X_0 \rightarrow Y_0$ for which $S_0 \times_{Y_0} X_0 \simeq T_0 \times_{Y_0} X_0$ and we are done. \square

5. THE DUBOULOZ-KISHIMOTO THEOREM

The aim of this section is to use the Kraft-Russell Generic Equivalence theorem to get a rather simple proof of the following result which has a strong overlap with the Dubouloz-Kishimoto theorem.

Theorem 5.1. *Let \mathbf{k} be an algebraically closed field of infinite transcendence degree over the prime field and let $f : X \rightarrow Y$ be a dominant morphism between geometrically integral algebraic \mathbf{k} -varieties. Suppose that for general closed points $y \in Y$, the fiber X_y contains a Zariski dense subvariety U_y of the form $U_y \simeq Z_y \times \mathbb{A}_{\mathbf{k}}^m$ over a $\kappa(y)$ -variety Z_y . Then there exists a dominant morphism $T \rightarrow Y$ of finite degree such that $X_T = X \times_Y T$ contains a variety $W \simeq Z \times \mathbb{A}_{\mathbf{k}}^m$ over a T -variety Z .*

We start with the following.

Lemma 5.2. *Let the notation of Lemma 2.4 hold, φ be an open immersion, μ be a maximal ideal of the ring C contained in the image of the morphism $S_{0,\omega} \times_{\text{Spec } \mathbf{k}_0} \text{Spec } C \rightarrow \text{Spec } C$ and $\mathbf{k}' = C/\mu$ (i.e. the field \mathbf{k}' is a finite extension of \mathbf{k}_0). Suppose that $X' = \tilde{X} \times_{\text{Spec } C} \text{Spec } \mathbf{k}'$, $Z' = \tilde{Z} \times_{\text{Spec } C} \text{Spec } \mathbf{k}'$, and $\varphi' : Z' \rightarrow X'$ is the morphism obtained from $\tilde{\varphi}$ by the base extension $C \rightarrow \mathbf{k}'$. Let $Z_0 = V_0 \times \mathbb{A}_{\mathbf{k}_0}^m$ (i.e. $Z = V \times \mathbb{A}_{\mathbf{k}}^m$) where V_0 is affine. Then $Z' = V' \times \mathbb{A}_{\mathbf{k}'}^m$ and $\dim_{\mathbf{k}'} \varphi'(Z') = \dim_{\mathbf{k}} Z$.*

Proof. By the assumption the algebra $\mathbf{k}_0[Z_0]$ of regular functions on $Z_0 = V_0 \times \mathbb{A}_{\mathbf{k}_0}^m$ coincides with a polynomial ring $B_0[x_1, \dots, x_m]$ where B_0 is some \mathbf{k}_0 -algebra. Hence for $\tilde{Z} = Z_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } C$ the algebra of regular functions is $\tilde{B}[x_1, \dots, x_m]$ where $\tilde{B} = B_0 \otimes_{\mathbf{k}_0} C$. This implies that the algebra $\mathbf{k}'[Z']$ of regular functions on Z' coincides with $B'[x_1, \dots, x_m]$ where $B' = \tilde{B} \otimes_C \mathbf{k}'$ which yields the equality $Z' = V' \times \mathbb{A}_{\mathbf{k}'}^m$.

Furthermore, by the Noether's normalization lemma there are algebraically independent elements $y_1, \dots, y_n \in B_0$ such that B_0 is a finitely generated module over $\mathbf{k}_0[y_1, \dots, y_n]$ and, hence, the natural embedding $\iota : \mathbf{k}_0[x_1, \dots, x_m, y_1, \dots, y_n] \rightarrow \mathbf{k}_0[Z_0]$ is an integral homomorphism (in particular, $\dim Z_0 = \dim Z = n + m =: d$). Note that $\mathbf{k}'[Z'] = (\mathbf{k}_0[Z_0] \otimes_{\mathbf{k}_0} C) \otimes_C \mathbf{k}' = \mathbf{k}_0[Z_0] \otimes_{\mathbf{k}_0} \mathbf{k}'$ and ι induces a homomorphism $\mathbf{k}'[x_1, \dots, x_n, y_1, \dots, y_m] \rightarrow \mathbf{k}'[Z']$ which is integral by [AM, Exercise 5.3]. Thus $\dim Z' = d$. Enlarging \mathbf{k}_0 in this construction we can suppose that the morphism $\varphi^{-1} : \varphi(Z) \rightarrow Z$ can be also obtained from a morphism of \mathbf{k}_0 -varieties. This implies that φ' is an immersion and we are done. \square

