LACK OF NULL-CONTROLLABILITY FOR THE FRACTIONAL HEAT EQUATION AND RELATED EQUATIONS*

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Abstract. We consider the equation $(\partial_t + \rho(\sqrt{-\Delta}))f(t,x) = \mathbb{1}_\omega u(t,x)$, $x \in \mathbb{R}$ or $\mathbb{T}$. We prove it is not null-controllable if ρ is analytic on a conic neighborhood of $\mathbb{R}_+$ and $\rho(\xi) = o(|\xi|)$. The proof relies essentially on geometric optics, i.e. estimates for the evolution of semiclassical coherent states.

The method also applies to other equations. The most interesting example might be the Kolmogorov-type equation $(\partial_t - \partial_v^2 + v^2\partial_x)f(t,x,v) = \mathbb{1}_\omega u(t,x,v)$ for $(x,v) \in \Omega_x \times \Omega_v$ with $\Omega_x = \mathbb{R}$ or $\mathbb{T}$ and $\Omega_v = \mathbb{R}$ or $(-1,1)$. We prove it is not null-controllable in any time if ω is a vertical band $\omega_x \times \Omega_v$. The idea is to remark that, for some families of solutions, the Kolmogorov equation behaves like the rotated fractional heat equation $(\partial_t + \sqrt{i}(-\Delta)^{1/4})g(t,x) = \mathbb{1}_\omega u(t,x)$, $x \in \mathbb{T}$.

Key words. null controllability, observability, fractional heat equation, degenerate parabolic equations

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1. Introduction.

1.1. Problem of the null-controllability. Consider the following equation, which is called the *fractional heat equation*, where $\Omega = \mathbb{R}$ or $\mathbb{T}$, ω is an open subset of Ω, $\alpha \geq 0$:

$$(\partial_t + (-\Delta)^{\alpha/2})f(t,x) = \mathbb{1}_\omega u(t,x) \quad t \in [0,T], x \in \Omega$$

Here, we define $(-\Delta)^{\alpha/2}$ with the functional calculus, that is, $(-\Delta)^{\alpha/2} f = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}(f))$ if $\Omega = \mathbb{R}$, where $\mathcal{F}$ is the Fourier transform; and $c_n((-\Delta)^{-\alpha/2} f) = |n|^\alpha c_n(f)$ if $\Omega = \mathbb{T}$, where $c_n(f)$ is the nth Fourier coefficient of f.

It is a control problem with state $f \in L^2(\Omega)$ and control u supported in ω. More precisely, we are interested in the exact null-controllability of this equation.

Definition 1.1. *We say that the fractional heat equation is null-controllable on ω in time $T > 0$ if for all f_0 in $L^2(\Omega)$, there exists u in $L^2([0,T] \times \omega)$ such that the solution f of:*

$$(1.1) \qquad \begin{aligned} (\partial_t + (-\Delta)^{\alpha/2})f(t,x) &= \mathbb{1}_\omega u(t,x) \qquad && t \in [0,T], x \in \Omega \\ f(0,x) &= f_0(x) && x \in \Omega. \end{aligned}$$

satisfies $f(T,x,v) = 0$ for all (x,v) in Ω.

The main motivation for this study, apart from studying the fractional heat equation itself, is the null-controllability of a Kolmogorov-type equation. More specifically, we are interested in the following equation, where $\Omega = \Omega_x \times \Omega_v$ with $\Omega_x = \mathbb{R}$ or $\mathbb{T}$, $\Omega_v = \mathbb{R}$ or $(-1,1)$ and ω is an open subset of Ω:

$$(\partial_t + v^2\partial_x - \partial_v^2)f(t,x,v) = \mathbb{1}_\omega u(t,x,v) \quad t \in [0,T], (x,v) \in \Omega.$$

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For convenience, we will just say in this paper "the Kolmogorov equation". Note that thanks to Hörmander's bracket condition [21, Section 22.2], the operator $v^2\partial_x - \partial_v^2$ is hypoelliptic. Also, this equation is well-posed. This can be proved by Hille-Yosida's theorem (see [2, Section 4] in the case $\Omega = \mathbb{T}\times(-1,1)$). As we will see, this Kolmogorov equation is related to the rotated fractional heat equation.

Definition 1.2. *We say that the Kolmogorov equation is null-controllable on ω in time $T>0$ if for all f_0 in $L^2(\Omega)$, there exists u in $L^2([0,T]\times\omega)$ such that the solution f of:*

$$
\begin{aligned}
(\partial_t + v^2\partial_x - \partial_v^2)f(t,x,v) &= \mathbb{1}_\omega u(t,x,v) && t\in[0,T], (x,v)\in\Omega\\
f(0,x,v) &= f_0(x,v) && (x,v)\in\Omega\\
f(t,x,v) &= 0 && t\in[0,T], (x,v)\in\partial\Omega \text{ (if non-empty)}
\end{aligned}
\tag{1.2}
$$

satisfies $f(T,x,v)=0$ for all (x,v) in Ω.

1.2. Statement of the results. We will prove that the rotated fractional heat equation is never null controllable if $\Omega\setminus\omega$ has nonempty interior, and that the Kolmogorov equation is never null-controllable if $\omega = \omega_x\times\Omega_v$ where $\Omega_x\setminus\omega_x$ has nonempty interior.

Theorem 1.3. *Let $0\le\alpha<1$ and $\Omega=\mathbb{R}$ or $\Omega=\mathbb{T}$. Let ω be a strict open subset of Ω. The fractional heat equation (1.1) is not null controllable in any time on ω.*

This Theorem still holds in higher dimension, with $\Omega = \mathbb{R}^d\times\mathbb{T}^{d'}$, but our method seems ineffective to treat the case where Ω is, say, an open subset of $\mathbb{R}$. This may be because we are using the spectral definition of the fractional Laplacian, and our method might be adapted if instead we used a singular kernel definition of the fractional Laplacian.

Actually, we prove the non-null controllability of a class of equations of the form $(\partial_t+\rho(\sqrt{-\Delta}))f(t,x) = \mathbb{1}_\omega(t,x)$.

Theorem 1.4. *Let $K>0$, $\mathcal{C} = \{\xi\in\mathbb{C}, \Re(\xi)>K, |\Im(\xi)|<K^{-1}\Re(\xi)\}$ and $\rho\colon\mathcal{C}\cup\mathbb{R}_+\to\mathbb{C}$ such that*

1. *ρ is holomorphic on $\mathcal{C}$,*
2. *$\rho(\xi)=o(|\xi|)$ in the limit $|\xi|\to+\infty$, $\xi\in\mathcal{C}$,*
3. *ρ is measurable on $\mathbb{R}_+$ and $\inf_{\xi\in\mathbb{R}_+}\Re(\rho(\xi))>-\infty$.*

Let $\Omega=\mathbb{R}$ or $\mathbb{T}$, ω be a strict open subset of Ω and $T>0$. Then the equation

$$
(\partial_t+\rho(\sqrt{-\Delta}))f(t,x) = \mathbb{1}_\omega u(t,x), \quad t\in[0,T],\ x\in\Omega \tag{1.3}
$$

is not null-controllable on ω in time T.

For lack of a better name, we will call the equation (1.3) the *generalized fractional heat equation*. This Theorem can be generalized to the case $\Omega=\mathbb{R}^d\times\mathbb{T}^{d'}$. The hypothesis $\inf_{\mathbb{R}_+}\Re(\rho)>-\infty$ is only used to ensure that the equation is well-posed.

The fractional heat equation is the case $\rho(\xi)=\xi^\alpha$. Note that if $\alpha=0$, then the fractional heat equation is just a family of decoupled ordinary differential equations, and the conclusion of Theorem 1.3 is unimpressive. At the other end, the method used in this article does not work as-is if $\alpha=1$, but we still expect non-null-controllability, even if this remains a conjecture if Ω is not the one-dimensional torus.

Some equations behave like the fractional heat equation, at least in some regimes. This is the case of the Kolmogorov equation, and if the control acts on a vertical band, we will prove it is not null-controllable with the same method.

THEOREM 1.5. *Let $\Omega_x = \mathbb{R}$ or $\mathbb{T}$, let $\Omega_v = \mathbb{R}$ or $(-1,1)$, and let $\Omega = \Omega_x \times \Omega_v$. Let $T > 0$ and ω_x be a strict open subset of Ω_x. The Kolmogorov equation (1.2) is not null-controllable on $\omega = \omega_x \times \Omega_v$ in time T.*

This Theorem can be extended to higher dimension in x and v if $\Omega_v = \mathbb{R}^d$. If we want, say $\Omega_v = (-1,1)^d$, we lack information on the eigenvalues and eigenfunctions of $-\partial_v^2 + inv^2$ on $(-1,1)^d$, but this is the only obstacle to the generalization of the Theorem to this case. We also give a non-null-controllability result in small time for more general control region.

THEOREM 1.6. *Let $\Omega_x = \mathbb{R}$ or $\Omega_x = \mathbb{T}$ and $\Omega_v = \mathbb{R}$ or $\Omega_v = (-1,1)$. Let $\Omega = \Omega_x \times \Omega_v$. Let $\omega \subset \Omega$. Assume that there exist $x_0 \in \Omega_x$ and $a > 0$ such that the symmetric vertical interval $\{(x_0, v), -a < v < a\}$ is disjoint from $\overline{\omega}$. Then, the Kolmogorov equation (1.2) is not null-controllable on ω in time $T < a^2/2$.*

Whether this condition $T < a^2/2$ is optimal or not is an open question, but we conjecture that it is optimal, at least for some geometries. If $\omega = \mathbb{T} \times (a,b)$ with $0 < a < b$, Theorem 1.6 proves that null-controllability does not hold in time $T < a^2/2$. This special case was already known [6, Theorem 1.3], it is also proved in the same reference that null-controllability holds for some $T > 0$. Our Theorem 1.6 sharpens the lower-bound on the minimal time of null-controllability if the geometry of ω is different than a cartesian product.

While the fractional heat equation and the Kolmogorov equation are the main focus of this article, the method can be used to treat other equations: those that behave like the fractional heat equation for $\alpha < 1$. In Appendix A, we briefly discuss the fractional Schrödinger equation, and sketch the proof for the Kolmogorov-type equation $(\partial_t - \partial_v^2 - v\partial_x) f(t,x,v) = \mathbb{1}_\omega u(t,x,v)$ (notice the v instead of the v^2), and the *improved Boussinesq equation* $(\partial_t^2 - \partial_x^2 - \partial_x^2\partial_t^2) f(t,x) = \mathbb{1}_\omega u(t,x)$.

1.3. Bibliographical comments.

1.3.1. Control of partial differential equations. Let A be on operator on a Hilbert space H such that the equation $\partial_t f + Af = 0$ is well-posed (i.e. $-A$ generates a strongly continuous semigroup of bounded linear operators on H, see for instance [34, Ch. 2] or [15, Sec. 2.3 and Appendix A] for the definition).

Let U be a Hilbert space and $B\colon U \to H$ a bounded linear operator. With the right choice of H, A, U and B, the problems we are interested in can be stated the following way: for every $f_0 \in H$, does there exist $u \in L^2(0,T;U)$ such that the solution of $\partial_t f + Af = Bu$, $f(0) = f_0$ satisfies $f(T) = 0$?

For the fractional heat equation (1.1) on $\mathbb{R}^n$, we choose $H = L^2(\mathbb{R}^n)$, $A = (-\Delta)^{\alpha/2}$ with domain $H^\alpha(\mathbb{R}^n)$, $U = L^2(\omega)$ and $B\colon u \mapsto u\mathbb{1}_\omega$. For the Kolmogorov equation (1.2) on $\mathbb{R}^2$, we choose $H = L^2(\mathbb{R}^2)$, $A = -\partial_v^2 + v^2\partial_x$ (the domain of A is a bit complicated to define, see [2, Sec. 4]), $U = L^2(\omega)$ and $B\colon u \mapsto u\mathbb{1}_\omega$.

Whether there exists a $u \in L^2(0,T;U)$ such that the solution of $\partial_t f + Af = Bu$, $f(0) = f_0$ satisfies $f(T) = 0$ depends of course of A, B and on the spaces H and U. Let us discuss existing results when A is a parabolic operator related to the fractional heat equation or the Kolmogorov equation.

1.3.2. Null-controllability of the (fractional) heat equation in dimension one: the moment method. Let us first look at the heat equation in dimension one with Dirichlet boundary conditions, i.e. $H = L^2(0,\pi)$, $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$ and $A\colon f \in D(A) \mapsto -\partial_x^2 f \in L^2(0,\pi)$. Let us also denote λ_n the eigenvalues of A, and assume that λ_n is increasing, so that $\lambda_n = n^2$.

A possible strategy to control the heat equation in dimension one is to look for a control of the form $\tilde{u}(t,x) = u(t)v(x)$. In the framework of subsection 1.3.1, this is the choice $U = \mathbb{R}$ and Bu the function $x \in (0,\pi) \mapsto uv(x)$. We will call this kind of controls *shaped controls*. This is the strategy pioneered by Fattorini and Russel [18]. Let us describe it briefly.

Let $v\colon (0,\pi) \to \mathbb{R}$. Let $f_0 \in L^2(0,\pi)$, let $u \in L^2(0,T)$ and let f be the solution of $(\partial_t - \partial_x^2)f(t,x) = u(t)v(x)$, $f(t,0) = f(t,\pi) = 0$ with initial condition $f(0,x) = f_0(x)$. Finally, for every $g \in L^2(0,\pi)$, let $c_n(g) \coloneqq \int_0^\pi g(x)\sin(nx)\,\mathrm{d}x$ be the n-th Fourier coefficient of g. Then, the relation $f(T,\cdot) = 0$ is equivalent to the *moment problem*

$$\forall n \in \mathbb{N}\setminus\{0\},\ \int_0^T e^{(t-T)\lambda_n} u(t)\,\mathrm{d}t = -\frac{e^{-T\lambda_n} c_n(f_0)}{c_n(v)}.$$

Fattorini and Russel prove such a u exists by constructing a biorthogonal family to $(e^{-t\lambda_n})_{n\in\mathbb{N}\setminus\{0\}}$, i.e. a family of functions $(g_n)_{n\in\mathbb{N}\setminus\{0\}}$ such that $\int_0^T g_n(t)e^{-\lambda_m t}\,\mathrm{d}t = 1$ if $n = m$ and 0 if $n \neq m$ (see also [33] for a more streamlined proof that this family exists). Then the function u defined by

$$u(t) = -\sum_{n>0} \frac{e^{-T\lambda_n} c_n(f_0)}{c_n(v)} g_n(T-t)$$

formally solves the moment problem. Moreover, we can prove some estimates on the functions $(g_n)_{n>0}$, and if $c_n(v)$ does not decay too fast when $n \to +\infty$, the series that defines u actually converges.

