RATIONAL HOMOLOGY MANIFOLDS AND HYPERSURFACE NORMALIZATIONS

BRIAN HEPLER

Abstract. We prove a criterion for determining whether the normalization of a complex analytic space on which the shifted constant sheaf is perverse is a rational homology manifold, using a perverse sheaf known as the multiple-point complex. This perverse sheaf is naturally associated to any "parameterized space", and has several interesting connections with the Milnor monodromy and mixed Hodge Modules.

1. Introduction

Let $\mathcal{U}$ be an open neighborhood of the origin in $\mathbb{C}^N$, let $X \subseteq \mathcal{U}$ be a complex analytic space containing $\mathbf{0}$ of pure dimension n, on which the (shifted) constant sheaf $\mathbb{Q}_X^\bullet[n]$ is perverse (e.g., if X is a local complete intersection), and let $\pi : Y \to X$ be the normalization of X.

There is then a surjection of perverse sheaves $\mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0$, where $\mathbf{I}_X^\bullet$ is the intersection cohomology complex on X with constant $\mathbb{Q}$ coefficients.

Remark 1.1. It is a classic result (see, e.g., [2] page 111) that there always exists a morphism $\mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet$ in the derived category $D_c^b(X)$ (where $\mathbf{I}_X^\bullet$ has constant $\mathbb{Q}$ coefficients). In our situation, $\mathbb{Q}_X^\bullet[n]$ is a perverse sheaf on X, so this descends to a morphism of perverse sheaves. Since we are working with field coefficients, $\mathbf{I}_X^\bullet$ is a semi-simple object in the category of perverse sheaves on X, so one easily concludes that the cokernel of this morphism must be zero, since the morphism $\mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet$ is an isomorphism when restricted to the smooth part of X.

It is worth noting that this morphism also exists with $\mathbb{Z}$ coefficients (and, for a local complete intersection, $\mathbb{Z}_X^\bullet[n]$ is still perverse), and the morphism is still surjective, but we can no longer use the fact that $\mathbf{I}_X^\bullet$ is a semi-simple object. Instead, we use that $\mathbf{I}_X^\bullet$ is also the intermediate extension of the local system $\mathbb{Z}_{X\backslash\Sigma X}^\bullet[n]$ to all of X (where ΣX denotes the singular locus of X), and therefore has no perverse quotient objects with support contained in ΣX. Since $\mathbb{Z}_X^\bullet[n] \to \mathbf{I}_X^\bullet$ is an isomorphism when restricted to $X\backslash\Sigma X$, it follows that the cokernel of this morphism is zero.

Since the category of perverse sheaves on X is Abelian, there is a perverse sheaf $\mathbf{N}_X^\bullet$ on X such that

$$0 \to \mathbf{N}_X^\bullet \to \mathbb{Q}_X^\bullet[n] \to \mathbf{I}_X^\bullet \to 0 \quad (\dagger)$$

is a short exact sequence of perverse sheaves.

Thus, if $\mathbf{I}_Y^\bullet$ is intersection cohomology on Y with constant $\mathbb{Q}$ coefficients, we have $\pi_*\mathbf{I}_Y^\bullet \cong \mathbf{I}_X^\bullet$ (π is a small resolution, in the sense of Goresky and Macpherson

[2]), and we obtain the short exact sequence of perverse sheaves

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to \pi_*\mathbf{I}^\bullet_Y \to 0$$

on X. We refer to this exact sequence as the **fundamental short exact sequence of the normalization**. This short exact sequence, and the perverse sheaf $\mathbf{N}^\bullet_X$ in particular, have been examined recently in several papers by the author and D. Massey in the case where the normalization Y is smooth ([4], [3]), where $\mathbf{N}^\bullet_X$ is called the **multiple-point complex** of the normalization (see Section 2).

Disregarding the normalization, if one just examines the short exact sequence (†), D.Massey has recently shown in [6] that, in the case where $X = V(f)$ is a hypersurface,

$$\mathbf{N}^\bullet_X \cong \ker\{\mathrm{id} - \widetilde{T}_f\},$$

where $\widetilde{T}_f$ is the monodromy action on the vanishing cycles $\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$, and the kernel takes place in the category of perverse sheaves on $X = V(f)$. In this context, Massey refers to $\mathbf{N}^\bullet_X$ as the **comparison complex** on X. It also seems that one may obtain this result in the algebraic setting (with $\mathbb{Q}$ coefficients) using the language of mixed Hodge modules.