Proof of Theorem 5.1. Let $\mathbf{k}_0, X_0, Y_0, \mathbf{K}_0$ and $X_{0,\omega}$ be as in Notation 2.1 and Lemma 2.2. That is, $\mathbf{k}_0 \subset \mathbf{k}$ is finitely generated over the prime field, $X = X_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$, $Y = Y_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{k}$, \mathbf{K}_0 is the field of rational functions on Y_0 , $X_{0,\omega} = X_0 \times_{\text{Spec } \mathbf{k}_0} \text{Spec } \mathbf{K}_0$, and, choosing an embedding $\mathbf{K}_0 \hookrightarrow \mathbf{k}$, we have an isomorphism $X_{0,\omega} \times_{\text{Spec } \mathbf{K}_0} \text{Spec } \mathbf{k} \rightarrow X_y$ for some closed point $y \in Y$. Enlarging \mathbf{k}_0 (and, therefore, \mathbf{K}_0) and treating

\mathbf{K}_0 as a subfield of \mathbf{k} we can suppose that the natural immersion $\varphi : Z_y \rightarrow X_y$ is obtained from an immersion $\varphi_0 : Z_y^0 \rightarrow X_y^0$ of \mathbf{K}_0 -varieties Z_y^0 and X_y^0 via the base extension $\text{Spec } \mathbf{k} \rightarrow \text{Spec } \mathbf{K}_0$. By Lemma 2.4 there exist a finitely generated \mathbf{K}_0 -algebra $C \subset \mathbf{k}$, ringed spaces \tilde{X}_y and \tilde{Z}_y with structure sheaves consisting of C -rings and a C -morphism $\tilde{\varphi} : \tilde{Z}_y \rightarrow \tilde{X}_y$ such that $X_y = \tilde{X}_y \times_{\text{Spec } C} \text{Spec } \mathbf{k}$, $Z_y = \tilde{Z}_y \times_{\text{Spec } C} \text{Spec } \mathbf{k}$ and $\varphi = \tilde{\varphi} \times_{\text{Spec } C} \text{id}_{\text{Spec } \mathbf{k}}$. Let μ be a maximal ideal of C and $\mathbf{L}_0 = C/\mu$. By Lemma 5.2 we get an immersion $\varphi' : Z' \rightarrow X'$ of the \mathbf{L}_0 -varieties $X'_y = \tilde{X}_y \times_{\text{Spec } C} \text{Spec } \mathbf{L}_0$ and $Z'_y = \tilde{Z}_y \times_{\text{Spec } C} \text{Spec } \mathbf{L}_0$ such that $Z' = V' \times \mathbb{A}_{\mathbf{L}_0}^m$ and $\dim_{\mathbf{L}_0} \varphi'(Z') = \dim_{\mathbf{k}} Z$.

Put $\mathbf{L} = \mathbf{K} \otimes_{\mathbf{K}_0} \mathbf{L}_0$ where \mathbf{K} is a field of rational functions on Y . Let $\check{Z} = Z' \times_{\text{Spec } \mathbf{L}_0} \text{Spec } \mathbf{L}$ (in particular $\check{Z} = \check{V} \times \mathbb{A}_{\mathbf{L}}^m$), $\check{X} = X' \times_{\text{Spec } \mathbf{L}_0} \text{Spec } \mathbf{L}$ and $\check{\varphi} : \check{Z} \rightarrow \check{X}$ be the open immersion induced by φ' . By construction the field \mathbf{L}_0 is a finite extension of \mathbf{K}_0 and, hence, \mathbf{L} is a finite extension of \mathbf{K} . This implies that there is a dominant morphism $T \rightarrow Y$ of finite degree such that the field of rational functions on T is \mathbf{L} and $X \times_Y T$ is a \mathbf{k} -variety for which the generic fiber of the projection $X \times_Y T$ is \check{X} . Hence $X \times_Y T$ contains a Zariski open dense subset of the form $W \times \mathbb{A}_{\mathbf{k}}^m$ which is the desired conclusion. \square

Remark 5.3. Note that unlike in the original formulation of the Dubouloz-Kishimoto theorem the field \mathbf{k} may be countable and it may have a positive characteristic. If \mathbf{k} has characteristic zero then similar to Remark 3.12 we can suppose that $T \rightarrow Y$ is étale as in the Dubouloz-Kishimoto theorem.

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