### Multipoint Cauchy problem for Schrödinger type equations with general elliptic part Veli Shakhmurov

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Abstract

In this paper, the existence, uniqueness and regularity properties, Strichartz type estimates for solution of multipoint Cauchy problem for linear and nonlinear Schrödinger equations with general elliptic leading part is obtained.

Key Word: Schrödinger equations, elliptic operators, Semigroups of operators, local solutions

# AMS 2010: 35Q41, 35K15, 47B25, 47Dxx, 46E40 1. Introduction

Consider the multipoint Cauchy problem for nonlinear Schrödinger equations (NLS)

$$i\partial_t u + Lu + F(u) = 0, \ x \in \mathbb{R}^n, \ t \in [0, T],$$
 (1.1)

$$u(0,x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x), \text{ for a.e. } x \in \mathbb{R}^n,$$
(1.2)

where L is an elliptic operator defined by

$$Lu = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \ a_{ij} \in \mathbb{C},$$
(1.3)

*m* is a positive integer,  $\alpha_k$  are complex numbers,  $\lambda_k \in (0, T]$ , *F* is a nonlinear operator,  $\mathbb{C}$ -denotes the set of complex numbers and u = u(t, x) is the unknown function. If  $F(u) = \lambda |u|^p u$  in (1.1) we get the multipoint Cauchy problem nonlinear equation

$$i\partial_t u + Lu + \lambda |u|^p u = 0, \ x \in \mathbb{R}^n, \ t \in [0, T],$$

$$u(0, x) = \varphi(x) + \sum_{k=1}^m \alpha_k u(\lambda_k, x) \text{ for a.e. } x \in \mathbb{R}^n,$$
(1.4)

where  $p \in (1, \infty)$ ,  $\lambda$  is a real number.

By rescaling the values of u it is possible to restrict attention to the cases  $\lambda = 1$  or  $\lambda = -1$ . These call as the focusing and defocusing Schrödinger equations,

respectively. The equation (1.1) also contain two critical case. These are the mass-critical Schrödinger equation,

$$i\partial_t u + Lu + \lambda |u|^{\frac{4}{n}} = 0, \ x \in \mathbb{R}^n, \ t \in [0, T],$$

which is associated with the conservation of mass,

$$M(u(t)) := \int_{R^n} \|u(t,x)\|_E^2 dx$$

and the energy-critical Schrödinger equation (in dimensions n > 2),

$$i\partial_t u + Lu + \lambda |u|^{\frac{4}{n-2}} = 0, \ x \in \mathbb{R}^n, \ t \in [0,T],$$
 (1.5)

which is associated with the conservation of energy,

$$H(u(t)) := \int_{R^n} \left[ \frac{1}{2} \left| (Lu, u)(t, x) \right|^2 + \lambda \left( \frac{1}{2} - \frac{1}{n} \right) \left| u(t, x) \right|^{\frac{2n}{n-2}} \right] dx,$$

where (Lu, u) denotes scalar product of Lu and u in  $L^{2}(\mathbb{R}^{n})$ .

The existence of solutions and regularity properties of Cauchy problem for NLS equations studied e.g in [2 - 10], [14, 16] and the references therein. In contrast, to the mentioned above results we will study the existence, uniqueness and the regularity properties of the multipoint Cauchy problem (1.1) - (1.2).

## 2. Definitions and background

Let  $L_t^q L_x^r((a, b) \times \Omega)$  denotes the space of strongly measurable functions that are defined on the measurable set  $(a, b) \times \Omega$  with the norm

$$\|f\|_{L^q_t L^r_x((a,b)\times\Omega)} = \left(\int\limits_a^b \left(\int\limits_\Omega \left|f\left(t,x\right)\right|^r dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}, \ 1 \le q, r < \infty \ .$$

Let F denotes the Fourier transformation,  $\hat{u} = F u$  and

$$s \in \mathbb{R}, \ \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \ |\xi|^2 = \sum_{k=1}^n \xi_k^2,$$
  
 $\langle \xi \rangle = \left(1 + |\xi|^2\right)^{\frac{1}{2}}.$ 

 $S = S(\mathbb{R}^n)$  denotes the Schwartz class, i.e. the space of all complex-valued rapidly decreasing smooth functions on  $\mathbb{R}^n$  equipped with its usual topology generated by seminorms.  $S(\mathbb{R}^n)$  denoted by just S. Let  $S'(\mathbb{R}^n)$  denote the space of all continuous linear operators,  $L: S \to \mathbb{C}$ , equipped with the bounded convergence topology. Recall  $S(\mathbb{R}^n)$  is norm dense in  $L^p(\mathbb{R}^n)$  when 1 . $Let <math>D'(\Omega)$  denote the class of generalized functions on  $\Omega \subset \mathbb{R}^n$ . Consider Sobolev space  $W^{s,p}(\mathbb{R}^n)$  and homogeneous Sobolev spaces  $\mathring{W}^{s,p}(\mathbb{R}^n)$  defined by respectively,

$$W^{s,p}(R^{n}) = \{u : u \in S'(R^{n}), \\ \|u\|_{W^{s,p}(R^{n})} = \left\| F^{-1} \left(1 + |\xi|^{2}\right)^{\frac{s}{2}} \hat{u} \right\|_{L^{p}(R^{n})} < \infty \right\},$$

$$\mathring{W}^{s,p}(R^{n}) = \left\{ u : u \in S'(R^{n}), \|u\|_{\mathring{W}^{s,p}(R^{n})} = \left\| \mathcal{F}^{-1} \left| \xi \right|^{s} \right\|_{L^{p}(R^{n})} < \infty \right\}.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say  $\alpha$ , we write  $C_{\alpha}$ .

