arXiv:1804.10268v1 [math.CV] 26 Apr 2018

An Ikehara-type theorem for functions convergent to zero

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November 15, 2021

Abstract

We prove an analogue of the Ikehara theorem for positive non-increasing functions convergent to zero, generalising the results postulated in Diekmann, Kaper (1978) [3] and Carr, Chmaj (2004) [1].

Keywords: Ikehara theorem, complex Tauberian theorem, Laplace transform, asymptotic behavior, traveling waves

2010 Mathematics Subject Classification: 40E05, 44A10, 35B40

1 Introduction

The Ikehara theorem and its extensions are the so-called complex Tauberian theorems, inspired, in particular, by the number theory, see e.g. the review [7]. The following version of the Ikehara theorem can be found in [4, Subsection 2.5.7]:

Theorem 1. *Let ϕ be a positive monotone increasing function, and let there exist $\mu > 0$, $j > 0$, such that*

$$\int_0^\infty e^{-tz}\phi(t)dt = \frac{F(z)}{(z-\mu)^j}, \quad \operatorname{Re} z > \mu, \tag{1.1}$$

where F is holomorphic on $\{\operatorname{Re} z \geq \mu\}$. Then, for some $D > 0$,

$$\phi(t) \sim \frac{D}{\Gamma(j)} t^{j-1} e^{\mu t}, \quad t \to \infty.$$

Alternatively, Theorem 1 may be formulated for the Stieltjes measure $d\phi(t)$ instead of $\phi(t)dt$ obtaining similar asymptotic for ϕ (see e.g. Proposition 2.1 below).

In Theorem 1, ϕ increases to ∞. In [3, Lemma 6.1] (for $j = 1$) and in [1, Proposition 2.3] (for $j > 0$), the similar results were stated for positive monotone decreasing φ (cf., correspondingly, Propositions 2.4 and 2.2 below). The aim of both generalizations was to find an *a priori* asymptotics for solutions to a class of nonlinear integral equations. In [1], in particular, it was applied to the

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study of the uniqueness of traveling wave solutions to certain nonlocal reaction-diffusion equations; see also e.g. [2, 8, 10, 11, 13]. Note also that then, the case $j = 2$ corresponded to the traveling wave with the minimal speed.

In both papers [1,3] no proof was given, mentioning that it is supposed to be analogous to the case of increasing ϕ without any further details. In Theorem 2 below, we prove an analogue of Theorem 1 for non-increasing function, and in Proposition 2.2 we apply it to prove the mentioned result of [1]. We require, however, an *a priori* regular decaying of φ, namely, we assume that there exists $\nu > 0$, such that $\varphi(t)e^{\nu t}$ is an increasing function. We require also the convergence of $\int_0^\infty e^{zt} d\varphi(t)$ for $0 < \operatorname{Re} z < \mu$ instead of the weaker corresponding assumption for $\int_0^\infty e^{zt}\varphi(t)dt$.

Beside the aim to present a proof, the reason for the generalization we provide was to omit the requirement on the function F to be analytical on the line $\{\operatorname{Re} z = \mu\}$ keeping the general case $j > 0$. We were motivated by the integro-differential equation we studied in [6] (which covers the equations considered in [1]), where the Laplace-type transform of the traveling wave with the minimal speed (that requires, recall, $j = 2$) might be not analytical at $z = \mu$.

Our result is based on a version of the Ikehara–Ingham theorem proposed in [9], see Proposition 2.1 below. Using the latter result, we prove also in Proposition 2.4 a generalization of [3, Lemma 6.1] (under the regularity assumptions on φ mentioned above).

2 Main results

Let, for any $D \subset \mathbb{C}$, $\mathcal{H}(D)$ be the class of all holomorphic functions on D.

Theorem 2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ := [0, \infty)$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,*

$$\text{the function } e^{\nu t}\varphi(t) \text{ is non-decreasing,} \tag{2.1}$$

and

$$\int_0^\infty e^{zt} d\varphi(t) < \infty, \quad 0 < \operatorname{Re} z < \mu. \tag{2.2}$$

Let also the following assumptions hold.