Looking at (†), one notices immediately that $\mathbb{Q}^\bullet_X[n] \cong \mathbf{I}^\bullet_X$ if and only if $\mathbf{N}^\bullet_X = 0$; that is, the LCI X is a rational homology manifold (or, a **$\mathbb{Q}$-homology manifold**) precisely when the complex $\mathbf{N}^\bullet_X$ vanishes (for this criterion, see for example [1], [7]). We will recall $\mathbb{Q}$-homology manifolds and their properties in Section 2. It is then natural to ask that, given the normalization Y of X and the resulting fundamental short exact sequence, is there a similar result relating $\mathbf{N}^\bullet_X$ to whether or not Y is a $\mathbb{Q}$-homology manifold?

We answer this question in our main result:

Main Theorem 1 (Theorem 2.3)**.** *Y is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree $-n+1$; i.e., for all $p \in X$, $H^k(\mathbf{N}^\bullet_X)_p$ is non-zero only possibly when $k = -n+1$.*

In general, it is quite difficult to compute these stalk cohomology groups, even in the "next simplest" case where the normalization of a hypersurface has an isolated singularity, e.g., the normalization of a surface with a curve singularity, which we will work out in detail in Section 4.

Remark 1.2. M. Saito has recently drawn interesting connections with the multiple-point complex $\mathbf{N}^\bullet_X$ to the setting of mixed Hodge modules in a recent preprint [9]. In particular, Saito shows, for an arbitrary reduced complex algebraic variety X of pure dimension n, that the weight zero part of the cohomology group $H^1(X;\mathbb{Q})$ is given by

$$W_0H^1(X;\mathbb{Q}) \cong \mathrm{coker}\{H^0(Y;\mathbb{Q}) \to H^0(X;\mathcal{F}_X)\},$$

where $\pi : Y \to X$ is the normalization of X, and $\mathcal{F}_X$ is a certain constructible sheaf on X, given by the cokernel of the natural morphism of sheaves $\mathbb{Q}_X \to \pi_*\mathbb{Q}_Y$. The algebraic setting is necessary here, in order to endow $H^0(X;\mathcal{F}_X)$ with a mixed Hodge structure, and for working in the derived category of mixed Hodge modules.

This constructible sheaf $\mathcal{F}_X$ is none other than the cohomology sheaf $H^{-n+1}(\mathbf{N}^\bullet_X)$; this follows immediately from taking the long exact sequence in cohomology of the fundamental short exact sequence of the normalization. If, as in Saito's case, the sheaf $\mathbb{Q}^\bullet_X[n]$ is not perverse, one can obtain this isomorphism from the distinguished triangle

$$\mathcal{F}_X[n] \to \mathbf{N}^\bullet_X[1] \to \pi_* \mathbf{N}^\bullet_Y[1] \xrightarrow{+1}$$

obtained via the octahedral axiom in the derived category $D^b_c(X)$, together with the fact that Y is normal.

Consequently, we can interpret Saito's result as an isomorphism

$$W_0 H^1(X;\mathbb{Q}) \cong \operatorname{coker}\{H^0(Y;\mathbb{Q}) \to \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)\},$$

since $H^0(X; H^{-n+1}(\mathbf{N}^\bullet_X)) \cong \mathbb{H}^{-n+1}(X;\mathbf{N}^\bullet_X)$. It would seem to be an interesting question in the local analytic case to relate this result with the isomorphism $\mathbf{N}^\bullet_X \cong W_{n-1}\mathbb{Q}^\bullet_X[n]$ obtained in Remark 2.5, where $\mathbf{N}^\bullet_X$ is endowed with the natural structure of a mixed Hodge module on X.

We would like to thank the Referee for suggesting the content of Remark 2.5, and Jörg Schürmann for many helpful discussions regarding mixed Hodge modules in general and Remark 1.2 and Remark 2.5 in particular.