This strategy can be adapted for the fractional heat equation (1.1) when $\alpha > 1$, as Micu and Zuazua [26] already remarked. Indeed, the construction and estimate on the biorthogonal family relies on the hypotheses $\sum_{n>0} |\lambda_n|^{-1} < +\infty$ and $\lambda_{n+1} - \lambda_n \geq c > 0$. These hypotheses still hold if we replace the operator $A = -\partial_x^2$ on $(0,\pi)$ by $(-\partial_x^2)^{\alpha/2}$ as long as $\alpha > 1$. Indeed, the eigenvalues are now $\lambda_n = n^\alpha$.

On the other hand, if $\alpha \leq 1$, this proof does not work anymore. In fact, Micu and Zuazua [26, Sec. 5] proved that if $\alpha \leq 1$, the fractional heat equation (1.1) is not null-controllable with controls of the form $u(t)v(x)$. Miller [27, Sec. 3.3] (see also [17, Appendix]) also gets similar results, with similar methods.

But these negative results, based on Müntz Theorem, only concern *scalar controls*, i.e. the case where the control space is $U = \mathbb{R}$ (or $\mathbb{C}$). If the control space is larger, say $U = L^2(\omega)$, we cannot rule out the existence of a control with Müntz Theorem. Indeed, there are many equations which are not null-controllable with scalar controls, but that are null-controllable with a larger control space. One of them is the heat equation in dimension larger than one. Let us discuss it now.

1.3.3. Null-controllability of the (fractional) heat equation and the spectral inequality. Let Ω be a bounded open subset of $\mathbb{R}^d$. Let $(\lambda_n)_{n\geq 0}$ be the sequence of the eigenvalues of $-\Delta$ on Ω with Dirichlet boundary conditions.[1] According to Weyl's law, if the dimension d is greater than 1, then $\sum_{n\geq 0} \lambda_n^{-1} = +\infty$, so we cannot prove null-controllability with the moment method. To build a control for the heat equation we have to choose a control space that is infinite dimensional [15, Th. 2.79]. We choose $U = L^2(\omega)$ and $B\colon u \in L^2(\omega) \mapsto u\mathbb{1}_\omega \in L^2(\Omega)$. We call this kind of controls *internal controls*.

To prove the null-controllability of the heat equation in any dimension, Fursikov and Immananuvilov [20] use *parabolic Carleman inequalities*, which are weighted energy estimates, to prove

[1] I.e. $D(A) = \{f \in H_0^1(\Omega),\ \Delta f \in L^2(\Omega)\}$ where Δf is in the sense of distributions.

(more or less) directly the observability inequality that is equivalent to the null-controllability (see for instance [15, Th. 2.44 or Th. 2.66] for the equivalence between null-controllability and observability).

Independently, Lebeau and Robbiano [25, 23] developped another strategy to prove the null-controllability of the heat equation. This strategy yields more insight for our purpose, so let us give more details.

By means of *elliptic Carleman inequalities*, Lebeau and Robbiano proved a *spectral inequality*, which is the following: let M be a connected compact riemannian manifold with boundary, let ω be an open subset of M, and let $(\phi_i)_{i\in\mathbb{N}}$ be an orthonormal basis of eigenfunctions of $-\Delta$ with associated eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$, then there exists $C>0$ and $K>0$ such that for every sequence of complex numbers $(a_i)_{i\in\mathbb{N}}$ and every $\mu>0$

$$\Big|\sum_{\lambda_i<\mu} a_i\phi_i\Big|_{L^2(M)} \leq Ce^{K\sqrt{\mu}}\Big|\sum_{\lambda_i<\mu} a_i\phi_i\Big|_{L^2(\omega)}. \tag{1.4}$$

The key point to deduce the null-controllability of the heat equation from this spectral inequality is that if one takes an initial condition of the form $f_0=\sum_{\lambda_i\geq\mu} a_i\phi_i$ with no component along frequencies less than μ, the solution of the heat equation decays like $e^{-T\mu}|f_0|_{L^2(M)}$, and the exponent in μ in this decay (i.e. 1) is larger than the one appearing in the spectral inequality (i.e. 1/2).[2]

For the fractional heat equation (1.1), the dissipation stays stronger that the spectral inequality as long as $\alpha>1$. Thus, for $\alpha>1$, we can prove the null-controllability with Lebeau and Robbiano's method, as already mentioned by Micu and Zuazua [26] and Miller [27] (see also [28, 17]).

Our Theorem 1.3 proves that the threshold $\alpha>1$ is optimal: if $\alpha<1$, then the fractional heat equation is not null-controllable (at least for $\Omega=\mathbb{T}^n$). Note that the case $\alpha=1$ and $\Omega=\mathbb{T}$ has already been proved to lack null-controllability with internal controls [22, Th. 4]. Let us also mention an article where the null-controllability of an equation closely related to our fractional heat equation have been investigated [11].

So it seems that the Lebeau-Robbiano method is in some sense optimal: if the dissipation is not stronger than the spectral inequality, then we do not have null-controllability. Let us finish with another class of parabolic equations for which Lebeau and Robbiano's method does not work: degenerate parabolic equations.

1.3.4. Null-controllability of degenerate parabolic partial differential equations. Degenerate parabolic equations are equations of the form $\partial_t f(t,x)+Af(t,x)=\mathbb{1}_\omega u(t,x)$, $0\leq t\leq T$, $x\in\Omega$, where A is a second-order differential operator which is degenerate elliptic, i.e. its principal symbol $P(x,\xi)$ satisfies $P(x,\xi)\geq 0$ but is zero for some $x\in\Omega$ and $\xi\neq 0$.

The interest in the null-controllability of degenerate parabolic equations is more recent. We now understand the null-controllability of parabolic equations degenerating at the boundary in dimension one [12] and two [13] (see also references therein), where the authors found that these equations where null-controllable if the degeneracy is not too strong, but might not be if the degeneracy is too strong. To the best of our knwoledge, the only other general family of degenerate parabolic equations whose null-controllability has been investigated is hypoelliptic quadratic differential equations [8, 7].

Other equations have been studied on a case-by-case basis. For instance, the Kolmogorov equation has been investigated since 2009 [9, 2, 6]. In these papers, the authors found that if

[2]The exponent $\frac{1}{2}$ of μ in the spectral inequality is optimal if ω is a strict open subset of M [23, Proposition 5.5].

$\Omega = \mathbb{T}\times(-1,1)$ and $\omega = \mathbb{T}\times(a,b)$ with $0<a<b<1$, the Kolmogorov equation is null-controllable in large times, but not in time smaller than $a^2/2$, and that if $-1<a<0<b<1$, it is null-controllable in arbitrarily small time [2]. If in the Kolmogorov equation (1.2) we replace v^2 by v, the null-controllability holds in arbitrarily small time if $\omega = \mathbb{T}\times(a,b)$ [9, 2]. On the other hand, if we replace v^2 by v^γ where γ is an integer greater than 2 and $\omega = \mathbb{T}\times(a,b)$, it is never null-controllable [6]. In this last article, the null-controllability of a model of the equation we are interested in, namely the equation $(\partial_t + iv^2(-\Delta_x)^{1/2} - \partial_v^2)g = 0$, is also investigated.

Another degenerate parabolic equation is the Grushin equation $(\partial_t - \partial_x^2 - x^2\partial_y^2)f(t,x,y) = \mathbb{1}_\omega u(t,x,y)$ on $\Omega = (-1,1)\times(0,\pi)$ with Dirichlet boundary conditions. If the control domain is a vertical band $\omega = (a,b)\times(0,\pi)$ with $0<a<b$, there exists a minimum time for the null-controllability to hold [4]. This minimum time has since been computed [5]. On the other end, if the domain control is an horizontal band $\omega = (0,\pi)\times(a,b)$ with $(a,b)\subsetneq(0,\pi)$, then the Grushin equation is not null controllable [22].

Let us finally just mention an article on the heat equation on the Heisenberg group since 2017 [3], and that some parabolic equations on the real half-line, some of them related to the present work, have been shown to strongly lack controllability [16].

1.4. Outline of the proof, structure of the article. As usual in controllability problems, we focus on *observability inequalities* on the *adjoint systems*, that are equivalent to the null-controllability (see [15, Theorem 2.44]).

Specifically, the null-controllability of the fractional heat equation (1.1) is equivalent to the existence of $C>0$ such that for every solution g of

$$(\partial_t + (-\Delta)^{\alpha/2})g(t,x) = 0 \quad t\in(0,T), x\in\Omega \tag{1.5}$$

we have

$$|g(T,\cdot)|_{L^2(\Omega)} \leq C|g|_{L^2((0,T)\times\omega)}. \tag{1.6}$$

So, to disprove the null controllability, we only have to find solutions of (1.5) that are concentrated outside ω. To construct such solutions, we consider initial states that are (essentially) *semiclassical coherent states*, i.e. initial states of the form $g_{0,h}\colon x\mapsto h^{-1/4}e^{-(x-x_0)^2/2h+ix\xi_0/h}$. We will prove that solutions of Eq. (1.5) with these initial conditions stay concentrated around x_0. More precisely, we get asymptotic expansion of these solutions thanks to the saddle point method. We do this informally at first, in section 2, then rigorously in subsection 4.1 in the case $\Omega=\mathbb{R}$ and in subsection 4.2 in the case $\Omega=\mathbb{T}$. This proof relies on some technical computations that are done in section 3. These computations are carried over in a slightly general framework, that allows to directly treat the other equations, namely the Kolmogorov-type equations and the improved Boussinesq equation. We also sketch the proof of the generalization of Theorem 1.4 in higher dimension in subsection 4.3.

Let us finish this introduction by explaining how the Kolmogorov equation for $\Omega_x = \Omega_v = \mathbb{R}$ and the fractional heat equation are related. The first eigenfunction of $-\partial_v^2 + i\xi v^2$ on $\mathbb{R}$, is[3] $e^{-\sqrt{i\xi}v^2/2}$ (up to a normalization constant), with eigenvalue $\sqrt{i\xi}$. So, $\Phi_\xi\colon (x,v)\in\mathbb{R}^2\mapsto e^{i\xi x-\sqrt{i\xi}v^2/2}$ is a generalized eigenfunction of the Kolmogorov operator $v^2\partial_x - \partial_v^2$, with eigenvalue $\sqrt{i\xi}$. So, the solution of the Kolomogorov equation $(\partial_t + v^2\partial_x - \partial_v^2)f = 0$ with initial condition $f(0,x,v) = \int_\mathbb{R} a(\xi)\Phi_\xi(x,v)\,\mathrm{d}\xi$ is $f(t,x,v) = \int_\mathbb{R} a(\xi)\Phi_\xi(x,v)e^{-\sqrt{i\xi}t}\,\mathrm{d}\xi$. This suggests that, dropping the v variable for the moment,

[3]Here and in all this paper, we choose the branch of the square root with positive real part.

the Kolmogorov equation behaves like an equation where the eigenfunctions are the $e^{i\xi x}$ with eigenvalue $\sqrt{i\xi}$, i.e. the equation $(\partial_t + \sqrt{i}(-\Delta_x)^{1/4})f(t,x) = 0$ with $x \in \mathbb{R}$.

Based on this observation and the non-null-controllability result of the rotated fractional heat equation on the whole real line, we prove in subsection 5.2 that the Kolmogorov equation is not null-controllable in the case $\Omega = \mathbb{R}\times\mathbb{R}$. For Kolmogorov's equation on $\Omega = \Omega_x \times (-1,1)$, we need some information on the eigenvalues and the eigenfunctions, which are not explicit anymore. We already proved most of what we need in another article [22, Section 4]. We prove the non-null-controllability of Kolmogorov equation with $\Omega_v = (-1,1)$ in subsection 5.4.

Finally, we sketch the proof for the Kolmogorov equation with v instead of v^2 and for the improved Boussinesq equation in Appendix A.

2. Informal presentation of the proof. As we explained in subsection 1.4, we will try to disprove the observability inequality (1.6). We only discuss here the case $\Omega = \mathbb{R}$.

Since the fractional heat equation is invariant by translation, we may assume that $\omega \subset \{|x| > \delta\}$ for some $\delta > 0$. Then, for $h > 0$, we consider the solution g_h of the fractional heat equation (1.5) with initial condition $g_{0,h}(x) = e^{-x^2/2h + i\xi_0/h}$ with some $\xi_0 > 0$. The solution of the fractional heat equation is then

$$g_h(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t|\xi|^\alpha + ix\xi}\mathcal{F}(g_{0,h})(\xi)\,\mathrm{d}\xi,$$

where $\mathcal{F}$ is the Fourier transform defined by $\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}\,\mathrm{d}x$. But the Fourier transform of $g_{0,h}$ has a closed-form expression. Indeed, $\mathcal{F}(e^{-x^2/2})(\xi) = \sqrt{2\pi}e^{-\xi^2/2}$, and using the properties of the Fourier transform (scaling and translation), we find $\mathcal{F}(g_{0,h})(\xi) = \sqrt{2\pi h}e^{-(h\xi-\xi_0)^2/2h}$. Thus

$$g_h(t,x) = \sqrt{\frac{h}{2\pi}}\int_{\mathbb{R}} e^{-(h\xi-\xi_0)^2/2h + ix\xi - t|\xi|^\alpha}\mathrm{d}\xi.$$

If we make the change of variables $\xi' = h\xi$, we find

$$g_h(t,x) = \frac{1}{\sqrt{2\pi h}}\int_{\mathbb{R}} e^{-(\xi-\xi_0)^2/2h + ix\xi/h - t|\xi|^\alpha/h^\alpha}\mathrm{d}\xi. \tag{2.1}$$

Notice that if h is small, the term $e^{-(\xi-\xi_0)^2/2h}$ is concentrated around $\xi = \xi_0$, and so is the integrand. Thus, the major part of this integral comes from a neighborhood of ξ_0. In this situation, we can compute asymptotic expansion with the saddle point method.

