CONFORMAL PSEUDO-SUBRIEMANNIAN FUNDAMENTAL GRADED LIE ALGEBRAS OF SEMISIMPLE TYPE

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Abstract. We introduce the notion of a conformal pseudo-subriemannian fundamental graded Lie algebra of semisimple type. Moreover we give a classification of conformal pseudo-subriemannian fundamental graded Lie algebras of semisimple type and their prolongations.

1. Introduction and notation

This paper is the sequel to the previous one [**16**]. We first recall the notion of fundamental graded Lie algebras. Moreover we define the notion of conformal pseudo-subriemannian fundamental graded Lie algebras, which is a generalization of conformal subriemannian fundamental graded Lie algebras.

A graded Lie algebra (GLA) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a fundamental graded Lie algebra (FGLA) if it is a finite dimensional graded Lie algebra generated by nonzero subspace $\mathfrak{g}_{-1}$. An FGLA $\mathfrak{m}$ is said to be of the μ-th kind if $\mathfrak{g}_{-\mu} \neq \{0\}$ and $\mathfrak{g}_p = \{0\}$ for $p < -\mu$. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA over $\mathbb{R}$ such that $\mathfrak{g}_{-2} \neq \{0\}$, and let $[g]$ be the conformal class of a nondegenerate symmetric bilinear form g on $\mathfrak{g}_{-1}$. Then the pair $(\mathfrak{m}, [g])$ is called a conformal pseudo-subriemannian FGLA. In particular if g is positive definite, then $(\mathfrak{m}, [g])$ is called a conformal subriemannian FGLA. Also if the signature of g has the form (r, r), then $(\mathfrak{m}, [g])$ is called a conformal neutral-subriemannian FGLA. Note that if $(\mathfrak{m}, [g])$ is a conformal pseudo-subriemannian FGLA, so is $(\mathfrak{m}, [-g])$. Given two conformal pseudo-subriemannian FGLAs $(\mathfrak{m}_1, [g_1])$ and $(\mathfrak{m}_2, [g_2])$ we say that $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ if there exists a graded Lie algebra isomorphism φ of $\mathfrak{m}_1$ onto $\mathfrak{m}_2$ such that $[\varphi^* g_2] = [g_1]$. Also we say that $(\mathfrak{m}_1, [g_1])$ is equivalent to $(\mathfrak{m}_2, [g_2])$ if $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ or $(\mathfrak{m}_2, [-g_2])$.

Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g}_0$ be the Lie algebra consisting of all the derivations D of $\mathfrak{m}$ satisfying the following conditions: (1) $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ for all $p < 0$; (2) $D|\mathfrak{g}_{-1} \in \mathfrak{co}(\mathfrak{g}_{-1}, g)$. There exists a GLA $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ such that: (i) $\mathfrak{g}_p = \mathfrak{l}_p$ for $p \leqq 0$; (ii) $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ is transitive, i.e., for $X \in \mathfrak{l}_p$, $p \geqq 0$, if $[X, \mathfrak{l}_{-1}] = \{0\}$, then $X = 0$; (iii) $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ is maximum among GLAs satisfying conditions (i) and (ii) above, which is called the prolongation of $(\mathfrak{m}, [g])$ (For more details on the prolongation, see [**13,** §5]). Note that the prolongation of $(\mathfrak{m}, [g])$ is finite dimensional (Lemma 3.2). Clearly the prolongation of $(\mathfrak{m}, [g])$ coincides with that of $(\mathfrak{m}, [-g])$.

It is known ([**5**], [**16**]) that the prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of a conformal subriemannian FGLA $(\mathfrak{m}, [g])$ satisfying the condition $\mathfrak{g}_1 \neq \{0\}$ is a real rank one simple graded Lie algebra . In contrast, there exists a conformal neutral-subriemannian FGLA $(\mathfrak{m}, [g])$ such that the prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, [g])$ is nonsemisimple and such that $\mathfrak{g}_1 \neq \{0\}$ (cf. Example 5.1). A conformal pseudo-subriemannian FGLA is said to be of semisimple type if the prolongation is semisimple. In this paper we give a classification of conformal pseudo-subriemannian FGLAs of semisimple type and their prolongations (Theorem 5.2). In particular we prove that the prolongation of a conformal pseudo-subriemannian FGLA of semisimple type is simple. Also we give a classification of conformal neutral-subriemannian FGLAs of semisimple type (Corollary 5.1).

Notation and conventions.

(1) Blackboard bold is used for the standard systems $\mathbb{Z}$ (the ring of integers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{C}'$ (split complex numbers), the real division rings $\mathbb{H}$ (Hamilton's quaternions), $\mathbb{H}'$ (split quaternions), $\mathbb{O}$ (Cayley's [nonassociative] octonions) and $\mathbb{O}'$ (split octonions). We denote by $\mathbb{R}_{>0}$ (resp. $\mathbb{R}_{\geqq 0}$) the set consisting of all the positive real numbers (resp. non-negative real numbers). For $\mathbb{K} = \mathbb{C}, \mathbb{C}', \mathbb{H}, \mathbb{H}', \mathbb{O}$ or $\mathbb{O}'$, we set $\operatorname{Im}\mathbb{K} = \{ z \in \mathbb{K} : \operatorname{Re} z = 0 \}$.

(2) For any real vector space V we denote by $V(\mathbb{C})$ the complexification of V.

(3) Let V be a finite dimensional real vector space, and let g be a nondegenerate symmetric bilinear form on V. We set

$$\mathfrak{so}(V,g) = \{ A \in \mathfrak{gl}(V) : A \cdot g = 0 \},$$
$$\mathfrak{co}(V,g) = \{ A \in \mathfrak{gl}(V) : A \cdot g = \eta_A g \text{ for some } \eta_A \in \mathbb{R} \},$$

where $A \cdot g$ is a symmetric bilinear form on V defined by $(A\cdot g)(x,y) = g(Ax,y) + g(x,Ay)$ $(x, y \in V)$. We define a linear mapping $g^\flat : V \to V^*$ by $g^\flat(x)(y) = g(x,y)$ $(x,y \in V)$. Since g is non-degenerate, $g^\flat$ is a linear isomorphism. We denote by $g^\sharp$ the inverse mapping of $g^\flat$.

(4) For a graded vector space $V = \bigoplus_{p\in\mathbb{Z}} V_p$ and $k \in \mathbb{Z}$ we denote subspaces $\bigoplus_{p\leqq k} V_p$ and $\bigoplus_{p\geqq k} V_p$ by $V_{\leqq k}$ and $V_{\geqq k}$ respectively. Also we denote the subspace $\bigoplus_{p<0} V_p$ by V_-. We call V_- the negative part of V.

(5) For a reductive Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{g}^{ss}$ the semisimple part of $\mathfrak{g}$.

(6) For a GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ we denote by $\operatorname{Aut}_0(\mathfrak{g})$ the group consisting of all the automorphisms a of $\mathfrak{g}$ such that $a(\mathfrak{g}_p) = \mathfrak{g}_p$ for all $p \in \mathbb{Z}$.

2. Finite dimensional semisimple graded Lie algebras

2.1. **Finite dimensional complex semisimple graded Lie algebras.** Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a complex semisimple GLA such that the negative part $\mathfrak{g}_-$ is an FGLA. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_0$; then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $E \in \mathfrak{h}$, where E is the characteristic element of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ (i.e., E is an element of $\mathfrak{g}_0$ such that $[E,X] = pX$ for $X \in \mathfrak{g}_p$). Let Δ be a root system of $(\mathfrak{g},\mathfrak{h})$. For $\alpha \in \Delta$, we denote by $\mathfrak{g}^\alpha$ the root space corresponding to α. We associate to any set of roots $Q \subset \Delta$ a subspace

$$\mathfrak{g}(Q) = \sum_{\alpha\in Q} \mathfrak{g}^\alpha \subset \mathfrak{g}.$$

There exists a simple root system $\Pi = \{\alpha_1, \dots, \alpha_l\}$ of $(\mathfrak{g},\mathfrak{h})$ such that $\mathfrak{g}(\Pi) \subset \bigoplus_{p\geqq 0} \mathfrak{g}_p$ ([**15,** p. 441]). Clearly $\alpha_i(E)$ is a non-negative integer. Since the negative part $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, $\alpha_i(E)$ is 0 or 1 ([**15,** Lemma 3.8]). We put $\Delta_p = \{ \alpha \in \Delta : \alpha(E) = p \}$ and $\Pi_p = \Delta_p \cap \Pi$; then $\Pi = \Pi_0 \cup \Pi_1$. When $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a simple graded Lie algebra (SGLA), we enumerate simple roots of $\mathfrak{g}$ as in [**3**]. Moreover if $\mathfrak{g}$ has the Dynkin diagram of type X_l, then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of type (X_l, Π_1).