2. Main result

Before we prove our main result, we first recall a theorem of Borho and MacPherson [1] giving us several equivalent characterizations of rational homology manifolds:

Theorem 2.1. *([B-M]) The following are equivalent:*

1. *X is a $\mathbb{Q}$-homology manifold (i.e., $\mathbf{I}^\bullet_X \cong \mathbb{Q}^\bullet_X[n]$);*
2. *$\mathcal{D}(\mathbb{Q}^\bullet_X[n]) \cong \mathbb{Q}^\bullet_X[n]$, where $\mathcal{D}$ is the Verdier duality functor;*
3. *For all $p \in X$, for all k, $H^k(X, X\backslash\{p\};\mathbb{Q}) = 0$ unless $k = 2n$, and $H^{2n}(X, X\backslash\{p\};\mathbb{Q}) \cong \mathbb{Q}$.*

The proof of Theorem 2.3 relies on the following well-known lemma.

Lemma 2.2. *Let X be a complex analytic space of pure dimension n. Then, for $p \in X$, the rank of $H^{-n}(\mathbf{I}^\bullet_X)_p$ is equal to the number of irreducible components of X at p.*

Proof. This result is well-known to experts, see e.g. Theorem 1G (pg. 74) of [10], or Theorem 4 (pg. 217) [5] □

Note that taking stalk cohomology at $p \in X$ of the fundamental short exact sequence yields the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_*\mathbf{I}^\bullet_Y)_p \to H^{-n+1}(\mathbf{N}^\bullet_X)_p \to 0,$$

and isomorphisms $H^k(\pi_*\mathbf{I}^\bullet_Y)_p \cong H^{k+1}(\mathbf{N}^\bullet_X)_p$ for $-n+1 \le k \le -1$. With this in mind, we claim that:

Theorem 2.3. *Y is a $\mathbb{Q}$-homology manifold if and only if $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree $-n+1$.*

Proof. ($\Longrightarrow$) Suppose that Y is a $\mathbb{Q}$-homology manifold, and let $p \in X$ be arbitrary. Since Y is a $\mathbb{Q}$-homology manifold, $\mathbb{Q}_Y[n] \cong \mathbf{I}^\bullet_Y$ in $D^b_c(Y)$, from which it follows $H^k(\mathbf{N}^\bullet_X)_p = 0$ for $k \neq -n+1$ by the above isomorphisms.

($\Longleftarrow$) Suppose that, for all $p \in X$, $H^k(\mathbf{N}_X^\bullet)_p \neq 0$ only possibly when $k = -n+1$. We wish to show that the natural morphism $\mathbb{Q}_Y[n] \to \mathbf{I}_Y^\bullet$ is an isomorphism in $D_c^b(Y)$.

There is still the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_*\mathbf{I}_Y^\bullet)_p \to H^{-n+1}(\mathbf{N}_X^\bullet)_p \to 0$$

and $H^k(\pi_*\mathbf{I}_Y^\bullet)_p = 0$ for $k \neq -n$, since $H^k(\pi_*\mathbf{I}_Y^\bullet)_p \cong H^{k+1}(\mathbf{N}_X^\bullet)_p$ for all $p \in X$ and $-n+1 \leq k \leq -1$. In degree $-n$, we have

$$H^{-n}(\pi_*\mathbf{I}_Y^\bullet)_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-n}(\mathbf{I}_Y^\bullet)_q.$$

This then implies that, for all $q \in Y$, $H^k(\mathbf{I}_Y^\bullet)_q = 0$ for $k \neq -n$. Our goal is to calculate this stalk cohomology in degree $-n$. Since Y is normal, and thus locally irreducible, it follows by Lemma 2.2 that $H^{-n}(\mathbf{I}_Y^\bullet)_q \cong \mathbb{Q}$ for all $q \in Y$.

Finally, we claim that the natural morphism $\mathbb{Q}_Y^\bullet[n] \to \mathbf{I}_Y^\bullet$ is an isomorphism in $D_c^b(Y)$. In stalk cohomology at any point $q \in Y$, both $H^k(\mathbb{Q}_Y^\bullet[n])_q$ and $H^k(\mathbf{I}_Y^\bullet)_q$ are non-zero only in degree $k = -n$, with stalks isomorphic to $\mathbb{Q}$. Consequently, the natural morphism is an isomorphism in $D_c^b(Y)$ provided that the morphism

$$\mathbb{Q} \cong H^{-n}(\mathbb{Q}_Y^\bullet[n])_q \to H^{-n}(\mathbf{I}_Y^\bullet)_q \cong \mathbb{Q}$$

is not the zero morphism. But this is just the "diagonal" morphism from a single copy of $\mathbb{Q}$ to the number of connected components of $Y \backslash \{p\}$, which is clearly non-zero. Thus, Y is a $\mathbb{Q}$-homology manifold.