1. *There exist a constant $j > 0$ and complex-valued functions*

$$H \in \mathcal{H}(0 < \operatorname{Re} z \le \mu), \qquad F \in \mathcal{H}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \le \mu),$$

such that the following representation holds

$$\int_0^\infty e^{zt}\varphi(t)dt = \frac{F(z)}{(\mu - z)^j} + H(z), \quad 0 < \operatorname{Re} z < \mu. \tag{2.3}$$

2. *For any $T > 0$,*

$$\lim_{\sigma \to 0+} g_j(\sigma) \sup_{|\tau| \le T} \bigl|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)\bigr| = 0, \tag{2.4}$$

where, for $\sigma > 0$,

$$g_j(\sigma) := \begin{cases} \sigma^{j-1}, & 0 < j < 1, \\ \log \sigma, & j = 1, \\ 1, & j > 1. \end{cases} \tag{2.5}$$

Then φ has the following asymptotic

$$\varphi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \to \infty. \tag{2.6}$$

The proof of Theorem 2 is based on the following Tenenbaum's result.

Proposition 2.1 ("Effective" Ikehara–Ingham Theorem, cf. [9, Theorem 7.5.11])**.** *Let $\alpha(t)$ be a non-decreasing function such that, for some fixed $a > 0$, the following integral converges:*

$$\int_0^\infty e^{-zt} d\alpha(t), \quad \operatorname{Re} z > a. \tag{2.7}$$

Let also there exist constants $D \geq 0$ and $j > 0$, such that for the functions

$$G(z) := \frac{1}{a+z} \int_0^\infty e^{-(a+z)t} d\alpha(t) - \frac{D}{z^j}, \quad \operatorname{Re} z > 0, \tag{2.8}$$

$$\eta(\sigma, T) := \sigma^{j-1} \int_{-T}^{T} \bigl|G(2\sigma + i\tau) - G(\sigma + i\tau)\bigr| d\tau, \quad T > 0, \tag{2.9}$$

one has that

$$\lim_{\sigma \to 0+} \eta(\sigma, T) = 0, \quad T > 0. \tag{2.10}$$

Then

$$\alpha(t) = \left\{ \frac{D}{\Gamma(j)} + O\bigl(\rho(t)\bigr) \right\} e^{at} t^{j-1}, \quad t \geq 1, \tag{2.11}$$

where

$$\rho(t) := \inf_{T \geq 32(a+1)} \Bigl\{ T^{-1} + \eta\bigl(t^{-1}, T\bigr) + (Tt)^{-j} \Bigr\}. \tag{2.12}$$

Proof of Theorem 2. We first express $\int_0^\infty e^{\lambda t} \varphi(t) dt$ in the form (2.7). Fix any $a > 0$ such that $\mu + a > \nu$. Then, by (2.1), the function

$$\alpha(t) := e^{(\mu+a)t} \varphi(t), \quad t > 0, \tag{2.13}$$

is increasing. Since φ is monotone, then, for any $0 < \operatorname{Re} z < \mu$, one has

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = (\mu + a) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt + \int_0^\infty e^{(\mu-z)t} d\varphi(t), \tag{2.14}$$

where both integrals in the right hand side of (2.14) converge, for $0 < \operatorname{Re} z < \mu$, because of (2.2)–(2.3).

Then, by [12, Corollary II.1.1a], the integral in the left hand side of (2.14) converges, for *all* $\operatorname{Re} z > 0$. Therefore, by [12, Theorem II.2.3a], one gets another representation for the latter integral, for $\operatorname{Re} z > 0$:

$$\int_0^\infty e^{-(a+z)t} d\alpha(t) = -\varphi(0) + (a+z) \int_0^\infty e^{(\mu-z)t} \varphi(t) dt. \tag{2.15}$$

Let G be given by (2.8) with $\alpha(t)$ as above and $D := F(\mu)$. Combining (2.15) with (2.3) (where we replace z by $\mu - z$), we obtain, for $0 < \operatorname{Re} z < \mu$,

$$G(z) = \frac{F(\mu - z)}{z^j} + K(z), \tag{2.16}$$

$$K(z) := H(\mu - z) - \frac{\varphi(0)}{a+z} - \frac{F(\mu)}{z^j}. \tag{2.17}$$

Check the condition (2.10); one can assume, clearly, that $0 < \sigma < \frac{\mu}{2}$. Since $K \in \mathcal{H}(0 < \operatorname{Re} z < \mu)$, one easily gets that

