

On modal logics of model-theoretic relations*

Denis I. Saveliev^{a,b} and Ilya B. Shapirovsky^{a,b}

^aSteklov Mathematical Institute of Russian Academy of Sciences

^bInstitute for Information Transmission Problems of Russian Academy of Sciences

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Abstract

Given a class \mathcal{C} of models, a binary relation \mathcal{R} between models, and a model-theoretic language L , we consider the modal logic and the modal algebra of the theory of \mathcal{C} in L where the modal operator is interpreted via \mathcal{R} . We discuss how modal theories of \mathcal{C} and \mathcal{R} depend on the model-theoretic language, their Kripke completeness, and expressibility of the modality inside L . We calculate such theories for the submodel and the quotient relations. We prove a downward Löwenheim–Skolem theorem for first-order language expanded with the modal operator for the extension relation between models.

Keywords: *modal logic, modal algebra, robust modal theory, logic of submodels, logic of quotients, logic of forcing, provability logic, model-theoretic logic*

Introduction

We consider modal systems in which the modal operator is interpreted via a binary relation on a class of models. Many instances of such systems can be found in the literature. During the last years, modal logics of various relations between models of set theory have been studied, see, e.g., [15, 18, 7, 17, 21]. A well established area in provability logic deals with modal axiomatizations of relations between models of arithmetic (and between arithmetic theories), see, e.g., [27, 5, 20, 34, 35, 19, 16]. In another extensively studied area modalities are interpreted by relations between Kripke and temporal models, see, e.g., [33, 11, 4] or the monograph [32]. In [3], the consequence along an abstract relation between models is studied, which is closely related to our consideration.

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Let f be a unary operation on sentences of a model-theoretic language L , and T a set of sentences of L (e.g., the set of theorems in a given calculus, or the set of sentences valid in a given class of models). Using the propositional modal language, one can consider the following “fragment” of T : variables are evaluated by sentences of L , and f interprets the modal operator; the *modal theory of f on T* , or just the *f -fragment of T* , is defined as the set of those modal formulas which are in T under every valuation. A well-known example of this approach is a complete modal axiomatization of formal provability in Peano arithmetic given by Solovay [31]. Another important example is the theorem by Hamkins and Löwe axiomatizing the modal logic of forcing (introduced earlier by Hamkins in [15]) where the modal operator expresses satisfiability in forcing extensions [18]. Both these modal systems have good semantic and algorithmic properties; in particular, they have the finite model property, are finitely axiomatizable, and hence decidable.

These examples inspire the following observation. Let \mathcal{C} be an arbitrary class of models of the same signature, $T = Th^L(\mathcal{C})$ the theory of \mathcal{C} in a model-theoretic language L , and \mathcal{R} a binary relation on \mathcal{C} . Assuming that the satisfiability in \mathcal{R} -images of models in \mathcal{C} can be expressed by an operation f on sentences of L , i.e., for every sentence φ of L , and every $\mathfrak{A} \in \mathcal{C}$,

$$\mathfrak{A} \models f(\varphi) \text{ (“}\varphi \text{ is possible at } \mathfrak{A}\text{”) iff } \mathfrak{B} \models \varphi \text{ for some } \mathfrak{B} \text{ with } \mathfrak{A} \mathcal{R} \mathfrak{B},$$

we can define the *modal theory of \mathcal{R} in L* as the f -fragment of T . In the general frame semantics, this modal theory is characterized by an enormous structure $(\mathcal{C}, \mathcal{R}, \mathcal{C}_\varphi : \varphi \text{ is a sentence of } L)$ where \mathcal{C}_φ is the class of models in \mathcal{C} validating φ . We can also define the *modal Lindenbaum algebra of $Th^L(\mathcal{C})$ and \mathcal{R}* , i.e., the Boolean algebra of sentences of L modulo the equivalence on \mathcal{C} , endowed with the modal operator induced by f .¹ In Section 2, we provide formal definitions and basic semantic tools for such modal theories. In particular, the algebra of $Th^L(\mathcal{C})$ and \mathcal{R} can be represented as the modal algebra of a general frame consisting of complete theories $\{Th^L(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}\}$. We use this in Section 3, where we calculate modal logics of the submodel relation \sqsubseteq ; to express the satisfiability in submodels, we use second-order language.

In Section 4,² we discuss the situation when the \mathcal{R} -satisfiability is not expressible in a language K (for example, this situation is typical when K is first-order). In this case the modal algebra of $Th^K(\mathcal{C})$ can be defined as the subalgebra of the modal algebra of $Th^L(\mathcal{C})$ generated by the sentences of K ,

¹Modal algebras of theories are broadly used in provability logic, where they are called *Magari* (or *diagonalizable*) *algebras*; they are known to keep a lot of information about theories containing arithmetic (see, e.g., [28]).

²A significant part of Sections 4 and 5 was motivated by reviewers’ questions on an earlier version of this paper.

for any L stronger than K where the \mathcal{R} -satisfiability on \mathcal{C} is expressible: the resulting modal algebra (and hence, its modal logic) does not depend on the way how we extend the language K . Under a natural assumption on \mathcal{C} and \mathcal{R} , such an L can always be constructed, and hence, *the* modal algebra of $Th^K(\mathcal{C})$ and \mathcal{R} is well-defined for arbitrary K (however, the resulting modal logic is not necessarily a “fragment” of the theory $Th^K(\mathcal{C})$ anymore). Then we consider the finitary first-order language expanded with the modal operator for the extension relation \sqsubseteq and prove a version of the downward Löwenheim–Skolem theorem for this language.

In general, modal theories of \mathcal{R} depend on the model-theoretic language we consider. We say that a modal theory of \mathcal{R} is *robust* iff making the language stronger does not alter this theory (intuitively, the robust theory can be considered as the “true” modal logic of the model-theoretic relation \mathcal{R}). We discuss this notion in Section 5. In Theorem 29, we show that under a certain assumption on \mathcal{C} and \mathcal{R} , the robust logic is Kripke complete. Then we use this theorem to describe robust theories of the quotient and the submodel relations on certain natural classes.

A preliminary report on some results in Sections 2 and 3 can be found in [26].

1 Preliminaries

To simplify reading, as a rule we denote the syntax of our object languages differently: we use inclined letters if they are related to model theory (x, y, \dots for individual variables, φ, ψ, \dots for formulas), and upright letters if they are related to modal logic (p, q, \dots for propositional variables, φ, ψ, \dots for formulas).

Model-theoretic languages The languages we use for model theory are model-theoretic languages in sense of [2] (where they are called “model-theoretic logics”). We are pointing out here only two of their features, which will be essential for the further investigations: if L is a model-theoretic language under our consideration, then:

- (i) Satisfiability in L is preserved under isomorphisms;
- (ii) L includes $L_{\omega, \omega}$, the standard first-order language with usual finitary connectives and quantifiers.

Also, we assume that L is a set unless otherwise specified. For a model \mathfrak{A} , $Th^L(\mathfrak{A})$, or just $Th(\mathfrak{A})$, denotes its theory in L , i.e., the set of all sentences of L holding in \mathfrak{A} ; likewise for a class of models.

Modal logic By a *modal logic* we shall mean a normal propositional uni-modal logic (see, e.g., [6] or [9]). Modal formulas are built from a countable set PV of propositional variables p, q, \dots , the Boolean connectives, and the modal connectives \diamond, \square . We assume that $\perp, \rightarrow, \diamond$ are basic, and others are abbreviations; in particular, \square abbreviates $\neg \diamond \neg$. Let MF denote the set of modal formulas. A set of modal formulas is called a *normal modal logic* iff it contains all classical tautologies, two following modal axioms:

$$\neg \diamond \perp \text{ and } \diamond(p \vee q) \rightarrow \diamond p \vee \diamond q,$$

and is closed under three following rules of inference:

$$\text{Modus Ponens } \frac{\varphi, \varphi \rightarrow \psi}{\psi}, \text{ Substitution } \frac{\varphi(p)}{\varphi(\psi)}, \text{ Monotonicity } \frac{\varphi \rightarrow \psi}{\diamond \varphi \rightarrow \diamond \psi}.$$

A *general frame* \mathfrak{F} is a triple (W, R, \mathcal{V}) where W is a nonempty set, $R \subseteq W \times W$, and \mathcal{V} is a subalgebra of the powerset algebra $\mathcal{P}(W)$ closed under R^{-1} (i.e., such that $R^{-1}(A) = \{x \in W : \exists y \in A \ x R y\}$ is in \mathcal{V} whenever A is in \mathcal{V}). By a *Kripke frame* (W, R) we mean $(W, R, \mathcal{P}(W))$. We shall often use the term *frame* for a general frame (and never for a Kripke frame). A *modal algebra* is a Boolean algebra endowed with a unary operation that distributes w.r.t. finite disjunctions. The *modal algebra of a frame* (W, R, \mathcal{V}) is (\mathcal{V}, R^{-1}) .

A *Kripke model* \mathfrak{M} on a frame $\mathfrak{F} = (W, R, \mathcal{V})$ is a pair (\mathfrak{F}, θ) where $\theta : PV \rightarrow \mathcal{V}$ is a *valuation*. The *truth relation* $\mathfrak{M}, x \models \varphi$ (“ φ is true at x in \mathfrak{M} ”) is defined in the standard way (see, e.g., [9]); in particular, $\mathfrak{M}, x \models \diamond \varphi$ iff $\mathfrak{M}, y \models \varphi$ for some y with $x R y$. A formula $\varphi \in MF$ is *true in a model* \mathfrak{M} iff it is true at every x in \mathfrak{M} , and φ is *valid in a frame* \mathfrak{F} iff it is true in every model on \mathfrak{F} . A modal formula φ is *valid in a modal algebra* \mathfrak{A} iff $\varphi = \top$ holds in \mathfrak{A} . In particular, it follows that if \mathfrak{A} is the modal algebra of a frame \mathfrak{F} , then φ is valid in \mathfrak{A} iff φ is valid in \mathfrak{F} . The set $MLog(\mathfrak{F})$ of all valid in \mathfrak{F} formulas is called the (*modal*) *logic of the frame* \mathfrak{F} ; likewise for algebras. It is well-known (see, e.g., [6] or [9]) that a set of formulas is a consistent normal logic iff it is the logic of a general frame iff it is the logic of a non-trivial modal algebra.

In our paper we also consider modal logics of general frames $(\mathcal{C}, \mathcal{R}, \mathcal{V})$ with a proper class \mathcal{C} ; see Remarks 1 and 3 in the next section.

S4 is the smallest logic containing the formulas $\diamond \diamond p \rightarrow \diamond p$ and $p \rightarrow \diamond p$, S4.2 is $S4 + \diamond \square p \rightarrow \square \diamond p$, and S4.2.1 is $S4 + \square \diamond p \leftrightarrow \diamond \square p$, where $\Lambda + \varphi$ is the smallest logic that includes $\Lambda \cup \{\varphi\}$.

2 Modal theories of model-theoretic relations

Definitions Fix a signature Ω and a language $L = L(\Omega)$ based on Ω . Let L_s be the set of all sentences of L .

Consider a unary operation f on sentences of L . A (*propositional*) *valuation in L* is a map from PV to L_s . A valuation $\|\cdot\|$ extends to the set MF of modal formulas as follows:

$$\begin{aligned}\|\perp\| &= \perp, \\ \|\varphi \rightarrow \psi\| &= \|\varphi\| \rightarrow \|\psi\|, \\ \|\diamond\varphi\| &= f(\|\varphi\|).\end{aligned}$$

Given a set of sentences $T \subseteq L_s$, we define the *modal theory of f on T* $\text{MTh}(T, f)$ as the set of modal formulas φ such that for every valuation $\|\cdot\|$ in L , $\|\varphi\|$ is in T . Thus, modal formulas can be viewed as axiom schemas (for sentences) and we can think of the modal theory of f on T as a fragment of T .

