EXPLICIT BOUNDS FOR L-FUNCTIONS ON THE EDGE OF THE CRITICAL STRIP

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Abstract. Assuming GRH and the Ramanujan-Petersson conjecture we prove explicit bounds for $L(1,f)$ for a large class of L-functions $L(s,f)$, which includes L-functions attached to automorphic cuspidal forms on $GL(n)$. The proof generalizes work of Lamzouri, Li and Soundararajan. Furthermore, the main results improve the classical bounds of Littlewood

$$(1+o(1))\left(\frac{12e^{\gamma}}{\pi^2}\log\log C(f)\right)^{-d} \le |L(1,f)| \le (1+o(1))\Big(2e^{\gamma}\log\log C(f)\Big)^{d},$$

where $C(f)$ is the analytic conductor of $L(s,f)$.

1. Introduction

In analytic number theory, and increasingly in other surprising places, L-functions show up as a tool for describing interesting algebraic and geometric phenomena. In particular, understanding the value of L-functions on the 1-line has a number of applications. For example, the non-vanishing of the Riemann zeta function for $\zeta(1+it)$, $t\in\mathbb{R}$, proves the celebrated Prime Number Theorem. Additionally, understanding the value $L(1,\chi)$ for certain Dirichlet characters, provides us with insight to the order of the class group of imaginary quadratic fields through Dirichlet's Class Number Formula. Unconditionally, for any non-trivial Dirichlet character χ with conductor q, we have

$$\frac{1}{q^{\epsilon}} \ll |L(1,\chi)| \ll \log q.$$

In fact, we can improve the lower bound to $(\log q)^{-1}$, excluding some exceptional cases related to Landau-Siegel zeros (see [2, Chapter 14]). Louboutin [8] proves an explicit upper bound of this shape. Under the assumption of the Generalized Riemann Hypothesis (GRH), we have the much stronger bounds due to Littlewood [7]:

$$\frac{\zeta(2)(1+o(1))}{2e^{\gamma}\log\log q} \le |L(1,\chi)| \le (2e^{\gamma}+o(1))\log\log q,$$

where $o(1)$ tends to 0 as $q\to\infty$. Recently, Lamzouri, Li and Soundararajan gave the following explicit refinement

Theorem 1.1. [6, Theorem 1.5] *Asume GRH. Let q be a positive integer and χ be a primitive character modulo q. For $q \ge 10^{10}$ we have*

$$|L(1,\chi)| \le 2e^{\gamma}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q}\right)$$

and

$$\frac{1}{|L(1,\chi)|} \le \frac{12e^{\gamma}}{\pi^2}\left(\log\log q - \log 2 + \frac{1}{2} + \frac{1}{\log\log q} + \frac{14\log\log q}{\log q}\right).$$

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The goal of this paper is to provide explicit upper and lower bounds for a large class of L-functions, including L-functions attached to automorphic cuspidal forms on $GL(n)$. More precisely, we bound the quantity $|L(1,f)|$, where L is a degree $d \geq 1$ L-function and f is some arithmetic or geometric object. The results will be valid under the assumption of GRH and the Ramanujan-Petersson conjecture. Additionally, we improve on the bound that comes from generalizing Littlewood's technique, which under both GRH and Ramanujan-Petersson conjecture provides

$$(1+o(1))\left(\frac{12e^\gamma}{\pi^2}\log\log C(f)\right)^{-d} \leq |L(1,f)| \leq (1+o(1))\Big(2e^\gamma \log\log C(f)\Big)^d,$$

where $o(1)$ is a quantity that tends to 0 as $C(f) \to \infty$. Here $C(f)$ denotes the analytic conductor of the L-function. A precise definition of $C(f)$ along with what the term L-function describes will be provided after another example. Other works discussing explicit bounds for higher degree L-functions focus on bounding $L(\frac{1}{2}, f)$, we refer the reader to [1] for details.

We provide a degree 2 example before appealing to the precise definitions. Let $k, q \geq 1$ be integers and let χ be a Dirichlet character modulo q. Take f to be a Hecke cusp form of weight k, level q, and character χ, with the following Fourier expansion at the cusp ∞,

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} e(nz),\ e(z) = e^{2\pi i z}.$$

Then

$$L(s,f) = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

is a degree 2 L-function. By works of Deligne [3] and Deligne and Serre [4], it is known that $L(s,f)$ satisfies Ramanujan-Petersson for all weights $k \geq 1$. In this situation, the analytic conductor is given by

$$C(f) = \frac{q}{\pi^2}\left(\frac{1+(k-1)/2}{2}\right)\left(\frac{1+(k+1)/2}{2}\right) \asymp qk^2.$$

We deduce the following corollary from our main results Thereom 2.1 and Theorem 2.2 below.

Corollary 1.2. *Under the assumption of GRH, if* $\log C(f) \geq 46$*, we have*

$$|L(1,f)| \leq (2e^\gamma)^2\left((\log\log C(f))^2 - (2\log 4 - 1)\log\log C(f) + (\log 4)^2 - \log 4 + 2.51\right),$$

and

$$\frac{1}{|L(1,f)|} \leq \left(\frac{12e^\gamma}{\pi^2}\right)^2 \Bigg((\log\log C(f))^2 - (2\log 4 - 1)\log\log C(f) + (\log 4)^2 - \log 4 + 2.67 + \frac{89.40((\log\log C(f))^2 - 2\log 4\log\log C(f) + \log^2 4)}{\log C(f)}\Bigg).$$

1.1. **Definitions and Notation.** To begin, let $d \geq 1$ be a fixed positive integer, and let $L(s,f)$ be given by the Dirichlet series and Euler product