$\square$

Corollary 2.4. *Suppose that $\mathbf{N}_X^\bullet$ has stalk cohomology concentrated in degree $-n+1$. Then, for all $p \in X$, if $j_p : \{p\} \hookrightarrow X$ is the inclusion map, we have*

$$H^k(j_p^! \mathbf{N}_X^\bullet) \cong \begin{cases} \widetilde{H}^{n+k-1}(K_{X,p};\mathbb{Q}), & \text{for } 0 \leq k \leq n-1; \\ 0, & \text{else.} \end{cases},$$

where $K_{X,p}$ denotes the real link of X at p, i.e., the intersection of X with a sphere of sufficiently small radius, centered at p.

This follows by applying $j_p^!$ to the fundamental short exact sequence of the normalization, and taking stalk cohomology.

When the normalization $Y \xrightarrow{\pi} X$ is a $\mathbb{Q}$-homology manifold, the short exact sequence

$$0 \to \mathbb{Q} \to H^{-n}(\pi_*\mathbf{I}_Y^\bullet)_p \to H^{-n+1}(\mathbf{N}_X^\bullet)_p \to 0$$

allows us to identify, given Lemma 2.2, that

$$m(p) := \dim_{\mathbb{Q}} H^{-n+1}(\mathbf{N}_X^\bullet)_p = |\pi^{-1}(p)| - 1.$$

Consequently, we conclude that the support of $\mathbf{N}_X^\bullet$ is none other than the **image multiple-point set** of the morphism π, which we denote by D; precisely, we have

$$D := \overline{\{p \in X \mid |\pi^{-1}(p)| > 1\}}.$$

For this reason, we have referred to the perverse sheaf $\mathbf{N}_X^\bullet$ as the **multiple-point complex** of X (or, of the morphism π, as we do in [3] and [4]). It is immediate from the fundamental short exact sequence that one always has the inclusion $D \subseteq \Sigma X$.

In such cases (see Section 4), it is useful to partition X into subsets $X_k = m^{-1}(k)$ for $k \geq 1$; clearly, one has

$$D = \overline{\bigcup_{k>1} X_k}.$$

Finally, since D is the support of a perverse sheaf which, on an open dense subset of D, has non-zero stalk cohomology only in degree $-n+1$, it follows that D is purely $(n-1)$-dimensional.

Remark 2.5. When Y is a $\mathbb{Q}$-homology manifold, in fact, both $\mathbf{I}^\bullet_X$ and $\mathbf{N}^\bullet_X$ are just sheaves (up to a shift); moreover, the short exact sequence of perverse sheaves

$$0 \to \mathbf{N}^\bullet_X \to \mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$$

can be rewritten as a short exact sequence of (constructible) sheaves

$$0 \to \mathbb{Q}_X \to \mathbf{I}^\bullet_X[-n] \to \mathbf{N}^\bullet_X[1-n] \to 0.$$

We then ask, is it ever the case that $\mathbf{N}^\bullet_X$ is a semi-simple perverse sheaf, so that $\mathbb{Q}^\bullet_X[n]$ is an extension of semi-simples? One can find a Whitney stratification $\mathfrak{S}$ of X for which the sets X_k are finite unions of strata for all k. Then, for each stratum $S \subset X_k$, the monodromy of the local system $\mathbf{N}^\bullet_{X|_S}$ is determined by the monodromy of the set $\pi^{-1}(p)$ for $p \in S$; since this is a finite set with k elements, it follows immediately that $\mathbf{N}^\bullet_{X|_S}$ is semi-simple as a local system on S (since the monodromy action is semi-simple).

Since $\mathbf{N}^\bullet_{X|_S}$ is semi-simple as a local system for any stratum $S \subset X_k$, is $\mathbf{N}^\bullet_X$ semi-simple as a perverse sheaf? If one has a Whitney stratification of X for which the sets X_k are finite unions of strata, and for which the subset $D = \operatorname{supp} \mathbf{N}^\bullet_X$ is a union of closed strata, then the above argument demonstrates (together with [9] Section 2.4) that $\mathbf{N}^\bullet_X$ is semi-simple as a perverse sheaf. In general, however, this fails to be the case (see Section 4).