Example 1. Let L be the usual first-order finitary language $L_{\omega, \omega}$.

1. Let Ω be the signature of arithmetic and $f(\varphi)$ express the consistency of a sentence φ in Peano arithmetic PA. By the well-known Solovay's results [31], the modal theory of f on PA is the Gödel–Löb logic GL, and the modal theory of f on the true arithmetic TA, the set of all sentences that are true in the standard model of arithmetic, is the (quasi-normal but not normal) Solovay logic S. A survey on provability logics can be found in [1].

2. Let Ω be the signature of set theory (i.e., $\Omega = \{\in\}$) and $f(\varphi)$ express that φ holds in a generic extension. By Hamkins and Löwe [18], the modal theory of f on ZFC is S4.2.

In our work, we are interested in the case when T is the theory of some class \mathcal{C} of models of Ω . An easy observation shows that in this case the modal theory is closed under the rules of Modus Ponens and Substitution. Notice that it is not necessarily closed under Monotonicity, as the instance of the Solovay logic S shows.³ However, as we shall see, the modal theory is a normal logic whenever f expresses the satisfiability in images of a binary relation between models.

Definition 1. Let \mathcal{R} be a binary relation on \mathcal{C} (or, perhaps, on a larger class $\mathcal{C}' \supseteq \mathcal{C}$). The *\mathcal{R} -satisfiability on \mathcal{C} is expressible in L* iff there exists $f : L_s \rightarrow L_s$ such that for every sentence $\varphi \in L_s$ and every model $\mathfrak{A} \in \mathcal{C}$,

$$\mathfrak{A} \models f(\varphi) \Leftrightarrow \exists \mathfrak{B} \in \mathcal{C} (\mathfrak{A} \mathcal{R} \mathfrak{B} \ \& \ \mathfrak{B} \models \varphi).$$

Some examples of such expressibility and non-expressibility will be given below.

The following is straightforward:

³Let us also mention the logic of *pure provability* introduced by Buss in [8], which is not closed under the rule of substitution.

Proposition 1. *If f and f' both express the \mathcal{R} -satisfiability on \mathcal{C} , and $T = Th(\mathcal{C})$, then $MTh(T, f) = MTh(T, f')$.*

Thus, in this case, $MTh(T, f)$ does not depend on the choice of f .

Definition 2. Let the \mathcal{R} -satisfiability be expressible in L on a class \mathcal{C} of models. The *modal theory of $(\mathcal{C}, \mathcal{R})$ in L* , denoted by $MTh^L(\mathcal{C}, \mathcal{R})$, is the set of modal formulas φ such that $\mathcal{C} \models \|\varphi\|$ for every propositional valuation in L .

In this section, we just write $MTh(\mathcal{C}, \mathcal{R})$ assuming L is fixed.

Given a propositional valuation $\|\cdot\|$ in L , consider $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, \theta)$, the enormous “Kripke model” with $\theta(\mathfrak{p}) = \{\mathfrak{A} \in \mathcal{C} : \mathfrak{A} \models \|\mathfrak{p}\|\}$. By a straightforward induction on a modal formula φ , for every \mathfrak{A} in \mathcal{C}

$$\mathfrak{A} \models \|\varphi\| \Leftrightarrow \mathfrak{M}, \mathfrak{A} \models \varphi,$$

where \models on the left-hand and on the right-hand side of the equivalence denotes the satisfiability relation in model theory and the truth relation of a modal formula in a Kripke model, respectively. It follows that

$$\mathcal{C} \models \|\varphi\| \Leftrightarrow \mathfrak{M}, \mathfrak{A} \models \varphi \text{ for all } \mathfrak{A} \in \mathcal{C}.$$

Let $Mod(\psi)$ be the class of models of $\psi \in L_s$, and $\mathcal{C}_\psi = Mod(\psi) \cap \mathcal{C}$. Then validity of $\|\varphi\|$ in \mathcal{C} for all propositional valuations in L can be considered as validity of the modal formula φ in an enormous “general frame of models” $(\mathcal{C}, \mathcal{R}, \mathcal{V})$ where \mathcal{V} “consists” of \mathcal{C}_ψ with $\psi \in L_s$.

Remark 1. The collection \mathcal{V} looks like a “class of classes”. In fact, however, \mathcal{V} is a usual class defined by a formula of set theory with two parameters: if \mathcal{C}, L_s are defined by formulas Φ, Φ' , respectively, then what we understand by \mathcal{V} is the class of pairs (\mathfrak{A}, ψ) defined by a formula \mathcal{Y} that is constructed from Φ, Φ' and expresses the satisfaction relation between models in \mathcal{C} and sentences in L_s . Thus subclasses \mathcal{C}_ψ of \mathcal{C} playing the role of “elements” of \mathcal{V} are in fact defined by \mathcal{Y} with a fixed second argument ψ . This allows us to formally extend the definition of validity in a (set) frame to the case of the “frame of models” $(\mathcal{C}, \mathcal{R}, \mathcal{V})$.

It follows that the modal logic $MLog(\mathcal{C}, \mathcal{R}, \mathcal{V})$, i.e., the set of all modal formulas that are valid in $(\mathcal{C}, \mathcal{R}, \mathcal{V})$, coincides with the modal theory of $(\mathcal{C}, \mathcal{R})$ in L . Namely, we have

Theorem 2. *If the \mathcal{R} -satisfiability on \mathcal{C} is expressible in L , then*

$$MTh(\mathcal{C}, \mathcal{R}) = MLog(\mathcal{C}, \mathcal{R}, \mathcal{V}). \quad (1)$$

Consequently, $MTh(\mathcal{C}, \mathcal{R})$ is a normal logic.

Proposition 3. *Assume that f expresses the \mathcal{R} -satisfiability on \mathcal{C} in L .*

1. If $\mathcal{D} \subseteq \mathcal{C}$ is \mathcal{R} -upward closed (i.e., $\mathfrak{A} \in \mathcal{D}$ & $\mathfrak{A} \mathcal{R} \mathfrak{B}$ & $\mathfrak{B} \in \mathcal{C}$ implies $\mathfrak{B} \in \mathcal{D}$), then the \mathcal{R} -satisfiability on \mathcal{D} is expressible in L by f .
2. If ψ is a sentence of L , then the \mathcal{R} -satisfiability on $\mathcal{C} \cap \text{Mod}(\psi)$ is expressible in L by $g : \varphi \mapsto f(\varphi \wedge \psi)$.

Let $\mathcal{R}^*(\mathfrak{A})$ denote $\bigcup_{n < \omega} \mathcal{R}^n(\mathfrak{A})$, the least \mathcal{R} -closed $\mathcal{D} \subseteq \mathcal{C}$ containing \mathfrak{A} . From (1) and the generated subframe construction (see, e.g., [9, Section 8.5]), we obtain

Corollary 4. *If the \mathcal{R} -satisfiability on \mathcal{C} is expressible in L , then*

$$\text{MTh}(\mathcal{C}, \mathcal{R}) = \bigcap_{\mathfrak{A} \in \mathcal{C}} \text{MTh}(\mathcal{R}^*(\mathfrak{A}), \mathcal{R}).$$

Frames of theories Assume that the \mathcal{R} -satisfiability is expressible on \mathcal{C} in L by some $f : L_s \rightarrow L_s$.

We have observed in Theorem 2 that $\text{MTh}(\mathcal{C}, \mathcal{R})$ can be viewed as the modal logic of a general frame of models. Now we provide other semantic characterizations of $\text{MTh}(\mathcal{C}, \mathcal{R})$.

Put $\mathcal{T} = \{\text{Th}(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}\}$. Note that \mathcal{T} is a set since L is assumed to be a set. For theories $T_1, T_2 \in \mathcal{T}$, let

$$\begin{aligned} T_1 \mathcal{R}_{\mathcal{T}} T_2 &\quad \text{iff} \quad \exists \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C} (\mathfrak{A}_1 \models T_1 \ \& \ \mathfrak{A}_2 \models T_2 \ \& \ \mathfrak{A}_1 \mathcal{R} \mathfrak{A}_2), \\ T_1 \mathcal{R}_{\mathcal{T}}^{\circ} T_2 &\quad \text{iff} \quad \forall \varphi \in T_2 \ f(\varphi) \in T_1. \end{aligned}$$

Observe that $\mathcal{R}_{\mathcal{T}}^{\circ}$ does not depend on the choice of f , and $\mathcal{R}_{\mathcal{T}} \subseteq \mathcal{R}_{\mathcal{T}}^{\circ}$.

For $\varphi \in L_s$, put $\mathcal{T}_{\varphi} = \{T \in \mathcal{T} : \varphi \in T\}$. Clearly, $\mathcal{T}_{\varphi \wedge \psi} = \mathcal{T}_{\varphi} \cap \mathcal{T}_{\psi}$ and $\mathcal{T}_{\neg \varphi} = \mathcal{T} \setminus \mathcal{T}_{\varphi}$. Consider an arbitrary binary relation $\overline{\mathcal{R}}$ such that

$$\mathcal{R}_{\mathcal{T}} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{R}_{\mathcal{T}}^{\circ}.$$

It follows from the definitions that $\overline{\mathcal{R}}^{-1}(\mathcal{T}_{\varphi}) = \mathcal{T}_{f(\varphi)}$. Therefore,

$$\mathcal{A} = \{\mathcal{T}_{\varphi} : \varphi \in L_s\}$$

is a subalgebra of the powerset algebra $\mathcal{P}(\mathcal{T})$ closed under $\overline{\mathcal{R}}^{-1}$, and $(\mathcal{T}, \overline{\mathcal{R}}, \mathcal{A})$ is a general frame. We call $(\mathcal{T}, \mathcal{R}_{\mathcal{T}}, \mathcal{A})$ and $(\mathcal{T}, \mathcal{R}_{\mathcal{T}}^{\circ}, \mathcal{A})$ the *minimal* and the *maximal frames of theories for \mathcal{C}, \mathcal{R} , and L* .

Remark 2. The relation $\overline{\mathcal{R}}$ can be viewed as a *filtration* of \mathcal{R} (see, e.g., [14, Section 4]). The frame $(\mathcal{T}, \mathcal{R}_{\mathcal{T}}^{\circ}, \mathcal{A})$ is known as the *refinement* of $(\mathcal{C}, \mathcal{R}, \mathcal{V})$ (cf. [9, Chapter 8]).

For sentences φ, ψ , put $\varphi \approx \psi$ iff $\mathcal{C} \models \varphi \leftrightarrow \psi$. Let L/\approx be the Lindenbaum algebra of the theory $\text{Th}(\mathcal{C})$, i.e., the set L_s of sentences of L modulo \approx with operations induced by Boolean connectives. We have $f(\varphi) \approx f(\psi)$ whenever $\varphi \approx \psi$, hence, f induces the operation f_{\approx} on L/\approx . It is easy to see that $(L/\approx, f_{\approx})$ is a modal algebra. It is called the *modal (Lindenbaum) algebra of the language L on $(\mathcal{C}, \mathcal{R})$* .

The above arguments yield

Theorem 5. *If $\mathcal{R}_{\mathcal{T}} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{R}_{\mathcal{T}}^{\circ}$, then the modal algebras $(L/\approx, f_{\approx})$ and $(\mathcal{A}, \overline{\mathcal{R}}^{-1})$ are isomorphic.*

Proof. The isomorphism takes the \approx -class of a sentence φ to \mathcal{T}_{φ} . \square

Theorem 6. *Assume that the \mathcal{R} -satisfiability on \mathcal{C} is expressible in L . Let $\mathcal{R}_{\mathcal{T}} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{R}_{\mathcal{T}}^{\circ}$. Then*

$$\text{MTh}(\mathcal{C}, \mathcal{R}) = \text{MLog}(\mathcal{T}, \overline{\mathcal{R}}, \mathcal{A}) = \text{MLog}(L/\approx, f_{\approx}).$$

Proof. Every valuation $\|\cdot\|$ in L can be viewed as a valuation θ in the frame $\mathfrak{F} = (\mathcal{T}, \overline{\mathcal{R}}, \mathcal{A})$, and vice versa. By induction on φ , for every \mathfrak{A} in \mathcal{C} we have $\mathfrak{A} \models \|\varphi\|$ iff $(\mathfrak{F}, \theta), \text{Th}(\mathfrak{A}) \models \varphi$. Thus $\text{MTh}(\mathcal{C}, \mathcal{R}) = \text{MLog}(\mathcal{T}, \overline{\mathcal{R}}, \mathcal{A})$.