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j,f}(p)}{p^s}\right)^{-1},$$

where $\lambda_f(1) = 1$, and both the series and product are absolutely convergent in $\mathrm{Re}(s) > 1$. We shall assume that $L(s, f)$ satisfies the Ramanujan-Petersson conjecture which states that $|\alpha_{j,f}(p)| \leq 1$ for all primes p and $1 \leq j \leq d$. Further, we define the gamma factor

$$\gamma(s, f) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right),$$

where κ_j are complex numbers. These κ_j are called the local parameters at infinity and may be referred to as such throughout. In general, it is assumed that $\mathrm{Re}(\kappa_j) > -1$, in our case the Ramanujan-Petersson conjecture guarantees that $\mathrm{Re}(\kappa_j) \geq 0$. This last condition ensures that $\gamma(s, f)$ has no pole in $\mathrm{Re}(s) > 0$. Furthermore, there exists a positive integer $q(f)$ (called the conductor of $L(s, f)$), such that the completed L-function,

$$\xi(s, f) = q(f)^{s/2} \gamma(s, f) L(s, f),$$

has an analytic continuation to the entire complex plane, and has finite order. This completion satisfies a functional equation

$$\xi(s, f) = \epsilon(f) \xi(1 - s, \overline{f}),$$

where $\epsilon(f)$ is a complex number of absolute value 1, and $\xi(s, \overline{f}) = \overline{\xi(\overline{s}, f)}$ ($\overline{f}$ is called the dual of f). Uniform estimates for analytic quantities associated to $L(s, f)$, when $L(s, f)$ is varying rely on a number of parameters, it is therefore convenient to state the results in terms of the analytic conductor which we define as follows: For $s \in \mathbb{C}$,

$$C(f, s) := \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{s + \kappa_j}{2} \right|.$$

In this article we are interested in studying the value of $L()$

$$C(f) := C(f, 1) = \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{1 + \kappa_j}{2} \right|.$$

We note that in [1] the author uses $C(f) = C(f, 1/2)$. This definition is very similar to the one given in Iwaniec and Kowalski [5] and only differs by a constant factor to the power of the degree of the L-function. To help orient the reader, we give an example in the form of the analytic conductor of a Dirichlet L-function. Let χ be a Dirichlet character modulo q then the associated L-function has analytic conductor:

$$C(\chi) = q \frac{1 + \mathfrak{a}}{2\pi}, \text{ where } \mathfrak{a} = \begin{cases} 1 & \text{if } \chi(-1) = -1 \\ 0 & \text{if } \chi(-1) = 1. \end{cases}$$

2. Results

Here we detail the theorems and make some remarks about how they fit into the general context of what is already known.

Theorem 2.1. *Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an L-function of degree d with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for*

$L(s,f)$. Then for $C(f)$ chosen such that $\log C(f) \geq 23d$ we have

$$|L(1,f)| \leq 2^d e^{d\gamma} \left((\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1} + \frac{dK(d)}{4}(\log\log C(f) - \log 2d)^{d-2} \right),$$

where

$$K(d) = 2.31 + \frac{22.59}{d}(e^{0.31d} - 1 - 0.31d). \tag{2.1}$$

Remark 1. This result is asymptotically better than the classical bound as it has the shape

$$|L(1,f)| \leq (2e^\gamma)^d \left((\log\log C(f))^d - (d\log(2d) - \tfrac{d}{2})(\log\log C(f))^{d-1} + O_d((\log\log C(f))^{d-2}) \right),$$

and $(d\log(2d) - \frac{d}{2}) > 0$ for all $d \geq 1$.

Remark 2. If we take $d = 1$, we may take $C(f) \geq 10^{10}$ and we obtain $K(1)/4 \leq 0.88$ which gives essentially Theorem 1.1

$$|L(1,\chi)| \leq 2e^\gamma \left(\log\log C(f) - \log 2 + \frac{1}{2} + \frac{0.88}{\log\log C(f) - \log 2} \right).$$

Theorem 2.2. *Let $d \geq 1$ be a fixed positive integer and let $L(s,f)$ be an L-function of degree d with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. Then for $C(f)$ chosen such that $\log C(f) \geq 23d$ we have*

$$\frac{1}{|L(1,f)|} \leq \left(\frac{12e^\gamma}{\pi^2}\right)^d \left((\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1} \right.$$
$$\left. + \frac{dJ_1(d)}{4}(\log\log C(f) - \log 2d)^{d-2} + \frac{d^2 J_2(d)(\log\log C(f) - \log 2d)^d}{\log C(f)} \right)$$

where

$$J_1(d) \leq 2 + \frac{4.18}{d}(e^{0.69d} - 1 - 0.69d). \tag{2.2}$$

and

$$J_2(d) = 9 + \frac{16.74}{d}(e^{0.69d} - 1 - 0.69d). \tag{2.3}$$

We notice that lower bound also provides something asymptotically better as in Remark 1.

Remark 3. If $d = 1$ we may take $C(f) \geq 10^{10}$ then $J_1(1)/4 \leq 0.82$ and $J_2(1) \leq 14.09$ this provides essentially Theorem 1.1

$$\frac{1}{|L(1,\chi)|} \leq \frac{12e^\gamma}{\pi^2} \left(\log\log C(f) + \frac{1}{2} - \log 2 + \frac{0.82}{\log\log C(f) - \log 2} + \frac{14.09(\log\log C(f) - \log 2)}{\log C(f)} \right).$$

As an easy corollary to these theorems we may a bound degree d L-functions in the t aspect as follows. Let t be a real number and define $L_t(1,f) := L(1+it, f)$, then the analytic conductor of $L_t(s,f)$ is given by

$$C_t(f) := \frac{q(f)}{\pi^d} \prod_{j=1}^{d} \left| \frac{1 + it + \kappa_j}{2} \right| \asymp_f |t|^d.$$

Corollary 2.3. *Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an L-function of degree d with conductor $q(f)$ and analytic conductor $C(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. If $\log C_t(f) \geq 23d$ then*

$$|L(1+it, f)| \leq (2e^\gamma)^d \left((\log\log C_t(f) - \log 2d)^d + \frac{d}{2}(\log\log C_t(f) - \log 2d)^{d-1} + \frac{dK(d)}{4}(\log\log C_t(f) - \log 2d)^{d-2}\right),$$

and

$$\frac{1}{|L(1+it, f)|} \leq \left(\frac{12e^\gamma}{\pi^2}\right)^d \left((\log\log C_t(f) - \log 2d)^d + \frac{d}{2}(\log\log C_t(f) - \log 2d)^{d-1} + \frac{dJ_1(d)}{4}(\log\log C_t(f) - \log 2d)^{d-2} + \frac{d^2 J_2(d)(\log\log C_t(f) - \log 2d)^d}{\log C_t(f)}\right).$$

The definitions of $K(d)$, $J_1(d)$ and $J_2(d)$ are given by equations (2.1), (2.2) *and* (2.3) *respectively.*