More generally, when $\mathbb{Q}^\bullet_X[n]$ is a perverse sheaf, one may use the general machinery of M. Saito (see [8], page 325 (4.5.9)) to obtain an isomorphism of perverse sheaves

$$\operatorname{Gr}^W_n \mathbb{Q}^\bullet_X[n] \xrightarrow{\sim} \mathbf{I}^\bullet_X.$$

underlying the corresponding isomorphism of mixed Hodge modules. Since $\dim_{\mathbf{0}} X = n$, the induced weight filtration on $\mathbb{Q}^\bullet_X[n]$ terminates after degree n, so that $W_n\mathbb{Q}^\bullet_X[n] \cong \mathbb{Q}^\bullet_X[n]$. Consequently, the above isomorphism yields a short exact sequence

$$0 \to W_{n-1}\mathbb{Q}^\bullet_X[n] \to \mathbb{Q}^\bullet_X[n] \to \mathbf{I}^\bullet_X \to 0$$

of perverse sheaves on X, implying $\mathbf{N}^\bullet_X \cong W_{n-1}\mathbb{Q}^\bullet_X[n]$. From this identification, it follows that $\mathbf{N}^\bullet_X$ is semi-simple as a perverse sheaf provided that the weight filtration $W_i\mathbb{Q}^\bullet_X[n]$ of $W_{n-1}\mathbb{Q}^\bullet_X[n] \cong \mathbf{N}^\bullet_X$ for $i < n$ is concentrated in one degree $k < n$, i.e., $W_i\mathbb{Q}^\bullet_X[n] = 0$ for $i < k$ and $W_i\mathbb{Q}^\bullet_X[n] \cong W_k\mathbb{Q}^\bullet_X[n]$ for $k < i < n$. Then, $\mathbf{N}^\bullet_X \cong \operatorname{Gr}^W_k \mathbb{Q}^\bullet_X[n]$ underlies a pure polarizable Hodge module, which is therefore by construction a semi-simple perverse sheaf.

We anticipate that the reverse implication will be more difficult, and be outside the scope of this paper.

3. Interpretation in terms of Comparison Complex

Recall that, by D. Massey, if $X = V(f)$ is a hypersurface, $\mathbf{N}^\bullet_X = \ker\{\mathrm{id} - \widetilde{T}_f\}$ is the perverse eigenspace of the eigenvalue 1 of the monodromy action on $\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1]$, where $\mathcal{U}$ is an open neighborhood of the origin in $\mathbb{C}^{n+1}$.

Since the content of this paper is interesting only in the case where $\dim_{\mathbf{0}} \Sigma f = n-1$ (otherwise, X is its own normalization), we will assume throughout that this is the case; consequently, the stalk cohomology $H^k(\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1])_p$ is possibly non-zero only for $-n+1 \leq k \leq 0$.

In general, it is **not the case** that, given a morphism of perverse sheaves, the cohomology of the stalk of the kernel of G is isomorphic to the kernel of the cohomology on the stalks; that is, there may exist points $p \in \Sigma f$ such that

$$H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \not\cong \ker\{\mathrm{id} - \widetilde{T}^k_{f,p}\}.$$

However, this isomorphism **does hold** in degree $-n+1$ for all $p \in \Sigma f$ (See Lemma 5.1 of [6]):

Proposition 3.1. *Let $\pi : Y \to V(f)$ be the normalization of $V(f)$, and suppose Y is a $\mathbb{Q}$-homology manifold. Then, the following isomorphisms hold for all $p \in \Sigma f$:*

$$H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \cong \begin{cases} \ker\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\}, & \text{if } k = -n+1; \\ 0, & \text{if } k \neq -n+1. \end{cases}$$

$$H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \cong \mathrm{im}\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\},$$

$$H^{-n+1}(\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\})_p \cong \mathrm{coker}\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\},$$

where $\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}$ is the Milnor monodromy action on $H^1(F_{f,p};\mathbb{Q})$.