Let L is differential operator defined by (1.4).

**Condition 2.1.** Assume  $a_{ij} = a_{ji}$  and there are positive constants  $M_1$  and  $M_2$  such that  $M_1 |\xi|^2 \leq L(\xi) \leq M_2 |\xi|^2$  for  $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$ , where

$$|\xi|^2 = \sum_{k=1}^n \xi_k^2, L(\xi) = \sum_{i,j=1}^2 a_{ij}\xi_i\xi_j.$$

**Definition 2.2.** Consider the initial value problem (1.1) - (1.2) for  $\varphi \in \overset{\circ}{W}^{s,p}(\mathbb{R}^n)$ . This problem is critical when  $s = s_c := \frac{n}{2} - \frac{2}{p}$ , subcritical when  $s > s_c$ , and supercritical when  $s < s_c$ .

We write  $a \leq b$  to indicate that  $a \leq Cb$  for some constant C, which is permitted to depend on some parameters.

## 3. Dispersive and Strichartz type inequalities for linear Schrödinger equation

Let the operator iL generates a continious  $C_0$  group  $e^{itL(\xi)}$ . It can be shown that the fundamental solution of the free Schrödinger equation

$$i\partial_t u + Lu = 0, \ t \in [0, T], \ x \in \mathbb{R}^n$$
 (3.1)

can be expressed as  $U_L(t) = F^{-1} e^{itL(\xi)}$ , i.e.

$$U_{L}(t) f(x) = \int_{R^{n}} U_{L}(t) (x - y) f(y) dy.$$
(3.2)

Lemma 3.1. The following dispersive inequalities hold

$$\|U_L(t)f\|_{L^p_x(R^n)} \lesssim t^{-\left[n\left(\frac{1}{2} - \frac{1}{p}\right)\right]} \|f\|_{L^{p'}_x(R^n)}, \qquad (3.3)$$

$$\|U_L(t-s)f\|_{L^{\infty}(\mathbb{R}^n)} \lesssim |t-s|^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}$$
(3.4)

for  $t \neq 0, 2 \leq p \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1$ . **Proof.** Indeed, by using Young's integral inequality from (3.2) we get

$$\|U_L(t)f\|_{L^p_x(R^n)} \lesssim |t|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)} \|f\|_{L^{p'}_x(R^n)}, \qquad (3.5)$$

$$\|U_L(t)f\|_{L^{\infty}_x(R^n)} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L^1_x(R^n)}.$$
(3.6)

Condition 3.1. Assume  $n \geq 1$ ,

$$\frac{2}{q} + \frac{n}{r} \le \frac{n}{2}, \ 2 \le q, r \le \infty \text{ and } (n, q, r) \ne (2, 2, \infty).$$

**Remark 3.1.** If  $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$ , then (q, r) is called sharp admissible, otherwise (q, r) is called nonsharp admissible. Note in particular that when n > 2 the endpoint  $\left(2, \frac{2n}{n-2}\right)$  is called sharp admissible.

For a space-time slab  $[0,T] \times \mathbb{R}^n$ , we define the *E*-valued Strichartz norm

$$\|u\|_{S^0(I)} = \sup_{(q,r) \text{ admissible}} \|u\|_{L^q_t L^r_x(I \times R^n)} \,,$$

where  $S^0([0,T])$  is the closure of test functions under this norm and  $N^0([0,T])$ denotes the dual of  $S^0([0,T])$ .

Assume H is an abstract Hilbert space and Q is a Hilbert space of function. Suppose for each  $t \in \mathbb{R}$  an operator  $U(t): Q \to L^2(\Omega)$  obeys the following estimates:

$$\left\| U\left(t\right)f\right\|_{L^{2}_{x}(\Omega)} \lesssim \left\|f\right\|_{H} \tag{3.7}$$

for all  $t, \Omega \subset \mathbb{R}^n$  and all  $f \in Q$ ;

$$\|U(s) U^{*}(t) g\|_{L_{x}^{\infty}(\Omega)} \lesssim |t-s|^{-\frac{n}{2}} \|g\|_{L_{x}^{1}(\Omega)}$$
(3.8)

$$\|U(s) U^{*}(t) g\|_{L^{\infty}_{x}(\Omega)} \lesssim \left(1 + |t - s|^{-\frac{n}{2}}\right) \|g\|_{L^{1}_{x}(\Omega)}$$
(3.9)

for all  $t \neq s$  and all  $g \in L^{1}_{x}(\Omega)$ .

For proving the main theorem of this section, we will use the following bilinear interpolation result (see [1], Section 3.13.5(b)).

Lemma 3.2. Assume  $A_0$ ,  $A_1$ ,  $B_0$ ,  $B_1$ ,  $C_0$ ,  $C_1$  are Banach spaces and T is a bilinear operator bounded from  $(A_0 \times B_0, A_0 \times B_1, A_1 \times B_0)$  into  $(C_0, C_1, C_1)$ , respectively. Then whenever  $0 < \theta_0, \theta_1 < \theta < 1$  are such that  $1 \leq \frac{1}{p} + \frac{1}{q}$ and  $\theta = \theta_0 + \theta_1$ , the operator is bounded from

$$(A_0, A_1)_{\theta_0 pr} \times (B_0, B_1)_{\theta_1 qr}$$

to  $(C_0, C_1)_{\theta_r}$ .