$$\begin{aligned}
&\lim_{\sigma \to 0+} \sigma^{j-1} \int_{-T}^{T} \big|G(2\sigma + i\tau) - G(\sigma + i\tau)\big| d\tau \\
&\le \lim_{\sigma \to 0+} \sigma^{j-1} \int_{-T}^{T} \Big| \frac{F(\mu - 2\sigma - i\tau) - F(\mu)}{(2\sigma + i\tau)^j} - \frac{F(\mu - \sigma - i\tau) - F(\mu)}{(\sigma + i\tau)^j} \Big| d\tau \\
&\le \lim_{\sigma \to 0+} \sigma^{j-1} \int_{-T}^{T} \Big| \frac{F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)}{(\sigma + i\tau)^j} \Big| d\tau \\
&\quad + \lim_{\sigma \to 0+} \sigma^{j-1} \int_{-T}^{T} \big|F(\mu - 2\sigma - i\tau) - F(\mu)\big| \Big| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \Big| d\tau, \\
&=: \lim_{\sigma \to 0+} A_j(\sigma) + \lim_{\sigma \to 0+} B_j(\sigma).
\end{aligned} \tag{2.18}$$

Prove that both limits in (2.18) are equal to 0. For each $j > 0$, we define the function

$$h_j(\sigma) := \sigma^{j-1} \int_{-T}^{T} \frac{1}{(\sigma^2 + \tau^2)^{\frac{j}{2}}} d\tau, \qquad \sigma > 0. \tag{2.19}$$

We have then

$$A_j(\sigma) \le \sup_{|\tau| \le T} \big|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)\big| \; h_j(\sigma). \tag{2.20}$$

It is straighforward to check that

$$h_1(\sigma) = 2 \log \frac{\sqrt{T^2 + \sigma^2} + T}{\sigma} \sim -2 \log \sigma, \quad \sigma \to 0+. \tag{2.21}$$

For $j \neq 1$, we make the substitution $\tau = \sigma \tan t$ in (2.19), then

$$h_j(\sigma) = \int_{-\arctan \frac{T}{\sigma}}^{\arctan \frac{T}{\sigma}} (\cos t)^{j-2}\, dt. \tag{2.22}$$

Therefore,

$$h_j(\sigma) \leq 2 \arctan \frac{T}{\sigma} < \pi, \qquad \sigma > 0,\ j \geq 2. \tag{2.23}$$

Let now $0 < j < 2$, $j \neq 1$. Then replacing t by $\frac{\pi}{2} - t$ in (2.22), we obtain

$$h_j(\sigma) = 2 \int_{\frac{\pi}{2} - \arctan \frac{T}{\sigma}}^{\frac{\pi}{2}} (\sin t)^{j-2}\, dt \leq 2^{3-j} \int_{\frac{\pi}{2} - \arctan \frac{T}{\sigma}}^{\frac{\pi}{2}} t^{j-2}\, dt, \tag{2.24}$$

since $\sin t > \frac{t}{2}$, $t \in (0, \frac{\pi}{2}]$. Therefore,

$$h_j(\sigma) \leq 2^{3-j} \int_0^{\frac{\pi}{2}} t^{j-2}\, dt = \frac{2^{4-2j}\pi^{j-1}}{j-1}, \quad \sigma > 0,\ 1 < j < 2. \tag{2.25}$$

Finally, for $0 < j < 1$, we obtain from (2.24), that

$$h_j(\sigma) \leq \frac{2^{3-j}}{1-j} \left(\left(\frac{\pi}{2} - \arctan \frac{T}{\sigma} \right)^{j-1} - \left(\frac{\pi}{2} \right)^{j-1} \right) \tag{2.26}$$

Since $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$, $x > 0$, we get

$$\arctan \frac{T}{\sigma} = \frac{\pi}{2} - \arctan \frac{\sigma}{T} \sim \frac{\pi}{2} - \frac{\sigma}{T}, \quad \sigma \to 0+.$$

Therefore, (2.26) implies that

$$h_j(\sigma) = O(\sigma^{j-1}), \quad \sigma \to 0+,\ 0 < j < 1. \tag{2.27}$$

Combining (2.21), (2.23), (2.25), (2.27) with (2.5), we have that

$$h_j(\sigma) = O(g_j(\sigma)), \quad \sigma \to 0+,\ j > 0, \tag{2.28}$$

that, together with (2.20) and (2.4), yield $\lim\limits_{\sigma \to 0+} A_j(\sigma) = 0$.