3. Lemmata

In this section we will outline a number of results which are necessary for proving the final bound. Additionally, we will disclose a few more properties of the L-functions we are studying. First, the logarithmic derivative of $L(s, f)$ is given by

$$-\frac{L'}{L}(s, f) = \sum_{n \geq 2} \frac{a_f(n)\Lambda(n)}{n^s} \text{ for } \mathrm{Re}(s) > 1,$$

where $a_f(n) = 0$ unless $n = p^k$ is a prime power in which case $a_f(n) = \sum_{j=1}^{d} \alpha_{j,f}(p)^k$. Since $L(s, f)$ satisfies the Ramanujan-Petersson conjecture, then $|a_f(n)| \leq d$. Further, let $\{\rho_f\}$ be the set of the nontrivial zeros of $L(s, f)$. Then we have the Hadamard factorization formula ([5, Theorem 5.6]),

$$\xi(s, f) = e^{A(f)+sB(f)} \prod_{\rho_f} \left(1 - \frac{s}{\rho_f}\right) e^{s/\rho_f}, \tag{3.1}$$

where $A(f)$ and $B(f)$ are constants. We note that $\mathrm{Re}B(f) = -\mathrm{Re}\sum_{\rho_f} 1/\rho_f$ and taking the logarithmic derivatives of both sides of (3.1) gives

$$\mathrm{Re}\frac{\xi'}{\xi}(s, f) = \mathrm{Re}\sum_{\rho_f} \frac{1}{s - \rho_f}. \tag{3.2}$$

3.1. Explicit Formulas for $\log|L(1, f)|$ and $|\mathrm{Re}(B(f))|$.

Lemma 3.1. *Let $d \geq 1$ be a fixed positive integer and let $L(s, f)$ be an L-function of degree d with conductor $q(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s, f)$. For any $x \geq 2$ there exists a real number $|\theta| \leq 1$ such that*

$$\log|L(1, f)| = \mathrm{Re}\sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \frac{\log(\frac{x}{n})}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d} \frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right) - \left(\frac{1}{\log x} - \frac{2\theta}{\sqrt{x}\log^2 x}\right)|\mathrm{Re}B(f)| + \frac{2d\theta}{x\log^2 x}.$$

Proof. We have for any fixed $\sigma \geq 1$ that

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s+\sigma, f)\frac{x^s}{s^2} ds = \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n^\sigma} \log(\tfrac{x}{n}).$$

Shifting the contour to the left, we see this integral is also equal to

$$-\left(\frac{L'}{L}\right)'(\sigma,f)-\frac{L'}{L}(\sigma,f)\log x-\sum_{\rho_f}\frac{x^{\rho_f-\sigma}}{(\rho_f-\sigma)^2}-\sum_{j=1}^{d}\sum_{m=0}^{\infty}\frac{x^{-2m-\kappa_j-\sigma}}{(2m+\kappa_j+\sigma)^2}.$$

Thus, since $\mathrm{Re}(\kappa_j)\geq 0$ we have

$$\begin{aligned}-\frac{L'}{L}(\sigma,f)&=\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n^\sigma}\frac{\log(\frac{x}{n})}{\log x}+\frac{1}{\log x}\left(\frac{L'}{L}\right)'(\sigma,f)\\&\quad+\frac{\theta x^{\frac12-\sigma}}{\log x}\sum_{\rho_f}\frac{1}{|\rho_f|^2}+\frac{\theta x^{-\sigma}}{\log x}\sum_{j=1}^{d}\sum_{m=0}^{\infty}\frac{x^{-2m}}{(2m+1)^2}.\end{aligned}$$

We integrate both sides with respect to σ from 1 to ∞, then take real parts to obtain

$$\log|L(1,f)|=\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x}-\frac{1}{\log x}\mathrm{Re}\frac{L'}{L}(1,f)+\frac{\theta}{\sqrt{x}\log^2 x}\sum_{\rho_f}\frac{1}{|\rho_f|^2}+\frac{2d\theta}{x\log^2 x}.$$

We note that $\sum_{\rho_f}\frac{1}{|\rho_f|^2}=2|\mathrm{Re}B(f)|$. Now we have

$$-\frac{L'}{L}(1,f)=\frac12\log q(f)-\frac{d}{2}\log\pi+\frac12\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)-\frac{\xi'}{\xi}(1,f)$$

Hence, after taking real parts we have the desired result. □

Lemma 3.2. *Let $d\geq 1$ be a fixed positive integer and let $L(s,f)$ be an L-function of degree d with conductor $q(f)$. Suppose that GRH and Ramanujan-Petersson hold for $L(s,f)$. Define $0\leq l(f)\leq d$ to be the number of κ_j in the gamma factor of $L(s,f)$ which equal 0. For any $x>1$ there exists a real number $|\theta|\leq 1$ such that*

$$\begin{aligned}&-\frac{\xi'}{\xi}(0,\overline{f})-\frac{1}{x}\frac{\xi'}{\xi}(0,f)+\frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(f))|=\\&\qquad\frac12\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac1x\right)-\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)+E(f,x),\end{aligned}$$

where

$$\begin{aligned}E(f,x)&=l(f)\left(-\log 2-\frac{\gamma}{2}\left(1-\frac1x\right)+\frac{\log x+1}{x}-\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}\right)\\&\quad+\sum_{i=1}^{d-l(f)}\left(\frac12\frac{\Gamma'}{\Gamma}(\frac{1+\kappa_i}{2})-\frac{1}{2x}\frac{\Gamma'}{\Gamma}(\frac{\kappa_i}{2})-\sum_{n=0}^{\infty}\frac{x^{-2n-\kappa_j-1}}{(2n+\kappa_i)(2n+\kappa_i+1)}\right),\end{aligned}$$

In particular, $\left(1+\frac1x+\frac{2\theta}{\sqrt{x}}\right)|\mathrm{Re}(B(f)|$ *equals*

$$\begin{aligned}&\frac12\left(1-\frac1x\right)\left(\log\left(\frac{q(f)}{\pi^d}\right)+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)-\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\\&\quad-(d\theta-(1+\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}+l(f)\frac{\log x+1}{x}+\frac{(d-2l(f))\log 2}{x}\\&\quad-\frac1x\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right).\end{aligned}$$

In both of the above expressions, the terms inside $\sum_{i=1}^{d-l(f)}$ are ranging over the local parameters at infinity, $\kappa_i \neq 0$.