More precisely, the saddle point method (see for instance [32, Ch. 2]) is a way to compute asymptotic expansion of integrals of the form $I(h) = \int e^{\phi(x)/h}u(x)\,\mathrm{d}x$ in the limit $h\to 0^+$, where ϕ and u are entire functions.

If the main contribution in the integral $I(h)$ comes from a nondegenerate critical point of ϕ at $x = 0$, the "standard" saddle point method gives

$$I(h) \sim e^{\phi(0)/h}\sum_k \sqrt{2\pi}\,\tilde{u}_{2k}h^{k+1/2} \tag{2.2}$$

where the $\tilde{u}_{2k}$ are of the form $A_{2k}u(0)$, and A_{2k} are differential operators of order $2k$, with in particular $\tilde{u}_0 = u(0)|\phi''(0)|^{-1/2}$.

The term $e^{-|\xi|^\alpha/h^\alpha}$ prevents us from applying the saddle point method to Eq. (2.1) as-is, but let us pretend we can do it anyway (the rigorous computation will be carried in section 3). Since the critical point of $\xi \mapsto -(\xi-\xi_0)^2/2 + ix\xi$ is $\xi_c = \xi_0 + ix$, we get from the saddle point method

$$g_h(t,x) \approx e^{-(\xi_c-\xi_0)^2/2h + ix\xi_c/h - t\xi_c^\alpha/h^\alpha} = e^{-x^2/2h + ix\xi_0/h - t(\xi_0+ix)^\alpha/h^\alpha}.$$

By integrating this asymptotic expansion, we get the following lower bound on the left-hand side of the observability inequality (1.6):

$$|g_h(T,\cdot)|^2_{L^2(\mathbb{R})} \geq |g_h(T,\cdot)|_{L^2(|x|<\delta)} \geq ch^{-1}\int_{|x|<\delta} e^{-x^2/h - Ch^{-\alpha}}\,\mathrm{d}x \geq c' e^{-C'h^{-\alpha}}. \tag{2.3}$$

Again by integrating the asymptotic expansion on g_h, we get the following upper on the right-hand side of the observability inequality

$$|g_h|_{L^2([0,T]\times\omega)} \leq |g_h|^2_{L^2([0,T]\times\{|x|>\delta\})} \leq Ch^{-1}\int_{[0,T]\times\{|x|>\delta\}} e^{-x^2/h - cth^{-\alpha}}\,\mathrm{d}t\,\mathrm{d}x \leq C'e^{-\delta^2/h}. \tag{2.4}$$

Comparing this upper bound (2.4) and the lower bound (2.3), and taking the limit $h \to 0^+$, we see that the observability inequality (1.6) cannot be true if $\alpha < 1$. Thus the fractional heat equation is not null-controllable.

3. Some technical computations. Before we make rigorously the proof outlined in section 2, we carry here some computations as a technical preparation.

3.1. Perturbation of the saddle point method. The "standard" saddle point method can be stated in the following way [32, Th. 2.1].

THEOREM 3.1. *Let U be an open bounded neighborhood of 0 in $\mathbb{R}^d$. For every $N \in \mathbb{N}$, there exists $C_N > 0$ such that for every $h > 0$ and every holomorphic function u on a complex neighborhood V of $\overline{U}$,*

$$\int_U e^{-x^2/2h} u(x)\,\mathrm{d}x = \sum_{j=0}^{N-1} \frac{h^{d/2+j}}{(2\pi)^{d/2} j! 2^j} (\Delta)^j u(0) + R_N(h),$$

where

$$|R_N(h)| \leq C_N \lambda h^{d/2+N} \sup_{z\in V} |u(z)|.$$

By using the Morse lemma (see for instance [21, Lemma C.6.1]), one can often transform integral of the form $\int e^{\phi(x)/h} u(x)\,\mathrm{d}x$ into integrals of the form of Theorem 3.1, plus some exponentially small error. Notice that in this theorem, the phase $-x^2/2$ does not depend on h. However, to rigorously treat the integral of Eq. (2.1), we need to allow the phase to depend on h.

PROPOSITION 3.2 (Perturbation of the saddle point method). *Let $h_0 > 0$ and $\epsilon\colon (0, h_0] \to \mathbb{R}_+$ such that $\epsilon(h) \to 0$ as $h \to 0$. Let $U \subset R$ be an open interval around 0. Let V be a complex open bounded neighborhood of $\overline{U}$ in $\mathbb{C}$.*

For every $0 < h \leq h_0$, let $r_h\colon V \to \mathbb{C}$ such that

1. *$\forall 0 < h \leq h_0$, r_h is holomorphic on V,*
2. *there exists $C > 0$ such that for every $0 < h \leq h_0$ and $\xi \in V$, $|r_h(\xi)| \leq C\epsilon(h)$.*

For such r_h, we define $|r|_\epsilon := \inf\{C > 0, \forall 0 < h \leq h_0, \forall \xi \in V, |r_h(\xi)| \leq C\epsilon(h)\}$.

For every $0 < h \leq h_0$, let u_h be a holomorphic bounded function on V. We consider

$$I_{h,r}(u) := \int_U e^{-\xi^2/2h + r_h(\xi)/h} u_h(\xi)\,\mathrm{d}\xi.$$

Let $M > 0$. We have uniformly in $|r|_\epsilon < M$, and uniformly in u_h holomorphic bounded on V

$$I_{h,r}(u) = \sqrt{2\pi h}\, e^{r_h(0)/h + \mathcal{O}(\epsilon(h)^2/h)}\Big(u_h(0) + \mathcal{O}\big((h+\epsilon(h))|u_h|_{L^\infty(V)}\big)\Big).$$

Proof of Proposition 3.2. The strategy is to see $\varphi_{h,r}(\xi) = -\xi^2/2 + r_h(\xi)$ as the phase, and to change the integration variable and the integration path to rewrite $I_{h,r}(u)$ in the form $I_{h,r}(u) = \int e^{-\xi^2/2h} v_h(\xi)\,\mathrm{d}\xi$, even if this change of variables depends on h, and then apply Theorem 3.1.

In this proof, $M > 0$ is fixed. We also choose $V' \subset \mathbb{C}$ convex such that[4] $U \Subset V' \Subset V$. Also, we use the convention that C denotes a constant, that depends only on ϵ, M, V and V', but not on h small enough, $|r|_\epsilon \le M$ or $\xi \in V'$, and that may be different each line.

Step 1: finding the critical point. We claim that for h small enough, for every $|r_h|_\epsilon \le M$, there exists a unique critical point $\xi_{c,h,r}$ of $\varphi_{h,r}$ in $\overline{V'}$, and that this critical point is non-degenerate.

Indeed, $\xi \in \overline{V'}$ is a critical point of $\varphi_{h,r}$ if and only if $\partial_\xi r_h(\xi) = \xi$. But for every[5] $\xi \in V'$, $|\partial_\xi r_h(\xi)| \le C|r_h|_{L^\infty(V)} \le C|r|_\epsilon \epsilon(h)$. Moreover, since V' is convex, according to the mean value inequality, for $\xi, \xi' \in \overline{V'}$, $0 < h \le h_0$ and $|r|_\epsilon \le M$,

$$|\partial_\xi r_h(\xi) - \partial_\xi r_h(\xi')| \le \sup_{V'} |\partial_\xi^2 r_h||\xi - \xi'| \le C \sup_V |r_h||\xi - \xi'| \le C|r|_\epsilon \epsilon(h)|\xi - \xi'| \le CM\epsilon(h)|\xi - \xi'|.$$

Thus, if h is small enough such that $CM\epsilon(h) < 1$, $\xi \mapsto \partial_\xi r_h(\xi)$ takes its value in $\overline{V'}$ and is a contraction. Then, according to the contraction mapping theorem, there exists a unique $\xi_{c,h,r} \in \overline{V'}$ such that $\partial_\xi r_h(\xi_{c,h,r}) = \xi_{c,h,r}$. This is the unique critical point of $\varphi_{h,r}$ in $\overline{V'}$.

Note that we have $|\xi_{c,h,r}| \le C|r|_\epsilon \epsilon(h)$, where C depends only on ϵ, M, V and V'. Also, the critical value $c_{h,r} \coloneqq \varphi_{h,r}(\xi_{c,h,r})$ satisfies

$$|c_{h,r} - r_h(0)| = \left|\frac{-\xi_{c,h,r}^2}{2} + r_h(\xi_{c,h,r}) - r_h(0)\right| \le \frac{|\xi_{c,h,r}|^2}{4} + |\xi_{c,h,r}| \sup_{V'} |\partial_\xi r_h| \le C\epsilon(h)^2 |r|_\epsilon.$$

Moreover, we have $|\partial_\xi^2 \varphi_{h,r}(\xi_{c,h,r}) + I| = |\partial_\xi^2 r_h(\xi_{c,h,r})| \le C|r|_\epsilon \epsilon(h)$. So, if h is small enough, the critical point $\xi_{c,h,r}$ is nondegenerate.

Step 2: change of variables. Now that we know where the critical point is, and what the critical value of the phase is, we want to change the intergation variables to rewrite $I_{h,r}(u)$ in the form $I_{h,r}(u) = \int e^{-\xi^2/2h} \tilde{u}_h(\xi)\,\mathrm{d}\xi$.

We define $\psi_{h,r}(\xi)$ on V', for h small enough by

$$\psi_{h,r}(\xi + \xi_{c,h,r}) = \xi \left(2\int_0^1 \partial_x^2 \varphi_{h,r}(s\xi)\,\mathrm{d}s\right)^{1/2}.$$

According to Taylor's formula, for every $\xi \in V'$, we have $\varphi_{h,r}(\xi) = c_{h,r} + \psi_{h,r}(\xi)^2/2$.

So, by the change of variables/integration path $\eta = \psi_{h,r}(\xi)$, we have:

$$I_{h,r}(u) = e^{c_{h,r}/h} \int_{\psi_{h,r}(U)} e^{-\eta^2/2h} u_h(\xi(\eta)) \frac{\mathrm{d}\xi}{\mathrm{d}\eta}\,\mathrm{d}\eta = e^{c_{h,r}/h} \int_{\psi_{h,r}(U)} e^{-\eta^2/2h} \tilde{u}_h(\xi(\eta))\,\mathrm{d}\eta, \tag{3.1}$$

[4]We denote $A \Subset B$ for "$\overline{A}$ compact and $\overline{A} \subset B$".

[5]We will frequently use the following standard consequence of the integral Cauchy formula: if f is holomorphic on $D(z_0, r)$, then $|f^{(n)}(z_0)| \le C_{n,r} |f|_{L^\infty(D(z_0,r))}$.

where $\tilde{u}_h(\eta) \coloneqq u_h(\xi(\eta))\,\mathrm{d}\xi/\mathrm{d}\eta$.

Note that according to the definition of $\psi_{h,r}$, for $\xi \in V'$, we have $|\psi_{h,r}(\xi) - \xi| \leq C\epsilon(h)|r|_\epsilon$. Thus, we have for every $\eta \in \psi_{h,r}(V')$, $|\psi_{h,r}^{-1}(\eta) - \eta| \leq C\epsilon(h)|r|_\epsilon$. Thus, $\mathrm{d}\eta/\mathrm{d}\xi = 1 + \mathcal{O}(\epsilon(h)|r|_\epsilon)$.

Step 3: conclusion. So, by the standard saddle point method (Theorem 3.1):

$$\begin{aligned} I_{h,r}(u) &= e^{c_{h,r}/h}\sqrt{2\pi h}\left(\tilde{u}_h(0) + \mathcal{O}\big(h|\tilde{u}_h|_{L^\infty(V)}\big)\right) \\ &= e^{c_{h,r}/h}\sqrt{2\pi h}\left(u_h(0) + \mathcal{O}\big((h+\epsilon(h))|u_h|_{L^\infty(V)}\big)\right) \end{aligned} \tag{3.2}$$

and since $c_{h,r}/h = r_h(0)/h + \mathcal{O}(\epsilon(h)^2/h)$ the Proposition is proved. ∎

3.2. The framework: truncated coherent states. As we said in the introduction, our aim is to prove the lack of null-controllability of several equations, that behave in some sense like the fractional heat equation. However, treating these equations requires more precise estimates than the fractional heat equation does.

To avoid making similar computations several times, we do them in a somewhat general framework. This way, we will be able to treat the other equations (Kolmogorov, etc.) directly.