For $\gamma \in \Pi_1$, we put

$$\Delta_{-1}(-\gamma) = \{ -\gamma + (\Delta_0 \cup \{0\}) \} \cap \Delta = \{ \alpha = -\gamma + \beta \in \Delta : \beta \in \Delta_0 \cup \{0\} \}.$$

Proposition 2.1 ([**9**, Ch.3, §3.5] and [**1**, Proposition 7.3]). *The decomposition of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ into irreducible submodules is given by*

$$\mathfrak{g}_{-1} = \bigoplus_{\gamma \in \Pi_1} \mathfrak{g}(\Delta_{-1}(-\gamma)).$$

In particular the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is completely reducible. Moreover $\mathfrak{g}(\Delta_{-1}(-\gamma))$ is an irreducible $\mathfrak{g}_0$-module with highest weight $-\gamma$.

From [**4**, Ch.VIII, §7, Propositions 11 and 12] and the table of [**3**] we obtain the following proposition.

Proposition 2.2. *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional complex SGLA satisfying the following conditions:* (i) *the negative part $\mathfrak{m}$ is an FGLA;* (ii) *$\mathfrak{g}_{-2}$ and the semisimple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$ are both nonzero;* (iii) *there exists a $\mathfrak{g}_0^{ss}$-invariant nondegenerate symmetric bilinear form $g : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \to \mathbb{C}$.*

(1) *If the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is of type $(C_l, \{\alpha_2\})$ $(l \geqq 3)$ or $(F_4, \{\alpha_4\})$.*

(2) *If the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible and if $\mathfrak{g}_{-1}$ is the direct sum of two irreducible $\mathfrak{g}_0$-submodules of $\mathfrak{g}_{-1}$ which are totally isotropic with respect to g, then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is of type $(A_l, \{\alpha_1, \alpha_l\})$ or $(B_l, \{\alpha_1, \alpha_l\})$ $(l \geqq 3)$.*

Remark 2.1. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a complex SGLA of type $(A_2, \{\alpha_1, \alpha_2\})$, $(B_2, \{\alpha_1, \alpha_2\})$ or $(G_2, \{\alpha_1, \alpha_2\})$. Then the semisimple part of $\mathfrak{g}_0$ is $\{0\}$. We can easily construct a nondegenerate symmetric bilinear form g on $\mathfrak{g}_{-1}$ satisfying the following condition: for any $A \in \mathfrak{g}_0$ there exists a $\lambda_A \in \mathbb{C}$ such that

$$g([A, X], Y) + g(X, [A, Y]) = \lambda_A g(X, Y) \quad \text{for all } X, Y \in \mathfrak{g}_{-1}$$

(cf. Examples 4.1, 4.2, 4.4).

2.2. **Finite dimensional real semisimple graded Lie algebras.** In this subsection we describe gradations of finite dimensional real semisimple GLAs. We first notice the following proposition.

Proposition 2.3 ([**15**, Proposition 3.3]). *The finite dimensional real SGLAs $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ fall into the following two distinct classes:*

(A) *The complex SGLAs, considered as real Lie algebras;*

(B) *The real form of complex simple Lie algebra so that $\mathfrak{g}(\mathbb{C}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathbb{C})$ is a complex SGLA.*

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional real semisimple GLA such that the negative part $\mathfrak{g}_-$ is an FGLA. Let E be the characteristic element of $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, and $\mathfrak{a}$ a maximal $\mathbb{R}$-diagonalizable commutative subalgebra of $\mathfrak{g}$ containing E. Clearly $\mathfrak{a}$ is contained in $\mathfrak{g}_0$. There exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{a} \subset \mathfrak{p}$ ([**9**, Proposition 4.1]). Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. The complexification $\mathfrak{h}(\mathbb{C})$ of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}(\mathbb{C})$. Let Δ be the root system of $(\mathfrak{g}(\mathbb{C}), \mathfrak{h}(\mathbb{C}))$. We set

$$\Delta_k = \{ \alpha \in \Delta : \alpha(E) = k \} \quad (k \in \mathbb{Z}),$$
$$\Delta^\bullet = \{ \alpha \in \Delta : \alpha(\mathfrak{a}) = \{0\} \}, \quad \Delta^\circ = \Delta \setminus \Delta^\bullet.$$

Let σ be the conjugation of $\mathfrak{g}(\mathbb{C})$ defined by its real form $\mathfrak{g}$. For $\lambda \in \mathfrak{h}(\mathbb{C})^*$ we define the element $\lambda^\sigma \in \mathfrak{h}(\mathbb{C})^*$ by $\lambda^\sigma = \overline{\lambda \circ \sigma}$. If $\alpha \in \Delta$, then $\alpha^\sigma \in \Delta$. We can choose a simple root system Π of $(\mathfrak{g}(\mathbb{C}), \mathfrak{h}(\mathbb{C}))$ such that: (i) the corresponding system of positive roots Δ^+ satisfies the following conditions: $\Delta^+ \cap \Delta^\circ$ is σ-invariant; (ii) $\mathfrak{g}(\Pi) \subset \mathfrak{g}(\mathbb{C})_{\geqq 0}$. There exists an involutive permutation ν of the set Π° such that

$$\gamma^\sigma = \nu(\gamma) + \sum_{\beta \in \Pi^\bullet} k_\beta \beta \quad (\gamma \in \Pi^\circ, k_\beta \in \mathbb{Z}_{\geqq 0}).$$

We set $\Pi^\bullet = \Delta^\bullet \cap \Pi$, $\Pi^\circ = \Delta^\circ \cap \Pi$ and $\Pi_k = \Delta_k \cap \Pi$. We shall identify the vertices of the Dynkin diagram X_l with the elements of Π. The Satake diagram S_l is obtained from X_l as follows: Firstly we paint the vertices $\alpha \in \Pi^\bullet$ (resp. $\alpha \in \Pi^\circ$) into black (resp. white). Secondly for $\alpha \in \Pi^\circ$, if $\alpha^\sigma \neq \alpha$, then we connect the pair $\{\alpha, \alpha^\sigma\}$ by a curved arrow. When this is done for all such pairs, we obtain the Satake diagram S_l.

Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional real semisimple GLA with Satake diagram S_l, and let Δ_k, Π and Π_k be as in the above. Since $\mathfrak{g}_-$ is an FGLA, $\Pi = \Pi_0 \cup \Pi_1$. Furthermore the following properties hold: (i) $\Pi^\bullet \subset \Pi_0$; (ii) $\Pi_1 \subset \Pi^\circ$; (iii) If $\alpha \in \Pi_1$, then $\alpha^\sigma \in \Delta_1$ ([**1,** Theorem 8.1]). The semisimple GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is said to be of type (S_l, Π_1) ([**7,** §2] and [**15,** §3.4]). For simplicity we denote by $\mathfrak{g}^{\mathbb{C}}_{-1}(-\gamma)$ the subspace $\mathfrak{g}(\mathbb{C})(\Delta_{-1}(-\gamma))$ of $\mathfrak{g}_{-1}(\mathbb{C})$, where $\gamma \in \Pi_1$.

Proposition 2.4 ([**1,** Proposition 8.3])**.** *Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional real semisimple GLA of type (S_l, Π_1). For $\gamma \in \Pi_1$, there are two possibilities:*

(1) *$\nu(\gamma) = \gamma$. Then $-\gamma^\sigma \in \Delta_{-1}(-\gamma)$ and the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}^{\mathbb{C}}_{-1}(-\gamma)$ is σ-invariant.*

(2) *$\nu(\gamma) \neq \gamma$. Then $-\gamma^\sigma \in \Delta_{-1}(-\nu(\gamma))$ and the two irreducible $\mathfrak{g}_0(\mathbb{C})$-modules $\mathfrak{g}^{\mathbb{C}}_{-1}(-\gamma)$ and $\mathfrak{g}^{\mathbb{C}}_{-1}(-\nu(\gamma))$ determine one irreducible $\mathfrak{g}_0$-submodule $\mathfrak{g} \cap (\mathfrak{g}^{\mathbb{C}}_{-1}(-\gamma) + \mathfrak{g}^{\mathbb{C}}_{-1}(-\nu(\gamma)))$ of $\mathfrak{g}_{-1}$.*

3. Conformal pseudo-subriemannian fundamental graded Lie algebras

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA of the μ-kind over $\mathbb{R}$, where $\mu \geqq 2$. Let g_1 and g_2 be two nondegenerate real symmetric bilinear forms on $\mathfrak{g}_{-1}$. We say that g_1 is equivalent to g_2 if there exists an $\eta \in \mathbb{R}_{>0}$ such that $g_2 = \eta g_1$. We denote by $[g]$ the equivalence class of a nondegenerate real symmetric bilinear form g on $\mathfrak{g}_{-1}$, which is called the conformal class of g.