Proof. We consider

$$I(f) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\xi'}{\xi}(s,f)\frac{x^{s-1}}{s(s-1)}ds.$$

Pulling the contour to the left we collect the residues of the poles at $s = 0, 1$ and ρ_f the nontrivial zeros of $L(s, f)$. Hence,

$$I(f) = \frac{\xi'}{\xi}(1,f) - \frac{1}{x}\frac{\xi'}{\xi}(0,f) + \sum_{\rho_f}\frac{x^{\rho_f-1}}{\rho_f(\rho_f-1)}.$$

Thus applying GRH we have for some $|\theta| \leq 1$

$$I(f) = -\frac{\xi'}{\xi}(0,\overline{f}) - \frac{1}{x}\frac{\xi'}{\xi}(0,f) + \frac{2\theta}{\sqrt{x}}|\mathrm{Re}(B(f))|.$$

On the other hand, we can also write

$$\begin{aligned}
I(f) &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left(\frac{1}{2}\log(q(f)) + \frac{\gamma'}{\gamma}(s,f) + \frac{L'}{L}(s,f)\right)\frac{x^{s-1}}{s(s-1)}ds \\
&= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\frac{x^{s-1}}{s(s-1)}ds + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa_j}{2}\right)\frac{x^{s-1}}{s(s-1)}ds \\
&\quad + \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{L'}{L}(s,f)\frac{x^{s-1}}{s(s-1)}ds \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

The contribution from I_1 and I_3 is

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right) - \sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right).$$

We rewrite I_2 as

$$I_2 = \sum_{j=1}^{d}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\kappa_j}{2}\right)\frac{x^{s-1}}{s(s-1)}ds.$$

Fix j, if $\kappa_j \neq 0$ then the j-th term of the summand will have simple poles at $s = 0, 1$ and $s = -2n-\kappa_j$ for $n \geq 0$. Thus the contribution will be

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right) - \frac{1}{2x}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j}{2}\right) - \sum_{n=0}^{\infty}\frac{x^{-2n-\kappa_j-1}}{(2n+\kappa_j)(2n+1+\kappa_j)}.$$

On the other hand, if $\kappa_j = 0$ then the j-th term of the summand will have simple poles at $s = 1$ and $s = -2n$ for $n \geq 1$, which contribute

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) - \sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}.$$

Additionally, we know that

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = -\frac{1}{s} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right),$$

so the residue of the double pole at $s = 0$ is given by

$$\frac{1 + \log x + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1)}{x}.$$

Using the fact that $\frac{\Gamma'}{\Gamma}(1) = -\gamma$ and $\frac{\Gamma'}{\Gamma}(1/2) = -2\log 2 - \gamma$ we see the overall contribution will be

$$-\log 2 - \frac{\gamma}{2}\left(1 - \frac{1}{x}\right) + \frac{\log x + 1}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}.$$

Let $l(f)$ be as in the statement of the lemma. Then, reordering the κ_j so that $\kappa_1, \kappa_2, \ldots, \kappa_{d-l(f)}$ are all nonzero and summing over j we get the desired expression for $E(f, x)$.

Finally, since $-\text{Re}\frac{\xi'}{\xi}(0, \overline{f}) = -\text{Re}\frac{\xi'}{\xi}(0, f) = |\text{Re}(B(f))|$, we see that taking real parts of the established identity we obtain

$$\left(1 + \frac{1}{x} + \frac{2\theta}{\sqrt{x}}\right)|Re(B(f))| = \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) - \text{Re}\sum_{n \le x} \frac{a_f(n)\Lambda(n)}{n}\left(1 - \frac{n}{x}\right) + \text{Re}(E(f, x)).$$

We find an explicit expression for the right hand side as follows:

Start by noting that for $\kappa_j = 0$ we have

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) = \frac{\Gamma'}{\Gamma}\left(\frac{1 + \kappa_j}{2}\right),$$

so that

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) + E(f, x) = \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) + \sum_{j=1}^{d} \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1 + \kappa_j}{2}\right)$$
$$+ l(f)\left(\frac{\log x + 1 + \gamma/2}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)}\right) - \sum_{i=1}^{d-l(f)}\left(\frac{1}{2x}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) + \sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_i-1}}{(2n + \kappa_i)(2n + \kappa_i + 1)}\right).$$

We note that for some $|\theta| \le 1$

$$\sum_{n=0}^{\infty} \frac{x^{-2n-\kappa_j-1}}{(2n + \kappa_i)(2n + \kappa_i + 1)} = \frac{x^{-\kappa_i}}{x\kappa_i(\kappa_j + 1)} + \theta\sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)},$$

hence

(3.3)

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) + E(f, x) = \frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1 - \frac{1}{x}\right) + \sum_{j=1}^{d} \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1 + \kappa_j}{2}\right)$$
$$-(d\theta - (1 + \theta)l(f))\sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x + \gamma/2 + 1}{x} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) + \frac{x^{-\kappa_i}}{\kappa_i(\kappa_i + 1)}\right).$$

Now, from the functional equation of $\Gamma(s)$ we see that

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2}\right) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2} + 1\right) - \frac{1}{\kappa_i},$$

we recall Legendre's duplication formula

$$\Gamma(s)\Gamma(s + \tfrac{1}{2}) = 2^{1-2s}\log(\sqrt{\pi})\Gamma(2s),$$

so we have

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i}{2} + 1\right) = -\log 2 + \frac{\Gamma'}{\Gamma}(\kappa_i + 1) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i + 1}{2}\right).$$

Finally, we note

$$\frac{\Gamma'}{\Gamma}(s) = -\gamma - \frac{1}{s} - \sum_{n=1}^{\infty}\left(\frac{1}{s + n} - \frac{1}{n}\right),$$

so that

$$\left[\frac{\Gamma'}{\Gamma}(\kappa_i + 1) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i + 1}{2}\right)\right] - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i + 1}{2}\right) = \frac{1}{\kappa_i + 1} + \sum_{n=1}^{\infty} \frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n}.$$

Combinging these facts gives (3.3) as

$$\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right)+E(f,x)=\frac{1}{2}\log\left(\frac{q(f)}{\pi^d}\right)\left(1-\frac{1}{x}\right)+\sum_{j=1}^{d}\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)$$
$$-(d\theta-(1-\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}+l(f)\frac{\log x+\gamma/2+1}{x}$$
$$-\frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)-\log 2+\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right). \tag{3.4}$$

Then since $\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)=\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{1}{2})=-\log 2-\gamma/2$ when $\kappa_i=0$ we add $\frac{-l(f)\frac{\Gamma'}{\Gamma}(\frac{1}{2})+l(f)\frac{\Gamma'}{\Gamma}(\frac{1}{2})}{2x}$ so that the RHS of (3.4) is given by

$$\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right)+\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)-(d\theta-(1+\theta)l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}+l(f)\frac{\log x+1}{x}$$
$$+\frac{(d-2l(f))\log 2}{x}-\frac{1}{x}\sum_{i=1}^{d-l(f)}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right).$$

Taking real parts gives the desired result. □

3.2. **Bounds for the Digamma Function.** The following are some technical lemmas which help to shorten the proof of the main results. The first is taken from V. Chandee.