Hypothesis 3.3. We consider the following domain, constants and functions:

1. let $K > 0$ and $\mathcal{C} = \{\xi \in \mathbb{C}, \Re(\xi) > K, |\Im(\xi)| < K^{-1}\Re(\xi)\}$,
2. let $\xi_0 > 0$ large enough (for instance $\xi_0 = 4(K+1)$),
3. let $\delta > 0$ small enough such that for every $\xi \in \mathbb{R}$ and $x \in \mathbb{R}$, $|\xi - \xi_0| < \delta$ and $|x| < \delta$ implies $\xi + \xi_0 + ix \in \mathcal{C}$,
4. let $\chi \in C_c^\infty(-\delta,\delta)$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on a neighborhood of 0, say $(-\delta_2, \delta_2)$,
5. let X be a topological space, and $0 < h_0 < 1$, and for every $\gamma \in X$ and $0 < h \leq h_0$, let $\rho_{\gamma,h}\colon \mathcal{C} \to \mathbb{C}$ be a holomorphic function,
6. we assume that uniformly in $\gamma \in X$ and $0 < h \leq h_0$, $\rho_{\gamma,h}(\xi) = o(|\xi|)$ in the limit $|\xi| \to +\infty$, $\xi \in \mathcal{C}$,
7. finally, for every $0 < h \leq h_0$, we define

$$\epsilon(h) \coloneqq \sup_{|\xi|<\delta, |x|<\delta, \gamma \in X} h\left|\rho_{\gamma,h}\left(\frac{\xi + \xi_0 + ix}{h}\right)\right|.$$

The goal of the next subsections is to get upper and lower bounds on the following integral, where (v_h) is a family of bounded holomorphic functions on $\mathcal{C}$:

$$I_{\gamma,h,v}(x) = \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-(\xi-\xi_0)^2/2h + ix\xi/h + \rho_{\gamma,h}(\xi/h)}v_h(\xi)\,\mathrm{d}\xi. \tag{3.3}$$

Remark 3.4.
1. The hypothesis 6 implies that $\epsilon(h) \to 0$ as $h \to 0$.
2. For instance, if ρ is independent of γ and h, with $\rho(\xi) = |\xi|^\alpha$ and $0 \leq \alpha < 1$, then we have $C^{-1}h^{1-\alpha} < \epsilon(h) < Ch^{1-\alpha}$ for some $C > 0$.
3. In the applications, we will typically choose $X = [0,T]$ and for $t \in X$, $\rho_{t,h}(\xi) = -t\rho(\xi)$ with some $\rho\colon \mathcal{C} \to \mathbb{C}$ such that $\rho(\xi) = o(|\xi|)$. We will also usually choose $v_h = 1$. In that case, $g_h(t,x) \coloneqq I_{t,h,1}(x)$ is solution of $(\partial_t + \rho(\sqrt{-\Delta}))g_h = 1$ with initial condition $g_{0,h}(x) = \sqrt{2\pi h}\chi(-ih\partial_x - \xi_0)e^{-x^2/2h + ix\xi_0/h}$, which belongs in $L^2(\mathbb{R})$. However, some applications will require a larger parameter space X and $v_h \neq 1$.

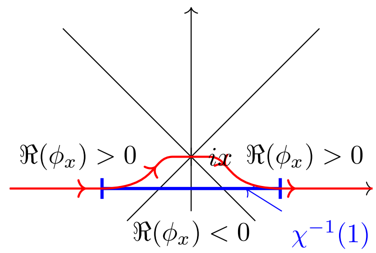

FIGURE 1. *In blue, the interval where $\chi = 1$. The diagonal lines define four sectors; in the left and right ones, $\Re(\varphi_x) > 0$ and in the top and bottom ones, $\Re(\phi_x) < 0$. In red, the path of integration we chose in the integral defining $I_{\gamma,h,v}(x)$ (Eq. 3.3). If $|x|$ is small enough, we choose a path that goes through the saddle point ix, but that stays in $\{\Re(\varphi_x) > 0\}$.*

3.3. Asymptotics for the evolution of coherent states.

PROPOSITION 3.5. *Assuming Hypothesis 3.3, we have uniformly in $\gamma \in X$, $|x|$ small enough, $\gamma \in X$ and $v_h\colon \mathcal{C} \to \mathbb{C}$ holomorphic bounded*

$$
\begin{aligned}
I_{\gamma,h,v}(x) &= \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-(\xi-\xi_0)^2/2h + ix\xi/h + \rho_{\gamma,h}(\xi/h)} v_h(\xi)\, \mathrm{d}\xi \\
&= \sqrt{2\pi h}\, e^{ix\xi_0/h - x^2/2h + \rho_{\gamma,h}\left(\frac{\xi_0+ix}{h}\right) + \mathcal{O}\left(\frac{\epsilon(h)^2}{h}\right)} \Big(v_h\left(\xi_0 + ix\right) + \mathcal{O}\big((h+\epsilon(h))|v_h|_{L^\infty(\mathcal{C})}\big)\Big),
\end{aligned}
$$

in the limit $h \to 0^+$.

Remark 3.6. For most of the applications, we don't care about the term $\rho((\xi_0 + ix)/h)$, apart from the fact it is $\mathcal{O}(\epsilon(h)/h)$. Moreover, we usually have $v_h(\xi) = 1$, or at least $v_h(\xi) \to 1$ as $h \to 0$, uniformly in $\xi \in \mathcal{C}$. Under this condition, Proposition 3.2 implies the slightly less precise asymptotic expansion

$$
I_{\gamma,h,v}(x) = \sqrt{2\pi h}\, e^{ix\xi_0/h - x^2/2h + \mathcal{O}\left(\frac{\epsilon(h)}{h}\right)} \left(1 + o(1)\right),
$$

which will be enough in most cases.

Proof. The idea is that this integral has almost the form of Proposition 3.2. Let us actually rewrite it as such.

Step 1: change of integration path. Note that $\chi(\xi - \xi_0) \equiv 1$ for $\xi_0 - \delta_2 < \xi < \xi_0 + 2\delta_2$, so $\chi(\xi - \xi_0)$ extends holomorphically to $|\Re(\xi) - \xi_0| < \delta_2$ by 1. Moreover, if $\xi \in \mathbb{C}$ and $|\Re(\xi) - \xi_0| < \delta_2$, $|\Im(\xi)| < \delta_2$, then, according to Hypothesis 3.3 item 3, $\xi \in \mathcal{C}$. We deduce that the integrand of $I_{\gamma,h,v}(x)$ is holomorphic on $\{\xi \in \mathbb{C}, |\Re(\xi) - \xi_0| < \delta_2, |\Im(\xi)| < \delta_2\}$.

Thus, we can change the integration path of $I_{\gamma,h,v}(x)$, as long as we modify it only between $\xi_0 - \delta_2$ and $\xi_0 + \delta_2$, and that the modified part of the integration path stays inside $\{|\Re(\xi) - \xi_0| < \delta_2, |\Im(\xi)| < \delta_2\}$.

Let $\xi_c = \xi_0 + ix$ be the critical point of $\varphi_x(\xi) \coloneqq -(\xi - \xi_0)^2/2 + ix\xi$. We choose an integration path Γ parametrized by $\Gamma(t) = t + \xi_0 + ix\chi_2(t)$, where $\chi_2 \in C_c^\infty(-\delta_2, \delta_2)$ with $0 \le \chi_2 \le 1$ and

$\chi_2 \equiv 1$ on a neighborhood of 0 (say $(-\delta_3,\delta_3)$). Then, we have

$$\begin{aligned} I_{\gamma,h,v}(x) &= \int_\Gamma \chi(\Re(z)) e^{\varphi_x(z)/h - \rho_{\gamma,h}(z/h)} v_h(z)\,\mathrm{d}z \\ &= \int_{-\delta_3}^{\delta_3} e^{-t^2/2h - x^2/2h + ix\xi_0/h - \rho_{\gamma,h}\left(\frac{\xi_0+ix+t}{h}\right)} v_h(\xi_0+ix+t)\,\mathrm{d}t + R_{\gamma,h,v}(x), \end{aligned}$$

where we used that for $\delta_3 < t < \delta_3$, $\varphi_x(\Gamma(t)) = -(t+ix)^2/2 + ix(t+\xi_0+ix) = -t^2/2 - x^2/2 + ix\xi_0$, and where $R_{\gamma,h,v}(x)$ is the part of the integral out of $(-\delta_3,\delta_3)$.

Step 2: upper-bound for the remainder. Since $\mathrm{support}(\chi) \subset (-\delta,\delta)$, so that the integrand is 0 for $|t| > \delta$, the remainder $R_{\gamma,h,v}(x)$ is upper-bounded by

$$|R_{\gamma,h,v}(x)| \le 2\delta e^{-\delta_3^2/2h + \epsilon(h)/h} |\chi_2'|_{L^\infty}.$$

where we used the definition of ϵ (Hypothesis 3.3 item 7) to bound $\rho_{\gamma,h}$. Moreover, $\epsilon(h) \to 0$, so we have (for instance)

$$|R_{\gamma,h,v}(x)| \le C e^{-\delta_3^2/4h} |v_h|_{L^\infty}. \tag{3.4}$$

Step 3: asymptotic expansion for the integral in $(-\delta_3,\delta_3)$. To get an asymptotic expansion of the part of the integral between $-\delta_3$ and δ_3, we can apply Proposition 3.2. Indeed, for $0 < h \le h_0$, $\gamma \in X$, $|x| < \delta_3/2$ and ξ in a small enough complex neighborhood U of $[-\delta_3,\delta_3]$, let

$$r_{\gamma,x,h}(\xi) = h\rho_{\gamma,h}\left(\frac{\xi_0+ix+\xi}{h}\right).$$

By definition of $\epsilon(h)$, we have $|r_{\gamma,h,x}| < \epsilon(h)$, or, in other words, $|r_{\gamma,x,h}|_\epsilon \le 1$. For the same parameters, we also define

$$u_{x,h}(\xi) = v_h(\xi_0+ix+\xi).$$

Then, according to Proposition 3.2,

$$\begin{aligned} I_{\gamma,h,v}(x) &= \int_{-\delta_3}^{\delta_3} e^{-x^2/2h + ix\xi_0/h - t^2/2h + r_{\gamma,x,h}(t)/h} u_{x,h}(t)\,\mathrm{d}t + R_{\gamma,h,v}(x) \\ &= \sqrt{2\pi h}\, e^{-x^2/2h + ix\xi_0/h + r_{\gamma,x,h}(0)/h}\Big(u_{x,h}(0) + \mathcal{O}((h+\epsilon(h))|u_{x,h}|_{L^\infty(U)})\Big) + R_{\gamma,h,v}(x) \end{aligned}$$

uniformly in $\gamma \in X$, $|x| < \delta_3/2$ and v_h holomorphic bounded on $\mathcal{C}$.

Step 4: conclusion. With the upper-bound (3.4) on $R_{\gamma,h,v}(x)$, the claimed asymptotic expansion follows. ∎

3.4. Upper bounds for the evolution of coherent states.

We will also need upper bounds for $I_{\gamma,h,v}(x)$ that hold for large x.

PROPOSITION 3.7. *We assume Hypothesis 3.3, except that the item 6 is only assumed to hold* locally *uniformly in $\gamma \in X$ (and uniformly in $0 < h \le h_0$). We define for $\gamma \in X$ and $0 < h \le h_0$*

$$\epsilon_\gamma(x) \coloneqq \sup_{|\xi|<\delta,|x|<\delta} h\Re\left(\rho_{\gamma,h}\left(\frac{\xi+\xi_0+ix}{h}\right)\right).$$

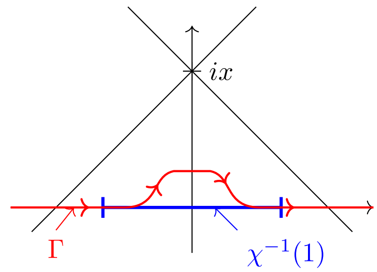

FIGURE 2. *As in Figure 1, the interval where $\chi = 1$ in blue. If x is not too small, we deform a bit the integration path toward ix.*

Let $\eta > 0$. For every $N > 0$, there exist $c, C > 0$ such that for every $|x| > \eta$, $\gamma \in X$ and v_h holomorphic bounded on $\mathcal{C}$,

$$|I_{\gamma,h,v}(x)| \leq \frac{C}{|x|^N} e^{-c/h+\epsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}.$$

Remark 3.8.
1. In the applications, we will typically choose $X = \mathbb{R}_+$ and $\rho_{t,h}(\xi) = -t\rho(\xi)$.
2. For instance, consider the case $X = \mathbb{R}_+$ and $\rho_{t,h}(\xi) = -tz\xi^\alpha$, where $\Re(z) > 0$. This choice is relevant to the equation $(\partial_t + z(-\Delta)^{\alpha/2})f(t,x) = \mathbb{1}_\omega u(t,x)$. If $K > 0$ is large enough, for every $\xi \in \mathcal{C}$, $\Re(z\rho(\xi)) \geq c|\xi|^\alpha$ for some $c > 0$.[6] Then $\epsilon_t(h) \leq -cth^{1-\alpha}$.

Proof. Step 1: integration by parts. First, we integrate by parts to get the decay in x. As in the previous proof, we denote $\varphi_x(\xi) = -(\xi - \xi_0)^2/2 + ix\xi$. We remark that $h\partial_\xi e^{\varphi_x(\xi)/h} = -(\xi - \xi_0 - ix)e^{\varphi_x(\xi)/h}$. Thus, with $L_x = \frac{1}{h}\partial_\xi \frac{1}{\xi - \xi_0 - ix}$, we have

$$I_{\gamma,h,v}(x) = \int_{-\infty}^{+\infty} e^{-\varphi_x(\xi)} L_x^N \left(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}\left(\frac{\xi}{h}\right)} v_h(\xi)\right) \mathrm{d}\xi.$$

Step 2: change of integration path. Next, as in the proof of Proposition 3.5, we can change the integration path between $\xi_0 - \delta_2$ and $\xi_0 + \delta_2$, as long as this modification stays inside $\{|\Re(\xi) - \xi_0| < \delta_2, |\Im(\xi)| < \delta_2\}$. We choose χ_2 as in the proof of Proposition 3.5, i.e. $\chi_2 \in C_c^\infty(-\delta_2, \delta_2)$, $0 \leq \chi_2 \leq 1$, $\chi_2 \equiv 1$ on $(-\delta_3, \delta_3)$. Then we choose the path $\Gamma(t) = t + i\eta_2 \operatorname{sgn}(x)\chi_2(t - \xi_0)$, where $\eta_2 > 0$ is small enough, for instance $\eta_2 = \min(\eta/2, \delta_2/2)$ (see Figure 2).

Step 3: upper-bound for $e^{\varphi_x(\xi)}$. On this path Γ, for every $|x| > \eta$ and $0 < h \leq h_0$,

$$\begin{aligned} |e^{\varphi_x(\Gamma(t))/h}| &= e^{\Re(\varphi_x(\Gamma(t)))} \\ &= e^{(-t^2/2 + \eta_2^2\chi_2^2(t-\xi_0)/2 - |x|\eta_2\chi_2(t-\xi_0))/h} \\ &\leq e^{-t^2/2h - |x|\eta_2\chi_2(t-\xi_0)/2h}, \end{aligned}$$

where we used that $|x| > \eta_2$ and $0 \leq \chi_2 \leq 1$. Thus, for some $c > 0$, we have for every $|x| > \eta_2$ and $t \in \mathbb{R}$

$$|e^{\varphi_x(\Gamma(t))/h}| \leq e^{-c/h}. \tag{3.5}$$

[6] Indeed, if $z = r_0 e^{i\theta_0}$, then $|\theta_0| < \pi/2$. And if $\xi = re^{i\theta}$, then $\Re(z\xi^\alpha) = r_0 r^\alpha \cos(\alpha\theta + \theta_0)$. But if $\xi \in \mathcal{C}$, then $r|\sin(\theta)| \leq rK^{-1}\cos(\theta)$, or $|\theta| \leq \arctan(K^{-1})$. So, if K is large enough, for every $\xi = re^{i\theta} \in \mathcal{C}$, $|\alpha\theta + \theta_0| \leq \pi/2 - \tau$ for some $\tau > 0$. Then, $\Re(z\xi^\alpha) = r_0 r^\alpha \cos(\alpha\theta + \theta_0) \geq r_0 r^\alpha \cos(\pi/2 - \tau) = c|\xi|^\alpha$.