By following [9, Theorem 1.2] we have:

**Theorem 3.1.** Assume U(t) obeys (3.8) and (3.9). Then the following estimates are hold

$$\|U(t)f\|_{L^{q}_{t}L^{r}_{x}} \lesssim \|f\|_{H},$$
 (3.10)

$$\left\| \int U^*(s) F(s) \, ds \right\|_Q \lesssim \|F\|_{L_t^{q'} L_x^{r'}}, \tag{3.11}$$

$$\int_{s < t} \|A^{\alpha}U(t) U^{*}(s) F(s) ds\|_{L^{q}_{t}L^{r}_{x}} \lesssim \|F\|_{L^{\bar{q}'}_{t}L^{\bar{r}'}_{x}}, \qquad (3.12)$$

for all sharp admissible exponent pairs (q, r),  $(\tilde{q}, \tilde{r})$ . Furthermore, if the decay hypothesis is strengthened to (3.9), then (3.10), (3.11) and (3.12) hold for all admissible (q, r),  $(\tilde{q}, \tilde{r})$ .

**Proof.** The first step: Consider the nonendpoint case, i.e.  $(q, r) \neq (2, \frac{2n}{n-2})$  and will show firstly, the estimates (3.10), (3.11). By duality, (3.10) is equivalent to (3.11). By the  $TT^*$  method, (3.11) is in turn equivalent to the bilinear form estimate

$$\left| \int \int \langle U^*(s) F(s), U^*(t) G(t) \rangle ds dt \right| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}}.$$
(3.13)

By symmetry it suffices to show the to the retarded version of (3.13)

$$|T(F,G)| \lesssim \|F\|_{L_t^{q'}L_x^{r'}} \|G\|_{L_t^{q'}L_x^{r'}}, \qquad (3.14)$$

where T(F,G) is the bilinear form defined by

$$T\left(F,G\right) = \int \int_{s < t} \langle U\left(s\right)^{*} F\left(s\right), \left(U\left(t\right)\right)^{*} G\left(t\right) \rangle ds dt$$

By real interpolation between the bilinear form of (3.7) we get

$$|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle| \lesssim ||F(s)||_{L^2_x} ||G(t)||_{L^2_x}$$

By using the bilinear form of (3.8) we have

$$\left| \left\langle \left( U\left( s\right) \right)^{*}F\left( s\right) ,\left( U\left( t\right) \right)^{*}G\left( t\right) \right\rangle \right| \lesssim$$

$$(3.15)$$

$$|t-s|^{-\frac{n}{2}} \|F(s)\|_{L^{1}_{x}(\Omega)} \|G(t)\|_{L^{1}_{x}(\Omega)}$$

In a similar way, we obtain

$$\left| \langle (U(s))^* F(s), (U(t))^* G(t) \rangle \right| \lesssim$$

$$(3.16)$$

$$t - s|^{-1 - \beta(r,r)} \|F(s)\|_{L_x^{r'}(\Omega)} \|G(t)\|_{L_x^{r'}(\Omega)},$$

where  $\beta(r, \tilde{r})$  is given by

$$\beta(r,\tilde{r}) = \frac{n}{2} - 1 - \frac{n}{2} \left(\frac{1}{r} - \frac{1}{\tilde{r}}\right).$$
(3.17)

It is clear that  $\beta(r, r) \leq 0$  when n > 2. In the sharp admissible case we have

$$\frac{1}{q} + \frac{1}{q'} = -\beta(r, r),$$

and (3.14) follows from (3.16) and the Hardy-Littlewood-Sobolev inequality ([20]) when q > q'.

If we are assuming the truncated decay (3.9), then (3.16) can be improved to

$$\left| \langle (U(s))^* F(s), (U(t))^* G(t) \rangle \right| \lesssim$$

$$(1 + |t - s|)^{-1 - \beta(r, r)} \|F(s)\|_{L_x^{r'}(\Omega)} \|G(t)\|_{L_x^{r'}(\Omega)}$$
(3.18)

and now Young's inequality gives (3.14) when

$$-\beta(r,r) + \frac{1}{q} > \frac{1}{q'},$$

i.e. (q, r) is nonsharp admissible. This concludes the proof of (3.10) and (3.11) for nonendpoint case.

The second step; It remains to prove (3.10) and (3.11) for the endpoint case, i.e. when

$$(q, r) = \left(2, \frac{2n}{n-2}\right), n > 2.$$

It suffices to show (3.14). By decomposing T(F,G) dyadically as  $\sum_{j} T_{j}(F,G)$ , where the summation is over the integers  $\mathbb{Z}$  and

where the summation is over the integers  $\mathbbm{Z}$  and

$$T_{j}(F,G) = \int_{t-2^{j-1} < s \le t-2^{j}} \langle (U(s))^{*} F(s), (U(t))^{*} G(t) \rangle ds dt$$
(3.19)

we see that it suffices to prove the estimate

$$\sum_{j} |T_{j}(F,G)| \lesssim \|F\|_{L_{t}^{2}L_{x}^{r'}(H)} \|G\|_{L_{t}^{2}L_{x}^{r'}}.$$
(3.20)