Take now an arbitrary $\beta \in (0, \mu)$ and consider, for each $T > 0$, the set

$$K_{\beta,\mu,T} := \left\{ z \in \mathbb{C} \;\middle|\; \beta \leq \operatorname{Re} z \leq \mu,\ |\operatorname{Im} z| \leq T \right\}. \tag{2.29}$$

Let $0 < \sigma < \mu^2$; since $F \in C(K_{\sqrt{\sigma},\mu,T})$, there exists $C_1 > 0$ such that $|F(z)| \leq C_1$, $z \in K_{\sqrt{\sigma},\mu,T}$. Therefore,

$$\begin{aligned} B_j(\sigma) \leq \sigma^{j-1} \sup_{|\tau| \leq \sqrt{\sigma}} \left| F(\mu - 2\sigma - i\tau) - F(\mu) \right| \int\limits_{|\tau| \leq \sqrt{\sigma}} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau \\ + 2C_1 \sigma^{j-1} \int\limits_{\sqrt{\sigma} \leq |\tau| \leq T} \left| \frac{1}{(2\sigma + i\tau)^j} - \frac{1}{(\sigma + i\tau)^j} \right| d\tau. \end{aligned} \tag{2.30}$$

Next, since

$$j\left|\frac{1}{(2\sigma+i\tau)^j}-\frac{1}{(\sigma+i\tau)^j}\right|=\left|\int_{\sigma+i\tau}^{2\sigma+i\tau}\frac{1}{z^{j+1}}\,dz\right|=\left|\int_0^1\frac{\sigma}{\bigl((1+t)\sigma+i\tau\bigr)^{j+1}}\,dt\right|$$
$$\leq\int_0^1\frac{\sigma}{\bigl((1+t)^2\sigma^2+\tau^2\bigr)^{\frac{j+1}{2}}}\,dt\leq\frac{\sigma}{\bigl(\sigma^2+\tau^2\bigr)^{\frac{j+1}{2}}},$$

we can continue (2.30) as follows, cf. (2.19),

$$jB_j(\sigma)\leq\sup_{|\tau|\leq\sqrt{\sigma}}\bigl|F(\mu-2\sigma-i\tau)-F(\mu)\bigr|\;h_{j+1}(\sigma)$$
$$+4C_1\int\limits_{\sqrt{\sigma}\leq\tau\leq T}\frac{\sigma^j}{\bigl(\sigma^2+\tau^2\bigr)^{\frac{j+1}{2}}}d\tau. \tag{2.31}$$

By (2.23) and (2.25), functions h_{j+1} are bounded on $(0,\infty)$ for all $j>0$. Next, since F is uniformly continuous on $K_{\sqrt{\sigma},\mu,T}$, we have that, for any $\varepsilon>0$ there exists $\delta>0$ such that $f(\mu,\sigma,\tau):=\bigl|F(\mu-2\sigma-i\tau)-F(\mu)\bigr|<\varepsilon$, if only $4\sigma^2+\tau^2<\delta$. Therefore, if $\sigma>0$ is such that $4\sigma^2+\sigma<\delta$ then $\sup_{|\tau|\leq\sqrt{\sigma}}f(\mu,\sigma,\tau)<\varepsilon$ hence

$$\sup_{|\tau|\leq\sqrt{\sigma}}\bigl|F(\mu-2\sigma-i\tau)-F(\mu)\bigr|\;h_{j+1}(\sigma)\to 0,\quad \sigma\to 0+. \tag{2.32}$$

Finally, making again the substitution $\tau=\sigma\tan t$ in the integral in (2.31), we obtain that it is equal to

$$I_j:=\int_{\arctan\frac{\sqrt{\sigma}}{\sigma}}^{\arctan\frac{T}{\sigma}}(\cos t)^{j-1}\,dt.$$

Similarly, to the above, for $j\geq 1$,

$$I_j\leq\arctan\frac{T}{\sigma}-\arctan\frac{\sqrt{\sigma}}{\sigma},$$

and, for $0<j<1$,

$$I_j=\int_{\frac{\pi}{2}-\arctan\frac{T}{\sigma}}^{\frac{\pi}{2}-\arctan\frac{\sqrt{\sigma}}{\sigma}}\frac{1}{(\sin t)^{1-j}}\,dt\leq\frac{2^{1-j}}{j}\biggl(\Bigl(\frac{\pi}{2}-\arctan\frac{\sqrt{\sigma}}{\sigma}\Bigr)^j-\Bigl(\frac{\pi}{2}-\arctan\frac{T}{\sigma}\Bigr)^j\biggr).$$

As a result, $I_j\to 0$ as $\sigma\to 0+$, that, together with (2.32) and (2.31), proves that $B_j(\sigma)\to 0$, $\sigma\to 0+$.