Step 4: upper bound for the rest of the integrand. We claim that there exists $C_N > 0$ such that for every f C^∞ on Γ, for every $|x| > \eta$ and $\xi \in \Gamma$,

$$|L_x^N f(\xi)| \leq \frac{C_N}{|hx|^N} \sum_{k \leq N} |\partial_\xi^k f(\xi)|. \tag{3.6}$$

Indeed, according to Leibniz' rule, for any $k \geq 0$, $\xi \in \Gamma$ and f C^∞ on Γ,

$$\partial_\xi^k L_x f(\xi) = h^{-1} \partial_\xi^{k+1} \frac{f(\xi)}{\xi - \xi_0 - ix} = h^{-1} \sum_{\ell \leq k+1} C_{k,\ell} \frac{\partial_\xi^\ell f(\xi)}{(\xi - \xi_0 - ix)^{k+2-\ell}}.$$

So, reminding that $|x| > \eta$ and $|\Im(\xi)| < \frac{\eta}{2}$

$$|\partial_\xi^k L_x f(\xi)| \leq \frac{C_k}{|hx|} \sum_{\ell \leq k+1} |\partial_\xi^\ell f(\xi)|.$$

By iterating this estimate, we get the upper bound (3.6). Now, choosing $f(\xi) = \chi(\xi - \xi_0) e^{-\rho_{\gamma,h}(\xi/h)} v_h(\xi)$, we get

$$\begin{aligned} |L_x^N(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi))| &\leq \frac{C_N}{|hx|^N} \sum_{k \leq N} \left| \partial_\xi^k \left(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right) \right| \\ &\leq \frac{C_N'}{|hx|^N} \sum_{k \leq N} \left| \partial_\xi^k \left(e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right) \right|. \end{aligned}$$

Moreover, for any f holomorphic on $\mathcal{C}$, and for any $\xi \in \Gamma$ and $r > 0$ such that $D(\xi, r) \subset \mathcal{C}$, the Cauchy integral formula implies that $|\partial_\xi^k f(\xi)| \leq C_r |f|_{L^\infty(D(\xi,r))}$. So,

$$\begin{aligned} |L_x^N(\chi(\xi - \xi_0) e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi))| &\leq \frac{C_N''}{|hx|^N} \left| e^{\rho_{\gamma,h}(\xi/h)} v_h(\xi) \right|_{L^\infty(D(\xi,r))} \\ &\leq \frac{C_N''}{|hx|^N} e^{\epsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}. \end{aligned} \tag{3.7}$$

Step 5: conclusion. Putting together the bounds (3.7) and (3.5), we get

$$|I_{\gamma,h,v}(x)| \leq \frac{C_N}{|x|^N} h^{-N} e^{-c/h + \epsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})},$$

which implies the claimed estimate. ◻

We also have the following upper-bound, which is weaker but valid for any x, even small. We will need it for some applications.

Proposition 3.9. *Under the same hypotheses as Proposition 3.7, there exist $c, C > 0$ such that for every $x \in \mathbb{R}$, $\gamma \in X$ and v_h holomorphic bounded on $\mathcal{C}$,*

$$|I_{\gamma,h,v}(x)| \leq C e^{\epsilon_\gamma(h)/h} |v_h|_{L^\infty(\mathcal{C})}.$$

Proof. It is only the integral triangle inequality. ◻

4. Non-null-controllability of the generalized fractional heat equation.

4.1. The generalized fractional heat equation on the whole real line.

Proof of Theorem 1.4 in the case $\Omega = \mathbb{R}$. Well-posedness. Let us recall that $\inf_{\mathbb{R}_+} \Re(\rho) < +\infty$. So, denoting M this infinimum, we have for every $t \geq 0$, $\sup_{\xi\in\mathbb{R}} |e^{-t\rho(|\xi|)}| \leq e^{tM}$. Thus, for every $t \geq 0$, we can define a linear bounded operator on $L^2(\mathbb{R})$ by

$$\forall f_0 \in L^2(\mathbb{R}),\ \forall x \in \mathbb{R},\ S(t)f_0(x) = \mathcal{F}^{-1}(e^{-t\rho(|\xi|)}\mathcal{F}f_0)(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi - t\rho(|\xi|)}\mathcal{F}f_0(\xi)\,\mathrm{d}\xi.$$

We can see that $S(t)$ is a strongly continuous semigroup of bounded operators on $L^2(\mathbb{R})$. Moreover, the infinitesimal generator of $S(t)$ is $\rho(\sqrt{-\Delta})$. Thus, the equation (1.3) is well-posed, in the sense of semigroups (see for instance [15, Def. 2.36 and Th. 2.37]).

Construction of the counterexample to the observability inequality. We remind that the null-controllability of the generalized fractional heat equation on ω and in time T is equivalent to the following *observability inequality* [15, Th. 2.44]: there exists $C > 0$ such that for every $g_0 \in L^2(\mathbb{R})$, the solution g of

$$\partial_t g + \overline{\rho}(\sqrt{-\Delta})g = 0, \quad g(0,\cdot) = g_0 \tag{4.1}$$

satisfies

$$|g(T,\cdot)|_{L^2(\mathbb{R})} \leq C|g|_{L^2([0,T]\times\omega)}. \tag{4.2}$$

In Hypothesis 3.3 item 1, we choose K and $\mathcal{C}$ to be those of the statement of Theorem 1.4. Let ξ_0, δ and χ as in Hypothesis 3.3 item 2–3. Then, for $h > 0$, we consider $g_{0,h} \in L^2(\mathbb{R})$ defined by

$$g_{0,h}(x) = \sqrt{2\pi h}\,\chi(-ih\partial_x - \xi_0)e^{-x^2/2h + ix\xi_0/h} = h\int_{\mathbb{R}} \chi(h\xi - \xi_0)e^{-(h\xi-\xi_0)^2/2h + ix\xi}\,\mathrm{d}\xi.$$

The solution g_h of the generalized fractional heat equation (4.1) with this initial condition is

$$\begin{aligned} g_h(t,x) &= h\int_{\mathbb{R}} \chi(h\xi - \xi_0)e^{-(h\xi-\xi_0)^2/2h + ix\xi - t\overline{\rho}(\xi)}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-(\xi-\xi_0)^2/2h + ix\xi_0/h - t\overline{\rho}(\xi/h)}\,\mathrm{d}\xi \end{aligned} \tag{4.3}$$

(let us remind that according to Hypothesis 3.3, $\chi(h\xi - \xi_0)$ is zero for $|h\xi - \xi_0| \geq \delta$, and in particular for $\xi < 0$ if δ is chosen small enough, as in Hypothesis 3.3).

Conclusion. In Hypothesis 3.3 item 5, we choose $X = [0,T]$. For $t \in X$ and $h > 0$, we choose $\rho_{t,h}\colon \xi \in \mathcal{C} \mapsto -t\overline{\rho}(\overline{\xi})$. Since ρ is holomorphic on $\mathcal{C}$ with $\rho(\xi) = o(|\xi|)$, so is $\rho_{t,h}$ for every $t \in X$ and $h > 0$. In other words Hypothesis 3.3 item 5–6 are satisfied. Moreover, with the notations of Eq. (3.3), the function g_h given by (4.3) can be writen as $g_h(t,x) = I_{t,h,1}(x)$.

So, according to Proposition 3.5 (or more precisely Remark 3.6), there exists $C, c > 0$ such that for $t \in [0,T]$, x small enough (say $|x| < \eta'$) and $h > 0$ small enough

$$|g_h(t,x)| \geq \frac{1}{2}e^{-x^2/2h - C\epsilon(h)/h}. \tag{4.4}$$

Moreover, according to Proposition 3.7, there exists $C, c > 0$ such that for $t \geq 0$, $|x| > \eta$ and $h > 0$ small enough,

$$g_h(t,x) \leq \frac{C}{|x|^2} e^{-c/h}. \tag{4.5}$$

Thus, we have according to the lower bound (4.4)

$$|g_h(T,\cdot)|_{L^2(\mathbb{R})} \geq |g_h(T,\cdot)|_{L^2(|x|<\eta')} \geq ch^{1/4} e^{-C\epsilon(h)/h} \tag{4.6}$$

and according to the upper bound (4.5),

$$|g_h|^2_{L^2((0,T)\times\omega)} \leq T \int_{|x|>\eta} \frac{C}{x^4} e^{-c/h}\,\mathrm{d}x \leq Ce^{-c/h}. \tag{4.7}$$

and since $\epsilon(h) \to 0$ as $h \to 0$, taking $h \to 0$ disproves the observability inequality. ☐

Remark 4.1. We implicitly looked at the generalized fractional heat equation with complex valued solutions. This means that we proved that there exists an initial condition f_0 of the generalized fractional heat equation that we cannot steer to 0, but this initial condition might not be real valued. In the case where $\rho(\mathbb{R}_+) \subset \mathbb{R}_+$, we might be more interested in real valued solutions. But our results actually implies there exists a real valued initial condition that cannot be steered to 0, for if both the real part $\Re(f_0)$ and the imaginary part $\Im(f_0)$ could be steered to 0, then f_0 itself could be steered to 0. A similar arguments stays valid for the Kolmogorov-type equation.

4.2. The generalized fractional heat equation on the torus. The case of the generalized fractional heat equation on the torus is a bit different because we are not dealing with integrals, but sums. Therefore, tools like the saddle point method do not seem to be of much use. Nonetheless, with a trick, we can deduce the theorem on the torus from the theorem on the whole real line.

Proof of Theorem 1.4 in the case $\Omega = \mathbb{T}$. The basic idea is the trick of the proof of Poisson summation formula, namely the fact that the Fourier coefficients of a function of the form $g_{0\text{per}}(x) = \sum_{k\in\mathbb{Z}} g_0(x + 2\pi k)$ are the values of the Fourier transform of g_0 evaluated at the integers (up to a multiplication by $\sqrt{2\pi}$).

So, let $g_h \in C^\infty(\mathbb{R})$ be as in the previous section. Since the Fourier transform of $g_h(t,\cdot)$ is C^∞ with compact support,[7] $g_h(t,x)$ decays faster than any polynomials as $|x| \to \infty$ and we can define $g_{h\text{per}}(t,x) = \sum_{k\in\mathbb{Z}} g_h(t, x+2\pi k)$. According to the trick described before, $c_n(g_{h\text{per}}(t,\cdot)) = (2\pi)^{-1/2}\mathcal{F}(g_h)(t,\cdot)(n)$. But, by definition of g_h as the solution of the rotated fraction heat equation, $\mathcal{F}(g_h)(t,\cdot)(\xi) = \mathcal{F}(g_h)(0,\cdot)(\xi)e^{-t\overline{\rho}(|\xi|)}$, so, using the trick again:

$$c_n(g_{h\text{per}}(t,\cdot)) = c_n(g_{h\text{per}}(0,\cdot))e^{-t\overline{\rho}(|n|)}. \tag{4.8}$$

So $g_{h\text{per}}$ is a solution to the generalized fractional heat equation (4.1) on the torus. Now we prove that the terms for $k \neq 0$ are negligible. Indeed, we have by definition of $g_{h\text{per}}$

$$|g_{h\text{per}}(T,\cdot)|_{L^2(\mathbb{T})} = \Big|\sum_{k\in\mathbb{Z}} g_h(T, \cdot + 2\pi k)\Big|_{L^2(\mathbb{T})} \tag{4.9}$$

[7]We added the cutoff function χ just to localize the Fourier transform away from the singularity of $|\xi|^\alpha$ at $\xi = 0$.

and by singling out to term for $k = 0$ and thanks to the triangle inequality

$$|g_{h\,\mathrm{per}}(T,\cdot)|_{L^2(\mathbb{T})} \geq |g_h(T,\cdot)|_{L^2(-\pi,\pi)} - \sum_{k\neq 0} |g_h(T,\cdot)|_{L^2((2k-1)\pi,(2k+1)\pi)} \tag{4.10}$$

and thanks to the pointwise estimates on g_h (Eq. (4.5) and (4.4))

$$|g_{h\,\mathrm{per}}(T,\cdot)|_{L^2(\mathbb{T})} \geq ch^{1/4}e^{-C\epsilon(h)/h} - \sum_{k\neq 0} \frac{C}{k^2}e^{-c/h} \geq ch^{1/4}e^{-C\epsilon(h)/h} - Ce^{-c/h}. \tag{4.11}$$

In the same spirit, we have thanks to the triangle inequality, and identifying $\omega = \mathbb{T} \setminus [-\epsilon, \epsilon]$ with $(-\pi, \pi) \setminus [-\epsilon, \epsilon] \subset \mathbb{R}$

$$|g_{h\,\mathrm{per}}|_{L^2([0,T]\times\omega)} \leq \sum_{k\in\mathbb{Z}} |g_h|_{L^2([0,T]\times(\omega+2\pi k))} \tag{4.12}$$

and according to the estimate (4.5),

$$|g_{h\,\mathrm{per}}|_{L^2([0,T]\times\omega)} = \mathcal{O}(e^{-c/h}). \tag{4.13}$$

Taking $h \to 0^+$ disproves the observability inequality (4.2) and proves the Theorem. □

4.3. Higher dimension. Theorem 1.4 can be generalized to take into account the case $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$. Indeed, the Propositions of section 3 are still valid in higher dimension. The computations are carried essentially the same way, only with the added technicalities of the higher dimension, for instance:

- in Proposition 3.2, U and V are assumed to be open, bounded and convex subset of $\mathbb{R}^d$ and $\mathbb{C}^d$ respectively,
- in the proof of Proposition 3.2, the change of variables of step 2 is given by a Morse Lemma with parameter, in the spirit of [21, Lemma C.6.1],
- in Hypothesis 3.3 χ is chosen to be $C_c^\infty(B(0,\delta))$ (open ball in $\mathbb{R}^d$),
- $\rho(|\xi|)$ has to be replaced by $\rho\Big[\big(\sum_i \xi_i^2\big)^{1/2}\Big]$ (i.e. what happens to be holomorphic in $\xi \in \mathbb{C}^d$ and that is equal to $\rho(|\xi|)$ if $\xi \in \mathbb{R}^d$),
- in all the complex integrals that follows, we integrate against $\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_d$,
- also, the power of $2\pi h$ in front of the asymptotic expansion is $(2\pi h)^{d/2}$ (but it does not matter).