For this aim, before we will show the following estimate

$$|T_j(F,G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_t L^{a'}_x} \|G\|_{L^2_t L^{b'}_x}$$
(3.21)

for all  $j \in \mathbb{Z}$  and all  $(\frac{1}{a}, \frac{1}{b})$  in a neighborhood of  $(\frac{1}{r}, \frac{1}{r})$ . For proving (3.21) we will use the real interpolation of Lebesque space and sequence spaces  $l_q^s$  (see e.g [15], § 1.18.2). Indeed, by [15, § 1.18.4.] we have

$$\left(L_t^2 L_x^{p_0}, L_t^2 L_x^{p_1}\right)_{\theta,2} = L_t^2 L_x^{p,2}$$
(3.22)

whenever  $p_0, p_1 \in [1, \infty]$ ,  $p_0 \neq p_1$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  and  $(l_{\infty}^{s_0}, l_{\infty}^{s_1})_{\theta,1} = l_1^s$  for  $s_0$ ,  $s_1 \in \mathbb{R}, s_0 \neq s_1$  and  $\frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1},$ 

where

$$l_{q}^{s} = \left\{ u = \{u_{j}\}_{j=1}^{\infty}, u_{j} \in \mathbb{C}, \|u\|_{l_{q}^{s}} = \left(\sum_{j=1}^{\infty} 2^{jsq} |u_{j}|^{q}\right)^{\frac{1}{q}} < \infty \right\}.$$

By (3.22) the estimate (3.21) can be rewritten as

$$T: L_t^2 L_x^{a'} \times L_t^2 L_x^{b'} \to l_{\infty}^{\beta(a,b)},$$
(3.23)

where  $T = \{T_j\}$  is the vector-valued bilinear operator corresponding to the  $T_j$ . We apply Lemma 3.2 to (3.23) with r = 1, p = q = 2 and arbitrary exponents  $a_0, a_1, b_0, b_1$  such that

$$\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0)$$

Using the real interpolation space identities we obtain

$$T:L^2_tL^{a',2}_x\times L^2_tL^{b',2}_x\to l_1^{\beta(a,b)}$$

for all (a, b) in a neighborhood of (r, r). Applying this to a = b = r and using the fact that  $L^{r'} \subset L^{r',2}$  we obtain

$$T: L_t^2 L_x^{a',2} \times L_t^2 L_x^{b',2} \to l_1^0$$

which implies (3.21).

Consider the multipoint Cauchy problem for forced Schrodinger equation

$$i\partial_t u + Lu = F, \ t \in [0, T], \ x \in \mathbb{R}^n, \tag{3.24}$$

$$u(t_0, x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x), \ x \in \mathbb{R}^n, \ t_0, \ \lambda_k \in [0, \ T), \ \lambda_k > t_0.$$

We are now ready to state the standard Strichartz estimates:

**Lemma 3.3.** Assume the Condition 2.1 are satisfied,  $\varphi \in \mathring{W}^{\gamma,p}(\mathbb{R}^n)$  for  $\gamma \geq \frac{n}{p}$  and  $p \in [1, \infty]$ . Then problem (3.24) has a unique generalized solution. **Proof.** By using the Fourier transform we get from (3.24):

$$i\hat{u}_{t}\left(t,\xi\right)+L\left(\xi\right)\hat{u}\left(t,\xi\right)=F\left(t,\xi\right),$$

$$\hat{u}(0,\xi) = \hat{\varphi}(\xi) + \sum_{k=1}^{m} \alpha_k \hat{u}(\lambda_k,\xi), \text{ for a.e. } \xi \in \mathbb{R}^n.$$
(3.25)

where  $\hat{u}(t,\xi)$  is a Fourier transform of u(t,x) with respect to x.

Consider the problem

$$\hat{u}_{t}(t,\xi) - iL(\xi)\hat{u}(t,\xi) = \hat{F}(t,\xi), \qquad (3.26)$$
$$\hat{u}(0,\xi) = u_{0}(\xi), \ \xi \in \mathbb{R}^{n}, \ t \in [0,T],$$

where  $u_0(\xi) \in \mathbb{C}$  for  $\xi \in \mathbb{R}^n$ . By Condition 2.1 and by [11, § 1.10, § 4.1],  $iL(\xi)$  is a generator of a strongly continuous  $C_0$  semigroups  $U_L(t,\xi =) e^{itL(\xi)}$  and the Cauchy problem (3.26) has a unique solution for all  $\xi \in \mathbb{R}^n$ , moreover, the solution can be expressed as

$$\hat{u}(t,\xi) = e^{itL(\xi)} u_0(\xi) + \int_{t_0}^t e^{itL(\xi)|t-\tau|} \hat{F}(\tau,\xi) \, d\tau, \ t \in (0,T).$$
(3.27)

Using the formula (3.27) and the condition (3.25) we get

$$u_{0}(\xi) = \hat{\varphi}(\xi) + \sum_{k=1}^{m} \alpha_{k} U_{L}(\lambda_{k},\xi) u_{0}(\xi) +$$

$$\sum_{k=1}^{m} \alpha_{k} \int_{t_{0}}^{\lambda_{k}} U_{L}(\lambda_{k}-\tau,\xi) \hat{F}(\tau,\xi) d\tau, \ \tau \in (0,T).$$
(3.28)

From (3.27) and (3.28) we obtain that the solution of problem (3.25) can be expressed as:

$$\hat{u}(t,\xi) = U_L(t,\xi)\,\hat{\varphi}(\xi) + \sum_{k=1}^m \alpha_k U_L(\lambda_k,\xi)\,u_0(\xi) +$$
(3.29)

$$\sum_{k=1}^{m} \alpha_{k} \int_{t_{0}}^{\lambda_{k}} U_{L} \left(\lambda_{k} - \tau, \xi\right) \hat{F} \left(\tau, \xi\right) d\tau + \int_{t_{0}}^{t} U_{L} \left(t - \tau, \xi\right) \hat{F} \left(\tau, \xi\right) d\tau, \ \tau \in (0, T).$$