Combining this with $A_j(\sigma)\to 0$, one gets (2.10) from (2.18); and we can apply Proposition 2.1. Namely, by (2.11), there exist $C>0$ and $t_0\geq 1$, such that

$$\frac{D}{\Gamma(j)}e^{at}t^{j-1}\leq\varphi(t)e^{(\mu+a)t}\leq\left\{\frac{D}{\Gamma(j)}+C\rho(t)\right\}e^{at}t^{j-1},\quad t\geq t_0.$$

By (2.10) and (2.12), $\rho(t)\to 0$ as $t\to\infty$. Therefore,

$$\varphi(t)e^{(\mu+a)t}\sim\frac{D}{\Gamma(j)}e^{at}t^{j-1},\quad t\to\infty,$$

that is equivalent to (2.6) and finishes the proof. □

The following simple Proposition show that if F in (2.3) is holomorphic on the line $\{\mathrm{Re}\, z = \mu\}$, then (2.4) holds.

Proposition 2.2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,* (2.1)–(2.2) *hold. Suppose also that there exist $j > 0$ and $F, H \in \mathcal{A}(0 < \mathrm{Re}\, z \leq \mu)$, such that* (2.3) *holds. Then φ has the asymptotic* (2.6).

Proof. Take any $\beta \in (0, \mu)$ and $T > 0$. Let $K_{\beta,\mu,T}$ be defined by (2.29). Since $F \in \mathcal{A}(0 < \mathrm{Re}\, z \leq \mu)$, then $F' \in C(K_{\beta,\mu,T})$, and hence F' is bounded on $K_{\beta,\mu,T}$. Then one can apply a mean-value-type theorem for complex-valued functions, see e.g. [5, Theorem 2.2], to get that, for some $K > 0$,

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| \leq K|\sigma|, \qquad 2\sigma < \mu - \beta,$$

that yields (2.4) for all $j > 0$, cf. (2.5). Hence we can apply Theorem 2. □

Remark 2.3. Note that, for $F \in \mathcal{A}(0 < \mathrm{Re}\, z \leq \mu)$ in (2.3), the holomorphic function H is redundant there, as we always can replace $F(z)$ by a holomorphic function $F(z) + H(z)(\mu - z)^j$. Therefore, Proposition 2.2 corresponds to [1, Proposition 2.3].

Proposition 2.4. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,* (2.1)–(2.2) *hold. Suppose also that there exist $j \geq 1$, $D > 0$, and $h : \mathbb{R} \to \mathbb{R}$ such that*

$$H(z) := \int_0^\infty e^{zt}\varphi(t)dt - \frac{D}{(\mu - z)^j} \to h\bigl(\mathrm{Im}\, z\bigr), \quad \mathrm{Re}\, z \to \mu-, \tag{2.33}$$

uniformly (in $\mathrm{Im}\, z$) on compact subsets of $\mathbb{R}$. Then the following asymptotic holds,

$$\varphi(t) \sim \frac{D}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \to \infty. \tag{2.34}$$

Proof. Let $a > \max\{0, \nu - \mu\}$ and $\alpha(t)$ be given by (2.13). Let G be given by (2.8). Similarly to the proof of Theorem 2, we will get from (2.15) and (2.33), that

$$G(z) = H(\mu - z) - \frac{\varphi(0)}{a + z}, \quad 0 < \mathrm{Re}\, z < \mu.$$

Next, (2.33) implies (2.10). Hence, by Lemma 2.1, (2.34) holds that fulfilled the proof. □

Note that the result in [3, Lemma 6.1] corresponds to $j = 1$ in Proposition 2.4.

Remark 2.5. It is worth noting that, for the case $j > 1$, we have, by (2.9), that if G is bounded, then (2.10) holds. Therefore, in this case, it is enough to assume that H in (2.33) is bounded to conclude (2.34).

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