Then, the construction of the counterexample to the observability inequality if $\Omega = \mathbb{R}^d$ is the same. For the case $\Omega = \mathbb{R}^d \times \mathbb{T}^{d'}$, we first consider a counter example in $\mathbb{R}^{d+d'}$, and we periodize the last d' components with the method of subsection 4.2.

5. Non-null-controllability of the Kolmogorov equation.

5.1. Introduction. Now, we look at the Kolmogorov equation (1.2). As for the fractional heat equation, the null-controllability of the Kolmogorov equation (1.2) is equivalent to the existence of $C > 0$ such that for every solution g of[8]

$$(\partial_t - v^2\partial_x - \partial_v^2)g(t,x,v) = 0 \quad t \in (0,T), (x,v) \in \Omega \tag{5.1}$$

[8]Note that this is the adjoint of the Kolmogorov equation where we reversed the time.

with Dirichlet boundary conditions if $\Omega_v = (-1,1)$,

$$|g(T,\cdot)|_{L^2(\Omega)} \leq C|g|_{L^2((0,T)\times\omega)}. \tag{5.2}$$

As hinted in the introduction, we look for counterexamples of the observability inequality among solutions of the adjoint of the Kolmogorov equation (5.1) of the form $g(t,x,v) = \int_{\mathbb{R}} a(\xi)e^{ix\xi}g_\xi(v)e^{-\lambda_\xi t}\,\mathrm{d}\xi$, where $g_\xi(v)$ is the first eigenfunction of $-\partial_v^2 - i\xi v^2$ and λ_ξ its associated eigenvalue. Let us remind that[9] $\lambda_\xi = \sqrt{-i\xi}$ if $\Omega_v = \mathbb{R}$, and is close to $\sqrt{-i\xi}$ if $\Omega_v = (-1,1)$.

We remark that apart from the $g_\xi(v)$ term, those solutions have the same form as solutions of the rotated fractional heat equation $(\partial_t + \sqrt{-i}(-\Delta)^{1/4})g = 0$. So, the strategy is to prove the same estimates we proved for the rotated fractional heat equation, but with some uniformity in the parameter v. Since the computations are essentially the same, we only tell what we need to care about in comparison with the rotated fractional heat equation, but we do not give the full details of the computations again.

5.2. The Kolmogorov equation with unbounded velocity.

Proof of Theorem 1.5 with $\Omega_x = \Omega_v = \mathbb{R}$. In the case $\Omega_v = \mathbb{R}$, the first eigenfunction of $-\partial_v^2 - i\xi v^2$ is $g_\xi(v) = e^{-\sqrt{-i\xi}v^2/2}$ with eigenvalue $\lambda_\xi = \sqrt{-i\xi}$. Without loss of generality, we may assume $\omega_x = \mathbb{R}\setminus[-\eta,\eta]$ (let us remind that $\omega = \omega_x \times \mathbb{R}$).

Thus, we consider the function $g_h\colon \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\begin{aligned} g_h(t,x,v) &= h\int_{\mathbb{R}} \chi(h\xi-\xi_0)e^{ix\xi-(h\xi-\xi_0)^2/2h-\sqrt{-i\xi}(v^2/2+t)}\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}} \chi(\xi-\xi_0)e^{ix\xi/h-(\xi-\xi_0)^2/2h-\sqrt{-i\xi/h}(v^2/2+t)}\,\mathrm{d}\xi. \end{aligned} \tag{5.3}$$

Since every $(t,x,v) \mapsto e^{ix\xi-\sqrt{-i\xi}(t+v^2/2)}$ is a generalized solution to the Kolmogorov equation (5.1), the function g_h is also solution to the Kolmogorov equation. We also remark that $g_h(0,\cdot,\cdot) \in L^2(\mathbb{R}^2)$. Notice that these solutions are of the form $\tilde g_h(t+v^2/2,x)$, where $\tilde g_h$ is solution to the "rotated fractional heat equation" $(\partial_t + \sqrt{-i}(-\Delta_x)^{1/4})\tilde g_h(t,x) = 0$. Thus we have asymptotic expansion on $\tilde g_h$ similar to (4.4) and (4.5).

For $t \geq 0$, $h > 0$ and $\Re(\xi) > 0$, we define $\rho_{t,h}(\xi) = -t\sqrt{-i\xi}$. Thus, with any $K > 0$, and with $X = [0, T+1/2]$, Hypothesis 3.3 holds. Thus, with the notations of Hypothesis 3.3 and Eq. (3.3), we have for h small enough

$$g_h(t,x,v) = I_{t+v^2/2,h,1}(x).$$

Moreover, still with the notations of Hypothesis 3.3, we have for some $C > 0$ and for every $0 < h < 1$, $C^{-1}h^{1/2} \leq \epsilon(h) \leq Ch^{1/2}$.

Thus, according to Proposition 3.5 (or more precisely Remark 3.6), there exist $C, c > 0$ such that for every $h > 0$ small enough, $0 \leq t \leq T$, $|x|$ small enough (say $|x| < \eta'$) and $|v| < 1$

$$|g_h(t,x,v)| \geq ce^{-x^2/2h-Ch^{-1/2}}. \tag{5.4}$$

Moreover, assuming K large enough and choosing $X = \mathbb{R}_+$, according to Proposition 3.7 (see also Remark 3.8), there exist $C, c > 0$ such that for every $h > 0$ small enough, $|x| > \eta$, $t \geq 0$ and $v \in \mathbb{R}$

$$|g_h(t,x,v)| \leq C|x|^{-2}e^{-c/h-c(t+v^2/2)h^{-1/2}}, \tag{5.5}$$

[9] We choose the branch of the square root with positive real part.

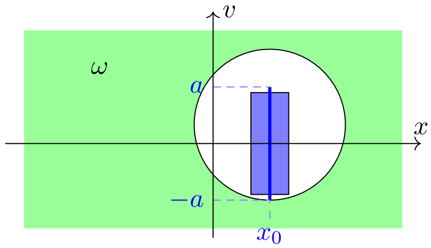

FIGURE 3. *In green, the control domain ω. If there is a vertical line, symmetrical with respect to $\{v=0\}$, that does not intersect $\bar\omega$ (in dark blue), for every $a'<a$, there exists a rectangle of the form $\{|x-x_0|<b, -a'<v<a'\}$ that does not intersect $\bar\omega$ (in lighter blue).*

So, integrating these estimates, we have

$$|g_h|_{L^2([0,T]\times\omega)} \leq Ce^{-c/h} \tag{5.6}$$

and

$$|g_h(T,\cdot,\cdot)|_{L^2(\Omega)} \geq |g_h(T,\cdot,\cdot)|_{L^2(|x|<\eta',|v|<1)} \geq ce^{-Ch^{-1/2}}. \tag{5.7}$$

Taking again $h\to 0$ disproves the observability inequality and proves the Theorem. ◻

For the Kolmogorov equation with $\Omega_x=\mathbb{T}$ and $\Omega_v=\mathbb{R}$, we define $g_{h\,\mathrm{per}}(t,x,v)=\sum_{k\in\mathbb{Z}} g_h(t,x+2\pi k,v)$, and as in subsection 4.2, all but the term for $k=0$ are $\mathcal{O}(e^{-c/h})$. We let the careful reader work out the details.

5.3. The Kolmogorov equation with non-rectangular control domain.

Proof of Theorem 1.6 in the case $\Omega_x=\Omega_v=\mathbb{R}$. If $0<a'<a$, there exists $b>0$ such that the rectangle $R=\{|x-x_0|<b, |v|<a'\}$ does not intersect $\overline{\omega}$ (see Figure 3). Since the equation is invariant by translation in the x direction, we may assume without loss of generality that $x_0=0$.

We will use the same functions g_h as in the previous proof. But while we used only the terms of order h^{-1} in the exponent of estimate of Proposition 3.5, we will now use the next term. More precisely, according to Proposition 3.5 with $X=[0,T+a^2/2]$ and $\rho_{t,h}(\xi)=-t\sqrt{-i\xi}$, we have uniformly in $|x|$ small enough, $0\leq t\leq T$ and $|v|<a$:

$$g_h(t,x,v)=I_{t+v^2/2,h,1}(x)=\sqrt{2\pi h}\,e^{\phi(t,x,v)/h}\big(1+\mathcal{O}_h(\sqrt{h})\big) \tag{5.8}$$

with

$$\phi(t,x,v)=ix\xi_0-\frac{x^2}{2}-\sqrt{-i\xi_0-x}\left(t+\frac{v^2}{2}\right)h^{1/2}+\mathcal{O}_h(h). \tag{5.9}$$

The idea is that, when computing $\int_\Omega |g_h(T,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v$, the dominant part of this integral is around $x=v=0$, and when computing $\int_{[0,T]\times\omega}|g_h(t,x,v)|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v$, the dominant part is around $t=0, x=x_0=0$ and $v=a$ or $-a$. So, noting $c_0=\Re(\sqrt{-i\xi_0})>0$ and ignoring the error terms for the moment, we have

$$\int_\Omega |g_h(T,x,v)|^2\,\mathrm{d}x\,\mathrm{d}v \approx 2\pi h\int_{\substack{|x|<\epsilon\\|v|<\epsilon}} e^{-x^2/h-c_0(2T+v^2)/\sqrt{h}}\,\mathrm{d}x\,\mathrm{d}v \approx Ch^{7/4}e^{-2Tc_0/\sqrt{h}} \tag{5.10}$$

and

$$\int_{[0,T]\times\omega} |g_h(t,x,v)|^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \approx 2\pi h \int_{\substack{|x|<\epsilon \\ a<|v|<a+\epsilon \\ t<\epsilon}} e^{-x^2/h - c_0(2t+v^2)/\sqrt{h}} \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v \approx Ch^{5/2}e^{-c_0 a^2/\sqrt{h}}. \tag{5.11}$$

So, if $2T < a^2$, the observability inequality cannot hold. Now, let us rigorously prove this. Let $\epsilon > 0$ and $T = a'^2/2 - \epsilon$. We have $\Re(\sqrt{-i\xi_0 - x}) = c_0 + \mathcal{O}_x(x)$. So, for x small enough, say, $|x| < \delta_{\xi_0}$, we have

$$c_0 - \epsilon \leq \Re\left(\sqrt{-i\xi_0 - x}\right) \leq c_0 + \epsilon. \tag{5.12}$$

So, we have locally uniformly in $|x| < \delta_{\xi_0}$, $t \geq 0$ and $v \in \mathbb{R}$:

$$\Re(\phi(t,x,v)) \geq -\frac{x^2}{2} - (c_0+\epsilon)\left(t + \frac{v^2}{2}\right)h^{1/2} - \mathcal{O}_h(h) \tag{5.13}$$

and

$$\Re(\phi(t,x,v)) \leq -\frac{x^2}{2} - (c_0-\epsilon)\left(t + \frac{v^2}{2}\right)h^{1/2} + \mathcal{O}_h(h). \tag{5.14}$$

Now, let us get a lower bound for the left-hand side of the observability inequality (5.2). We have:

$$|g_h(T,\cdot,\cdot)|^2_{L^2(\Omega)} \geq |g_h(T,\cdot,\cdot)|^2_{L^2(|x|<b,|v|<a')} \tag{5.15}$$

and thanks to the asymptotic of Eq. (5.8):

$$|g_h(T,\cdot,\cdot)|^2_{L^2(\Omega)} \geq 2\pi h \int_{\substack{|x|<b \\ |v|<a'}} e^{2\Re(\phi(T,x,v))/h} \,\mathrm{d}x\,\mathrm{d}v\big(1 + \mathcal{O}(\sqrt{h})\big) \tag{5.16}$$

and with the lower bound above (Eq. (5.13)):

$$|g_h(T,\cdot,\cdot)|^2_{L^2(\Omega)} \geq 2\pi h e^{\mathcal{O}_h(1)} \int_{\substack{|x|<b \\ |v|<a'}} e^{-x^2/h - (c_0+\epsilon)(2T+v^2)/\sqrt{h}} \,\mathrm{d}x\,\mathrm{d}v\big(1 + \mathcal{O}(\sqrt{h})\big). \tag{5.17}$$

The integral in x is

$$\int_{|x|<b} e^{-x^2/h} \,\mathrm{d}x = \sqrt{\pi h} + \mathcal{O}(e^{-c/h}) \tag{5.18}$$

while the integral in v is

$$\int_{|v|<a'} e^{-(c_0+\epsilon)v^2/\sqrt{h}} \,\mathrm{d}v = \sqrt{\frac{\pi}{c_0+\epsilon}}h^{1/4} + \mathcal{O}\big(e^{-c/\sqrt{h}}\big). \tag{5.19}$$

So, we have

$$|g_h(T,\cdot,\cdot)|^2_{L^2(\mathbb{R}^2)} \geq c\frac{2\pi^2}{\sqrt{c_0+\epsilon}} h^{7/4} e^{-(c_0+\epsilon)2T/\sqrt{h}}\big(1+\mathcal{O}(\sqrt{h})\big) \tag{5.20}$$

and for h small enough:

$$|g_h(T,\cdot,\cdot)|^2_{L^2(\Omega)} \geq ch^{7/4}e^{-(c_0+\epsilon)2T/\sqrt{h}}. \tag{5.21}$$