Lemma 3.3. [1, Lemma 2.3] *Let $z=x+iy$, where $x\ge\frac{1}{4}$. Then*

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}(z)\le\log|z|.$$

Lemma 3.4. *Let $\kappa=\sigma+it$ such that $\sigma\ge 0$, then*

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa+1+2n}-\frac{1}{\kappa+1+n}\right)\right)=\frac{1}{2}\log 4+\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}-\frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}+\frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right).$$

Proof. We take the real part inside the sum and focus on the individual partial sums given by

$$\sum_{n=1}^{N}\frac{2(\sigma+1+2n)}{(\sigma+1+2n)^2+t^2}\text{ and }\sum_{n=1}^{N}\frac{(\sigma+1+n)}{(\sigma+1+n)^2+t^2}.$$

Using partial summation we find

$$\sum_{n=1}^{N}\frac{2(\sigma+1+2n)}{(\sigma+1+2n)^2+t^2}=\frac{2N(\sigma+1+2N)+\sigma^2+2\sigma(N+1)+2Y+1+t^2}{(\sigma+1+2N)^2+t^2}-\frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}$$
$$+\frac{1}{2}\log((\sigma+1+2N)^2+t^2)-\frac{1}{2}\log((\sigma+3)^2+t^2)$$

and

$$\sum_{n=1}^{N}\frac{(\sigma+1+n)}{(\sigma+1+n)^2+t^2}=\frac{N(\sigma+1+N)+\sigma^2+\sigma(N+1)+\sigma+N+1+t^2}{(\sigma+1+N)^2+t^2}-\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}$$
$$+\frac{1}{2}\log((\sigma+1+N)^2+t^2)-\frac{1}{2}\log((\sigma+2)^2+t^2).$$

Taking the limit as $N\to\infty$ we see

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa+1+2n}-\frac{1}{\kappa+1+n}\right)\right)=\frac{1}{2}\log 4+\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}-\frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}+\frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right),$$

as was claimed. ☐

Lemma 3.5. *Let $\kappa = s + it$ such that $\sigma \geq 0$, and $x > 1$ then*
$$\left|\frac{x^{-\kappa}-1}{\kappa(\kappa+1)}\right| \leq \frac{2\log x}{\log 3}.$$

Proof. We consider two cases.

First suppose $|\kappa| \geq \frac{c}{\log x}$ then we can trivially bound the norm to obtain
$$\left|\frac{x^{-\kappa}-1}{\kappa(\kappa+1)}\right| \leq \frac{2\log x}{c}.$$
If $|\kappa| < \frac{c}{\log x}$ then
$$x^{-\kappa} - 1 = \sum_{k=1}^{\infty} \frac{(-\kappa\log x)^k}{k!},$$
so that
$$\left|\frac{x^{-\kappa}-1}{\kappa(\kappa+1)}\right| \leq \log x \sum_{k=1}^{\infty} \frac{c^{k-1}}{k!} = \frac{e^c-1}{c}\log x.$$
The choice of $c = \log 3$ gives the desired result. ☐

3.3. **Relevant Results from** [6]**.** Let
$$B = -\sum_{\rho} \mathrm{Re}\frac{1}{\rho} = \frac{1}{2}\log(4\pi) - 1 - \frac{\gamma}{2},$$
where the sum is taken over the non-trivial zeros of the Riemann zeta function.

Lemma 3.6. [6, Lemma 2.4] *Assume the Riemann Hypothesis. For $x > 1$ we have, for some $|\theta| \leq 1$,*
$$\sum_{n\leq x} \Lambda(n) n \left(1 - \frac{n}{x}\right) = \log x - (1+\gamma) + \frac{2\pi}{x} - \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} + 2\frac{\theta|B|}{\sqrt{x}}.$$

Lemma 3.7. [6, Lemma 2.6] *Assume the Riemann Hypothesis. For all $x \geq e$ we have*
$$\sum_{n\leq x} \frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x} = \log\log x - \gamma - 1 + \frac{\gamma}{\log x} + \frac{2|B|\theta}{\sqrt{x}\log^2 x} + \frac{\theta}{3x^3\log^2 x}.$$

We also prove the following lemma which is a slight generalization of [6, Lemma 5.1].

Lemma 3.8. *Assume the Ramanujan-Petersson conjecture. Then for $x \geq 100$ we have*
$$\mathrm{Re}\sum_{n\leq x} \alpha_{j,f}(n)\Lambda(n)\left(\frac{1}{n\log n} - \frac{1}{x\log x}\right) \geq \sum_{p^k\leq x} \Lambda(p^k)(-1)^k \left(\frac{1}{p^k\log p^k} - \frac{1}{x\log x}\right). \tag{3.5}$$
In particular, we have
$$\mathrm{Re}\sum_{n\leq x} a_f(n)\Lambda(n)\left(\frac{1}{n\log n} - \frac{1}{x\log x}\right) \geq d\sum_{p^k\leq x} \Lambda(p^k)(-1)^k \left(\frac{1}{p^k\log p^k} - \frac{1}{x\log x}\right). \tag{3.6}$$

Proof. Note that if x is a prime power then the summand at x on both sides of the inequality (3.5) contribute 0, so we assume x is not a prime power. We begin by recalling that $a_f(n) = 0$ unless $n = p^k$ is a prime power in which case $a_f(n) = \sum_{j=1}^{d} \alpha_{j,f}(p)^k$. So that (3.6) follows immediately, once we prove (3.5).