Then the solution of the problem (3.24) will be expressed as the following formula:

$$u(t,x) = V(t)\varphi(x) + \sum_{k=1}^{m} \alpha_k V_k(t,x) + \sum_{k=1}^{m} \alpha_k G_k(t,x) + G_0(t,x),$$

where

$$V(t) = F^{-1} [U_L(t,\xi) \hat{\varphi}(\xi)], \quad V_k(t,x) = F^{-1} [U_L(\lambda_k,\xi) \hat{\varphi}(\xi)], \quad (3.30)$$
$$G_k(t,x) = F^{-1} \left[ \int_{t_0}^{\lambda_k} U_L(\lambda_k - \tau,\xi) \hat{F}(\tau,\xi) d\tau \right],$$

$$G_0(t,x) = F^{-1} \left[ \int_{t_0}^t U_L(t-\tau,\xi) \hat{F}(\tau,\xi) d\tau \right].$$

**Theorem 3.2.** Assume the Conditions 2.1 and 3.1 are satisfied. Let  $0 \le s \le 1$ ,  $\varphi \in \mathring{W}^{s,2}(\mathbb{R}^n)$ ,  $F \in \mathbb{N}^0([0,T]; \mathring{W}^{s,2}(\mathbb{R}^n))$  and let  $u : [0,T] \times \mathbb{R}^n \to \mathbb{C}$  be a solution to (3.24). Then

$$\||\nabla|^{s} u\|_{S^{0}([0,T])} + \||\nabla|^{s} u\|_{C^{0}([0,T];L^{2}(R^{n}))} \lesssim$$

$$\||\nabla|^{s} \varphi\|_{L^{2}(R^{n})} + \||\nabla|^{s} F\|_{N^{0}([0,T])}.$$

$$(3.31)$$

**Proof.** Let  $2 \leq q, r, \tilde{q}, \tilde{r} \leq \infty$  with

$$\frac{2}{q} + \frac{n}{r} = \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2}.$$

If n = 2, we also require that  $q, \tilde{q} > 2$ . Consider first, the nonendpoint case. By Lemma 3.3 the problem has a solution. The linear operators in (3.10) and (3.11) are adjoint of one another; thus, by the method of  $TT^*$  both will follow once we prove

$$\left\| \int_{s < t} U_L \left( t - s \right) F \left( s \right) ds \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}.$$
(3.32)

Apply Theorem 3.1 with  $Q = L_x^2(\mathbb{R}^n) = L_x^2$ . The energy estiamate (3.10):

 $||U_L(t)f||_{L^2_x} \lesssim ||f||_{L^2_x}$ 

follows from Plancherel's theorem, the untruncated decay estimate

$$||U_L(t-s) f||_{L^{\infty}_x} \lesssim |t-s|^{-\frac{n}{2}} ||f||_{L^1_x},$$

and explicit representation of the Schrödinger evolution operator  $U_L(t) f(x)$ . In view of (3.30), due to properties gropes  $U_L(t)$  and by the dispersive estimate (3.4) we have

$$|\Phi| \lesssim \int_{s < t} |U_L(t-s) \, ds|_{B(H)} \, |F(s)| \, ds \lesssim \int_{\mathbb{R}} |t-s|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)} \, |F(s)| \, ds,$$

where

$$\Phi = \int_{s < t} U_L \left( t - s \right) F(s) \, ds.$$

Moreover, from above estimate by the Hardy-Littlewood-Sobolev inequality, we get

$$\|\Phi\|_{L^{q}_{t}L^{r}_{x}(R^{n+1})} \lesssim \left\| \int_{\mathbb{R}} |t-s|^{-n\left(\frac{1}{2}-\frac{1}{p}\right)} \|F(s)\|_{L^{r'}_{x}(R^{n})} ds \right\|_{L^{q}_{t}(\mathbb{R})} \lesssim (3.33)$$

$$\|F\|_{L_t^{q_1}L_x^{r'}}$$

where

$$\frac{1}{q_1} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2} - \frac{\alpha}{n}$$

The argument just presented also covers (3.33) in the case  $q = \tilde{q}, r = \tilde{r}$ . It allows to consider the estimate in dualized form:

$$\left| \int \int_{s < t} \langle U_L (t - s) F (s), G (t) \rangle ds \right| \lesssim \|F\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{\tilde{q}_1} L_x^{\tilde{r}'}}$$
(3.34)

when

$$\frac{1}{\tilde{q}_1} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} + \frac{1}{2} - \frac{\nu}{n}$$

The case  $\tilde{q}=\infty,\,\tilde{r}=2$  follows from (3.33), i.e.

.

$$K \lesssim \left\| \int_{s < t} U_L \left( t - s \right) F \left( s \right) ds \right\|_{L_t^\infty L_x^2} \|G\|_{L_t^1 L_x^2} \lesssim$$
(3.35)

$$\|F\|_{L_t^{q_1}L_x^{r'}} \|G\|_{L_t^1L_x^2},$$

where

$$K = \left| \int \int_{s < t} \langle U_L (t - s) F(s), G(t) \rangle ds \right|.$$

From (3.3.5) we obtain the esimate (3.34) when s = 0. The general case is obtained by using the same argument.