Let g be a nondegenerate real symmetric bilinear form on $\mathfrak{g}_{-1}$ with signature (r, s). We call the pair $(\mathfrak{m}, [g])$ a conformal pseudo-subriemannian FGLA of type (r, s). In particular, if $s = 0$ (resp. $r = s$), then $(\mathfrak{m}, [g])$ is called a conformal subriemannian FGLA (resp. a conformal neutral-subriemannian FGLA).

Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g}_0$ be the Lie algebra consisting of all the derivations D of $\mathfrak{m}$ satisfying the following conditions (i) and (ii): (i) $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ for all $p < 0$; (ii) $D|\mathfrak{g}_{-1} \in \mathfrak{co}(\mathfrak{g}_{-1}, g)$. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ (see [**13,** §5.2]). We call the transitive GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ the prolongation of $(\mathfrak{m}, [g])$. If $\mathfrak{g}$ is finite dimensional and semisimple, then $(\mathfrak{m}, [g])$ is said to be of semisimple type.

Let $(\mathfrak{m}_1, [g_1])$ and $(\mathfrak{m}_2, [g_2])$ be two conformal pseudo-subriemannian FGLAs. We say that $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ if there exists a graded Lie algebra isomorphism φ of $\mathfrak{m}_1$ onto $\mathfrak{m}_2$ such that $[\varphi^* g_2] = [g_1]$. Also we say that $(\mathfrak{m}_1, [g_1])$ is equivalent to $(\mathfrak{m}_2, [g_2])$ if $(\mathfrak{m}_1, [g_1])$ is isomorphic to $(\mathfrak{m}_2, [g_2])$ or $(\mathfrak{m}_2, [-g_2])$.

The following lemma can be proved by the same methods as in the case of conformal subriemannian FGLAs ([**16,** Lemma 3.1]).

Lemma 3.1. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$. Let ρ_{-1} be the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ defined by $\rho_{-1}(A)(X) = [A, X]$ $(A \in \mathfrak{g}_0, X \in \mathfrak{g}_{-1})$. We set $\hat{\mathfrak{g}}_0 = (\rho_{-1})^{-1}(\mathfrak{so}(\mathfrak{g}_{-1}, g))$. Then*

(1) $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \hat{\mathfrak{g}}_0$.
(2) *Let E be the characteristic element of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Then $\mathfrak{g}_0 = \mathbb{R}E \oplus \hat{\mathfrak{g}}_0$.*

Lemma 3.2. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$. Then $\mathfrak{g}$ is finite dimensional.*

Proof. We first assume that $\dim \mathfrak{g}_{-1} \geqq 3$. We define a subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ as follows:

$$\mathfrak{h}_0 = \{ X \in \mathfrak{g}_0 : [X, \mathfrak{g}_{\leqq -2}] = \{0\}\}.$$

Identifying $\mathfrak{h}_0$ with a subalgebra of $\mathfrak{gl}(\mathfrak{g}_{-1})$, we see that $\mathfrak{h}_0 \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$. Since the second algebraic prolongation $\mathfrak{co}(\mathfrak{g}_{-1}, g)^{(2)}$ of $\mathfrak{co}(\mathfrak{g}_{-1}, g)$ vanishes, we get $\mathfrak{h}_0^{(2)} = \{0\}$. From Corollary 1 of Theorem 11.1 in [**13**], it follows that $\mathfrak{g}$ is finite dimensional. Next we assume that $\dim \mathfrak{g}_{-1} = 2$. There exists a basis (e_1, e_2) of $\mathfrak{g}_{-1}$ such that $g(e_i, e_j) = \varepsilon_i \delta_{ij}$ for all $i, j = 1, 2$, where $\varepsilon_i \in \{-1, 1\}$. Note that $[e_1, e_2] \neq 0$. For $A \in \mathfrak{h}_0$, we set $\operatorname{ad} A(e_i) = \sum_{k=1}^{2} a_{ki} e_k$ $(i = 1, 2; a_{ki} \in \mathbb{R})$. Since $g([A, e_i], e_j) + g(e_1, [A, e_2]) = \lambda_A g(e_i, e_j)$, we see that $2a_{ii} = \lambda_A$ and $a_{ji}\varepsilon_j + a_{ij}\varepsilon_i = 0$. Also since $[A, [e_i, e_j]] = 0$, we get $a_{11} + a_{22} = 0$, so $\lambda_A = 0$. Hence $\mathfrak{h}_0$ is considered as a subalgebra of $\mathfrak{so}(\mathfrak{g}_{-1}, g)$. However since the first algebraic prolongation $\mathfrak{so}(\mathfrak{g}_{-1}, g)^{(1)}$ of $\mathfrak{so}(\mathfrak{g}_{-1}, g)$ vanishes, we see that $\mathfrak{g}$ is finite dimensional. □

Lemma 3.3. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA, and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$. If $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ is a transitive semisimple GLA such that $\mathfrak{g}_p = \mathfrak{l}_p$ for all $p < 0$ and $\operatorname{ad}(\mathfrak{l}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$, then $\mathfrak{g}$ coincides with $\mathfrak{l}$.*

Proof. Since $\operatorname{ad}(\mathfrak{l}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$, $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ is a graded subalgebra of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$; then $\mathfrak{r}$ is a graded ideal of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$: $\mathfrak{r} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{r}_p$, $\mathfrak{r}_p = \mathfrak{r} \cap \mathfrak{g}_p$. Since $\mathfrak{m} = \mathfrak{l}_-$, we see that $\mathfrak{r}_- = \{0\}$. By transitivity of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, we get $\mathfrak{r} = \{0\}$, so $\mathfrak{g}$ is semisimple. Since $\dim \mathfrak{g}_p = \dim \mathfrak{g}_{-p} = \dim \mathfrak{l}_{-p} = \dim \mathfrak{l}_p$ for $p > 0$, we get $\mathfrak{g}_p = \mathfrak{l}_p$ for $p \neq 0$. Since $\mathfrak{l}_0 = [\mathfrak{l}_{-1}, \mathfrak{l}_1] = [\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$, we obtain $\mathfrak{g} = \mathfrak{l}$. □

The following lemma is essentially due to the proof of [**2,** Lemma 4.1].

Lemma 3.4. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA of type (r, s), and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$. If $\mathfrak{a}$ is a maximal $\mathbb{R}$-diagonalizable commutative subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$, then $\dim \mathfrak{a} \leqq \min\{r, s\} + 1$. In particular, if $\mathfrak{g}$ is semisimple, then we have $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} \leqq \min\{r, s\} + 1$.*

Proof. Clearly $\mathfrak{a}$ contains the characteristic element E of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. By lemma 3.1, $\mathfrak{a}$ can be decomposed into the direct sum $\mathfrak{a}' \oplus \mathbb{R}E$, where $\mathfrak{a}'$ is a subalgebra of $\mathfrak{a}$ such that $\operatorname{ad}(\mathfrak{a}')|\mathfrak{g}_{-1} \subset \mathfrak{so}(\mathfrak{g}_{-1}, g)$. Then $\mathfrak{a}'$ is $\mathbb{R}$-diagonalizable in $\mathfrak{g}_{-1}$. Let λ, μ be weights of the $\mathfrak{a}'$-module $\mathfrak{g}_{-1}$ and let V^λ, V^μ be the corresponding weight spaces. For $x \in V^\lambda$, $y \in V^\mu$ and $t \in \mathfrak{a}'$, we get

$$0 = g([t, x], y) + g(x, [t, y]) = (\lambda + \mu)(t) g(x, y).$$

Hence if $\lambda + \mu \neq 0$, then $g(V^\lambda, V^\mu) = 0$. Let $\hat{\mathfrak{a}}$ be the subspace of $\mathfrak{a}'^*$ spanned by the weights of the $\mathfrak{a}'$-module $\mathfrak{g}_{-1}$. Since the $\mathfrak{a}'$-module $\mathfrak{g}_{-1}$ is faithful, the annihilator space $\{ h \in \mathfrak{a}' : \lambda(h) = 0 \quad \text{for all } \lambda \in \hat{\mathfrak{a}} \}$ vanishes, so $\dim \hat{\mathfrak{a}} = \dim \mathfrak{a}'$. Thus the weights of the module span $\mathfrak{a}'^*$. There exists a basis $(\lambda_1, \ldots, \lambda_l)$ of $\mathfrak{a}'^*$ such that each λ_i is a weight of the $\mathfrak{a}'$-module $\mathfrak{g}_{-1}$. Then $U = \bigoplus_{i=1}^{l} V^{\lambda_i}$

is a totally isotropic subspace of $(\mathfrak{g}_{-1}, g)$, so $\dim \mathfrak{a} - 1 = \dim U \leqq \min\{r, s\}$. If $\mathfrak{g}$ is semisimple, then $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}$ equals to $\dim \mathfrak{a}$, so we obtain $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} \leqq \min\{r, s\} + 1$. □