The second equality immediately follows from Theorem 5. \square

Corollary 7. *If \mathcal{T} is finite, then $\text{MTh}(\mathcal{C}, \mathcal{R})$ is the logic of the Kripke frame $(\mathcal{T}, \mathcal{R}_{\mathcal{T}})$.*

Proof. In this case $\mathcal{A} = \mathcal{P}(\mathcal{T})$. \square

The family \mathcal{T} of complete theories can be viewed as the quotient of \mathcal{C} by the L -equivalence \equiv where $\mathfrak{A} \equiv \mathfrak{B}$ iff $\text{Th}(\mathfrak{A}) = \text{Th}(\mathfrak{B})$. Theorem 6 can be generalized for the case of any equivalence \sim on \mathcal{C} finer than \equiv ; in particular, it holds for the isomorphism equivalence \simeq on models, or for the equivalence in a stronger model-theoretic language. Namely, we let:

$$\begin{aligned} [\mathfrak{A}]_{\sim} \mathcal{R}_{\sim} [\mathfrak{B}]_{\sim} &\Leftrightarrow \exists \mathfrak{A}' \sim \mathfrak{A} \exists \mathfrak{B}' \sim \mathfrak{B} \mathfrak{A}' \mathcal{R} \mathfrak{B}', \\ [\mathfrak{A}]_{\sim} \mathcal{R}_{\sim}^{\circ} [\mathfrak{B}]_{\sim} &\Leftrightarrow \forall \varphi \in L_s (\mathfrak{B} \models \varphi \Rightarrow \mathfrak{A} \models f(\varphi)); \end{aligned}$$

the algebra of valuations is defined as the collection \mathcal{V}_{\sim} “consisting” of classes \mathcal{C}_{ψ}/\sim for $\psi \in L_s$. (Again, since $[\mathfrak{A}]_{\sim}$ are in general proper classes, \mathcal{V}_{\sim} looks like a “class of classes of classes” but actually is nothing but a three-parameter formula; cf. Remarks 1 and 3.) As in the proof of Theorem 6, for $\mathcal{R}_{\sim} \subseteq \overline{\mathcal{R}} \subseteq \mathcal{R}_{\sim}^{\circ}$ one can obtain that

$$\text{MTh}(\mathcal{C}, \mathcal{R}) = \text{MLog}(\mathcal{C}/\sim, \overline{\mathcal{R}}, \mathcal{V}_{\sim}). \quad (2)$$

To the best of our knowledge, Theorem 6 (or its analogue (2)) has never been formulated explicitly before, although similar constructions related to frames of arithmetic theories were considered earlier (V. Yu. Shavrukov, an unpublished note, 2013; [19, Remark 3]).

We conclude this section with a continuation of Remark 1.

Remark 3. It is possible to give a general definition of frames that are classes and their logics (in our metatheory which is assumed to be ZFC where “classes” are shorthands for formulas). We outline the idea and postpone

details for a further paper. Below capital Greek letters Φ, Ψ, \dots denote formulas of the metatheory.

Assume that $\mathcal{C}, \mathcal{L}, \mathcal{R}$ are (arbitrary) classes defined by formulas Φ, Φ', Ψ , respectively: $\mathcal{C} = \{x : \Phi(x)\}$, $\mathcal{L} = \{y : \Phi'(y)\}$, and $\mathcal{R} = \{(x, v) : \Psi(x, v)\}$. Let us say that a class \mathcal{V} forms a *class modal algebra* (admissible for $\mathcal{C}, \mathcal{L}, \mathcal{R}$) iff it is defined by a formula Υ that fulfills the conditions expressing that classes $\mathcal{C}_y = \{x : \Upsilon(x, y)\}$ indexed by y in \mathcal{L} play the role of “elements” of \mathcal{V} . Namely, Υ implies that all \mathcal{C}_y are subclasses of \mathcal{C} and their collection is closed under Boolean operations and the modal operator given by \mathcal{R} ; e.g., the latter is expressed as follows:

$$\forall y \exists z \forall x (\Upsilon(x, z) \& \Phi'(y) \Leftrightarrow \Phi(x) \& \exists v (\Psi(x, v) \& \Upsilon(v, y))).$$

Further, let us say that θ is a *valuation* of propositional variables in \mathcal{V} iff θ is a (set) function with $\text{dom}(\theta) = \text{PV}$ and $\text{ran}(\theta) \subseteq \mathcal{L}$, and that $\mathfrak{F} = (\mathcal{C}, \mathcal{R}, \mathcal{V})$ is a *class general frame* and $\mathfrak{M} = (\mathcal{C}, \mathcal{R}, \mathcal{V}, \theta)$ a *class Kripke model* on \mathfrak{F} . To define the *truth* of modal formulas φ at a point $x \in \mathcal{C}$ in the model \mathfrak{M} , denoted by $\mathfrak{M}, x \models \varphi$, we first extend θ to a suitable $\bar{\theta}$ with $\text{dom}(\bar{\theta}) = \text{MF}$ and $\text{ran}(\bar{\theta}) \subseteq \mathcal{L}$. It can be shown that such an extension exists and is unique up to the equivalence \sim on \mathcal{L} defined by letting $y \sim z$ iff $\forall x (\Upsilon(x, y) \Leftrightarrow \Upsilon(x, z))$. Then we let

$$\mathfrak{M}, x \models \varphi \Leftrightarrow \Upsilon(x, \bar{\theta}(\varphi)).$$

This notion of truth has the expected properties; e.g., we have

$$\mathfrak{M}, x \models \diamond \varphi \Leftrightarrow \exists v (\Psi(x, v) \& \mathfrak{M}, v \models \varphi).$$

A formula $\varphi \in \text{MF}$ is *true* in \mathfrak{M} iff $\forall x (\Phi(x) \Rightarrow \mathfrak{M}, x \models \varphi)$, and *valid* in \mathfrak{F} iff it is true in all models \mathfrak{M} on \mathfrak{F} . The *modal logic* $\text{MLog}(\mathfrak{F})$ of the class frame \mathfrak{F} consists of those $\varphi \in \text{MF}$ that are valid in \mathfrak{F} . It can be verified that this logic is a set defined by a ZFC-formula constructed from the formulas $\Phi, \Phi', \Psi, \Upsilon$, and it is normal.

By using formulas with additional parameters, one can imitate higher order class algebras and their modal logics.

3 Logics of submodels

In this part, we apply Theorem 6 to calculate the modal theory of the *submodel relation* on the class of all models of a given signature.

Expressing the satisfiability Given models \mathfrak{A} and \mathfrak{B} of a signature Ω , let $\mathfrak{A} \sqsupseteq \mathfrak{B}$ mean “ \mathfrak{A} contains \mathfrak{B} as a submodel”. As the initial step, we find a model-theoretic language for expressing the \sqsupseteq -satisfiability. Observe first that first-order languages are generally too weak for this.

Proposition 8. *If Ω contains a predicate symbol of arity ≥ 2 , then the \sqsubseteq -satisfiability is not expressible in $L_{\omega,\omega}$ and moreover, in the infinitary language $L_{\infty,\omega}$.*

Proof. We can suppose w.l.g. that Ω contains a binary predicate symbol $<$ (otherwise mimic it by a predicate symbol of a bigger arity by fixing other arguments). Toward a contradiction, assume that some f mapping the class of $L_{\infty,\omega}$ -sentences into itself expresses the \sqsubseteq -satisfiability. Let φ be an obvious $L_{\omega,\omega}$ -sentence saying that there exists a $<$ -minimal element, and let ψ be the sentence $\neg f(\neg \varphi)$. Then ψ says that each submodel has a $<$ -minimal element (thus whenever Ω has no functional symbols then ψ says that $<$ is well-founded). Let κ be such that $\psi \in L_{\kappa,\omega}$. It follows from Karp's theorem (see, e.g., [24, Theorem 14.29]) that there are models \mathfrak{A}_0 and \mathfrak{B}_0 of $\Omega_0 = \{<\}$ such that \mathfrak{A}_0 is isomorphic to an ordinal while \mathfrak{B}_0 is not, and $\mathfrak{A}_0 \equiv_{L_{\kappa,\omega}} \mathfrak{B}_0$. Add a $<$ -last element to each of the models \mathfrak{A}_0 and \mathfrak{B}_0 and check that the resulting models \mathfrak{A}_1 and \mathfrak{B}_1 remain $L_{\kappa,\omega}$ -equivalent (e.g., by using [24, Lemma 14.24]).

Expand \mathfrak{A}_1 and \mathfrak{B}_1 to models \mathfrak{A} and \mathfrak{B} of Ω , respectively, by interpreting each predicate symbol other than $<$ by the empty set, each functional symbol of positive arity by the projection onto the first argument, and each constant symbol by the $<$ -last element of the model. It is easy to see that in both \mathfrak{A} and \mathfrak{B} any formula of Ω is equivalent to a formula of Ω_0 ; so we still have $\mathfrak{A} \equiv_{L_{\kappa,\omega}} \mathfrak{B}$. On the other hand, in both models every subset forms a submodel whenever it contains the $<$ -last element of the whole model, whence it easily follows that $\mathfrak{A} \models \psi$ and $\mathfrak{B} \models \neg \psi$. A contradiction. \square

However, second-order language suffices to express the \sqsubseteq -satisfiability by the relativization argument (see, e.g., [36, p. 242]). Given a second-order formula φ and a unary predicate variable U that does not occur in φ , let φ^U be the relativization of φ to U defined in the standard way; in particular, if P and F are second-order predicate functional variables of arity n , then

$$\begin{aligned} (\exists P \varphi)^U & \text{ is } \exists P (\forall x_0 \dots \forall x_{n-1} (P(x_0, \dots, x_{n-1}) \rightarrow \bigwedge_{i < n} U(x_i)) \wedge \varphi^U), \\ (\exists F \varphi)^U & \text{ is } \exists F (\forall x_0 \dots \forall x_{n-1} (\bigwedge_{i < n} U(x_i) \rightarrow U(F(x_0, \dots, x_{n-1}))) \wedge \varphi^U). \end{aligned}$$

Let $\psi(U)$ be the formula expressing that U is a submodel, i.e., saying that the interpretation of U is non-empty and is closed under interpretations of functional symbols in Ω . Then the map $\varphi \mapsto \exists U (\psi(U) \wedge \varphi^U)$ expresses the \sqsubseteq -satisfiability on the class of all models of Ω . In view of Proposition 3, we obtain:

Proposition 9. *Let $\kappa = |\{F \in \Omega : F \text{ is a functional symbol}\}|$.*

1. *Let \mathcal{C} be a class of models of Ω closed under submodels. Then the \sqsubseteq -satisfiability is expressible on \mathcal{C} in $L_{\lambda,\omega}^2$ whenever $\lambda > \kappa$ and $\lambda \geq \omega$.*

2. Let T be a set of sentences of $L_{\mu,\omega}^2$ of Ω . Then the \sqsubseteq -satisfiability is expressible on the class $\text{Mod}(T)$ in $L_{\lambda,\omega}^2$ whenever $\lambda > \max(\kappa, |T|)$ and $\lambda \geq \mu$.