Now, consider the endpoint case, i.e.  $(q,r) = \left(2, \frac{2n}{n-2}\right)$ . It is suffices to show the following estimates

$$\left\|U_{L}\left(t\right)\varphi\right\|_{L_{t}^{q}L_{x}^{r}} \lesssim \left\|\varphi\right\|_{W^{s,2}\left(R^{n}\right)},\tag{3.36}$$

.

$$\|U_L(t)\varphi\|_{C^0(L^2_x)} \lesssim \|\varphi\|_{W^{s,2}(R^n)},$$
 (3.37)

$$\left\| \int_{s < t} U_L(t - s) F(s) \, ds \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{3.38}$$

$$\left\| \int_{s < t} U_L \left( t - s \right) F \left( s \right) ds \right\|_{C^0 L^2_x} \lesssim \|F\|_{L^{q'}_t L^{\tilde{r}'}_x}.$$
(3.39)

Indeed, applying Theorem 3.1 for

$$Q = L^{2}(R^{n}), U(t) = \chi_{[0,T]}U_{L}(t)$$

with the energy estimate

$$\|U(t)f\|_{L^{2}(\mathbb{R}^{n})} \lesssim \|f\|_{L^{2}(\mathbb{R}^{n})}$$

which follows from Plancherel's theorem, the untruncated decay estimate (3.8) and by using of Lemma 3.1 we obtain the estimates (3.36) and (3.38). Let us temporarily replace the  $C_t^0 L_x^2$  norm in estimates (3.36), (3.38) by the  $L_t^{\infty} L_x^2$ . Then, all of the above the estimates will follow from Theorem 3.1, once we show that U(t) obeys the energy estimate (3.7) and the truncated decay estimate (3.9). The estimate (3.7) is obtain immediate from Plancherel's theorem, and (3.9) follows in a similar way as in [13, p. 223-224]. To show that the operator

$$GF(t) = \int_{s < t} U_L(t - s) F(s) \, ds$$

is continuous in  $L^{2}(\mathbb{R}^{n})$ , we use the the identity

$$GF(t+\varepsilon) = U(\varepsilon) GF(t) + G\left(\chi_{[t,t+\varepsilon]}F\right)(t),$$

the continuity of  $U(\varepsilon)$  as an operator on  $L^{2}(\mathbb{R}^{n})$ , and the fact that

$$\left\|\chi_{[t,t+\varepsilon]}F\right\|_{L^{\tilde{q}'}_tL^{\tilde{r}'}_x}\to 0 \text{ as } \varepsilon\to 0.$$

From the estimates (3.36) - (3.39) we obtain (3.31) for endpoint case.

## 4. Strichartz type estimates for solution to nonlinear Schrödinger equation

Consider the multipoint initial-value problem (1.1) - (1.2).

**Condition 4.1.** Assume that the function  $F : \mathbb{C} \to C$  is continuously differentiable and obeys the power type estimates

$$F(u) = O\left(|u|^{1+p}\right), \ F_u(u) = O\left(|u|^p\right),$$
 (4.1)

$$F_{u}(v) - F_{u}(w) = O\left(|v - w|^{\min\{p,1\}} + |w|^{\max\{0,p-1\}}\right)$$
(4.2)

for some p > 0, where  $F_u(u)$  denotes the derivative of operator function F with respect to u.

From (4.1) we obtain

$$|F(u) - F(v)| \lesssim |u - v| (|u|^p + |v|^p).$$
(4.3)

**Remark 4.1.** The model example of a nonlinearity obeying the conditions above is  $F(u) = |u|^p u$ ,  $p \in (1, \infty)$  for which the critical homogeneous Sobolev space is  $\mathring{W}_x^{s_c,2}(\mathbb{R}^n)$  with  $s_c := \frac{n}{2} - \frac{2}{p}$ . **Definition 4.1.** A function  $F : [0,T] \times \mathbb{R}^n \to \mathbb{C}$  is called a (strong) solution to (1.1) - (1.2) if it lies in the class

$$C_t^0\left([0,T]; \mathring{W}_x^{s,2}\left(R^n\right)\right) \cap L_t^{p+2} L_x^{\frac{np(p+2)}{4}}\left([0,T] \times R^n\right)$$

and obey:

$$u(t,x) = V(t)\varphi(x) + \sum_{k=1}^{m} \alpha_k V_k(t,x) + \sum_{k=1}^{m} \alpha_k G_k(t,x) + G_0(t,x), \quad (4.4)$$

where

$$V(t) = F^{-1}\left[U_L(t,\xi)\,\hat{\varphi}(\xi)\right], \ V_k(t,x) = F^{-1}\left[U_L(\lambda_k,\xi)\,\hat{\varphi}(\xi)\right],$$

$$G_{k}(t,x) = F^{-1} \left[ \int_{t_{0}}^{\lambda_{k}} U_{L}(\lambda_{k} - \tau, \xi) \hat{F}(\tau, \xi) d\tau \right], \qquad (4.5)$$
$$G_{0}(t,x) = F^{-1} \left[ \int_{t_{0}}^{t} U_{L}(t - \tau, \xi) \hat{F}(\tau, \xi) d\tau \right].$$

We say that u is a global solution if  $T = \infty$ .