4. Examples of conformal pseudo-subriemannian FGLAs of semisimple type

4.1. Conformal pseudo-subriemannian FGLAs of classical type.

Example 4.1 (cf. [**12,** §9] and [**6,** Example 3.1.2, p.241])**.** Let $\mathbb{K}$ be $\mathbb{C}$, $\mathbb{H}$, $\mathbb{C}'$ or $\mathbb{H}'$. Here we consider $\mathbb{K}$ as an $\mathbb{R}$-algebra. We put $\mathfrak{l} = \mathfrak{sl}(n, \mathbb{K})$ $(n \geqq 3)$; then $\mathfrak{l}$ is a real simple Lie algebra. Let K_m be the $m \times m$ matrix whose (i, j)-component is $\delta_{i,m+1-j}$. We define an $n \times n$ symmetric real matrix $S_{p,q}$ as follows:

$$S_{p,q} = \begin{bmatrix} 0 & 0 & K_p \\ 0 & 1_q & 0 \\ K_p & 0 & 0 \end{bmatrix} \qquad (p \geqq 1, q \geqq 0, 2p + q = n \geqq 3),$$

where 1_q denotes the $q \times q$ identity matrix. Here the center column and the center row of $S_{p,q}$ should be deleted when $q = 0$. Then $S_{p,q}$ is a symmetric real matrix with signature $(p + q, p)$ such that $S_{p,q}^2 = S_{p,q}$. We put $\mathfrak{g} = \{\, X \in \mathfrak{l} : X^* S_{p,q} + S_{p,q} X = O \,\}$; then

$$\mathfrak{g} = \left\{ X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & -S_{p-1,q} X_{12}^* \\ X_{31} & -X_{21}^* S_{p-1,q} & -\overline{X_{11}} \end{bmatrix} \in \mathfrak{l} : \begin{array}{l} X_{11} \in \mathbb{K},\ X_{12} \in M(1, n', \mathbb{K}), \\ X_{21} \in M(n', 1, \mathbb{K}), \\ X_{31}, X_{13} \in \operatorname{Im} \mathbb{K}, X_{22} \in \mathfrak{gl}(n', \mathbb{K}), \\ X_{22} + S_{p-1,q} X_{22}^* S_{p-1,q} = O \end{array} \right\},$$

where $n' = n - 2$ and we set $S_{0,m} = 1_m$. Here $M(p, q, \mathbb{K})$ denotes the set of $\mathbb{K}$-valued $p \times q$-matrices. We define subspaces $\mathfrak{g}_p$ of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_{-2} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ X_{31} & 0 & 0 \end{bmatrix} \in \mathfrak{g} : X_{31} \in \operatorname{Im} \mathbb{K} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ X_{21} & 0 & 0 \\ 0 & -X_{21}^* S_{p-1,q} & 0 \end{bmatrix} \in \mathfrak{g} : X_{21} \in M(n', 1, \mathbb{K}) \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & -\overline{X_{11}} \end{bmatrix} \in \mathfrak{g} : \begin{array}{l} X_{11} \in \mathbb{K}, X_{22} \in \mathfrak{gl}(n', \mathbb{K}), \\ X_{22} + S_{p-1,q} X_{22}^* S_{p-1,q} = O \end{array} \right\},$$

$$\mathfrak{g}_p = \{\, X \in \mathfrak{g} : {}^t X \in \mathfrak{g}_{-p} \,\} \quad (p = 1, 2), \quad \mathfrak{g}_p = \{0\} \quad (|p| > 2).$$

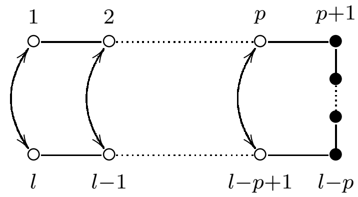
Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a GLA whose negative part $\mathfrak{m}$ is an FGLA of the second kind. We define a symmetric bilinear form g on $\mathfrak{g}_{-1}$ as follows:

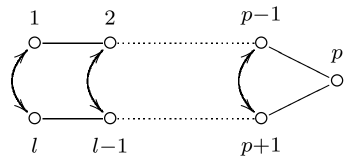
$$g(X, Y) = \operatorname{Re}(X_{21}^* S_{p-1,q} Y_{21}), \qquad X, Y \in \mathfrak{g}_{-1}.$$

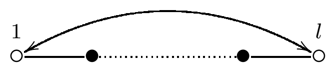
Then g is nondegenerate and for $A \in \mathfrak{g}_0$ we obtain $(\operatorname{ad}(A)|\mathfrak{g}_{-1}) \cdot g = -2(\operatorname{Re} A_{11}) g$. Hence $\operatorname{ad}(\mathfrak{g}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$. The conformal pseudo-subriemannian FGLA $(\mathfrak{m}, [g])$ is said to be of type $(\mathrm{H}\mathbb{K})_{p,q}$.

In case $\mathbb{K} = \mathbb{C}$ or $\mathbb{C}'$ we know that $\mathfrak{g}$ is denoted by $\mathfrak{su}(p + q, p, \mathbb{K})$ $(n = 2p + q)$. Note that $\mathfrak{su}(p + q, p, \mathbb{C}')$ is isomorphic to $\mathfrak{sl}(2p + q, \mathbb{R})$ for any p, q. If $\mathbb{K} = \mathbb{C}$ (resp. $\mathbb{K} = \mathbb{C}'$), then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a real SGLA of type $((\mathrm{AIIIa})_{l,p}, \{\alpha_1, \alpha_l\})$ $(l = n - 1 = 2p + q - 1, p \geqq 2, q \geqq 1)$, $((\mathrm{AIIIb})_l, \{\alpha_1, \alpha_l\})$ $(l = n - 1 = 2p - 1, p \geqq 2, q = 0)$ or $((\mathrm{AIV})_l, \{\alpha_1, \alpha_l\})$ $(l = n - 1 = q + 1, p = 1, q \geqq 1)$ (resp. $((\mathrm{AI})_l, \{\alpha_1, \alpha_l\})$ $(l = n - 1 = 2p + q - 1 \geqq 2)$), and g has the signature $(2p + 2q - 2, 2p - 2) =$

$(2l-2p, 2p-2)$ (resp. $(2p+q-2, 2p+q-2) = (l-1, l-1)$). Here $(\mathrm{AIIIa})_{l,p}$, $(\mathrm{AIIIb})_l$, $(\mathrm{AIV})_l$ and $(\mathrm{AI})_l$ are the following Satake diagrams.

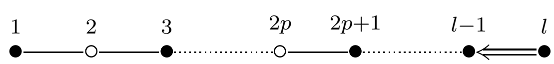
$(\mathrm{AIIIa})_{l,p}$:


$(\mathrm{AIIIb})_l$:


$(\mathrm{AIV})_l$:


$(\mathrm{AI})_l$:


In case $\mathbb{K} = \mathbb{H}$ or $\mathbb{H}'$ we know that $\mathfrak{g}$ is denoted by $\mathfrak{sp}(p+q, p, \mathbb{K})$. Note that $\mathfrak{sp}(p+q, p, \mathbb{H}')$ is isomorphic to $\mathfrak{sp}(2p+q, \mathbb{R})$ for any p, q and that $\mathfrak{sp}(n, \mathbb{R})$ is denoted by $\mathfrak{sp}(2n, \mathbb{R})$ or $\mathfrak{sp}_{2n}(\mathbb{R})$ in [**6**], [**9**] and [**11**]. If $\mathbb{K} = \mathbb{H}$ (resp. $\mathbb{K} = \mathbb{H}'$), then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a real SGLA of type $((\mathrm{CIIa})_{l,p}, \{\alpha_2\})$ $(l = n = 2p+q \geqq 3, p, q \geqq 1)$, or $((\mathrm{CIIb})_l, \{\alpha_2\})$ $(n = l = 2p \geqq 3, q = 0)$ (resp. $((\mathrm{CI})_l, \{\alpha_2\})$ $(l = n = 2p+q \geqq 3)$) and g has the signature $(4p+4q-4, 4p-4) = (4l-4p-4, 4p-4)$ (resp. $(4p+2q-4, 4p+2q-4) = (2l-4, 2l-4)$). Here $(\mathrm{CIIa})_{l,p}$, $(\mathrm{CIIb})_l$ and $(\mathrm{CI})_l$ are the following Satake diagrams.