Remark 4. These results on expressibility can be refined in several directions. In particular, the first statement of Proposition 9 remains true for monadic language $L_{\lambda,\omega}^2$; the assumption $\lambda > \kappa$ is necessary; for details and further results, see [25].

Axiomatization Henceforth in this section we assume that L expresses the \sqsubseteq -satisfiability on the class of models under consideration.

The next easy result is soundness for modal theories of the submodel relation.

Theorem 10. *Let \mathcal{C} be a class of Ω -models closed under submodels. Then $\text{MTh}^L(\mathcal{C}, \sqsubseteq)$ is a normal modal logic including S4. If moreover, Ω contains a constant symbol, then $\text{MTh}^L(\mathcal{C}, \sqsubseteq)$ includes S4.2.1.*

Proof. Let $\mathfrak{A} \in \mathcal{C}$, $\|\cdot\|$ a valuation in L . Trivially, $\mathfrak{A} \models \|\diamond\diamond p \rightarrow \diamond p\|$ and $\mathfrak{A} \models \|\Box p \rightarrow \diamond p\|$. If Ω contains a constant symbol, consider the submodel \mathfrak{B} of \mathfrak{A} generated by constants. It is straightforward that in this case $\mathfrak{A} \models \|\Box\diamond p\|$ iff $\mathfrak{B} \models \|\diamond p\|$ iff $\mathfrak{B} \models \|\Box p\|$ iff $\mathfrak{A} \models \|\diamond\Box p\|$. \square

We are going to prove completeness. Let Q_n be the lexicographic product of $(n^{<n}, \subseteq)$ (an n -ramified tree of height n) and $(n, n \times n)$ (a cluster of size n). Thus for $s, t \in n^{<n}$ and $i, j \in n$, in Q_n we have

$$(s, i) \leq (t, j) \Leftrightarrow s \subseteq t,$$

so Q_n is a pre-tree which is n -ramified, has height n and clusters of size n at each point. Let also Q'_n be the ordered sum of Q_n and a reflexive singleton, thus Q'_n adds to Q_n an extra top element. The following fact is standard (see, e.g., [9, p. 563]).

Proposition 11. *Let φ be a modal formula. If $\varphi \notin \text{S4}$, then φ is not valid in Q_n for some $n > 0$. If $\varphi \notin \text{S4.2.1}$, then φ is not valid in Q'_n for some $n > 0$.*

Let \equiv be the L -equivalence. For a model \mathfrak{A} , let $\text{Sub}(\mathfrak{A})$ be the set of all its submodels, $\text{Sub}(\mathfrak{A})_{\equiv}$ abbreviate $\text{Sub}(\mathfrak{A})/\equiv$.

Theorem 12. *Let Ω have a functional symbol of arity ≥ 2 . For every positive $n < \omega$, there exists a model \mathfrak{A}_n of Ω such that*

$$(\text{Sub}(\mathfrak{A}_n)_{\equiv}, \sqsubseteq_{\equiv}) \text{ is isomorphic to } \begin{cases} Q_n & \text{if } \Omega \text{ has no constant symbols,} \\ Q'_n & \text{otherwise.} \end{cases}$$

Proof. First, suppose that Ω has no constant symbols.

Without loss of generality we may assume that Ω has a binary functional symbol; we write \cdot for it.

Fix $n \geq 1$. Let $X_n = n^{<n} \times \omega$. We define the model $\mathfrak{A}_n = (X_n, \cdot, \dots)$ of Ω as follows. Let E be any injective map from $n^{<n}$ into ω . For $s, t \in n^{<n}$ and $i, j \in \omega$, we put

$$(s, i) \cdot (t, j) = \begin{cases} (s, i + 1) & \text{if } s = t, i = j, \\ (s \frown (i \bmod n), j) & \text{if } s = t, j < i, |s| < n - 1, \\ (s, j + E(s)) & \text{if } s = t, j = i + 1, i \equiv_n 0, \\ (\inf\{s, t\}, \inf\{i, j\}) & \text{otherwise,} \end{cases}$$

where \frown denotes the concatenation, $|s|$ the length of s , \bmod the remainder, and \equiv_n the congruence modulo n . For other operations F in \mathfrak{A}_n we put $F((s, i), \dots) = (s, i)$ and take the relations in \mathfrak{A}_n to be empty.

For $(s, i) \in X_n$ let $X_n(s, i) = \{(t, j) \in X_n : (s, i) \preceq (t, j)\}$, where we let $(s, i) \preceq (t, j)$ iff $s \subseteq t$ and $i \leq j$.

Lemma 13. *An $X \subseteq X_n$ is the universe of a submodel of \mathfrak{A}_n iff $X = X_n(s, i)$ for some $(s, i) \in X_n$.*

Proof. Straightforward from the definition of \mathfrak{A}_n . □

Let $\mathfrak{A}_n(s, i)$ be the submodel of \mathfrak{A}_n with the universe $X_n(s, i)$.

Lemma 14. *Let $(s, i), (s, j) \in X_n$. If $i \equiv_n j$, then the models $\mathfrak{A}_n(s, i)$ and $\mathfrak{A}_n(s, j)$ are isomorphic.*

Proof. The map $(t, l) \mapsto (t, l + n)$ is an isomorphism between $\mathfrak{A}_n(s, i)$ and $\mathfrak{A}_n(s, i + n)$. □

We define $p^0(x)$ as x , and $p^{k+1}(x)$ as $(p^k(x)) \cdot (p^k(x))$. Then for $k < \omega$

$$\mathfrak{A}_n \models (t, j) = p^k(s, i) \quad \text{iff} \quad s = t \text{ and } j = i + k.$$

For $s \in n^{<n}$, let $\varphi_s(x)$ be the following one-parameter formula:

$$x \cdot p(x) = p^{E(s)+1}(x).$$

Lemma 15. *Let \mathfrak{A} be a submodel of \mathfrak{A}_n and (t, i) an element of \mathfrak{A} . Then $\mathfrak{A} \models \varphi_s(t, i)$ iff $s = t$ and $i \equiv_n 0$.*

Proof. We have $p(t, i) = (t, i + 1)$. By the definition, $(t, i) \cdot (t, i + 1) = (t, i + E(t) + 1)$ if $i \equiv_n 0$, and $(t, i) \cdot (t, i + 1) = (t, i)$ otherwise. □

Lemma 16. *Let \mathfrak{A} be a submodel of \mathfrak{A}_n . For every $s \in n^{<n}$, we have: $\mathfrak{A} \models \exists x \varphi_s(x)$ iff (s, i) is in \mathfrak{A} for some $i \in \omega$.*

Proof. The ‘only if’ part is immediate from Lemma 15. For the ‘if’ part we use Lemmas 15 and 13. \square

For $S \subseteq n^{<n}$, let χ_S be the sentence $\bigwedge_{s \in S} \exists x \varphi_s(x) \wedge \bigwedge_{s \notin S} \neg \exists x \varphi_s(x)$, and let $\chi_{\geq s}$ be χ_S for $S = \{t \in n^{<n} : s \subseteq t\}$.

Lemma 17. *Let \mathfrak{A} be a submodel of \mathfrak{A}_n . Then $\mathfrak{A} \models \chi_{\geq s}$ iff $\mathfrak{A} = \mathfrak{A}_n(s, i)$ for some $i \in \omega$.*

Proof. Follows from Lemmas 16 and 13. \square

Let $\psi(x)$ be the formula $\neg \exists y (x = p(y))$. Then

$$\mathfrak{A}_n(s, i) \models \psi(t, j) \text{ iff } j = i.$$

For $s \in n^{<n}$ and $k < n$, let $\chi_{s,k}$ be the following sentence:

$$\exists x (\varphi_s(p^{n-k}(x)) \wedge \psi(x)) \wedge \chi_{\geq s}.$$

Lemma 18. *For every submodel \mathfrak{A} of \mathfrak{A}_n and every $k < n$, we have $\mathfrak{A} \models \chi_{s,k}$ iff $\mathfrak{A} = \mathfrak{A}_n(s, i)$ for some i such that $i \equiv_n k$.*

Proof. Follows from Lemmas 17 and 15. \square

Lemma 19. *Let $\mathfrak{A}, \mathfrak{B}$ be submodels of \mathfrak{A}_n . The following are equivalent:*

- (i) \mathfrak{A} and \mathfrak{B} are isomorphic,
- (ii) \mathfrak{A} and \mathfrak{B} are L -equivalent,
- (iii) \mathfrak{A} and \mathfrak{B} are elementarily equivalent, i.e., $L_{\omega, \omega}$ -equivalent,
- (iv) $\mathfrak{A} = \mathfrak{A}_n(s, i)$ and $\mathfrak{B} = \mathfrak{A}_n(s, j)$ for some $s \in n^{<n}$ and $i, j < \omega$ such that $i \equiv_n j$.

Proof. The implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are immediate from our basic assumptions on model-theoretic languages. The crucial step (iii) \Rightarrow (iv) follows from Lemmas 13 and 18. Finally, (iv) \Rightarrow (i) holds by Lemma 14. \square

From Lemma 13 we conclude that $(\text{Sub}(\mathfrak{A}_n), \sqsubseteq)$ is isomorphic to (X_n, \preceq) . Now it follows that $(\text{Sub}(\mathfrak{A}_n)_{\equiv}, \sqsubseteq_{\equiv})$ is isomorphic to Q_n , as required.

For the case when Ω has constant symbols, we add a new element c to X_n and define the model \mathfrak{A}'_n on the set $X_n \cup \{c\}$. We extend the above defined operation \cdot by letting $c \cdot x = x \cdot c = c$ for all x ; all constant symbols in Ω are interpreted by c . The same arguments as above prove that $(\text{Sub}(\mathfrak{A}'_n)_{\equiv}, \sqsubseteq_{\equiv})$ is isomorphic to Q'_n .

This completes the proof of Theorem 12. \square

Now the completeness result follows:

Theorem 20. *Let Ω contain a functional symbol of arity ≥ 2 , and let \mathcal{C} be the class of all models of Ω . Then*

$$\text{MTh}^L(\mathcal{C}, \sqsupseteq) = \begin{cases} \text{S4} & \text{if } \Omega \text{ has no constant symbols,} \\ \text{S4.2.1} & \text{otherwise.} \end{cases}$$

Proof. We have soundness by Theorem 10. On the other hand, if $\mathfrak{A} \in \mathcal{C}$, then $\text{MTh}^L(\mathcal{C}, \sqsupseteq)$ is contained in the logic of the Kripke frame $(\text{Sub}(\mathfrak{A})_{\equiv}, \sqsupseteq_{\equiv})$ by Corollaries 4 and 7. Now completeness follows from Theorem 12. \square

Notice that the binary operation used in the proof of Theorem 12 is not associative. Modal axiomatizations of \sqsupseteq in second-order language on semigroups, monoids, and groups are open questions.

Recall that Theorem 20 was formulated under the assumption that L expresses the \sqsupseteq -satisfiability. In the next section we discuss how to define modal theories without such requirements. The logics calculated in Theorem 20 do not depend on L , while in general, modal theories depend on a chosen model-theoretic language; we shall discuss this in Section 5.

4 Inexpressible modalities

In this section, we discuss the situation when the \mathcal{R} -satisfiability is not expressible in a model-theoretic language.

Definition The following question was raised by one of the reviewers on an earlier version of the paper: what is the modal theory of $(\mathcal{C}, \mathcal{R})$ if the \mathcal{R} -satisfiability on \mathcal{C} is not expressible in a given language? In particular, what is the modal theory of the relation \sqsupseteq in the first-order case? Another natural question concerns the definition of modal theories in the case of \sqsubseteq .

Assume that L is a language stronger than K , and the \mathcal{R} -satisfiability on \mathcal{C} is expressible in L . In this case, $\text{MTh}^K(\mathcal{C}, \mathcal{R})$ can be naturally defined as the logic of the subalgebra of the modal algebra of L on $(\mathcal{C}, \mathcal{R})$ generated by the sentences of K . Let us provide details.