Fix j and consider each $\alpha_{j,f}$ separately. From the definition we see $\alpha_{j,f}(n)$ is only nonzero if $n = p^k$ for some prime power. If $\alpha_{j,f}(p) = 0$ then the contribution is 0 while the value on the right hand side < 0. If $\alpha_{j,f}(p) \neq 0$ then, from Ramanujan-Petersson we have that $|\alpha_{j,f}(p)| \leq 1$, so we express $\alpha_{j,f}(p) = -re(\theta)$, for $0 < r \leq 1$ where $e(\theta) = e^{2\pi i\theta}$. Consider the difference of the left and right side of (3.5):

$$\log(p) \sum_{p^k \leq x} (-1)^{k-1}(1 - r^k \cos(k\theta)) \left(\frac{1}{p^k \log p^k} - \frac{1}{x \log x} \right). \tag{3.7}$$

If we establish this is non-negative, then we are finished.
Before we proceed we see that for all $k \geq 1$

$$1 - r^k \cos(k\theta) \leq k^2(1 - r\cos\theta). \tag{3.8}$$

The case $k = 1$ is trivial, for the remaining $k \geq 2$, the inequality follows from

$$k^2 - 1 \geq 3 > r^k + r \geq r^k \cos(k\theta) - r\cos(\theta).$$

If $p \geq 3$, then by (3.8) we have (3.7) is greater than

$$\log(p)(1 - r\cos\theta) \left(\frac{1}{p\log p} - \frac{1}{x\log x} - \sum_{j=1}^{\infty} \frac{(2j)^2}{p^{2j} \log p^{2j}} \right) \geq 0.$$

For $p = 2$, when $k \geq 6$ we apply (3.8) again. Otherwise, when $1 \leq k \leq 5$ we compute the trigonometric polynomial exactly. A little computer computation completes the result. □

4. Proof of Theorems 2.1 and 2.2

4.1. **Upper bounds for $L(1, f)$.** Let $C(f) \geq 10^{10}$ and $x \geq 132$, be a real number to be chosen later. Lemma 3.1 says

$$\begin{aligned}\log|L(1, f)| \leq &\mathrm{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n \log n} \frac{\log(x/n)}{\log x} + \frac{1}{2\log x} \left(\log \frac{q(f)}{\pi^d} + \mathrm{Re} \sum_{j=1}^{d} \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_i + 1}{2} \right) \right) \\ &+ \frac{2d}{x \log^2 x} - \left(\frac{1}{\log x} - \frac{2}{\sqrt{x} \log^2 x} \right) |\mathrm{Re} B(f)|.\end{aligned}$$

Applying Lemma 3.2 with the conditions on x as above, we see

$$\begin{aligned}|\mathrm{Re} B(f)| \geq &\left(1 + \frac{1}{\sqrt{x}}\right)^{-2} \left(\frac{1}{2}\left(1 - \frac{1}{x}\right) \left(\log \frac{q(f)}{\pi^d} + \mathrm{Re} \sum_{j=1}^{d} \frac{\Gamma'}{\Gamma} \left(\frac{\kappa_j + 1}{2} \right) \right) - \mathrm{Re} \sum_{n \leq x} \frac{a_f(n)\Lambda(n)}{n} \left(1 - \frac{n}{x}\right) \right. \\ &+ l(f)\frac{\log x + 1}{x} + \frac{(d - 2l(f))\log 2}{x} - (d - 2l(f)) \sum_{n=1}^{\infty} \frac{x^{-2n-1}}{2n(2n+1)} \\ &\left. - \frac{1}{x} \sum_{i=1}^{d-l(f)} \mathrm{Re} \left(\sum_{n=1}^{\infty} \left(\frac{2}{\kappa_i + 1 + 2n} - \frac{1}{\kappa_i + 1 + n} \right) + \frac{x^{-\kappa_i} - 1}{\kappa_i(\kappa_i + 1)} \right) \right).\end{aligned}$$

For $x \geq 132$ we bound

$$-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\Bigg(l(f)\frac{\log x+1}{x}+\frac{(d-2l(f))\log 2}{x}$$
$$-(d-2l(f))\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}-\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)+\frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right)\Bigg)+\frac{2d}{x\log^2 x}$$
$$=-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5)+\frac{2d}{x\log^2 x}.$$

First, we consider

$$A_4=\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)\right).$$

Fix i and study the inner sum, writing $\kappa_i=\sigma+it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma 3.4 so that

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n}-\frac{1}{\kappa_i+1+n}\right)\right)=\frac{1}{2}\log 4+\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}-\frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}+\frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right)$$
$$\leq \log 2-\frac{4}{5}(\sqrt{3}-2)\leq 1.$$

The inequality comes from the following facts. First, the last term is negative. Next, taking $\sigma \geq 0$, a maple calculation finds that $-\frac{4}{5}(\sqrt{3}-2)$ is a global maximum for

$$\frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2}-\frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2}.$$

Thus we may combine the terms A_2 and A_4 to obtain

$$-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_2-A_4)\leq\frac{(d-l(f))(1-\log 2)+l(f)\log 2}{(1+\sqrt{x})^2\log x}.$$

For A_5, fix i, then writing $\kappa_i=\sigma+it$, since we have $\sigma\geq 0$, we apply Lemma 3.5 to obtain

$$\mathrm{Re}\left(\frac{x^{-\kappa_i}-1}{\kappa_j(\kappa_i+1)}\right)\leq\frac{2\log x}{\log 3}.$$

Thus combining A_1 and A_5 we have

$$-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1-A_5)\leq\frac{(2d/\log 3-l(f)(1+2/\log 3))}{(1+\sqrt{x})^2}-\frac{l(f)}{(1+\sqrt{x})^2\log x}.$$

Finally, for $x\geq 132$ we have

$$-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5)+\frac{2d}{x\log^2 x}$$
$$\leq\frac{1}{(1+\sqrt{x})^2}\left(2d/\log 3-l(f)(1+2/\log 3)+\frac{(d-2l(f))(1-\log 2+\sum_{n=1}^{\infty}\frac{x^{-2n}}{2n(2n+1)})}{\log x}+\frac{2d(1+\sqrt{x})^2}{x\log^2 x}\right)$$
$$\leq\frac{2d}{(1+\sqrt{x})^2}.$$