$(\mathrm{CIIa})_{l,p}$:


$(\mathrm{CIIb})_l$:


$(\mathrm{CI})_l$:


By Lemma 3.3, the prolongation of a conformal pseudo-subriemannian FGLA of type $(\mathrm{H}\mathbb{K})_{p,q}$ coincides with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$.

Example 4.2 (cf.[**15,** §4.4 (3)]). We put $\mathfrak{g} = \{\, X \in \mathfrak{gl}(2l+1, \mathbb{R}) : {}^tXS + SX = 0\}$ $(l \geqq 2)$, where $S = S_{l,1}$. More explicitly

$$\mathfrak{g} = \left\{ X = \begin{bmatrix} A & a & B \\ \xi & 0 & -a' \\ C & -\xi' & -A' \end{bmatrix} \in \mathfrak{gl}(2l+1, \mathbb{R}) : \begin{array}{l} A, B, C \in \mathfrak{gl}(l, \mathbb{R}), \\ B = -B', C = -C', \\ a \in M(l, 1, \mathbb{R}), \xi \in M(1, l, \mathbb{R}) \end{array} \right\}.$$

Here for an $r \times s$-matrix X we put $X' = K_s{}^tXK_r$. The Lie algebra $\mathfrak{g}$ is a real simple Lie algebra $\mathfrak{so}(l+1, l, \mathbb{R})$ of type $(\mathrm{BI})_{l,l}$. Here $(\mathrm{BI})_{l,l}$ is the following Satake diagram:

$(\mathrm{BI})_{l,l}$:


We define subspaces $\mathfrak{g}_p$ of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_{-3} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ X_{41} & 0 & 0 & 0 & 0 \\ 0 & -X'_{41} & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g} : X_{41} \in M(l-1,1,\mathbb{R}) \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ X_{31} & 0 & 0 & 0 & 0 \\ 0 & X_{42} & 0 & 0 & 0 \\ 0 & 0 & -X_{31} & 0 & 0 \end{bmatrix} \in \mathfrak{g} : \begin{array}{l} X_{31} \in \mathbb{R}, X_{42} \in \mathfrak{gl}(l-1,\mathbb{R}), \\ X'_{42} = -X_{42} \end{array} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ X_{21} & 0 & 0 & 0 & 0 \\ 0 & X_{32} & 0 & 0 & 0 \\ 0 & 0 & -X'_{32} & 0 & 0 \\ 0 & 0 & 0 & -X'_{21} & 0 \end{bmatrix} \in \mathfrak{g} : \begin{array}{l} X_{21} \in M(l-1,1,\mathbb{R}), \\ X_{32} \in M(1,l-1,\mathbb{R}) \end{array} \right\},$$

$$\mathfrak{g}_{0} = \left\{ \begin{bmatrix} X_{11} & 0 & 0 & 0 & 0 \\ 0 & X_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -X'_{22} & 0 \\ 0 & 0 & 0 & 0 & -X_{11} \end{bmatrix} \in \mathfrak{g} : X_{11} \in \mathbb{R}, X_{22} \in \mathfrak{gl}(l-1,\mathbb{R}) \right\},$$

$$\mathfrak{g}_p = \{\, X \in \mathfrak{g} : {}^tX \in \mathfrak{g}_{-p} \,\} \quad (p = 1,2,3), \quad \mathfrak{g}_p = \{0\} \quad (|p| > 3).$$

Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a real SGLA of type $((\mathrm{BI})_{l,l}, \{\alpha_1, \alpha_l\})$ whose negative part $\mathfrak{m}$ is an FGLA of the third kind. We define a symmetric bilinear form g on $\mathfrak{g}_{-1}$ by

$$g(X,Y) = -\frac{1}{2}(X_{32}Y_{21} + Y_{32}X_{21}) \qquad (X, Y \in \mathfrak{g}_{-1}).$$

Then g is nondegenerate, and for $A \in \mathfrak{g}_0$ we see that $(\mathrm{ad}(A)|\mathfrak{g}_{-1}) \cdot g = -A_{11}g$. Hence $\mathrm{ad}(\mathfrak{g}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$. By Lemma 3.3 $(\mathfrak{m}, [g])$ is a conformal neutral-subriemannian FGLA such that $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, [g])$. The conformal pseudo-subriemannian FGLA $(\mathfrak{m}, [g])$ is said to be of type $(\mathrm{BI})_l$.

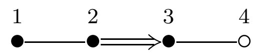
4.2. **Conformal pseudo-subriemannian FGLAs of exceptional type.**

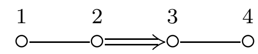
Example 4.3 ([**8,** §3])**.** Let $\mathbb{K} = \mathbb{O}$ or $\mathbb{O}'$. Here we consider $\mathbb{K}$ as an $\mathbb{R}$-algebra. We define a nondegenerate symmetric bilinear form g on $\mathbb{K}$ by $g(x,y) = \frac{1}{2}(\bar{x}y + \bar{y}x)$. We set

$$\mathfrak{g}_{-1} = \mathfrak{g}_1 = \mathbb{K}, \quad \mathfrak{g}_{-2} = \mathfrak{g}_2 = \mathrm{Im}\,\mathbb{K}, \quad \mathfrak{g}_0 = \mathfrak{g}'_0 \oplus \mathbb{R}, \quad \mathfrak{g}_p = \{0\} \quad \text{for } |p| > 2,$$

where $\mathfrak{g}'_0 = \{\, A \in \mathfrak{so}(\mathbb{K}, g) : A(1) = 0 \,\}$. Note that $\mathfrak{g}'_0$ is isomorphic to $\mathfrak{so}(\mathrm{Im}\,\mathbb{K}, g)$. We further put $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ and $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$. We define a bracket operation $[\,\cdot,\cdot\,]$ on $\mathfrak{g}$ as in [**8,** §3.2, p.444 and §3.4, pp.447–448]. By using [**8,** Lemma 3.1] we can prove that $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ becomes a GLA whose negative part $\mathfrak{m}$ is an FGLA of the second kind. For $A \oplus r \in \mathfrak{g}_0$ and $X, Y \in \mathfrak{g}_{-1}$ we see that $(\mathrm{ad}(A \oplus r)|\mathfrak{g}_{-1}) \cdot g = -2rg$, and hence $\mathrm{ad}(\mathfrak{g}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$. The conformal pseudo-subriemannian FGLA $(\mathfrak{m}, [g])$ is said to be of type $(\mathrm{H}\mathbb{K})$.

By [**8,** Theorem 3.5], in case $\mathbb{K} = \mathbb{O}$ (resp. $\mathbb{K} = \mathbb{O}'$) the GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is a real SGLA of type $(\mathrm{FII}, \{\alpha_4\})$ (resp. $(\mathrm{FI}, \{\alpha_4\})$). Here FII$(= \mathrm{F}_{4(-20)})$ and FI$(= \mathrm{F}_{4(4)})$ are the following Satake diagrams respectively.

FII : 

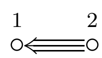
FI : 

Clearly $(\mathfrak{m}, [g])$ is a conformal subriemannian FGLA (resp. a conformal neutral-subriemannian FGLA) when $\mathbb{K} = \mathbb{O}$ (resp. $\mathbb{K} = \mathbb{O}'$). By Lemma 3.3, the prolongation of a conformal pseudo-subriemannian FGLA of type (H$\mathbb{K}$) coincides with $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$.

Example 4.4. Let V be a real vector space $\mathbb{R}^2$, and we set $\mathfrak{s} = \mathfrak{sl}(V)$. We define real vector spaces $\mathfrak{l}_p$ $(p \in \mathbb{Z})$ as follows:

$$\mathfrak{l}_{-2} = \mathfrak{l}_2 = \mathbb{R}, \quad \mathfrak{l}_{-1} = \mathfrak{l}_1 = S^3(V), \quad \mathfrak{l}_0 = \mathfrak{s} \oplus \mathbb{R}, \quad \mathfrak{l}_p = \{0\} \quad (p > |2|).$$

We define a bracket operation $[\cdot,\cdot]$ on $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ as in [**8,** §3.1, p.450]. Then $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ becomes a real SGLA of type $(\mathrm{G}_{2(2)}, \{\alpha_2\})$ ([**8,** Theorem 4.3]) and the negative part $\mathfrak{m}$ is an FGLA of the second kind. Here $\mathrm{G}_{2(2)}$ is the following Satake diagram:

$\mathrm{G}_{2(2)}$: 

We set $V_{-1} = \mathbb{R}e_1$ and $V_{-2} = \mathbb{R}e_2$, where (e_1, e_2) is the canonical basis of V. We put $\mathfrak{s}_p = \{X \in \mathfrak{s} : X(V_k) \subset V_{k+p}$ for all $k\}$; then $\mathfrak{s} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{s}_p$ is a real SGLA of the first kind.