First, we define an interim notion $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$. By Theorem 6, $\text{MTh}^L(\mathcal{C}, \mathcal{R})$ is the logic of the Lindenbaum modal algebra $\mathcal{A}(L) = (L/\approx, f_{\approx}^L)$. Let $\mathcal{A}(L[K])$ be the subalgebra of $\mathcal{A}(L)$ generated by the sentences of K (more formally, $\mathcal{A}(L[K])$ is generated by the set $\{[\varphi]_{\approx}^L : \varphi \in K_s\}$, where $[\varphi]_{\approx}^L = \{\psi \in L_s : \mathcal{C} \models \varphi \leftrightarrow \psi\}$). Define $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$ as the modal logic of $\mathcal{A}(L[K])$. (Formally, we have defined $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$ for the case $K \subseteq L$, but the same construction works for the case when K is weaker than L in \mathcal{C} , i.e., if every K -definable subclass of \mathcal{C} is definable in L .)

It is immediate that $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$ is a normal logic, which includes $\text{MTh}^L(\mathcal{C}, \mathcal{R})$. Another simple (but important) observation is that $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$ does not depend on the choice of L . Indeed, if M is another

language stronger than K , which expresses the \mathcal{R} -satisfiability on \mathcal{C} , then the algebras $\mathcal{A}(L[K])$ and $\mathcal{A}(M[K])$ are isomorphic: their elements can be thought as classes of models obtained from definable in K subclasses of \mathcal{C} via Boolean operations and \mathcal{R}^{-1} . Thus we have

Proposition 21. *If languages L and M express the \mathcal{R} -satisfiability on \mathcal{C} and are stronger than K , then:*

1. *the algebras $\mathcal{A}(L[K])$ and $\mathcal{A}(M[K])$ are isomorphic, and so*
2. $\text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R}) = \text{MTh}^{M[K]}(\mathcal{C}, \mathcal{R})$.

Definition 3 (tentative). If the \mathcal{R} -satisfiability on \mathcal{C} is expressible in some language L stronger than K , put $\text{MTh}^K(\mathcal{C}, \mathcal{R}) = \text{MTh}^{L[K]}(\mathcal{C}, \mathcal{R})$.

Trivially, $\text{MTh}^{L[L]}(\mathcal{C}, \mathcal{R}) = \text{MTh}^L(\mathcal{C}, \mathcal{R})$ whenever L expresses the \mathcal{R} -satisfiability on \mathcal{C} . Thus this definition generalizes Definition 2. Note that, however, the modal logic $\text{MTh}^K(\mathcal{C}, \mathcal{R})$ is not necessarily a “fragment” of the theory $\text{Th}^K(\mathcal{C})$ anymore.

Once we do not require the \mathcal{R} -satisfiability to be expressible in the language, we can give an “external” definition of modal theory and modal algebra for \mathcal{C} , \mathcal{R} , and K . Let K_\diamond be the set of modal K -sentences, which are built from sentences of K using Boolean connectives and \diamond ; namely, they are expressions of form $\varphi(\psi_1, \dots, \psi_n)$ where $\varphi(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is a modal formula and ψ_i are sentences of K . Such languages are regularly used in the context of modal logics of relations between models of arithmetic or set theory.⁴ Given \mathcal{C} and \mathcal{R} , the K_\diamond -satisfaction relation is defined in the straightforward way; in particular, $\mathfrak{A} \models \diamond\varphi(\psi_1, \dots, \psi_k)$ iff $\mathfrak{B} \models \varphi(\psi_1, \dots, \psi_k)$ for some $\mathfrak{B} \in \mathcal{C}$ with $\mathfrak{A} \mathcal{R} \mathfrak{B}$. Assume that \mathcal{R} and \mathcal{C} satisfy the following natural condition: for all $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ in \mathcal{C} ,

$$\mathfrak{A} \simeq \mathfrak{A}' \ \& \ \mathfrak{A} \mathcal{R} \mathfrak{B} \ \Rightarrow \ \exists \mathfrak{B}' \in \mathcal{C} \ (\mathfrak{B}' \simeq \mathfrak{B} \ \& \ \mathfrak{A}' \mathcal{R} \mathfrak{B}') \quad (3)$$

(this condition says that the isomorphism equivalence \simeq is a *bisimulation* w.r.t. \mathcal{R} on \mathcal{C}). In this case the K_\diamond -satisfaction relation is preserved under isomorphisms, and so K_\diamond satisfies our basic assumptions on model-theoretic languages.

The operation on sentences of K_\diamond that takes φ to $\diamond\varphi$ expresses the \mathcal{R} -satisfiability on \mathcal{C} in K_\diamond . Hence, we can apply the constructions described in Section 2 to K_\diamond . In particular, $\varphi \approx \psi$ implies $\diamond\varphi \approx \diamond\psi$ (recall that $\varphi \approx \psi$ means $\mathcal{C} \models \varphi \leftrightarrow \psi$), so the connective \diamond induces the operation \diamond_{\approx} on K_\diamond/\approx , and $\mathcal{A}(K_\diamond) = (K_\diamond/\approx, \diamond_{\approx})$ is a modal algebra. It is immediate that $\mathcal{A}(K_\diamond) = \mathcal{A}(K_\diamond[K])$, and hence

$$\text{MTh}^K(\mathcal{C}, \mathcal{R}) = \text{MTh}^{K_\diamond}(\mathcal{C}, \mathcal{R})$$

⁴In [16], the language K_\diamond is called *partial potentialist*.

by Definition 3. Therefore, assuming that \mathcal{C} and \mathcal{R} satisfy (3), we obtain the following generalization of our previous definitions for arbitrary K :

Definition 4. Let \mathcal{C} and \mathcal{R} satisfy (3). The *modal (Lindenbaum) algebra of a language K on $(\mathcal{C}, \mathcal{R})$* is the modal algebra $(K_\diamond / \approx, \diamond \approx)$. The *modal theory $\text{MTh}^K(\mathcal{C}, \mathcal{R})$ of $(\mathcal{C}, \mathcal{R})$ in K* is the modal logic of this algebra.

Let us emphasize that natural classes of models with relations between them always satisfy (3); in particular, so are the instances discussed in our paper. One might say that \mathcal{R} is a *model theoretic relation on \mathcal{C}* iff property (3) holds. E.g., the submodel relation on the class of models of a given signature is model-theoretic; hence, Theorem 20 remains true for arbitrary model-theoretic language.

Downward Löwenheim–Skolem theorem Our next result provides a version of the downward Löwenheim–Skolem theorem for first-order language enriched with the modal operator for the extension relation between models.

For a cardinal κ , put $\mathcal{C}_{\leq \kappa} = \{\mathfrak{A} \in \mathcal{C} : |\mathfrak{A}| \leq \kappa\}$.

Theorem 22. Let K be $L_{\omega, \omega}$ based on a signature Ω , let K_\diamond be K enriched with the modal operator for the extension relation on models \sqsubseteq , and let \mathcal{C} be an elementary (in K) class of models. For every $\kappa \geq \omega + |\Omega|$, the following statements hold:

1. Let X be a set of elements of $\mathfrak{A} \in \mathcal{C}$, and let λ be a cardinal such that $|X| + \kappa \leq \lambda \leq |\mathfrak{A}|$. Then \mathfrak{A} has a submodel \mathfrak{B} of cardinality λ such that $\text{Th}^{K_\diamond}(\mathfrak{A}) = \text{Th}^{K_\diamond}(\mathfrak{B})$ and \mathfrak{B} contains X .
2. $\{\text{Th}^{K_\diamond}(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}\} = \{\text{Th}^{K_\diamond}(\mathfrak{A}) : \mathfrak{A} \in \mathcal{C}_{\leq \kappa}\}$.
3. The modal algebras of K on $(\mathcal{C}, \sqsubseteq)$ and on $(\mathcal{C}_{\leq \kappa}, \sqsubseteq)$ coincide.
4. $\text{MTh}^K(\mathcal{C}, \sqsubseteq) = \text{MTh}^K(\mathcal{C}_{\leq \kappa}, \sqsubseteq)$.

First, we provide a general observation about any \mathcal{C} and \mathcal{R} satisfying the condition (3), and arbitrary K . Let \equiv be the K -equivalence on \mathcal{C} , and let \equiv_\diamond be the K_\diamond -equivalence on \mathcal{C} . Hence, $\mathfrak{A} \equiv \mathfrak{B}$ means that $\text{Th}^K(\mathfrak{A}) = \text{Th}^K(\mathfrak{B})$, and $\mathfrak{A} \equiv_\diamond \mathfrak{B}$ that $\text{Th}^{K_\diamond}(\mathfrak{A}) = \text{Th}^{K_\diamond}(\mathfrak{B})$. We recall that $\text{Th}^{K_\diamond}(\mathfrak{A})$ depends not only on \mathfrak{A} but also on \mathcal{C} and \mathcal{R} .

Proposition 23. Assume that \mathcal{C} and \mathcal{R} satisfy (3). Let \equiv be a bisimulation w.r.t. \mathcal{R} on \mathcal{C} , i.e., for all $\mathfrak{A}, \mathfrak{A}', \mathfrak{B} \in \mathcal{C}$ we have

$$\mathfrak{A} \equiv \mathfrak{A}' \ \& \ \mathfrak{A} \mathcal{R} \mathfrak{B} \ \Rightarrow \ \exists \mathfrak{B}' \in \mathcal{C} \ (\mathfrak{B}' \equiv \mathfrak{B} \ \& \ \mathfrak{A}' \mathcal{R} \mathfrak{B}'). \quad (4)$$

Then \equiv and \equiv_\diamond coincide on \mathcal{C} .

Proof. By the standard bisimulation argument: an easy induction on the construction of $\psi(p_1, \dots, p_n)$ shows that, whenever $\mathfrak{A} \equiv \mathfrak{A}'$ and $\varphi_1, \dots, \varphi_n \in K_s$ then we have $\mathfrak{A} \models \psi(\varphi_1, \dots, \varphi_n) \Leftrightarrow \mathfrak{A}' \models \psi(\varphi_1, \dots, \varphi_n)$. \square

Proposition 24. *Let $K = L_{\omega, \omega}$. If \mathcal{C} is an elementary class, then \equiv is a bisimulation w.r.t. \sqsubseteq on \mathcal{C} .*

Proof. Pick $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ in \mathcal{C} . If $\mathfrak{A} \equiv \mathfrak{A}'$, then there exists an ultrafilter D such that the ultrapowers \mathfrak{A}_D and \mathfrak{A}'_D are isomorphic, due to the Keisler–Shelah isomorphism theorem (see, e.g., [10, Theorem 6.1.15]). If $\mathfrak{A} \sqsubseteq \mathfrak{B}$, then \mathfrak{A}_D is embeddable in \mathfrak{B}_D , the ultrapower of \mathfrak{B} . Since \mathfrak{A}' is embeddable in \mathfrak{A}'_D , we obtain that \mathfrak{A}' is embeddable in \mathfrak{B}_D , and thus $\mathfrak{A}' \sqsubseteq \mathfrak{B}'$ for some $\mathfrak{B}' \simeq \mathfrak{B}_D$. But then we have $\mathfrak{B}' \in \mathcal{C}$ and $\mathfrak{B}' \equiv \mathfrak{B}$, which completes the proof. \square

Proof of Theorem 22. By Propositions 23 and 24, \equiv and \equiv_{\diamond} coincide on \mathcal{C} , i.e., for all $\mathfrak{A}', \mathfrak{A}$ in \mathcal{C} we have:

$$\mathfrak{A}' \text{ and } \mathfrak{A} \text{ are } K\text{-equivalent} \Leftrightarrow \mathfrak{A}' \text{ and } \mathfrak{A} \text{ are } K_{\diamond}\text{-equivalent.} \quad (5)$$

Now the first statement of the theorem follows from the downward Löwenheim–Skolem theorem for the first-order case: \mathfrak{A} has an elementary submodel \mathfrak{B} of cardinality λ such that \mathfrak{B} contains X (see, e.g., [10, Theorem 3.1.6]); hence $\mathfrak{A} \equiv_{\diamond} \mathfrak{B}$ by (5).