Hence,

$$\begin{aligned}\log|L(1,f)| \leq & \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x}+\frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)\right)\\ &+\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)+\frac{2d}{(1+\sqrt{x})^2}\\ &-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j+1}{2}\right)\right).\end{aligned}$$

Next, note that

$$0\leq\frac{1}{2\log x}-\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\frac{1}{2}\left(1-\frac{1}{x}\right)\leq\frac{1}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right),$$

and Lemma 3.3 gives

$$\mathrm{Re}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_j+1}{2}\right)\leq\log\left|\frac{1+\kappa_i}{2}\right|,$$

so

$$\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{\kappa_i+1}{2}\right)\leq\log C(f).$$

Therefore,

$$\begin{aligned}\log|L(1,f)| \leq & \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x}+\frac{\log C(f)}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right)\\ &+\left(\frac{1}{\log x}-\frac{2}{\sqrt{x}\log^2 x}\right)\left(1+\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)+\frac{2d}{(1+\sqrt{x})^2}.\end{aligned}$$

The right hand side of the above is largest when $a_f(p)=d$ for all $p\leq x$, thus

$$\log|L(1,f)|\leq d\mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n\log n}\frac{\log(x/n)}{\log x}+\frac{d}{\log x}\mathrm{Re}\sum_{n\leq x}\frac{\Lambda(n)}{n}\left(1-\frac{n}{x}\right)+\frac{\log C(f)}{(\sqrt{x}+1)\log x}\left(1+\frac{1}{\log x}\right)+\frac{2d}{(1+\sqrt{x})^2}.$$

So applying Lemmas 3.6 and 3.7 and choosing $x=\frac{\log^2 C(f)}{4d^2}$ (which implies $\frac{\log C(f)}{\sqrt{x}}=2d$ and allows us to factor d from each term) we obtain

$$\log|L(1,f)|\leq d\left(\log\log x+\gamma-\frac{1}{\log x}+\frac{2}{(1+\sqrt{x})^2}\right)+\frac{\log C(f)}{\sqrt{x}\log x}\left(1+\frac{1}{\log x}\right).$$

Thus for $x\geq 132$ we have

$$\begin{aligned}\log|L(1,f)| &\leq d\left(\log\log x+\gamma+\frac{1}{\log x}+\frac{2}{\log^2 x}+\frac{2}{(1+\sqrt{x})^2}\right)\\ &\leq d\left(\log\log x+\gamma+\frac{1}{\log x}+\frac{2.31}{\log^2 x}\right).\end{aligned}$$

Therefore,

$$|L(1,f)|\leq e^{d\gamma}\log^d x\left(1+\frac{d}{\log x}+\frac{dK(d)}{\log^2 x}\right)$$

where $K(d)=2.31+(1+\frac{4.62}{\log x}+\frac{(2.31)^2}{\log^2 x})\sum_{k=0}^{\infty}\frac{d^{k+1}}{(k+2)!}(\frac{1}{\log x}+\frac{2.31}{\log^2 x})^k$. Replacing x gives

$$|L(1,f)|\leq 2^d e^{d\gamma}\left((\log\log C(f)-\log 2d)^d+\frac{d}{2}(\log\log C(f)-\log 2d)^{d-1}+\frac{dK(d)}{4}(\log\log C(f)-\log 2d)^{d-2}\right),$$

which proves the result.

4.2. **Lower bounds for $L(1,f)$.** The argument proceeds similarly. As before we let $C(f)$ be chosen such that $x = \frac{\log^2 C(f)}{4d^2} \geq 132$, then from Lemma 3.1 we have

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x} + \frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d} + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)$$
$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)|\mathrm{Re}B(f)| - \frac{2d}{x\log^2 x}.$$

Applying Lemma 3.2 we see

$$|\mathrm{Re}(B(f))| \leq \left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right) + \mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right) - \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\right.$$
$$\left.-d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)} + l(f)\frac{\log x+1}{x} + \frac{(d-2l(f))\log 2}{x} - \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n} + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right)\right)\right).$$

For $x \geq 132$ we bound

$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(l(f)\frac{\log x+1}{x} + \frac{(d-2l(f))\log 2}{x} - d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}\right.$$
$$\left.-\frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right)\right) + \frac{x^{-\kappa_i}-1}{\kappa_i(\kappa_i+1)}\right) - \frac{2d}{x\log^2 x}$$
$$= -\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5) - \frac{2d}{x\log^2 x}.$$

First, we consider

$$A_4 = \frac{1}{x}\sum_{i=1}^{d-l(f)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right)\right).$$

Fix i and study the inner sum, writing $\kappa_i = \sigma + it$, and noting that Ramanujan-Petersson gives us $\sigma \geq 0$, we apply Lemma 3.4 so that

$$\mathrm{Re}\left(\sum_{n=1}^{\infty}\left(\frac{2}{\kappa_i+1+2n} - \frac{1}{\kappa_i+1+n}\right)\right) = \frac{1}{2}\log 4 + \frac{\sigma^2+3\sigma+2+t^2}{(\sigma+2)^2+t^2} - \frac{\sigma^2+4\sigma+3+t^2}{(\sigma+3)^2+t^2} + \frac{1}{2}\log\left(\frac{(\sigma+2)^2+t^2}{(\sigma+3)^2+t^2}\right)$$
$$\geq 2\log 2 - \log(3).$$

The inequality comes from the following facts. First, the combination of the second and third term is positive since $\sigma \geq 0$, and the last term has a global minimum at the point $(0,0)$ which gives $\log(2/3)$. Thus we may combine the terms A_2 and A_4 to obtain

$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_2-A_4) \geq 1.04\frac{d(\log 2-\log 3) + l(f)\log 3}{(\sqrt{x}-1)^2\log x}.$$

For A_5, fix j, then writing $\kappa_j = \sigma + it$ and invoking Ramanujan-Petersson, we can apply Lemma 3.5 to obtain

$$\mathrm{Re}\left(\frac{x^{-\kappa_j}-1}{\kappa_j(\kappa_j+1)}\right) \geq -\frac{2\log x}{\log 3}.$$

Thus combining the terms A_1 and A_5 we have

$$-\left(\frac{1}{\log x} + \frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1-A_5) \geq -1.04\left(\frac{2d/\log 3 + l(f)(2/\log(3)-1)}{(\sqrt{x}-1)^2} + \frac{l(f)}{(\sqrt{x}-1)^2\log x}\right).$$