Now, let us bound the right-hand side of the observability inequality (5.2). Let us remind that ω is a subset of $\Omega \setminus \{|x| < b, |v| < a'\}$. Let $a'' > a'$, that will be chosen large enough afterwards. We define

$$\omega_0 = \{|x| < \delta_{\xi_0}, a' < |v| < a''\} \tag{5.22}$$

$$\omega_1 = \{|x| \geq \delta_{\xi_0}\} \tag{5.23}$$

$$\omega_2 = \{|x| < \delta_{\xi_0}, |v| > a''\}. \tag{5.24}$$

With these definitions, if $\delta_{\xi_0} < b$, we have $\omega \subset \omega_0 \cup \omega_1 \cup \omega_2$. So,

$$|g_h|^2_{L^2([0,T]\times\omega)} \leq |g_h|^2_{L^2([0,T]\times\omega_0)} + |g_h|^2_{L^2([0,T]\times\omega_1)} + |g_h|^2_{L^2([0,T]\times\omega_2)} \tag{5.25}$$

First, according to Proposition 3.9, we have for every $t\geq 0, v\in\mathbb{R}$ and $x\in\mathbb{R}$,

$$|g_h(t,x,v)| \leq Ce^{-c'(t+v^2/2)h^{-1/2}} \tag{5.26}$$

So, integrating this estimate, we have

$$|g_h|^2_{L^2([0,T]\times\omega_2)} = \mathcal{O}\left(\int_{|v|\geq a''} e^{-c'v^2/\sqrt{h}}\,\mathrm{d}v\right) = \mathcal{O}\left(e^{-c'a''^2/\sqrt{h}}\right). \tag{5.27}$$

We choose a'' large enough so that $c'a''^2 > (c_0-\epsilon)a'^2$. That way, we have

$$|g_h|^2_{L^2([0,T]\times\omega_2)} = \mathcal{O}\left(e^{-(c_0-\epsilon)a'^2/\sqrt{h}}\right). \tag{5.28}$$

We have already seen in subsection 5.2 that

$$|g_h|^2_{L^2([0,T]\times\omega_1)} = \mathcal{O}(e^{-c/h}). \tag{5.29}$$

Finally, thanks to Eq. (5.8) with upper bound (5.14), we have uniformly in $0\leq t\leq T$, $|x| < \delta_{\xi_0}$ and $a' < |v| < a''$

$$|g_h(t,x,v)|^2 \leq 2\pi h e^{-x^2/h-(c_0-\epsilon)(2t+v^2)/\sqrt{h}+\mathcal{O}_h(1)}. \tag{5.30}$$

So, we have

$$|g_h|^2_{L^2([0,T]\times\omega_0)} = \mathcal{O}\left(\int_{a'<|v|<a''} e^{-(c_0-\epsilon)v^2/\sqrt{h}}\,\mathrm{d}v\right) = \mathcal{O}\left(e^{-(c_0-\epsilon)a'^2/\sqrt{h}}\right). \tag{5.31}$$

So, putting the three upper bounds (5.28) (5.29) and (5.31) together, we have

$$|g_h|^2_{L^2([0,T]\times\omega)} = \mathcal{O}\left(e^{-(c_0-\epsilon)a'^2/\sqrt{h}}\right). \tag{5.32}$$

Let us assume that $(c_0-\epsilon)a'^2 > 2T(c_0+\epsilon)$. Then, considering the previous upper bound (5.32) and the lower bound (5.21), and taking $h \to 0$ disproves the observability inequality. So, the Kolmogorov equation is not null-controllable in time $T < \frac{c_0-\epsilon}{c_0+\epsilon}a'^2/2$. This is true for every $a' < a$ and $\epsilon > 0$, so the Kolmogorov equation is not null-controllable in time $T < a^2/2$. ∎

The case $\Omega_x = \mathbb{T}$, $\Omega_v = \mathbb{R}$ is similar. We look at $g_{h\,\mathrm{per}}(t,x,v) = \sum_{k\in\mathbb{Z}} g_h(t,x+2\pi k,v)$. In this sum, as in subsection 4.2, only the term for $k=0$ matters, as the other are $\mathcal{O}(e^{-c/h})$.

5.4. The Kolmogorov equation with bounded velocity. To treat the Kolomogorov equation with $\Omega_v = (-1,1)$, we need some information on the first eigenfunction g_ξ of $-\partial_v^2 - i\xi v^2$ with Dirichlet boundary conditions on $(-1,1)$, and with associated eigenvalue $\lambda_\xi = \sqrt{-i\xi}+\rho_\xi$. Moreover, as we will use Theorems 3.5–3.9, we also need some analycity in ξ. We will denote $\tilde g_{\tilde\xi}$ the first[10] eigenfunction of $-\partial_v^2 + (\tilde\xi v)^2$, and $\tilde\lambda_{\tilde\xi} = \tilde\xi + \tilde\rho_{\tilde\xi}$ the associated eigenvalue, so that, with $\tilde\xi = \sqrt{-i\xi}$, we have $g_\xi = \tilde g_{\tilde\xi}$ and $\rho_\xi = \tilde\rho_{\tilde\xi}$, when this is defined.

In an article on the Grushin equation [22, Section 4] we proved that $\tilde\rho_{\tilde\xi}$ and $\tilde g_{\tilde\xi}$ exist if $\Re(\tilde\xi) > 0$ and $|\tilde\xi| > r(|\arg(\tilde\xi)|)$ for some non-decreasing function $r : (0,\pi/2) \to \mathbb{R}_+$. We also proved the next two theorems.

THEOREM 5.1 (Theorem 22 and Remark 23 of [22]). *Let $0<\theta<\pi/2$. With $\tilde\rho_{\tilde\xi}$ defined above, we have*

$$\tilde\rho_{\tilde\xi} \sim \frac{4}{\sqrt{\pi}}\tilde\xi^{3/2}e^{-\tilde\xi}$$

in the limit $|\tilde\xi| \to \infty, |\arg(\tilde\xi)| < \theta$.

PROPOSITION 5.2 (Proposition 25 of [22]). *Let $\tilde g_{\tilde\xi}$ be defined above and normalized by $\tilde g_{\tilde\xi}(0) = 1$ (instead of $|\tilde g_{\tilde\xi}|_{L^2} = 1$). Let $0<\theta<\pi/2$ and $\epsilon > 0$. We have for all $v \in (-1,1)$ and $|\tilde\xi| > r(\theta)$, $|\arg(\tilde\xi)| < \theta$:*

$$|e^{(1-\epsilon)\tilde\xi v^2/2}\tilde g_{\tilde\xi}(v)| \le C_{\epsilon,\theta}.$$

Theorem 5.1 gives us all we need to know on the eigenvalue, while Proposition 5.2 gives us an upper bound on the eigenfunction. We will also need the following lower bound, that we prove in Appendix B.

PROPOSITION 5.3. *Let $0<\theta<\pi/2$ and $\epsilon>0$. We normalize $\tilde g_{\tilde\xi}$ again by $\tilde g_{\tilde\xi}(0) = 1$ and define $\tilde u_{\tilde\xi}(v) = e^{\tilde\xi v^2/2}\tilde g_{\tilde\xi}(v)$. Then $\tilde u_{\tilde\xi}(v)$ converges exponentially fast to 1, as $|\tilde\xi| \to \infty$, $|\arg(\tilde\xi)| < \theta$, this convergence being uniform in $|v| < 1-\epsilon$.*

With this, we know all we need to adapt the proof of the non-null-controllability of the Kolmogorov equation with $\Omega_v = \mathbb{R}$ to the case of $\Omega_v = (-1,1)$.

Proof of Theorem 1.5 with $\Omega_v = (-1,1)$. We start with the case $\Omega_x = \mathbb{R}$.

Step 1: construction of the counterexample to the observability inequality. The counterexample we build to the observability inequality (5.2) is basically the same as in the case $\Omega_v = \mathbb{R}$, only with

[10]"First" in the sense that it is the analytic continuation in $\tilde\xi$ of the first eigenfunction of $-\partial_v^2 + (\tilde\xi v)^2$ for $\tilde\xi \in \mathbb{R}_+$, assuming it exists.

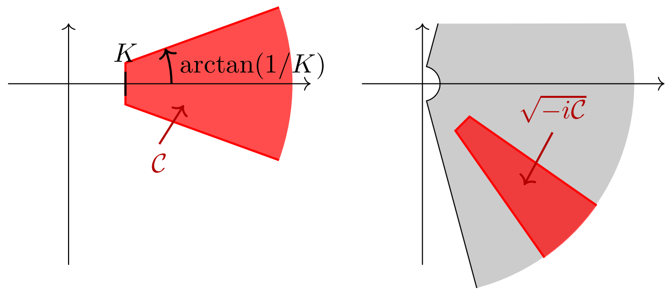

FIGURE 4. *Left figure: in red, shape of $\mathcal{C}$ for some K. Right figure: if $\tilde{\xi}$ is in the gray domain, the eigenvalue $\tilde{\rho}_{\tilde{\xi}}$ and the eigenfunction $\tilde{g}_{\tilde{\xi}}$ of $-\partial_v^2 + (\tilde{\xi}v)^2$ are defined. Thus, the eigenvalue λ_ξ and eigenfunction g_ξ of $-\partial_v^2 + i\xi v^2$ are defined for $\xi \in \mathcal{C}$ if $\sqrt{-i\mathcal{C}}$ lies inside the gray domain.*

the added corrections to the eigenvalues and eigenfunctions. We define $g_h(t,x,v)$ for $t \geq 0$, $x \in \mathbb{R}$, $v \in (-1,1)$ and $h > 0$ small enough by:

$$g_h(t,x,v) = \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-ix\xi/h - (\xi-\xi_0)^2/2h - \lambda_{\xi/h} t} g_{\xi/h}(v)\,\mathrm{d}\xi, \tag{5.33}$$

where $\xi_0 > 0$ and $\chi \in C_c^\infty(\mathbb{R})$ are chosen as follows.

First note that according to the discussion at the top of this subsection, λ_ξ and g_ξ are defined and holomorphic with respect to $\xi \in \mathbb{C}$ such that $|\arg(\xi)| < 3\pi/8$ (for instance) and $|\xi|$ large enough. Let then $K > 0$ be large enough so that for any $\xi \in \mathcal{C} := \{\Re(\xi) > K, |\Im(\xi)| < K^{-1}\Re(\xi)\}$, λ_ξ and g_ξ are defined and holomorphic with respect to $\xi \in \mathcal{C}$. Finally, let $\xi_0 > 0$ and χ as in Hypothesis 3.3 (see Figure 4). With these choices, g_h is well-defined for $0 < h \leq 1$, $t \geq 0$, $x \in \mathbb{R}$ and $v \in (-1,1)$.

We remark that each function $(t,x,v) \mapsto e^{-ix\xi - \lambda_\xi t} g_\xi(v)$ is solution to the Kolmogorov equation (5.1). So g_h is solution of the Kolmogorov equation.

Step 2: estimates on g_h. Note that Theorem 5.1 and Propositions 5.2 and 5.3 (with the choice $\epsilon = 1/2$) translate respectively into the estimates:

$$|e^{-t\rho_\xi} - 1| \leq Ce^{-c\sqrt{|\xi|}} \qquad \text{for } \xi \in \mathcal{C} \text{ and } 0 \leq t \leq T \tag{5.34}$$

$$|e^{\sqrt{-i\xi}v^2/4} g_\xi(v)| \leq C \qquad \text{for } \xi \in \mathcal{C} \text{ and } |v| < 1 \tag{5.35}$$

$$|e^{\sqrt{-i\xi}v^2/2} g_\xi(v) - 1| \leq Ce^{-c\sqrt{|\xi|}} \qquad \text{for } \xi \in \mathcal{C} \text{ and } |v| < 1/2. \tag{5.36}$$

for some $C, c > 0$.

Step 2a: lower bound on g_h. We want to write $g_h(t,x,v)$ in the form of Eq. (3.3). Let $X = [0, T+1/2]$, and for $t \in X$, $0 < h \leq 1$ and $\xi \in \mathcal{C}$, let $\rho_{t,h}(\xi) = -t\sqrt{-i\xi}$. Finally, for $0 < h \leq 1$, $t \geq 0$, $v \in (-1,1)$ and $\xi \in \mathcal{C}$, let

$$\delta_{h,t,v}(\xi) := e^{\sqrt{-i\xi/h}v^2/2} g_{\xi/h}(v) e^{-t\rho_{\xi/h}}. \tag{5.37}$$

Then, according to the definition of g_h (Eq. (5.33)),

$$g_h(t,x,v) = \int_{\mathbb{R}} \chi(\xi - \xi_0) e^{-ix\xi/h - (\xi-\xi_0)^2/2h - \sqrt{-i\xi/h}(t+v^2/2)} \delta_{h,t,v}(\xi)\,\mathrm{d}\xi = I_{t+v^2/2,h,\delta}(x). \tag{5.38}$$

Moreover, according to estimates Eq. (5.34) and (5.36), we have, for some $C, c > 0$:

$$|\delta_{h,t,v}(\xi) - 1| \leq Ce^{-ch^{-1/2}} \quad \text{for } \xi \in \mathcal{C},\ |v| < 1/2 \text{ and } 0 < h \leq 1. \tag{5.39}$$

So, according to Proposition 3.5, there exists $C, c > 0$ such that for every $t \in [0, T]$, $|v| < 1/2$, x small enough and h small enough,

$$|g_h(t, x, v)| \geq ce^{-x^2/2h - Ch^{-1/2}}. \tag{5.40}$$

Step 2b: upper bound on g_h. The estimate (5.36) does not extend up to the boundary. Thus, we have to use the less precise upper bound (5.35). To this end, we define $\tilde{\delta}_{h,t,v}(\xi) = e^{-\sqrt{-i\xi/h}v^2/2}\delta_{h,t,v}(\xi)$. Then, according to the definition of g_h (Eq. (5.33)) and δ (Eq. (5.37)),

$$g_h(t, x, v) = \int_{\mathbb{R}} \chi(\xi - \xi_0)e^{-ix\xi/h - (\xi-\xi_0)^2/2h - \sqrt{-i\xi/h}(t+v^2/4)}\tilde{\delta}_{h,t,v}(\xi)\,\mathrm{d}\xi = I_{t+v^2/4,h,\tilde{\delta}}(x). \tag{5.41}$$

Moreover, according to estimates (5.34) and (5.35), there exist $C, c > 0$ such that

$$|\tilde{\delta}_{h,t,v}(\xi)| \leq C \quad \text{for } \xi \in \mathcal{C},\ |v| < 1 \text{ and } 0 < h \leq 1. \tag{5.42}$$

So, according to Proposition 3.7, there exist $C, c > 0$ such that for every $t \in [0, T]$, $|v| < 1$, $|x| > \eta$ (where we assume without loss of generality $\omega = \{|x| > \eta\} \times (-1, 1)$) and h small enough,

$$|g_h(t, x, v)| \leq \frac{C}{|x|^2}e^{-c/h}. \tag{5.43}$$

Step 3: conclusion. From this point on, the proof is the same as in subsection 5.2: integrating the estimates (5.40) and (5.43) proves that g_h is a counterexample to the observability inequality (5.2) in the case $\Omega = \mathbb{R} \times (-1, 1)$ and $\omega = \omega_x \times (-1, 1)$.