Let $(\mathcal{T}, \sqsubseteq_{\mathcal{T}}, \mathcal{A})$ and $(\mathcal{T}^{(\kappa)}, \sqsubseteq_{\mathcal{T}}^{(\kappa)}, \mathcal{A}^{(\kappa)})$ be the minimal frames of theories for \mathcal{C} , \sqsubseteq , K_{\diamond} and for $\mathcal{C}_{\leq \kappa}$, \sqsubseteq , K_{\diamond} , respectively. Let us show that these two structures are equal.

We have $\mathcal{T} = \mathcal{T}^{(\kappa)}$, the second statement of the theorem, as an immediate corollary of the first one.

Suppose that $T_1 \sqsubseteq_{\mathcal{T}} T_2$, that is, $T_1 = Th^{K_{\diamond}}(\mathfrak{A})$ and $T_2 = Th^{K_{\diamond}}(\mathfrak{B})$ for some $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ with $\mathfrak{A} \sqsubseteq \mathfrak{B}$. Then \mathfrak{A} has a submodel \mathfrak{A}' of cardinality $\leq \kappa$ with $\mathfrak{A}' \equiv_{\diamond} \mathfrak{A}$ (indeed, if the cardinality of \mathfrak{A} is less than κ then we put $\mathfrak{A}' = \mathfrak{A}$, otherwise we use the first statement of the theorem). Likewise, \mathfrak{B} has a submodel \mathfrak{B}' of cardinality $\leq \kappa$ such that $\mathfrak{B}' \equiv_{\diamond} \mathfrak{B}$ and \mathfrak{B}' contains the universe of \mathfrak{A}' . Clearly, \mathfrak{A}' is submodel of \mathfrak{B}' . It follows that $T_1 \sqsubseteq_{\mathcal{T}}^{(\kappa)} T_2$. Hence $\sqsubseteq^{(\kappa)}$ includes \sqsubseteq . The converse inclusion is obvious, so $\sqsubseteq^{(\kappa)}$ equals \sqsubseteq .

Let $\varphi \in K_{\diamond}$. Since $\mathcal{T} = \mathcal{T}^{(\kappa)}$, it is immediate that $\{T \in \mathcal{T} : \varphi \in T\} = \{T \in \mathcal{T}^{(\kappa)} : \varphi \in T\}$. Hence, $\mathcal{A}^{(\kappa)} = \mathcal{A}$.

Thus $(\mathcal{T}, \sqsubseteq_{\mathcal{T}}, \mathcal{A}) = (\mathcal{T}^{(\kappa)}, \sqsubseteq_{\mathcal{T}}^{(\kappa)}, \mathcal{A}^{(\kappa)})$. Now the third and consequently, the forth, statements are immediate from Definition 4 and Theorem 5. \square

This result is based on the interplay between the downward Löwenheim–Skolem property of first-order language and the fact that \equiv is a bisimulation w.r.t. \sqsubseteq . The property of \equiv being a bisimulation w.r.t. a given \mathcal{R} seems interesting enough per se. Contrary to the case of the relation \sqsubseteq (Proposition 24), we have:

Proposition 25. *Let Ω contain a predicate symbol of arity ≥ 2 . Then the (usual) elementary equivalence is not a bisimulation w.r.t. \sqsubseteq on the class of models of Ω .*

Proof. Assume w.l.g. Ω contains a single binary predicate symbol \leq , and let \equiv denote $\equiv_{L_{\omega,\omega}}$. Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ be the linearly ordered sets $\mathbb{Q} \cdot \mathbb{Z}, \mathbb{Z}, \mathbb{Q}$, respectively (where \cdot denotes the lexicographical multiplication). We have $\mathfrak{A} \equiv \mathfrak{A}'$ (this fact can be established by using an Ehrenfeucht–Fraïssé game, see, e.g., [23, Proposition 2.4.10]) and \mathfrak{B} is embeddable in \mathfrak{A} . However, every \mathfrak{B}' satisfying $\mathfrak{B}' \equiv \mathfrak{B}$ is a dense linearly ordered set without endpoints. Clearly, no such \mathfrak{B}' can at the same time be embeddable in \mathfrak{A}' . \square

Let us say that \mathcal{R} is *image-closed under ultraproducts* iff for every $\mathfrak{A}, (\mathfrak{B}_i)_{i \in I}$, and ultrafilter D over I , if $\mathfrak{A} \mathcal{R} \mathfrak{B}_i$ for all $i \in I$ then $\mathfrak{A} \mathcal{R} \prod_D \mathfrak{B}_i$ (e.g., $\mathfrak{A} \mathcal{R} \mathfrak{B}$ may mean that, up to isomorphism, \mathfrak{B} is an extension of \mathfrak{A}).

Proposition 26. *Let \mathcal{C} and \mathcal{R} satisfy (3), let \mathcal{R} be image-closed under ultraproducts on \mathcal{C} , and let $K = L_{\omega,\omega}$. If \equiv and \equiv_{\diamond} coincide on \mathcal{C} , then \equiv is a bisimulation w.r.t. \mathcal{R} on \mathcal{C} .*

Proof. Let $\mathfrak{A}, \mathfrak{A}', \mathfrak{B}$ in \mathcal{C} be such that $\mathfrak{A} \equiv \mathfrak{A}'$ and $\mathfrak{A} \mathcal{R} \mathfrak{B}$. Observe that for all $\varphi \in K_s$ we have:

$$\begin{aligned} \mathfrak{B} \models \varphi &\Rightarrow \mathfrak{A} \models \diamond \varphi \\ &\Rightarrow \mathfrak{A}' \models \diamond \varphi \quad (\text{since } \mathfrak{A} \equiv_{\diamond} \mathfrak{A}') \\ &\Rightarrow \mathfrak{A}' \mathcal{R} \mathfrak{B}_{\varphi} \ \& \ \mathfrak{B}_{\varphi} \models \varphi \text{ for some } \mathfrak{B}_{\varphi} \in \mathcal{C}. \end{aligned}$$

Let $T = Th^K(\mathfrak{B})$. For any $\Gamma \in \mathcal{P}_{\omega}(T)$ choose a model \mathfrak{B}_{Γ} in \mathcal{C} as in the observation above, i.e., such that $\mathfrak{A}' \mathcal{R} \mathfrak{B}_{\Gamma}$ and $\mathfrak{B}_{\Gamma} \models \bigwedge \Gamma$. Pick any ultrafilter D extending the centered family of sets $S_{\varphi} = \{\Gamma \in \mathcal{P}_{\omega}(T) : \varphi \in \Gamma\}$ for all $\varphi \in T$ (i.e., a *fine* ultrafilter over $\mathcal{P}_{\omega}(T)$). Then the ultraproduct $\mathfrak{B}' = \prod_D \mathfrak{B}_{\Gamma}$ satisfies all $\varphi \in T$, thus $\mathfrak{B}' \models T$. Moreover, since \mathcal{R} is image-closed under ultraproducts, we have $\mathfrak{A}' \mathcal{R} \mathfrak{B}'$. This completes the proof. \square

Remark 5. It suffices to assume that \mathcal{R} is image-closed under ultraproducts by ultrafilters over $|K|$ only. The proposition remains true for $K = L_{\kappa,\lambda}$ with any strongly compact κ .

5 Robustness and Kripke completeness

The logics of submodels calculated in Section 3 do not depend on choosing particular language (Theorem 20). However, in general, modal theories depend on L .

Example 2. Let L be the first-order language $L_{\omega, \omega}$, T the theory of dense linear orders without end-points. Trivially, the \sqsubseteq -satisfiability is expressible on $\mathcal{C} = \text{Mod}(T)$ in L : put $f(\varphi) = \varphi$. Then $\text{MTh}^L(\mathcal{C}, \sqsubseteq)$ contains the “trivial” formula $\mathfrak{p} \leftrightarrow \diamond \mathfrak{p}$. Obviously, this formula is falsified if L is second-order.

Henceforth we assume that \mathcal{C} and \mathcal{R} satisfy (3).

Proposition 27. *Let L and K be two languages. Then $L \subseteq K$ implies $\text{MTh}^L(\mathcal{C}, \mathcal{R}) \supseteq \text{MTh}^K(\mathcal{C}, \mathcal{R})$.*

Proof. Follows from Definition 4, since on $(\mathcal{C}, \mathcal{R})$, the modal algebra of L is a subalgebra of the algebra of K . \square

One can think that L describes the properties of $(\mathcal{C}, \mathcal{R})$ in a robust way whenever the modal theory does not change under strengthening the language. Thus, we shall say that $\text{MTh}^L(\mathcal{C}, \mathcal{R})$ is *robust* iff for every language $K \supseteq L$ we have

$$\text{MTh}^K(\mathcal{C}, \mathcal{R}) = \text{MTh}^L(\mathcal{C}, \mathcal{R}).$$

Intuitively, the robust theory can be considered as a “true” modal logic of the model-theoretic relation \mathcal{R} on \mathcal{C} .

By Definition 4 and Theorem 6, modal theories are logics of general frames. The following construction shows that, under certain assumptions, robust theories are logics of Kripke frames (i.e., they are *Kripke complete*).

A relation \mathcal{S} is said to be *image-set* iff for every \mathfrak{A} the image $\mathcal{S}(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{A} \mathcal{S} \mathfrak{B}\}$ is a set; \mathcal{S} is *image-set on \mathcal{C}* iff its restriction to \mathcal{C} is image-set. Recall that $\mathfrak{A} \simeq \mathfrak{B}$ means that \mathfrak{A} and \mathfrak{B} are isomorphic, and that $[\mathfrak{A}]_{\simeq} \mathcal{R}_{\simeq} [\mathfrak{B}]_{\simeq}$ iff $\exists \mathfrak{A}' \simeq \mathfrak{A} \exists \mathfrak{B}' \simeq \mathfrak{B} (\mathfrak{A}' \mathcal{R} \mathfrak{B}')$. Let \mathcal{C}_{\simeq} abbreviate \mathcal{C}/\simeq . Also, recall that $\mathcal{R}^*(\mathfrak{A})$ denotes $\bigcup_{n < \omega} \mathcal{R}^n(\mathfrak{A})$, the least \mathcal{R} -closed $\mathcal{D} \subseteq \mathcal{C}$ containing \mathfrak{A} .

Proposition 28. *If \mathcal{R}_{\simeq} is image-set on \mathcal{C}_{\simeq} , then*

$$\text{MTh}^L(\mathcal{C}, \mathcal{R}) \supseteq \bigcap_{\mathfrak{A} \in \mathcal{C}} \text{MLog}(\mathcal{R}^*(\mathfrak{A})_{\simeq}, \mathcal{R}_{\simeq}).$$

Proof. Let $\mathfrak{A} \in \mathcal{C}$. The inclusion $\text{MTh}^L(\mathcal{R}^*(\mathfrak{A}), \mathcal{R}) \supseteq \text{MLog}(\mathcal{R}^*(\mathfrak{A})_{\simeq}, \mathcal{R}_{\simeq})$ follows from (2) at the end of Section 2. Now we apply Corollary 4. \square

Theorem 29. *If \mathcal{R}_{\simeq} is image-set on \mathcal{C}_{\simeq} , then*

$$\text{MTh}^L(\mathcal{C}, \mathcal{R}) \text{ is robust iff } \text{MTh}^L(\mathcal{C}, \mathcal{R}) = \bigcap_{\mathfrak{A} \in \mathcal{C}} \text{MLog}(\mathcal{R}^*(\mathfrak{A})_{\simeq}, \mathcal{R}_{\simeq}).$$

Proof. (\Rightarrow) Put $\Lambda = \text{MTh}^L(\mathcal{C}, \mathcal{R})$. In view of Proposition 28, we only have to show that Λ is valid in the Kripke frame $(\mathcal{R}^*(\mathfrak{A})_{\simeq}, \mathcal{R}_{\simeq})$ for every $\mathfrak{A} \in \mathcal{C}$.