Let  $B(x, \delta)$  denotes the boll in  $\mathbb{R}^n$  centered in x with radius  $\delta$  and M denote the Hardy-Littlewood type maximal operator that is defined as:

$$Mf(x) = \sup_{\delta > 0} \left[ \mu \left( B(x, \delta) \right) \right]^{-1} \int_{B(x, \delta)} \left| f(y) \right| dy.$$

For proving the main result of this section we need the following:

**Proposition 4.1** [12](Ch.2, § 1, Theorem 1) Let  $1 , <math>1 < q \le \infty$ . Then there exists a constant C(p,q) such that for all  $\{f\}_{k\geq 0} \in L^p(\mathbb{R}^n)$  one has

$$\left\| \{Mf\}_{k \ge 0} \right\|_{L^p(R^n)} \le C\left(p,q\right) \left\| \{f\}_{k \ge 0} \right\|_{L^p(R^n; l_q)}.$$

**Lemma 4.1** [4, Proposition 3.1]. Assume  $F \in C^{(1)}(\mathbb{R})$ . Suppose  $\alpha \in (0,1), 1 < p, q, r < \infty$  and  $r^{-1} = p^{-1} + q^{-1}$ . If  $u \in L^{\infty}(\mathbb{R}), D^{\alpha}u \in L^{q}(\mathbb{R})$  and  $F'(u) \in L^{p}(\mathbb{R})$ , then  $D^{\alpha}(F(u)) \in L^{r}(\mathbb{R})$  and

$$\left\|D^{\alpha}\left(F\left(u\right)\right)\right\|_{L^{r}(\mathbb{R})} \lesssim \left\|F'\left(u\right)\right\|_{L^{p}(\mathbb{R})} \left\|D^{\alpha}u\right\|_{L^{q}(\mathbb{R})}.$$

**Theorem 4.1.** Assume the Conditons 2.1, 3.1, 4.1 are satisfied. Let  $0 \le s \le 1$ ,  $\varphi \in \mathring{W}^{s,2}(\mathbb{R}^n)$  and  $n \ge 1$ . Then there exists  $\eta_0 = \eta_0(n) > 0$  such that if  $0 < \eta \le \eta_0$  such that

$$\left\|\left|\nabla\right|^{s} U_{L}\left(t\right)\varphi\right\|_{L_{t}^{p+2}L_{x}^{\sigma}\left([0,T]\times R^{n}\right)} \leq \eta,\tag{4.5}$$

then here exists a unique solution u to (1.1) - (1.2) on  $[0, T] \times \mathbb{R}^n$ . Moreover, the following estimates hold

$$\||\nabla|^{s} U_{L} u\|_{L^{p+2}_{t} L^{\sigma}_{x}([0,T] \times R^{n})} \le 2\eta,$$
(4.6)

$$\||\nabla|^{s} u\|_{S^{0}([0,T]\times R^{n})} + \|u\|_{C^{0}([0,T]; \mathring{W}^{s,2}(R^{n}))} \lesssim \||\nabla|^{s} \varphi\|_{L^{2}_{x}(R^{n})} + \eta^{1+p}, \quad (4.7)$$

$$\|u\|_{S^{0}([0,T]\times R^{n};H)} \lesssim \|\varphi\|_{L^{2}_{x}(R^{n};H)}, \ r = r(p,n) = \frac{2n(p+2)}{2(n-2)+np}.$$
 (4.8)

**Proof.** We apply the standard fixed point argument. More precisely, using the Strichartz estimates (3.31), we will show that the solution map  $u \to \Phi(u)$  defined by (4.4) – (4.5) is a contraction on the set  $B_1 \cap B_2$  under the metric given by

$$d(u,v) = \|u - v\|_{L_t^{p+2}L_x^r([0,T] \times R^n)},$$

where

$$B_1 = \left\{ u \in W^{\infty, s_c, 2} = L_t^{\infty} W_x^{s_c, 2} \left( [0, T] \times R^n \right) : \right.$$

$$\begin{aligned} \|u\|_{W^{\infty,s_{c},2}} &\leq 2 \, \|\varphi\|_{W^{s_{c},2}_{x}(R^{n})} + C\left(n\right) \left(2\eta\right)^{1+p} \Big\} \,, \\ B_{2} &= \left\{ u \in W^{p+2,s_{c},r} = L^{p+2}_{t} W^{s_{c},r}_{x}\left([0,T] \times R^{n}\right) : \\ \||\nabla|^{s_{c}} \, u\|_{L^{p+2}_{t} L^{r}_{x}([0,T] \times R^{n})} &\leq 2\eta, \ \|u\|_{L^{p+2}_{t} L^{r}_{x}([0,T] \times R^{n})} \leq 2C\left(n\right) \ \|\varphi\|_{L^{2}_{x}(R^{n})} \Big\} \,, \end{aligned}$$

here C(n) denotes the constant from the Strichartz inequality in (3.25).