Proof. Since $H^k(\ker\{\mathrm{id} - \widetilde{T}_f\})_p = 0$ for $k \neq -n+1$, the result follows from the short exact sequences

$$0 \to H^{-n+1}(\ker\{\mathrm{id} - \widetilde{T}_f\})_p \to H^1(F_{f,p};\mathbb{Q}) \to H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \to 0,$$

and

$$0 \to H^{-n+1}(\mathrm{im}\{\mathrm{id} - \widetilde{T}_f\})_p \to H^1(F_{f,p};\mathbb{Q}) \to H^{-n+1}(\mathrm{coker}\{\mathrm{id} - \widetilde{T}_f\})_p \to 0.$$

□

By taking stalk cohomology of the fundamental short exact sequence, we have

$$0 \to H^{-n}(\mathbb{Q}^\bullet_X[n])_p \to H^{-n}(\mathbf{I}^\bullet_X)_p \to \ker\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\} \to 0.$$

Since $\pi_*\mathbf{I}^\bullet_Y \cong \mathbf{I}^\bullet_X$, and $H^{-n}(\pi_*\mathbf{I}^\bullet_Y)_p \cong \mathbb{Q}^{|\pi^{-1}(p)|}$,

$$\ker\{\mathrm{id} - \widetilde{T}^{-n+1}_{f,p}\} \cong \mathbb{Q}^{|\pi^{-1}(p)|-1}$$

for all $p \in X$, yielding the following nice lower-bound:

Corollary 3.2.

$$\dim_{\mathbb{Q}} H^1(F_{f,p};\mathbb{Q}) \geq |\pi^{-1}(p)| - 1.$$

Remark 3.3. In the case where $X = V(f)$ is a hypersurface with smooth normalization in some open neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$, we prove in [3] that a strong relationship holds between the **characteristic polar multiplicities** of $\mathbf{N}^\bullet_X$ and the Lê numbers of the function f (Theorem 5.2).

A careful observation yields that the same result holds for hypersurfaces whose normalizations are $\mathbb{Q}$-homology manifolds (since all computations in the Theorem take place inside the hypersurface $V(f)$). Moreover, one can even use the same proof as Theorem 5.2 to obtain this more general result.

Remark 3.4. In the hypersurface case $X = V(f)$, Saito's calculation of $H^0(X; \mathcal{F}_X)$ via invariant cycles of the monodromy ([9] Section 2.4) is especially interesting to us.

Suppose the normalization of X is a rational homology manifold. Massey's result that $\mathbf{N}^\bullet_X \cong \ker\{\mathrm{id} - \widetilde{T}_f\}$ together with Proposition 3.1 allows us to identify $\mathcal{F}_X$ with the constructible sheaf $\mathbf{ker}\{\mathrm{id} - \widetilde{T}_f^{-n+1}\}$ whose stalk at a point p is

$$\ker\{\mathrm{id} - \widetilde{T}_{f,p}^{-n+1}\} \subseteq H^{-n+1}(\phi_f[-1]\mathbb{Q}^\bullet_{\mathcal{U}}[n+1])_p \cong H^1(F_{f,p}; \mathbb{Q}),$$

where $\widetilde{T}_{f,p}^{-n+1}$ is the Milnor monodromy operator on $H^1(F_{f,p}; \mathbb{Q})$. Consequently, Saito's calculation of $H^0(X; \mathcal{F}_X)$ via the internal monodromy of $\mathcal{F}_X$ allows us to compute information about the Milnor monodromy of f.

4. Example

We consider the following "trivial, non-trivial" example of the normalization of a surface X with one-dimensional singularity in $\mathbb{C}^3$, which nicely illustrates the content of Theorem 2.3.

Let $f(x, y, z) = xz^2 - y^2(y + x^3)$, so that $X = V(f) \subseteq \mathbb{C}^3$ has critical locus $\Sigma f = V(y, z)$. Then, if we let $Y = V(u^2 - x(y + x^3), uy - xz, uz - y(y + x^3)) \subseteq \mathbb{C}^4$, the projection map $\pi : Y \to X$ is the normalization of X.

It is easy to check that $\Sigma Y = V(x, y, z, u)$, and

$$\pi^{-1}(\Sigma f) = V(u^2 - x^4, y, z).$$

It then follows that $X_k = \emptyset$ if $k > 2$, and $X_2 = V(y, z)\backslash\{\mathbf{0}\}$, so that

$$\operatorname{supp} \mathbf{N}^\bullet_X = V(y, z) = \Sigma f.$$

For $p \in X$,

$$H^{-2}(\pi_*\mathbf{I}^\bullet_Y)_p \cong \bigoplus_{q \in \pi^{-1}(p)} H^{-2}(\mathbf{I}^\bullet_Y)_q \quad (\dagger 4.1)$$