Finally, for $x \geq 132$ we have

$$-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}(A_1+A_2-A_3-A_4-A_5)-\frac{2d}{x\log^2 x}$$
$$\geq \frac{1}{(\sqrt{x}-1)^2}1.04\left(-(2d/\log 3-l(f)(2d/\log(3)-1))-\frac{2d}{1.04x\log^2 x}(\sqrt{x}-1)^2\right.$$
$$\left.+\frac{d(\log 2-\log 3)+l(f)(\log(3)-1)-d\sum_{n=1}^{\infty}\frac{x^{-2n-1}}{2n(2n+1)}}{\log x}\right)\geq\frac{-2.05d}{(\sqrt{x}-1)^2}.$$

Thus, for $x \geq 132$

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x}+\frac{1}{2\log x}\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)$$
$$-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\left(\log\left(\frac{q(f)}{\pi^d}\right)+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\right.$$
$$\left.-\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\right)-\frac{2.05d}{(\sqrt{x}-1)^2}.$$

We note that

$$\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\left(\frac{1}{2\log x}-\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{2}\left(1-\frac{1}{x}\right)\right)\right)$$
$$\geq -\left(\log\frac{q(f)}{\pi^d}+\mathrm{Re}\sum_{j=1}^{d}\frac{\Gamma'}{\Gamma}\left(\frac{1+\kappa_j}{2}\right)\right)\frac{1}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)$$
$$\geq -\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right).$$

Where the last inequality comes from Lemma 3.3. So far, we have proven

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n\log n}\frac{\log(\frac{x}{n})}{\log x}-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)$$
$$+\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)-\frac{2.05d}{(\sqrt{x}-1)^2}. \tag{4.1}$$

To continue, we see from Lemma 3.6 if $x \geq 132$ we have

$$\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\geq d(1-\log x),$$

thus as in [6, pg 18 line 11] we have

$$\left(\left(1-\frac{1}{\sqrt{x}}\right)^{-2}\left(\frac{1}{\log x}+\frac{2}{\sqrt{x}\log^2 x}\right)-\frac{1}{\log x}\right)\mathrm{Re}\sum_{n\leq x}\frac{a_f(n)\Lambda(n)}{n}\left(1-\frac{n}{x}\right)\geq-\frac{2d}{\sqrt{x}}.$$

Using this in (4.1) we have

$$\log|L(1,f)| \geq \mathrm{Re}\sum_{n\leq x}a_f(n)\Lambda(n)\left(\frac{1}{n\log n}-\frac{1}{x\log x}\right)-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)-\frac{2d}{\sqrt{x}}-\frac{2.05d}{(\sqrt{x}-1)^2}$$
$$\geq \mathrm{Re}\sum_{n\leq x}a_f(n)\Lambda(n)\left(\frac{1}{n\log n}-\frac{1}{x\log x}\right)-\frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1+\frac{1+1/\sqrt{x}}{\log x}\right)-\frac{9d}{4\sqrt{x}}.$$

We apply Lemma 3.8, thus guaranteeing that the first term in the right hand side is smallest when $a_f(p) = -d$ for every prime $p \le x$. Therefore, we have

$$\log|L(1,f)| \ge d \sum_{p^k \le x} \Lambda(p^k)(-1)^k \left(\frac{1}{p^k \log p^k} - \frac{1}{x \log x}\right) - \frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1 + \frac{1+1/\sqrt{x}}{\log x}\right) - \frac{9d}{4\sqrt{x}}.$$

Following the discussion after [6, Equation 5.3] we see that

$$\log|L(1,f)| \ge d\left(-\log\log x - \gamma + \log\zeta(2) + \frac{1}{\log x} - \frac{8}{5\sqrt{x}}\right) - \frac{\log C(f)}{(\sqrt{x}-1)\log x}\left(1 + \frac{1+1/\sqrt{x}}{\log x}\right) - \frac{9d}{4\sqrt{x}}.$$

Our choice of $x = \frac{\log^2 C(f)}{4d^2}$ gives $-\log C(f) \ge -2d\sqrt{x} - 1$ so that with a little calculation one obtains

$$\log|L(1,f)| \ge d\left(-\log\log x - \gamma + \log\zeta(2) - \frac{1}{\log x} - \frac{2}{\log^2 x} - \frac{9}{2\sqrt{x}}\right).$$

Exponentiating both sides gives

$$\begin{aligned}\frac{1}{|L(1,f)|} &\le \left(e^{\gamma}\frac{6}{\pi^2}\right)^d \log^d x \exp\left(\frac{d}{\log x} + \frac{2d}{\log^2 x} + \frac{9d}{2\sqrt{x}}\right)\\ &\le \left(e^{\gamma}\frac{6}{\pi^2}\right)^d \log^d x\left(1 + \frac{d}{\log x} + \frac{dJ_1(d)}{\log^2 x} + \frac{dJ_2(d)}{2\sqrt{x}}\right),\end{aligned}$$

where

$$J_1(d) = 2 + \left(1 + \frac{4}{\log x} + \frac{4}{\log^2 x}\right)\sum_{k=0}^{\infty}\frac{d^{k+1}}{(k+2)!}\left(\frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{9}{2\sqrt{x}}\right)^k,$$

and

$$J_2(d) = 9 + \left(\frac{18}{\log x} + \frac{18}{\log^2 x} + \frac{81}{2\sqrt{x}}\right)\sum_{k=0}^{\infty}\frac{d^{k+1}}{(k+2)!}\left(\frac{1}{\log x} + \frac{2}{\log^2 x} + \frac{9}{2\sqrt{x}}\right)^k.$$

Replacing x we get

$$\begin{aligned}\frac{1}{|L(1,f)|} \le \left(2e^{\gamma}\frac{6}{\pi^2}\right)^d \Bigg(&(\log\log C(f) - \log 2d)^d + \frac{d}{2}(\log\log C(f) - \log 2d)^{d-1}\\ &+ \frac{dJ_1(d)}{4}(\log\log C(f) - \log 2d)^{d-2} + \frac{d^2 J_2(d)(\log\log C(f) - \log 2d)^d}{\log C(f)}\Bigg)\end{aligned}$$

Thus the theorem is proven.

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