We define subspaces W_k (resp. W_{-k}) $(k = 1, \dots, 4)$ of $\mathfrak{l}_1$ (resp. $\mathfrak{l}_{-1}$) as follows:

$$W_{\pm1} = S^3(V_{-1}), \quad W_{\pm2} = S^2(V_{-1}) \otimes V_{-2}, \quad W_{\pm3} = V_{-1} \otimes S^2(V_{-2}), \quad W_{\pm4} = S^3(V_{-2});$$

then $\mathfrak{l}_{\pm1} = \bigoplus_{k=1}^{4} W_{\pm k}$. We define subspaces $\mathfrak{g}_p$ of $\mathfrak{l}$ as follows:

$$\mathfrak{g}_{\pm5} = \mathfrak{l}_{\pm2}, \quad \mathfrak{g}_k = W_k \quad (k = \pm2, \pm3, \pm4), \quad \mathfrak{g}_{\pm1} = \mathfrak{s}_{\pm1} \oplus W_{\pm1},$$
$$\mathfrak{g}_0 = \mathfrak{s}_0 \oplus \mathbb{R}, \quad \mathfrak{g}_p = \{0\} \quad (p > |5|).$$

Then $\mathfrak{l} = \mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ becomes a real SGLA of type $(\mathrm{G}_{2(2)}, \{\alpha_1, \alpha_2\})$ such that the negative part $\mathfrak{m}$ is an FGLA of the 5-th kind. Let $(\cdot \mid \cdot)$ be an inner product on $S^3(V)$ induced by the canonical inner product on V. We define a symmetric bilinear form g on $\mathfrak{g}_{-1}$ as follows:

$$g(\mathfrak{s}_{-1}, \mathfrak{s}_{-1}) = g(W_{-1}, W_{-1}) = 0, \quad g(X, u) = g(u, X) = (Xu \mid e_1^2 e_2) \quad (X \in \mathfrak{s}_{-1}, u \in W_{-1}).$$

Then g is nondegenerate, and for $A = \lambda(E_{11} - E_{22}) \oplus r \in \mathfrak{g}_0$, $X \in \mathfrak{s}_{-1}$ and $u \in W_{-1}$ $(\lambda, r \in \mathbb{R})$, we see that $(\mathrm{ad}(A)|\mathfrak{g}_{-1}) \cdot g = (\lambda - r)g$, where E_{ij} is an element of $\mathfrak{s}$ such that $E_{ij}e_k = \delta_{jk}e_i$. Thus $\mathrm{ad}(\mathfrak{g}_0)|\mathfrak{g}_{-1} \subset \mathfrak{co}(\mathfrak{g}_{-1}, g)$. Hence by Lemma 3.3 $(\mathfrak{m}, [g])$ is a conformal neutral-subriemannian FGLA such that $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, [g])$. The conformal pseudo-subriemannian FGLA $(\mathfrak{m}, [g])$ is said to be of type (G).

5. Classification of conformal pseudo-subriemannian FGLAs of semisimple type

In this section we prove that a conformal pseudo-subriemannian FGLA of semisimple type is isomorphic to one of conformal pseudo-subriemannian FGLAs given in the previous section.

Proposition 5.1. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA of semisimple type, and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$.*

(1) *If the* $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *is irreducible and the* $\mathfrak{g}_0(\mathbb{C})$*-module* $\mathfrak{g}_{-1}(\mathbb{C})$ *is reducible, there exist* $\mathfrak{g}_0(\mathbb{C})$*-submodules* $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i=1,2)$ *such that* (i) $\mathfrak{g}_{-1}(\mathbb{C})=\mathfrak{g}_{-1}(\mathbb{C})^{(1)}\oplus\mathfrak{g}_{-1}(\mathbb{C})^{(2)}$; (ii) $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i=1,2)$ *are totally isotropic subspaces of* $(\mathfrak{g}_{-1}(\mathbb{C}),g)$; (iii) $\mathfrak{g}_{-1}(\mathbb{C})^{(1)}$ *is contragredient to* $\mathfrak{g}_{-1}(\mathbb{C})^{(2)}$ *as a* $\hat{\mathfrak{g}}_0(\mathbb{C})$*-module, where* $\hat{\mathfrak{g}}_0=(\rho_{-1})^{-1}(\mathfrak{so}(\mathfrak{g}_{-1},g))$.

(2) *If the* $\mathfrak{g}_0$*-module* $\mathfrak{g}_{-1}$ *is reducible, then there exist* $\mathfrak{g}_0$*-submodules* $\mathfrak{g}_{-1}^{(i)}$ $(i=1,2)$ *such that:* (i) $\mathfrak{g}_{-1}=\mathfrak{g}_{-1}^{(1)}\oplus\mathfrak{g}_{-1}^{(2)}$; (ii) $\mathfrak{g}_{-1}^{(i)}$ $(i=1,2)$ *are totally isotropic subspaces of* $(\mathfrak{g}_{-1},g)$; (iii) $\mathfrak{g}_{-1}^{(1)}$ *is contragredient to* $\mathfrak{g}_{-1}^{(2)}$ *as a* $\hat{\mathfrak{g}}_0$*-module;* (iv) *the* $\mathfrak{g}_0(\mathbb{C})$*-modules* $\mathfrak{g}_{-1}^{(i)}(\mathbb{C})$ *are irreducible.*

(3) $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ *is an SGLA of class* (B).

Proof. (1) and (2). We decompose $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_{-1}(\mathbb{C})$) into irreducible $\mathfrak{g}_0$-modules (resp. $\mathfrak{g}_0(\mathbb{C})$-modules) as follows:

$$\mathfrak{g}_{-1}=\bigoplus_{i=1}^{k}\mathfrak{g}_{-1}^{(i)},\quad \mathfrak{g}_{-1}(\mathbb{C})=\bigoplus_{i=1}^{k'}\mathfrak{g}_{-1}(\mathbb{C})^{(i)}.$$

Let E_i be an element of $\mathfrak{g}_0$ such that $[E_i,X_j]=-\delta_{ij}X_j$ for all $X_j\in\mathfrak{g}_{-1}^{(j)}$. We first assume that $\mathfrak{g}_{-1}^{(1)}$ is a nondegenerate subspace of $(\mathfrak{g}_{-1},g)$. There exist elements X_1,Y_1 of $\mathfrak{g}_{-1}^{(1)}$ such that $g(X_1,Y_1)\neq 0$. Since

$$g([E_1,X_1],Y_1)+g(X_1,[E_1,Y_1])=\eta_{E_1}g(X_1,Y_1),$$

we see that $\eta_{E_1}=-2$. For $X_i\in\mathfrak{g}_{-1}^{(i)}$ $(i\geqq 2)$,

$$g([E_1,X_1],X_i)+g(X_1,[E_1,X_i])=\eta_{E_1}g(X_1,X_i),$$

so $g(X_1,X_i)=0$. Thus we get $g(\mathfrak{g}_{-1}^{(1)},\mathfrak{g}_{-1}^{(i)})=0$ $(i\geqq 2)$. If there exists a $j\geqq 2$ such that $\mathfrak{g}_{-1}^{(j)}$ is a nondegenerate subspace of $(\mathfrak{g}_{-1},g)$, then

$$0=g([E_j,X_1],Y_1)+g(X_1,[E_j,Y_1])=\eta_{E_j}g(X_1,Y_1)=-2g(X_1,Y_1),$$

which is a contradiction. Hence $\mathfrak{g}_{-1}^{(i)}$ $(i\geqq 2)$ are totally isotropic subspaces of $(\mathfrak{g}_{-1},g)$. Assume that $\mathfrak{g}_{-1}^{(2)}\neq\{0\}$. There exists $j\geqq 3$ such that the restriction of g to the space $\mathfrak{g}_{-1}^{(1)}\times\mathfrak{g}_{-1}^{(j)}$ is nondegenerate. We set $E_2'=E_2+E_j$. Let X_2 (resp. X_j) be an element of $\mathfrak{g}_{-1}^{(2)}$ (resp. $\mathfrak{g}_{-1}^{(j)}$) such that $g(X_2,X_j)\neq 0$. Since