But $\pi^{-1}(p) \subseteq Y\backslash\Sigma Y$, and $(\mathbf{I}^\bullet_Y)_{|_{\pi^{-1}(p)}} \cong (\mathbb{Q}^\bullet_Y[2])_{|_{\pi^{-1}(p)}}$, so from ($\dagger$4.1), it follows that

$$H^{-2}(\pi_*\mathbf{I}^\bullet_Y)_p \cong \mathbb{Q}^2.$$

Similarly, since $(\mathbf{I}^\bullet_Y)_{Y\backslash\Sigma Y} \cong \mathbb{Q}^\bullet_{Y\backslash\Sigma Y}[2]$, it follows that

$$H^0(\mathbf{N}^\bullet_X)_p \cong H^{-1}(\pi_*\mathbf{I}^\bullet_Y)_p = 0.$$

When $p = \mathbf{0}$, we find

$$H^k(\mathbf{I}_Y^\bullet)_\mathbf{0} \cong \begin{cases} \mathbb{H}^k(K_{Y,\mathbf{0}}; \mathbf{I}_Y^\bullet), & \text{if } k \leq -1 \\ 0, & \text{if } k > -1 \end{cases}$$

Since Y has an isolated singularity at the origin in $\mathbb{C}^4$, we further have

$$\mathbb{H}^k(K_{Y,\mathbf{0}}; \mathbf{I}_Y^\bullet) \cong H^{k+2}(K_{Y,\mathbf{0}}, \mathbb{Q}).$$

For $0 < \epsilon \ll 1$, the sphere S_ϵ transversely intersects Y near $\mathbf{0}$, so the real link $K_{Y,\mathbf{0}} = Y \cap S_\epsilon$ is a compact, orientable, smooth manifold of (real) dimension 3. We are interested in computing the two integral cohomology groups $H^0(K_{Y,\mathbf{0}}; \mathbb{Q})$ and $H^1(K_{Y,\mathbf{0}}; \mathbb{Q})$.

Because $K_{Y,\mathbf{0}}$ is also connected, we can apply Poincaré duality to find $H^0(K_{Y,\mathbf{0}}; \mathbb{Q}) \cong \mathbb{Q}$.

Consider the standard parameterization of the twisted cubic $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ via

$$\nu([s:t]) = [s^3 : st^2 : t^3 : s^2t] = [x : y : z : u]$$

which lifts to a map $\nu : \mathbb{C}^2 \to \mathbb{C}^4$, parameterizing the affine cone over the twisted cubic, i.e., the normalization $Y = V(u^2 - xy, uy - xz, uz - y^2)$. Then, we claim that ν is a 3-to-1 covering map away from the origin. Clearly, since ν parameterizes Y, we see that ν is a surjective local diffeomorphism onto $\nu(\mathbb{C}^2) = Y$.

Suppose that $\nu(s,t) = \nu(s',t')$. Then, we must have $s^3 = (s')^3$ and $t^3 = (t')^3$, so that there are cube roots of unity η and ω for which $s = \eta s'$ and $t = \omega t'$. But then,

$$s^2 t = (s')^2(t') = \eta^2 \omega s^2 t,$$

so either $\eta^2\omega = 1$, or $st = 0$. Since η and ω are both cube roots of unity, if $\eta^2\omega = 1$, then $\eta = \omega$. Additionally, note that $st = 0$ implies $(s,t) = \mathbf{0}$. It then follows that ν is 3-to-1 away from the origin.

Consider then the (real analytic) function

$$r(x,y,z,u) = |x|^2 + 3|y|^2 + |z|^2 + 3|u|^2$$

on $\mathbb{C}^4$; r is proper, transversally intersects Y away from $\mathbf{0}$, and $Y \cap r^{-1}[0,\epsilon)$ gives a fundamental system of neighborhoods of the origin in Y. Consequently, $Y \cap r^{-1}(\epsilon)$ gives, up to homotopy, the real link $K_{Y,\mathbf{0}}$. The composition $r(\nu(s,t))$ then gives:

$$\begin{aligned} r(\nu(s,t)) &= |s^3|^2 + 3|st^2| + |t^3|^2 + 3|s^2t|^2 \\ &= |s|^6 + 3|s|^4|t|^2 + 3|s|^2|t|^2 + |t|^6 \\ &= \left(|s|^2 + |t|^2\right)^3 = \epsilon, \end{aligned}$$

provided that $|s|^2 + |t|^2 = \sqrt[3]{\epsilon}$; that is, ν maps the 3-sphere in $\mathbb{C}^2$ 3-to-1 onto the real link $K_{Y,\mathbf{0}}$. Since the 3-sphere is simply-connected, it is the universal cover of $K_{Y,\mathbf{0}}$. The group of deck transformations given by multiplying (s,t) by a cube root of unity then yields the isomorphism $\pi_1(K_{Y\mathbf{0}}) \cong \mathbb{Z}/3\mathbb{Z}$. Thus, $H_1(K_{Y,\mathbf{0}}; \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$.