Note that both  $B_1$  and  $B_2$  are closed in this metric. Using the Strichartz estimate (3.31), Proposition 4.1 and Sobolev embedding in fractional Sobolev spaces [[15], § 2.3] we get that for  $u \in B_1 \cap B_2$ ,

$$\begin{split} \|\Phi\left(u\right)\|_{L_{t}^{\infty}W^{s_{c},2}\left([0,T]\times R^{n}\right)} &\leq \|\varphi\|_{W_{x}^{s_{c},2}\left(R^{n}\right)} + C\left(n\right)\|\langle\nabla\rangle^{s_{c}}F\left(u\right)\|_{L_{t}^{\left(p+2\right)/\left(p+1\right)}L_{x}^{r_{1}}} \leq \\ &\|\varphi\|_{W_{x}^{s_{c},2}\left(R^{n}\right)} + C\left(n\right)\|\langle\nabla\rangle^{s_{c}}u\|_{L_{t}^{p+2}L_{x}^{\sigma}}\|u\|_{L_{t}^{p+2}L_{x}^{n}^{p\left(p+2\right)/4}} \leq \\ &\|\varphi\|_{W_{x}^{s_{c},2}\left(R^{n}\right)} + C\left(n\right)\left(2\eta + 2C\left(n\right)\|\varphi\|_{L_{x}^{2}\left(R^{n}\right)}\right)\||\nabla|^{s_{c}}u\|_{L_{t}^{p+2}L_{x}^{r}} \leq \\ &\|\varphi\|_{W_{x}^{s_{c},2}\left(R^{n}\right)} + C\left(n\right)\left(2\eta + 2C\left(n\right)\|\varphi\|_{L_{x}^{2}\left(R^{n}\right)}\right)\left(2\eta\right)^{p}, \end{split}$$

where

$$L_t^q L_x^r = L_t^q L_x^r \left( [0, T] \times \mathbb{R}^n \right), \, r_1 = r_1 \left( p, n \right) = \frac{2n \left( p + 2 \right)}{2 \left( n + 2 \right) + np}.$$

Similarly,

$$\|\Phi\left(u\right)\|_{L^{p+2}_{t}L^{r}_{x}} \leq C\left(n\right)\|\varphi\|_{L^{2}_{x}(R^{n};H)} + C\left(n\right)\|u\|_{L^{p+2}_{t}L^{r}_{x}} \leq$$

$$\|\varphi\|_{W^{s_c,2}_x(R^n)} + 2C^2(n) \|\varphi\|_{L^2_x(R^n)}(2\eta)^p$$

Arguing as above and invoking (4.5), we obtain

$$\| |\nabla|^{s_c} \Phi(u) \|_{L^{p+2}_t L^r_x} \le \eta + C(n) \| |\nabla|^{s_c} \Phi(u) \|_{L^{(p+2)/(p+1)}_t L^{r_1}_x} \le \eta + C(n) (2\eta)^{1+p}.$$

Thus, choosing  $\eta_0 = \eta_0(n)$  sufficiently small, we see that for  $0 < \eta \leq \eta_0$  the function  $\Phi$  maps the set  $B_1 \cap B_2$  to itself. To see that it is a contraction, we repeat the computations above and use (4.4) to obtain

$$\|F(u) - F(v)\|_{L_t^{p+2}L_x^r} \le C(n) \|F(u) - F(v)\|_{L_t^{(p+2)/(p+1)}L_x^{r_1}} \le C(n) (2\eta)^p \|u - (v)\|_{L_t^{p+2}L_x^r}.$$

Thus, choosing  $\eta_0 = \eta_0(n)$  small enough, we can guarantee that is a contraction on the set  $B_1 \cap B_2$ . By the contraction mapping theorem, it follows that has a fixed point in  $B_1 \cap B_2$ . Since  $\Phi$  maps into  $C_t^0 W_x^{s_c,2}([0,T] \times \mathbb{R}^n)$  we derive that the fixed point of  $\Phi$  is indeed a solution to (1.1) - (1.2).

In view of Definition 4.1, uniqueness follows from uniqueness in the contraction mapping theorem.

### References

- J. Bergh and J. Lofstrom, Interpolation spaces: An introduction, Springer-Verlag, New York, 1976.
- J. Bourgain, Global solutions of nonlinear Schrodinger equations. American Mathematical, Society Colloquium Publications, 46. American Mathematical Society, Providence, RI, 1999, MR1691575.
- T. Cazenave and F. B. Weissler, The Cauchy problem for the critical nonlinear Schrodinger equation in H<sup>s</sup>. Nonlinear Anal. 14 (1990), 807-836.
- M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), 87-109.
- J. Ginibre and G. Velo, Smoothing properties and retarded estimates for some dispersive evolution equations, Comm. Math. Phys. 123 (1989), 535–573.
- L. Escauriaza, C. E. Kenig, G. Ponce, and L. Vega, Hardy's uncertainty principle, convexity and Schrödinger evolutions, J. European Math. Soc. 10, 4 (2008) 883–907.
- G. Grillakis, On nonlinear Schrodinger equations. Comm. PDE 25 (2000), 1827{1844.

- 8. C. E. Kenig and F. Merle, Global well-posedness, scattering and blow up for the energycritical, focusing, nonlinear Schrodinger equation in the radial case. Invent. Math. 166 (2006), 645{675.
- M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), 955-980.
- R. Killip and M. Visan, Nonlinear Schrodinger equations at critical regularity, Clay Mathematics Proceedings, v. 17, 2013.
- A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer, Berlin, 1983.
- E. M. Stein, Singular Integrals and differentiability properties of functions, Princeton Univ. Press, Princeton. NJ, 1970.
- C. D. Sogge, Fourier Integrals in Classical Analysis, Cambridge University Press, 1993.
- R. S. Strichartz, Restriction of Fourier transform to quadratic surfaces and deay of solutions wave equations. Duke Math. J. 44 (1977), 705{714. MR0512086A.
- 15. H. Triebel, Interpolation theory, Function spaces, Differential operators, North-Holland, Amsterdam, 1978.
- T. Tao, Nonlinear dispersive equations. Local and global analysis. CBMS Regional conference series in mathematics, 106. American Mathematical Society, Providence, RI, 2006., MR2233925