$$g([E_2',X_2],X_j)+g(X_2,[E_2',X_j])=\eta_{E_2'}g(X_2,X_j),$$

we see that $\eta_{E_2'}=-2$. Also

$$0=g([E_2',X_1],Y_1)+g(X_1,[E_2',Y_1])=-2g(X_1,Y_1).$$

This is a contradiction. Therefore we obtain that $\mathfrak{g}_{-1}$ is an irreducible $\mathfrak{g}_0$-module. Next we assume that $\mathfrak{g}_{-1}^{(i)}$ is a totally isotropic subspace of $(\mathfrak{g}_{-1},g)$. Here we may assume that the restriction of g to $\mathfrak{g}_{-1}^{(1)}\times\mathfrak{g}_{-1}^{(2)}$ is nondegenerate. From the above result, $\mathfrak{g}_{-1}^{(2)}$ is a totally isotropic subspace of $(\mathfrak{g}_{-1},g)$ and is contragredient to $\mathfrak{g}_{-1}^{(1)}$ as a $\hat{\mathfrak{g}}_0$-module. If the restriction of g to $\mathfrak{g}_{-1}^{(1)}\times\mathfrak{g}_{-1}^{(3)}$ is nondegenerate, $\mathfrak{g}_{-1}^{(3)}$ is contragredient to $\mathfrak{g}_{-1}^{(1)}$ as a $\hat{\mathfrak{g}}_0$-module, so $\mathfrak{g}_{-1}^{(1)}$ is isomorphic to $\mathfrak{g}_{-1}^{(3)}$ as a $\mathfrak{g}_0$-module, which is a contradiction. Hence $g(\mathfrak{g}_{-1}^{(1)},\mathfrak{g}_{-1}^{(3)})=0$. Similarly we get $g(\mathfrak{g}_{-1}^{(2)},\mathfrak{g}_{-1}^{(3)})=0$. There exists $k\geqq 4$ such that the restriction of g to $\mathfrak{g}_{-1}^{(3)}\times\mathfrak{g}_{-1}^{(k)}$ is nondegenerate. We set $E_1'=E_1+E_2$. Let X_i $(i=1,2,3,k)$ be elements of $\mathfrak{g}_{-1}^{(i)}$ such that $g(X_1,X_2)\neq 0$ and $g(X_3,X_k)\neq 0$. Since

$$0=g([E_1',X_1],X_2)+g(X_1,[E_1',X_2])=\eta_{E_1'}g(X_1,X_2),$$

we get $\eta_{E_1'}=-2$. On the other hand, we see that

$$0=g([E_1',X_3],X_k)+g(X_3,[E_1',X_k])=\eta_{E_1'}g(X_3,X_k),$$

which is a contradiction. Hence $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^{(1)} \oplus \mathfrak{g}_{-1}^{(2)}$. Similarly we can prove that if the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is reducible, there exist $\mathfrak{g}_0(\mathbb{C})$-submodules $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i = 1, 2)$ such that (i) $\mathfrak{g}_{-1}(\mathbb{C}) = \mathfrak{g}_{-1}(\mathbb{C})^{(1)} \oplus \mathfrak{g}_{-1}(\mathbb{C})^{(2)}$; (ii) $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i = 1, 2)$ are totally isotropic subspaces of $(\mathfrak{g}_{-1}(\mathbb{C}), g)$; (iii) $\mathfrak{g}_{-1}(\mathbb{C})^{(1)}$ is contragredient to $\mathfrak{g}_{-1}(\mathbb{C})^{(2)}$ as a $\hat{\mathfrak{g}}_0(\mathbb{C})$-module. The assertions (1) and (2) follow from these results.

(3) We assume that $\mathfrak{g}$ is not simple. There exist ideals $\mathfrak{a}^{(1)}$ and $\mathfrak{a}^{(2)}$ of $\mathfrak{g}$ such that $\mathfrak{a}^{(1)}$ is a simple ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{a}^{(1)} \oplus \mathfrak{a}^{(2)}$. Both ideals $\mathfrak{a}^{(i)}$ $(i = 1, 2)$ are graded ideals of $\mathfrak{g}$; we write $\mathfrak{a}^{(i)} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{a}_p^{(i)}$. By transitivity of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, we see that $\mathfrak{a}_{-1}^{(i)} \neq \{0\}$ $(i = 1, 2)$. From the results of (1) and (2) $\mathfrak{a}_{-1}^{(2)}$ is contragredient to $\mathfrak{a}_{-1}^{(1)}$ as a $\hat{\mathfrak{g}}_0$-module, which is a contradiction. Hence $\mathfrak{g}$ is simple. Also from the results of (1) and (2) and from [**9**, p.157, Example 2], it follows that $\mathfrak{g}$ is of class (B). □

We decompose the conformal pseudo-subriemannian FGLAs of semisimple type into the following three classes:

(SI) The $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is irreducible.
(SII) The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible and the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is reducible.
(SIII) The $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is reducible.

Theorem 5.1. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA of semisimple type, and let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}, [g])$. Assume that $(\mathfrak{m}, [g])$ is of type (r, s) $(r \geqq s)$.*

(1) *If $(\mathfrak{m}, [g])$ is of class (SI), then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is an SGLA of type $((\mathrm{CI})_l, \{\alpha_2\})$, $((\mathrm{CIIa})_{l,p}, \{\alpha_2\})$, $((\mathrm{CIIb})_l, \{\alpha_2\})$ $(l \geqq 3, p \geqq 1)$, $(\mathrm{FI}, \{\alpha_4\})$, or $(\mathrm{FII}, \{\alpha_4\})$.*
(2) *If $(\mathfrak{m}, [g])$ is of class (SII), then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is an SGLA of type $((\mathrm{AIIIa})_{l,p}, \{\alpha_1, \alpha_l\})$, $((\mathrm{AIIIb})_l, \{\alpha_1, \alpha_l\})$ or $((\mathrm{AIV})_l, \{\alpha_1, \alpha_l\})$ $(l \geqq 2)$.*
(3) *If $(\mathfrak{m}, [g])$ is class (SIII), then $(\mathfrak{m}, [g])$ is conformal neutral-subriemannian and $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is an SGLA of type $((\mathrm{AI})_l, \{\alpha_1, \alpha_l\})$, $((\mathrm{BI})_{l,l}, \{\alpha_1, \alpha_l\})$ $(l \geqq 2)$ or $(\mathrm{G}_{2(2)}, \{\alpha_1, \alpha_2\})$.*

Proof. By Proposition 5.1 the complexification $\mathfrak{g}(\mathbb{C}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathbb{C})$ is an SGLA. We first assume that $(\mathfrak{m}, [g])$ is of class (SI); then $\mathfrak{g}(\mathbb{C}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathbb{C})$ be of type $(X_l, \{\alpha_i\})$. Furthermore $\mathfrak{g}_{-1}(\mathbb{C})$ is an irreducible $\mathfrak{g}_0(\mathbb{C})$-module with highest weight $-\alpha_i$ and there exists a $\mathfrak{g}_0^{ss}(\mathbb{C})$-invariant symmetric bilinear form g on $\mathfrak{g}_{-1}(\mathbb{C})$. By Proposition 2.2 (1) we obtain that $(X_l, \{\alpha_i\})$ is one of $(C_l, \{\alpha_2\})$ $(l \geqq 3)$, $(F_4, \{\alpha_4\})$. Next we assume that $(\mathfrak{m}, [g])$ is of class (SII) or (SIII). By Proposition 5.1 (2), the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is decomposed as follows: $\mathfrak{g}_{-1}(\mathbb{C}) = \mathfrak{g}_{-1}(\mathbb{C})^{(1)} \oplus \mathfrak{g}_{-1}(\mathbb{C})^{(2)}$, where $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i = 1, 2)$ are irreducible $\mathfrak{g}_0(\mathbb{C})$-submodule of $\mathfrak{g}_{-1}(\mathbb{C})$ such that: (i) each $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ is totally isotropic with respect to g; (ii) $\mathfrak{g}_{-1}(\mathbb{C})^{(1)}$ is contragredient to $\mathfrak{g}_{-1}(\mathbb{C})^{(2)}$ as a $\hat{\mathfrak{g}}_0(\mathbb{C})$-module. By Proposition 2.2 (2) and Remark 2.1, we obtain that $\mathfrak{g}(\mathbb{C}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p(\mathbb{C})$ is of type $(A_l, \{\alpha_1, \alpha_l\})$, $(B_l, \{\alpha_1, \alpha_l\})$ or $(G_2, \{\alpha_1, \alpha_2\})$. Hence the assertions (1)–(3) follow from Proposition 2.4, and the tables of [**11**, pp.79–82] and [**14**, pp.30–32]. □