Fix $\mathfrak{A} \in \mathcal{C}$ and put $Y = \mathcal{R}^*(\mathfrak{A})$. For a language M , let $(\mathcal{T}^M, \mathcal{R}_{\mathcal{T}^M}, \mathcal{A}^M)$ be the minimal frame of its theories in the class Y .

Since \mathcal{R}_{\simeq} is image-set on \mathcal{C}_{\simeq} , the quotient Y_{\simeq} is a set. Hence we can choose a language K stronger than L and such that the K -equivalence and

the isomorphism relation coincide on Y . Observe that for every M stronger than K , the Kripke frames $(\mathcal{T}^M, \mathcal{R}_{\mathcal{T}^M})$, $(\mathcal{T}^K, \mathcal{R}_{\mathcal{T}^K})$, and $(Y_{\simeq}, \mathcal{R}_{\simeq})$ are isomorphic. We choose such M that

$$\begin{aligned} \forall T \in \mathcal{T}^K \exists \varphi_T \in M_s \forall \mathfrak{A} \in Y (\mathfrak{A} \models \varphi_T \Leftrightarrow Th^K(\mathfrak{A}) = T), \text{ and moreover} \\ \forall \mathcal{S} \subseteq \mathcal{T}^K \exists \varphi_{\mathcal{S}} \in M_s \forall \mathfrak{A} \in Y (\mathfrak{A} \models \varphi_{\mathcal{S}} \Leftrightarrow Th^K(\mathfrak{A}) \in \mathcal{S}). \end{aligned}$$

It follows that \mathcal{A}^M is $\mathcal{P}(\mathcal{T}^M)$, the powerset of \mathcal{T}^M , and so $(\mathcal{T}^M, \mathcal{R}_{\mathcal{T}^M}, \mathcal{A}^M)$ is a Kripke frame. By Theorem 6, $\text{MTh}^M(Y, \mathcal{R})$ is the logic of this frame. Therefore, $\text{MTh}^M(Y, \mathcal{R})$ is the logic of the Kripke frame $(Y_{\simeq}, \mathcal{R}_{\simeq})$. Since Λ is robust, we have $\Lambda = \text{MTh}^M(\mathcal{C}, \mathcal{R})$. By Corollary 4, $\Lambda \subseteq \text{MTh}^M(Y, \mathcal{R})$.

(\Leftarrow) Immediate from Propositions 27 and 28. \square

Hence, when \mathcal{R}_{\simeq} is image-set on \mathcal{C}_{\simeq} , this theorem describes a unique modal logic, the *robust theory of* $(\mathcal{C}, \mathcal{R})$.

Remark 6. More generally, if \mathcal{K} is a class of model-theoretic languages, a modal theory $\text{MTh}^L(\mathcal{C}, \mathcal{R})$ is *robust in* \mathcal{K} iff it coincides with $\text{MTh}^K(\mathcal{C}, \mathcal{R})$ for every language $K \supseteq L$ in \mathcal{K} . It is easy to see that there exists at most one robust theory of $(\mathcal{C}, \mathcal{R})$ whenever \mathcal{K} is directed, and exactly one such theory whenever \mathcal{K} is countably directed, in the sense that for every $\{K_n \in \mathcal{K} : n \in \omega\}$ there is $K \in \mathcal{K}$ which is stronger than any K_n .

Notice that even if \mathcal{C} is a set, Theorem 29 does not mean that the robust theory of $(\mathcal{C}, \mathcal{R})$ is the logic of the Kripke frame $(\mathcal{C}, \mathcal{R})$.

Example 3. Let \mathfrak{A} be the algebra $(2^{<\omega}, \text{inf}, \text{l}, \text{r})$ with the binary operation $\text{inf}(x, y) = x \cap y$ and the unary operations $\text{l}(x) = x \hat{\ } 0$, $\text{r}(x) = x \hat{\ } 1$, where $\hat{\ }$ denotes the concatenation. It is easy to see that $(\text{Sub}(\mathfrak{A}), \sqsubseteq)$, the structure of submodels of \mathfrak{A} , is isomorphic to the binary tree $\mathfrak{T}_2 = (2^{<\omega}, \subseteq)$. The modal logic of the Kripke frame \mathfrak{T}_2 is known to be S4 (see [13]). However, for every L , the modal theory $\text{MTh}^L(\text{Sub}(\mathfrak{A}), \sqsubseteq)$ is the trivial logic given by the axiom $\text{p} \leftrightarrow \diamond \text{p}$ because every submodel of \mathfrak{A} is isomorphic to \mathfrak{A} .

More completeness results We apply Theorem 29 to describe robust theories of the quotient and the submodel relations on certain natural classes.

The following fact most probably was known since 1970s.

Proposition 30 (V.B. Shehtman, private communication). *The logic of the Kripke frame $(\mathcal{P}(\omega), \subseteq)$ is S4.2.1.*

Proof (sketch). It is not difficult to construct a family $\mathcal{V} \subset \mathcal{P}(\omega)$ such that \mathcal{V} is downward-closed (i.e., $U_1 \subseteq U_2 \in \mathcal{V}$ implies $U_1 \in \mathcal{V}$) and there exists a p-morphism of (\mathcal{V}, \subseteq) onto the binary tree $\mathfrak{T}_2 = (2^{<\omega}, \subseteq)$. (E.g., let f be a bijection $\omega \rightarrow 2^{<\omega}$, and let \mathcal{V} consist of $U \subset \omega$ such that $f(U)$ is a finite chain in \mathfrak{T}_2 ; the required p-morphism takes U to the greatest element

of this chain.) This p-morphism obviously extends to the p-morphism of $(\mathcal{P}(\omega), \subseteq)$ onto \mathfrak{T}'_2 , the ordered sum of \mathfrak{T}_2 and a reflexive singleton. Since $\text{MLog}(\mathfrak{T}'_2) = \text{S4.2.1}$ (see [29]), it follows that $\text{MLog}(\mathcal{P}(\omega), \subseteq)$ is included in S4.2.1. The converse inclusion is clear. \square

Theorem 31. *Let Ω contain a functional symbol and a constant symbol, and let \mathcal{C} be the class of all models of Ω . Then the robust theory of $(\mathcal{C}, \sqsupseteq)$ is S4.2.1.*

Proof. Soundness is straightforward. Let us check the converse inclusion.

Let a signature Ω_0 consist of a single unary functional symbol F ; models of this signature are called *unars*. For any $n \in \omega \setminus \{0\}$, let \mathfrak{A}_n be a cycle of length n , i.e., a one-generated unar of cardinality n that satisfies $F^n(x) = x$. As easy to see, each \mathfrak{A}_n has no proper submodels.

Let also \mathfrak{A} be the disjoint sum of all these \mathfrak{A}_n . Clearly, \mathfrak{A} is a countably generated unar, and any its submodel is given by a nonempty set $S \subseteq \omega$, and the map taking S to the disjoint sum of \mathfrak{A}_n for all $n \in S$ is an isomorphism between $(\mathcal{P}(\omega) \setminus \{\emptyset\}, \supseteq)$ and the frame $(\text{Sub}(\mathfrak{A}), \sqsupseteq)$ of submodels of \mathfrak{A} . Moreover, all submodels of \mathfrak{A} are pairwise non-isomorphic, and hence, the Kripke frame $(\text{Sub}(\mathfrak{A})_{\simeq}, \sqsupseteq_{\simeq})$ is also isomorphic to $(\mathcal{P}(\omega) \setminus \{\emptyset\}, \supseteq)$.

Without loss of generality we may assume that F is the only functional symbol in Ω (we can always mimic a unary functional symbol by any functional symbol of arity ≥ 1). Expanding Ω_0 to Ω , let \mathfrak{A}' be the model of Ω obtained from \mathfrak{A} by adding a single extra point a which interprets all constant symbols and assuming $F(a) = a$. Then the Kripke frame $(\text{Sub}(\mathfrak{A}')_{\simeq}, \sqsupseteq_{\simeq})$ is isomorphic to $(\mathcal{P}(\omega), \supseteq)$.

By Theorem 29, the robust theory of $(\mathcal{C}, \sqsupseteq)$ is included in the logic $\text{MLog}(\mathcal{P}(\omega), \supseteq)$, which is S4.2.1 by Proposition 30. \square

Remark 7. It follows from the above proof that for the class \mathcal{C} of unars without constant symbols, the robust theory of $(\mathcal{C}, \sqsupseteq)$ is included in the intersection of the logics of *Skvortsov frames* $(\mathcal{P}(\kappa) \setminus \{\emptyset\}, \supseteq)$ for all κ . However, it is unclear how the “soundness” part of the proof can be obtained. This question is connected to a long-standing open problem about properties of the modal Medvedev logic, see, e.g., [30].

Remark 8. The proof of Theorem 31 is much simpler than the proof of Theorem 20, and covers more signatures in the case with constants. However, it is unclear whether the completeness result holds for second- or first-order language. In other words, we do not know whether the resulting theories are robust in the case of second- or first-order language.

Let $\mathfrak{B} \leq \mathfrak{A}$ mean that \mathfrak{B} is a quotient of \mathfrak{A} (for the definition of quotients of arbitrary models, see [22, Section 2.4]), and $\text{Quot}(\mathfrak{A}) = \{\mathfrak{B} : \mathfrak{B} \leq \mathfrak{A}\}$. Quotients are, up to isomorphism, images under strong homomorphisms, hence the logic of quotients is the same that the logic of strong homomorphic images.

Theorem 32. *Let Ω have a functional symbol of arity ≥ 1 , and \mathcal{C} the class of all models of Ω . Then the robust theory of (\mathcal{C}, \geq) is S4.2.1.*

Proof. Soundness is straightforward. In particular, every model \mathfrak{A} has a unique single-point quotient \mathfrak{B} , and for any valuation $\|\cdot\|$ in L we have: $\mathfrak{A} \models \|\Box \Diamond p\|$ iff $\mathfrak{B} \models \|\Diamond p\|$ iff $\mathfrak{B} \models \|\Box p\|$ iff $\mathfrak{A} \models \|\Diamond \Box p\|$.

By Theorem 29, the robust theory of (\mathcal{C}, \geq) is included in the logic $(\text{Quot}(\mathfrak{A}_{\simeq}, \geq_{\simeq})$ for any model \mathfrak{A} of Ω . We shall construct \mathfrak{A} such that $\text{MLog}(\text{Quot}(\mathfrak{A}_{\simeq}, \geq_{\simeq})) \subseteq \text{S4.2.1}$. It suffices to handle the case when Ω consists of a single unary functional symbol F . As in Theorem 31, let \mathfrak{A}_n be a one-generated unar forming the cycle of length n . It is easy to see that all quotients of \mathfrak{A}_n are (up to isomorphism) exactly \mathfrak{A}_m for n divisible by m ; in particular, if p is prime, $\text{Quot}(\mathfrak{A}_p)$ is $\{\mathfrak{A}_1, \mathfrak{A}_p\}$.