By again applying Poincaré duality, we find $H^2(K_{Y,\mathbf{0}}; \mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as well. By the Universal Coefficient theorem for cohomology, we then have $H_2(K_{Y,\mathbf{0}}; \mathbb{Z}) = 0$ so that $H^1(K_{Y,\mathbf{0}}; \mathbb{Z}) = 0$ by Poincaré duality. Using $\mathbb{Q}$ coefficients, this implies:

$$H^k(K_{Y,p}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}, & \text{if } k = 0,3 \\ 0, & \text{else} \end{cases}$$

for all $p \in Y$, so that Y is a $\mathbb{Q}$-homology manifold.

Equivalently, we find:

$$H^k(\mathbf{N}^\bullet_X)_p \cong \begin{cases} \mathbb{Q}, & \text{if } k=-1 \text{ and } p \in \Sigma f \backslash \{\mathbf{0}\} \\ 0, & \text{if } k \neq -1,\, p \in \Sigma f \end{cases}$$

i.e., $\mathbf{N}^\bullet_X$ has stalk cohomology concentrated in degree -1.

It is not hard to show that the monodromy of the local system $H^{-1}(\mathbf{N}^\bullet_X)_{|_{\Sigma f \backslash \{\mathbf{0}\}}}$ is trivial; consequently, $\mathbf{N}^\bullet_{X|_{\Sigma f}}$ is isomorphic to the extension by zero of the constant sheaf on $\Sigma f \backslash \{\mathbf{0}\}$. That is, if $j : \Sigma f \backslash \{\mathbf{0}\} \hookrightarrow \Sigma f$ is the open inclusion, then $\mathbf{N}^\bullet_{X|_{\Sigma f}} \cong j_! \mathbb{Q}^\bullet_{\Sigma f \backslash \{\mathbf{0}\}}[1]$. In particular, we see that $\mathbf{N}^\bullet_X$ is not semi-simple as a perverse sheaf on X.

To compare with Remark 2.5, this failure to be a semi-simple perverse sheaf can be detected by the presence of $W_0 \mathbf{N}^\bullet_X \cong \mathbb{Q}^\bullet_{\{\mathbf{0}\}} \neq \mathbf{0}$.

References

[1] Borho, W. and MacPherson, R. “Partial Resolutions of Nilpotent Varieties”. In: *Astérisque* 101-102 (1982), pp. 23–74.

[2] Goresky, M. and MacPherson, R. “Intersection Homology II”. In: *Invent. Math.* 71 (1983), pp. 77–129.

[3] Hepler, B. “Deformation Formulas for Parameterizable Hypersurfaces”. In: *ArXiv e-prints* (2017). arXiv: 1711.11134 [math.AG].

[4] Hepler, B. and Massey, D. “Perverse Results on Milnor Fibers inside Parameterized Hypersurfaces”. In: Publ. RIMS Kyoto Univ. 52 (2016), pp. 413–433.

[5] Łojasiewicz, S. *Introduction to Complex Analytic Geometry*. Translated by Klimek, M. Birkhäuser, 1991.

[6] Massey, D. “Intersection Cohomology and Perverse Eigenspaces of the Monodromy”. In: *ArXiv e-prints* (2017). arXiv: 1801.02113 [math.AG].

[7] Massey, D. “Intersection Cohomology, Monodromy, and the Milnor Fiber”. In: *International J. of Math.* 20 (2009), pp. 491–507.

[8] Saito, M. “Mixed Hodge Modules”. In: *Publ. RIMS, Kyoto Univ.* 26 (1990), pp. 221–333.

[9] Saito, M. “Weight zero part of the first cohomology of complex algebraic varieties”. In: *ArXiv e-prints* (2018). arXiv: 1804.03632 [math.AG].

[10] Whitney, H. *Complex Analytic Varieties*. Addison-Wesley, 1972.