Proposition 5.2. *Let $(\mathfrak{m}, [g_1])$ and $(\mathfrak{m}, [g_2])$ be two conformal pseudo-subriemannian FGLAs of semisimple type. If the prolongation of $(\mathfrak{m}, [g_1])$ coincides with that of $(\mathfrak{m}, [g_2])$, then $(\mathfrak{m}, [g_1])$ is equivalent to $(\mathfrak{m}, [g_2])$.*

Proof. The mapping $\varphi = g_1^\sharp \circ g_2^\flat$ induces an isomorphism of $\mathfrak{g}_{-1}(\mathbb{C})$ onto itself as a $\hat{\mathfrak{g}}_0(\mathbb{C})$-module. If the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is reducible, then $\mathfrak{g}_{-1}(\mathbb{C})$ is the direct sum of two irreducible $\mathfrak{g}_0(\mathbb{C})$-modules $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i = 1, 2)$ and $\mathfrak{g}_{-1}(\mathbb{C})^{(1)}$ is not isomorphic to $\mathfrak{g}_{-1}(\mathbb{C})^{(2)}$ as a $\mathfrak{g}_0(\mathbb{C})$-module. In this case $\varphi(\mathfrak{g}_{-1}(\mathbb{C})^{(i)}) = \mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ $(i = 1, 2)$. By Schur's lemma, there exist two complex numbers λ_1, λ_2 such that $\varphi|\mathfrak{g}_{-1}(\mathbb{C})^{(i)} = \lambda_i id$ $(i = 1, 2)$. For $X \in \mathfrak{g}_{-1}(\mathbb{C})^{(1)}$ and $Y \in \mathfrak{g}_{-1}(\mathbb{C})^{(2)}$ we obtain $\lambda_1 g_1(X, Y) = g_1(\varphi(X), Y) = g_2(X, Y)$ and $\lambda_2 g_1(X, Y) = g_1(X, \varphi(Y)) = g_2(X, Y)$. Since $\mathfrak{g}_{-1}(\mathbb{C})^{(i)}$ are totally isotropic with respect to g_1 and g_2, we get $\lambda_1 = \lambda_2$. Hence $g_2 = \lambda_1 g_1$ and $\lambda_i \in \mathbb{R}$. Thus we see that $[g_1] = [g_2]$ or $[g_1] = [-g_2]$. Similarly we can prove that $[g_1] = [g_2]$ or $[g_1] = [-g_2]$ when the $\mathfrak{g}_0(\mathbb{C})$-module $\mathfrak{g}_{-1}(\mathbb{C})$ is irreducible. □

From Theorem 5.1, Proposition 5.2 and the results of §4 we obtain the following theorem.

Theorem 5.2. *Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA of semisimple type. Then $(\mathfrak{m}, [g])$ is equivalent to one of conformal pseudo-subriemannian FGLAs of types* $(\mathrm{H}\mathbb{C})_{p,q}$, $(\mathrm{H}\mathbb{C}')_{p,q}$, $(\mathrm{H}\mathbb{H})_{p,q}$, $(\mathrm{H}\mathbb{H}')_{p,q}$, $(\mathrm{H}\mathbb{O})$, $(\mathrm{H}\mathbb{O}')$, $(\mathrm{BI})_l$, (G). *The prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, [g])$ and the signature (r, s) of g are given in the following table.*

$(\mathfrak{m}, [g])$	$(\mathrm{H}\mathbb{C})_{p,q}$ $(p \geqq 2,\ q \geqq 1)$	$(\mathrm{H}\mathbb{C})_{p,0}$ $(p \geqq 2)$	$(\mathrm{H}\mathbb{C})_{1,q}$ $(q \geqq 1)$	$(\mathrm{H}\mathbb{C}')_{p,q}$ $(p \geqq 1,\ 2p+q \geqq 3)$
$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$	$((\mathrm{AIIIa})_{l,p}, \{\alpha_1, \alpha_l\})$ $(l = 2p+q-1)$	$((\mathrm{AIIIb})_l, \{\alpha_1, \alpha_l\})$ $(l = 2p-1)$	$((\mathrm{AIV})_l, \{\alpha_1, \alpha_l\})$ $(l = q+1)$	$((\mathrm{AI})_l, \{\alpha_1, \alpha_l\})$ $(l = 2p+q-1)$
(r, s)	$(2p+2q-2, 2p-2)$	$(2p-2, 2p-2)$	$(2q, 0)$	$(2p+q-2, 2p+q-2)$
$(\mathfrak{m}, [g])$	$(\mathrm{H}\mathbb{H})_{p,q}$ $(p, q \geqq 1)$	$(\mathrm{H}\mathbb{H})_{p,0}$ $(p \geqq 2)$	$(\mathrm{H}\mathbb{H}')_{p,q}$ $(p \geqq 1,\ 2p+q \geqq 3)$	$(\mathrm{H}\mathbb{O})$
$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$	$((\mathrm{CIIa})_{l,p}, \{\alpha_2\})$ $(l = 2p+q)$	$((\mathrm{CIIb})_l, \{\alpha_2\})$ $(l = 2p)$	$((\mathrm{CI})_l, \{\alpha_2\})$ $(l = 2p+q)$	$(\mathrm{FII}, \{\alpha_4\})$
(r, s)	$(4p+4q-4, 4p-4)$	$(4p-4, 4p-4)$	$(4p+2q-4, 4p+2q-4)$	$(8, 0)$
$(\mathfrak{m}, [g])$	$(\mathrm{H}\mathbb{O}')$	$(\mathrm{BI})_l$ $(l \geqq 2)$	(G)	
$\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$	$(\mathrm{FI}, \{\alpha_4\})$	$((\mathrm{BI})_{l,l}, \{\alpha_1, \alpha_l\})$	$(\mathrm{G}_{2(2)}, \{\alpha_1, \alpha_2\})$	
(r, s)	$(4, 4)$	$(l-1, l-1)$	$(1, 1)$	

Corollary 5.1. Let $(\mathfrak{m}, [g])$ be a conformal pseudo-subriemannian FGLA of semisimple type. Unless $(\mathfrak{m}, [g])$ is equivalent to one of $(\mathrm{H}\mathbb{C})_{p,q}$ $(p \geqq 2,\ q \geqq 1)$, $(\mathrm{H}\mathbb{C})_{1,q}$ $(q \geqq 1)$, $(\mathrm{H}\mathbb{H})_{p,q}$ $(p, q \geqq 1)$, $(\mathrm{H}\mathbb{O})$, it is conformal neutral-subriemannian.

Example 5.1. Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a real SGLA such that $\mathfrak{l}_{-1} \neq \{0\}$. We assume that $\mathfrak{l}$ is splittable and $\operatorname{rank} \mathfrak{l} \geqq 2$. Let $S = \bigoplus_{p<0} S_p$ be a faithful irreducible graded $\mathfrak{l}$-module such that S is isomorphic to $\mathfrak{l}$ as an $\mathfrak{l}$-module and such that $S_{-1} \neq \{0\}$. Let $\mathfrak{t}$ be the semidirect product of $\mathfrak{l}$ by S. Here S considers as a commutative Lie algebra. We define a gradation $(\mathfrak{t}_p)$ of $\mathfrak{t}$ as follows:

$$\mathfrak{t}_p = \mathfrak{l}_p \quad (p \geqq 0), \quad \mathfrak{t}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}, \quad \mathfrak{t}_q = S_q \quad (q \leqq -2).$$

Then $\mathfrak{t} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{t}_p$ becomes a GLA such that the negative part $\mathfrak{m}$ is an FGLA of the third kind. By assumption $\mathfrak{l}_{-1}$ is contragredient to S_{-1} as a $\mathfrak{t}_0$-module. That is, there exists a $\mathfrak{t}_0$-module isomorphism φ of $\mathfrak{l}_{-1}$ onto S_{-1}^*. We define a symmetric bilinear form g on $\mathfrak{t}_{-1}$ as follows:

$$g(X, Y) = g(Z, W) = 0, \quad g(X, Z) = g(Z, X) = \langle \varphi(X), Z \rangle \quad (X, Y \in \mathfrak{l}_{-1},\ Z, W \in S_{-1}).$$

Then g is nondegenerate, and hence $(\mathfrak{m}, [g])$ becomes a conformal neutral-subriemannian FGLA. Clearly $\mathfrak{t}$ is contained in the prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m}, [g])$. If $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is of type $((\mathrm{BI})_{l,l}, \{\alpha_1, \alpha_l\})$ $(l \geqq 3)$, then $\{\ X \in \mathfrak{g}_{-1} : [X, \mathfrak{g}_{-2}] = \{0\}\ \} = \{0\}$, which is a contradiction. By Theorem 5.2, $(\mathfrak{m}, [g])$ is not of semisimple type and $\mathfrak{g}_1 \neq \{0\}$.

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