Let \mathfrak{A} be the disjoint sum of the unars \mathfrak{A}_p for $p = 1$ or p prime. Let P denote the set of primes. Up to isomorphism, every quotient \mathfrak{B} of \mathfrak{A} is the disjoint sum of unars $\{\mathfrak{A}_p : p \in S\}$ and also n copies of \mathfrak{A}_1 for some $S \subseteq P$ and $n \leq 1 + |P \setminus S|$; we put $h(\mathfrak{B}) = h([\mathfrak{B}]_{\simeq}) = S$ and $g(\mathfrak{B}) = n$. We claim that h is a p-morphism of $(\text{Quot}(\mathfrak{A}_{\simeq}, \geq_{\simeq}))$ onto $(\mathcal{P}(P), \supseteq)$. Surjectivity and monotonicity are straightforward. Assume that $h([\mathfrak{B}]_{\simeq}) = S$ and pick any $S' \subseteq S$. Let \mathfrak{B}' be the disjoint sum of unars $\{\mathfrak{A}_p : p \in S'\}$ and $g(\mathfrak{B})$ copies of \mathfrak{A}_1 . Then \mathfrak{B}' is isomorphic to a quotient of \mathfrak{B} , and $h([\mathfrak{B}']_{\simeq}) = S'$, as required.

It follows that $\text{MLog}(\text{Quot}(\mathfrak{A}_{\simeq}, \geq_{\simeq}))$ is included in $\text{MLog}(\mathcal{P}(P), \supseteq)$. By Proposition 30, the latter logic is S4.2.1, which completes the proof. \square

Remark 9. This theorem remains true for (not necessarily strong) homomorphic images as well. That S4.2.1 includes the modal theory of homomorphisms follows from the proof above (the signature of \mathfrak{A} does not have predicate symbols). And S4.2.1 is sound since \mathcal{C} has a single-point model that gives a top element in \mathcal{C}/\simeq .

Remark 10. Similarly to the case of the submodel relation, the quotient relation is expressible in an appropriate second-order language. Given Ω and $\lambda > |\{F \in \Omega : F \text{ is a functional symbol}\}|$, for every φ of $L_{\lambda, \omega}^2$ in Ω we pick a fresh binary predicate variable U and define φ_U by induction on φ : if t_0, \dots, t_{n-1} are terms in Ω and $P(t_0, \dots, t_{n-1})$ is an atomic formula, then

$$P(t_0, \dots, t_{n-1})_U \text{ is } \exists x_0 \dots \exists x_{n-1} (\bigwedge_{i < n} U(x_i, t_i) \wedge P(x_0, \dots, x_{n-1}))$$

where x_i are fresh first-order variables; if F is a functional variable, then

$$(\exists F \varphi)_U \text{ is } \exists F (U \text{ is a congruence for } F \wedge \varphi_U);$$

we let the operation distribute w.r.t. Boolean connectives and quantifiers over first-order and predicate variables (e.g., $(\exists P \varphi)_U$ is $\exists P \varphi_U$ for every predicate variable P). If $\chi(U)$ says that the interpretation of U is a congruence for interpretations of all functional symbols in Ω , then the map $\varphi \mapsto \exists U (\chi(U) \wedge \varphi_U)$ expresses the \geq -satisfiability on Ω -models.

However, in second-order language, we do not know whether the modal theory of quotients is S4.2.1.

So far we have calculated modal theories only for classes consisting of all models of a given signature. In the following examples, we describe robust modal theories of classes consisting of models of a non-trivial theory; we only outline ideas and postpone complete proofs for a further paper.

Example 4. Let \mathcal{D} be the class of dense linearly ordered sets without endpoints. It was observed in Example 2 that the modal theory $\text{MTh}^{L_{\omega,\omega}}(\mathcal{D}, \sqsupseteq)$ is trivial: this is the logic of a reflexive singleton axiomatized by the formula $\mathbf{p} \leftrightarrow \diamond \mathbf{p}$. On the other hand, the robust theory of $(\mathcal{D}, \sqsupseteq)$ is S4.2.1.

By the classical Cantor theorem, every two countable orders in \mathcal{D} are isomorphic (see, e.g., [24, Theorem 2.8]). Therefore, $(\mathcal{D}_{\simeq}, \sqsupseteq_{\simeq})$ has a top element (the order type of rationals), so its logic includes S4.2.1 (in spite of the fact that we do not have constant symbols in the signature). To prove the converse inclusion, in view of Corollary 4, it suffices to show that S4.2.1 includes the logic of a generated subframe of the frame $(\mathcal{D}_{\simeq}, \sqsupseteq_{\simeq})$.

Given $S \subseteq \mathbb{R}$, let $D_S = \{[X]_{\simeq} : X \subseteq S \text{ is dense in } \mathbb{R}\}$. (Note that $[X]_{\simeq}$ is the order type of X .) First we observe that, whenever $S \subseteq \mathbb{R}$ is dense, then $(D_S, \sqsupseteq_{\simeq})$ forms an upper cone of $(\mathcal{D}_{\simeq}, \sqsupseteq_{\simeq})$. Hence it suffices to find S such that the logic of $(D_S, \sqsupseteq_{\simeq})$ coincides with S4.2.1. For this, we use the following result by Sierpiński: there are two disjoint sets E, F of reals both dense in \mathbb{R} , having cardinality $|E| = |F| = 2^{\aleph_0}$, and such that $f(E) \not\subseteq E \cup F$ for any non-identity order embedding $f : \mathbb{R} \rightarrow \mathbb{R}$ (see, e.g., [24, Chapter 9, § 2]). Let $S = E \cup F$, and for any dense subset X of S , let $\pi([X]_{\simeq}) = A$ if $X = E \cup A$ for some $A \subseteq F$, and $\pi([X]_{\simeq}) = \emptyset$ otherwise. It can be verified that π is a well-defined map and moreover, a p-morphism of $(D_S, \sqsupseteq_{\simeq})$ onto $(\mathcal{P}(F), \sqsupseteq)$. It induces a p-morphism of $(D_S, \sqsupseteq_{\simeq})$ onto $(\mathcal{P}(\omega), \sqsupseteq)$. Applying Proposition 30, we conclude that $\text{MLog}(D_S, \sqsupseteq_{\simeq})$ is S4.2.1, as required.

Let \mathcal{L} and \mathcal{O} be the classes of linearly ordered sets and partially ordered sets, respectively. The robust theories of $(\mathcal{L}, \sqsupseteq)$ and $(\mathcal{O}, \sqsupseteq)$ also coincide with S4.2.1. To see this, we note that there exists a p-morphism of $(\mathcal{L}, \sqsupseteq)$ onto $(\mathcal{D}, \sqsupseteq)$ (condensing scattered segments, see, e.g., [24, Chapter 4]) and that $(\mathcal{L}, \sqsupseteq)$ is an upper cone of $(\mathcal{O}, \sqsupseteq)$. This induces the same relationships between $(\mathcal{D}_{\simeq}, \sqsupseteq_{\simeq})$, $(\mathcal{L}_{\simeq}, \sqsupseteq_{\simeq})$, and $(\mathcal{O}_{\simeq}, \sqsupseteq_{\simeq})$.

Example 5. Let \mathcal{C} be the class of modal algebras. Similarly to the proof of Theorem 31, one can show that the robust theory of (\mathcal{C}, \geq) is S4.2.1. Namely, let \mathfrak{F} be the disjoint sum of Kripke frames $\mathfrak{F}_n = (n, n \times n)$, and \mathfrak{A} be the modal algebra of \mathfrak{F} . One can construct a p-morphism from $(\text{Quot}(\mathfrak{A})_{\simeq}, \geq_{\simeq})$ onto $(\mathcal{P}(\omega), \sqsupseteq)$. Thus, S4.2.1 is the robust theory of the quotient relation on the class \mathcal{C} of modal algebras. We have the same axiomatization in the case when \mathcal{C} is the class of S4-algebras or S5-algebras, because \mathfrak{A} is an S5-algebra. (More generally, this holds if \mathcal{C} contains \mathfrak{A} and is closed under quotients.)

The axiomatization in the case when \mathcal{C} is the class of Boolean algebras is an open question.

Example 6. Let \mathcal{C} be the class of free groups. Let \mathbb{F}_κ be a κ -generated free group. All \mathbb{F}_κ with $2 \leq \kappa \leq \aleph_0$ are pairwise embeddable and non-isomorphic, while on all other \mathbb{F}_κ their ordering by embedding coincides with their ordering by rank (the least cardinality of generators). Therefore, the Kripke frame $(\mathcal{C}_\simeq, \sqsubseteq_\simeq)$ has the top (corresponding to \mathbb{F}_1), a countable cluster immediate below the top (corresponding to \mathbb{F}_κ 's with $2 \leq \kappa \leq \aleph_0$), and a structure isomorphic to (Ord, \geq) (corresponding to \mathbb{F}_κ 's with $\kappa \geq \aleph_1$). By Theorem 29, it follows that the robust theory of $(\mathcal{C}, \sqsubseteq)$ is the modal logic of ordered sums $(\alpha, \geq) + S + S_0$ for $\alpha \in Ord$, a countable cluster S , and a singleton S_0 . By the standard filtration technique, this logic has the finite model property (more precisely, we may assume that α and S are finite) and decidable.

The class \mathcal{C} is closed under \sqsubseteq as any subgroup of a free group is free by the classical Nielsen–Schreier theorem. Hence, \sqsubseteq -satisfiability on \mathcal{C} is expressible in $L_{\omega, \omega}^2$ by Proposition 9. We do not know whether the modal theory of \sqsubseteq on \mathcal{C} in $L_{\omega, \omega}^2$ is robust.

We remark that in the first-order case the theory is not robust. Let \equiv be the usual elementary equivalence. Observe that $\mathbb{F}_1 \simeq \mathbb{Z} \not\equiv \mathbb{F}_2$ (obvious), and $\mathbb{F}_\kappa \equiv \mathbb{F}_2$ for all $\kappa \geq 2$ (by recently proved famous Tarski's conjecture; see, e.g., [12, Chapter 9]). It follows that $(\mathcal{C}_\equiv, \sqsubseteq_\equiv)$ is isomorphic to the ordinal 2 (with the top corresponding to \mathbb{F}_1 and the bottom to other free groups). Moreover, the logic $\text{MTh}^{L_{\omega, \omega}}(\mathcal{C}, \sqsubseteq)$ is the logic of the ordinal 2. This follows from Corollary 7 in view of the following observation made by one the reviewers on an earlier version of the paper: the function $f : L_s \rightarrow L_s$ defined by letting

$$f(\varphi) = \begin{cases} \top & \text{if } \mathbb{F}_1 \models \varphi, \\ \varphi & \text{if } \mathbb{F}_1 \not\models \varphi \text{ but } \mathbb{F}_2 \models \varphi, \\ \perp & \text{otherwise} \end{cases}$$

expresses the \sqsubseteq -satisfiability on \mathcal{C} . This observation can be generalized as follows: if the elementary equivalence \equiv is a bisimulation w.r.t. \mathcal{R} on \mathcal{C} and \mathcal{C}_\equiv is finite, then the \mathcal{R} -satisfiability on \mathcal{C} is first-order expressible.

Even in the case of all models of a given signature, the axiomatization of the robust theory can be a very difficult problem. Consider the theories of the relation \sqsubseteq on the class \mathcal{C} of all models of a signature Ω consisting of unary predicates and perhaps some constants.

The easiest is the degenerated case when Ω has no symbols other than constants. The theory of submodels on this class is Grz.3, the Grzegorzcyk logic with the linearity axiom. As well-known, this is the logic of conversely well-ordered sets, or of the set ω with the converse order, or else of all finite linearly ordered sets.

Furthermore, these theories include the logic Grz iff Ω contains finitely many predicates. We announce that in the case of $n < \omega$ predicates, the robust modal theory coincides with the modal logic of the direct product of 2^n copies of ω with the converse order if the signature has constants, and with the logic of this structure without the top element otherwise. These logics have the finite model property, decrease as n grows, and their intersection coincides with $\text{MLog}(\mathcal{P}_\omega(\omega) \setminus \{\emptyset\}, \supseteq)$, the modal Medvedev logic. Despite this clear semantical description, no complete axiomatizations for these logics are known.

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