In the case $\Omega_x = \mathbb{T}$, we again look at the periodic version of g_h, that is $g_{h\text{per}}(t, x, v) = \sum_{k\in\mathbb{Z}} g_h(t, x + 2\pi k, v)$. As in subsection 4.1 (and 5.2), $g_{h\text{per}}$ is a solution to the Kolmogorov equation, and it is a counterexample to the observability inequality. ◻

The proof of Theorem 1.6 in the case $\Omega_v = (-1, 1)$ is similar to the case $\Omega_v = \mathbb{R}$ with the adaptation of the previous proof. Let us just sketch it.

We choose $a' < a$ and $b > 0$ as in subsection 5.3. We consider the functions g_h of the previous proof (Eq. (5.33)).

We compute the next order in the estimate (5.40). With Proposition 3.5, we can prove that locally uniformly in $v \in (-1, 1)$, $|x|$ small enough and $t > 0$,

$$g_h(t, x, v) = \sqrt{2\pi h}\,e^{ix\xi_0/h - x^2/2h - \sqrt{\xi_0 + ix}(t+v^2/2)/\sqrt{h} + \mathcal{O}_h(1)}\left(1 + \mathcal{O}(\sqrt{h})\right). \tag{5.44}$$

Also, thanks to Proposition 3.7, we prove that uniformly in $|x| > b$, $t > 0$ and $v \in (-1, 1)$,

$$g_h(t, x, v) = \mathcal{O}(|x|^{-2}e^{-c/h}). \tag{5.45}$$

We choose $a' < a'' < 1$ and we define ω_0, ω_1 and ω_2 as in equations (5.22), (5.23) and (5.24) (the δ_{ξ_0} is the same in the cases $\Omega_v = (-1, 1)$ and $\Omega_v = \mathbb{R}$).

With the estimate (5.44), we can prove an estimate similar to the lower bound (5.21). We can also prove an upper bound similar to (5.31). With the estimate (5.45), we can prove an estimate similar to (5.29). And with the help of Proposition 5.2 to manage the terms for $|v| > a''$, we can prove an upper bound similar to (5.28). The rest of the proof is a copy-paste.

Appendix A. Other equations. In this appendix, we explain how we can use the method of section 4 to prove the lack of null-controllability for some other equations.

A.1. Fractional Schrödinger equations. Let $0 \le \alpha < 1$. If we consider $\rho(\xi) = i\xi^\alpha$ (defined e.g. for $\Re(\xi) \ge 0$), the hypotheses of Theorem 1.4 hold. Thus, we have:

Corollary A.1. *Let $0 \le \alpha < 1$. Let $T > 0$ and ω be a strict open subset of $\mathbb{R}$. The fractional Schrödinger equation $(\partial_t + i(-\Delta)^{\alpha/2})f(t,x) = \mathbb{1}_\omega u(t,x)$, $t \ge 0$, $x \in \mathbb{R}$ is not null-controllable on ω in time T.*

Since $i(-\Delta)^{\alpha/2}$ generates a strongly-continuous *group* of bounded operators on $L^2(\mathbb{T})$, it seems likely that this corollary can be extended to any riemannian manifold (and not only $\mathbb{R}^d \times \mathbb{T}^d$).[11] But this is outside the scope of this article. Also, we conjecture that the threshold $\alpha < 1$ is optimal. Indeed, it seems that if ω satisfies the Geometric Control Condition of Bardos, Lebeau and Rauch [1], then the methods of [24] could be used to prove exact controllability in any time if $\alpha > 1$ (see [19] for the case $\alpha \ge 2$) and with a minimal time if $\alpha = 1$ (better known as “the half-wave equation”), but this is again outside the scope of this article. See also [10] for a variant of the fractional Schrödinger equation.

A.2. Another Kolmogorov-type equation. Our method can also be used to prove that the Kolmogorov-type equation $(\partial_t - \partial_v^2 + v\partial_x)f(t,x,v) = \mathbb{1}_\omega u(t,x,v)$ is not null-controllable on vertical bands (notice that is is Eq. (1.2) where we replaced v^2 by v).

Theorem A.2. *Let $\Omega = (0,+\infty) \times \Omega_x$, and $\Omega_x = \mathbb{R}$ or $\mathbb{T}$. Let ω_x be a strict open subset of Ω_x and $\omega = \omega_x \times (0,+\infty)$, and let $T > 0$. Then the equation*

$$
\begin{aligned}
(\partial_t + v\partial_x - \partial_v^2)f(t,x,v) &= \mathbb{1}_\omega u(t,x,v) && t \in [0,T], (x,v) \in \Omega \\
f(t,x,0) &= 0 && t \in [0,T], x \in \Omega_x \\
f(0,x,v) &= f_0(x,v) && (x,v) \in \Omega
\end{aligned}
\tag{A.1}
$$

is not null-controllable on ω in time T.

Sketch of the proof. We consider Ai the standard Airy function (see for instance [30, Ch. 9]). Let $-\mu_0$ the first zero of Ai. We denote $\lambda_0 = e^{i\pi/3}\mu_0$. For $\xi > 0$, let $u_\xi\colon \mathbb{R} \to \mathbb{C}$ defined by $u_\xi(v) = \mathrm{Ai}(\xi^{1/3}e^{-i\pi/6}v - \mu_0)$. Using the ODE satisfied by Ai ([30, §9.2(i)]), we see that $(-\partial_v^2 - i\xi v)u_\xi = \xi^{2/3}\lambda_0 u_\xi$. Moreover, $u_\xi(0) = 0$, and according to the asymptotic expansion satisfied by Ai ([30, §9.7ii]), u_ξ decays exponentially at ∞, as well as its derivatives. So u_ξ is an eigenfunction of $-\partial_v^2 - i\xi v$ on $(0,+\infty)$ with Dirichlet boundary condition at $v = 0$.

Let $\xi_0 > 0$ and $\chi \in C_c^\infty(-\xi_0,\xi_0)$. For $h > 0$ we consider the function $g_h\colon \mathbb{R}_+ \times \mathbb{R} \times (0,+\infty) \to \mathbb{C}$ defined by

$$
\begin{aligned}
g_h(t,x,v) &= h\int_{\mathbb{R}_+} \chi(h\xi - \xi_0)e^{-(h\xi-\xi_0)^2/2h + ix\xi - t\lambda_0\xi^{2/3}} u_\xi(v)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}_+} \chi(\xi - \xi_0)e^{-(\xi-\xi_0)^2/2h + ix\xi/h - t\lambda_0\xi^{2/3}h^{-2/3}} u_{\xi/h}(v)\,\mathrm{d}\xi.
\end{aligned}
\tag{A.2}
$$

Since u_ξ is an eigenfunction of $-\partial_v^2 - i\xi v$, g_h is solution to $(\partial_t - v\partial_x - \partial_v^2)g_h = 0$. Moreover, using the asymptotic expansion of Ai [30, §9.7(ii)], we have uniformly in $v > 1$, in the limit $|\xi| \to +\infty$, $|\arg(\xi)| < \pi/2$ (for instance)

$$
u_\xi(v) = \exp\left(-\frac{2}{3}(e^{-i\pi/6}\xi^{1/3}v - \mu_0)^{3/2}\right)\tilde{u}_\xi(v) \quad \text{with} \quad \tilde{u}_\xi(v) = C\xi^{-1/12}v^{-1/4}(1 + \mathcal{O}(\xi^{-1/2})).
$$

[11] The fact that $i(-\Delta)^{\alpha/2}$ generates a strongly continuous group also implies that null-controllability is equivalent to *exact*-controllability [15, Th. 2.41].

Thus, we can rewrite Eq. (A.2) as

$$g_h(t,x,v) = \int_{\mathbb{R}_+} \chi(\xi-\xi_0) e^{-(\xi-\xi_0)^2/2h + ix\xi/h + \rho_{t,v,h}(\xi/h)} \tilde{u}_{\xi/h}(v)\,\mathrm{d}\xi \tag{A.3}$$

with

$$\rho_{t,v,h}(\xi) = -t\lambda_0\xi^{2/3} - \frac{2}{3}(e^{-i\pi/6}\xi^{1/3}v - \mu_0)^{3/2}.$$

Thus, $g_h(t,x,v)$ can be written in the form (3.3). Moreover, if we choose $K>0$, then we can choose $\xi_0>0$ and $\chi\in C_c^\infty(\mathbb{R})$ such that Hypothesis 3.3 holds with $X=\{(t,v), 0\le t\le T, 1\le v\le 2\}$. Then, Proposition 3.5 can be used to prove the lower-bound

$$|g_h(t,x,v)| \ge ce^{-x^2/2h - Ch^{-2/3}}, \quad t\in[0,T], |x| \text{ small enough}, v\in[1,2].$$

To get an upper-bound, we choose K large enough in Hypothesis 3.3 so that $|\tilde{u}_\xi(v)|\le C$ for some $C>0$ and every $\xi\in\mathcal{C}$ and $v\in(0,+\infty)$. Then, with the choice $X=[0,T]\times(0,+\infty)$ in Hypothesis 3.3, the Proposition 3.7 can be used to prove the upper-bound

$$|g_h(t,x,v)| \le \frac{C}{|x|^2} e^{-c/h - cv^{3/2}}.$$

As for the Kolmogorov equation (1.2), these two estimates prove that the observability inequality associated with the control problem (A.1) does not hold if $\Omega_x=\mathbb{R}$. For the case $\Omega_x=\mathbb{T}$, we periodize the solutions as in subsection 4.2. ◻

We refer to subsection 1.3.4 for references related to the equation (1.2). It seems Theorem A.2 could be extended to the case $\Omega=\Omega_x\times(a,b)$, as it is only a perturbation of the case $\Omega=\Omega_x\times(0,+\infty)$.

A.3. Improved Boussinesq equation. Finally, we mention another equation whose null-controllability can be treated with our method.

Proposition A.3. *Let $\Omega=\mathbb{R}$ or $\mathbb{T}$. Let ω be a strict open subset of Ω and let $T>0$. The equation*

$$(\partial_t^2 - \partial_x^2 - \partial_x^2\partial_t^2) f(t,x) = \mathbb{1}_\omega u(t,x), \quad t\in[0,T], x\in\Omega \tag{A.4}$$

is not null-controllable on ω in time T.

This equation has been studied by Cerpa and Crépeau [14], where it is called «improved Boussinesq equation». They prove that, when posed on $\Omega=(0,1)$, it is not null-controllable with *boundary control* at $x=1$. They also prove that if $\Omega=\mathbb{T}$, it is is null-controllable with *moving* internal control on $\omega+ct$[12] if the speed c is large enough. But while their results suggest the improved Boussinesq equation is not null-controllable with (non-moving) internal control, they do not prove it. Here, we provide a proof of this fact.

Sketch of the proof. Let $\xi_0>0$ and $\chi\in C_c^\infty$ to be chosen later. For $\xi\in\mathbb{R}$ we define $\lambda_\xi=\xi^2(1+\xi^2)^{-1}$. For $h>0$, we consider

$$g_h(t,x) = h\int_{\mathbb{R}} \chi(h\xi-\xi_0) e^{-(h\xi-\xi_0)^2/2h + ix\xi - it\sqrt{\lambda_\xi}}\,\mathrm{d}\xi.$$

[12] In other words, the right-hand side is $\mathbb{1}_\omega(x-ct)u(t,x)$ instead of $\mathbb{1}_\omega(x)u(t,x)$.

Elementary computations prove that g_h is solution of $(\partial_t^2 - \partial_x^2 - \partial_x^2\partial_t^2)g_h(t,x) = 0$ (it is related to the fact that this equation can be rewritten as $(\partial_t^2 - (I-\partial_x^2)^{-1}\partial_x^2)g_h(t,x) = 0$, and to the spectral analysis of this operator, see [14]).

With the notation of Eq. (3.3), $g_h(t,x) = I_{t,h,1}(x)$ with ρ independant of (t,h) defined by $\rho(\xi) = it\sqrt{\lambda_\xi}$. We can choose $K>0$, $\xi_0>0$, and $\chi \in C_c^\infty$ such that Hypothesis 3.3 holds with $X = [0,T]$.

Then, with Proposition 3.5 and Proposition 3.7, we prove that $(g_h)_{h>0}$ is a counterexample to the observability inequality associated to the control problem (A.4) in the case $\Omega = \mathbb{R}$. In the case $\Omega = \mathbb{T}$, we periodize g_h as in subsection 4.2. ◻

Appendix B. Precise estimation of the eigenfunctions. To prove Proposition 5.3, we will need the following theorem, which is a special case[13] of Theorem 18 in [22].

THEOREM B.1. *Let $\mathcal{S}$ be the space of holomorphic function on the domain $\Omega = \{\Re(z) > 1\}$ with sub-exponential growth at infinity, i.e. $\gamma \in \mathcal{S}$ if and only if for all $\epsilon > 0$, $p_\epsilon(\gamma) = \sup_{\Re(z)>1} |\gamma(z)e^{-\epsilon|z|}| < +\infty$. We endow $\mathcal{S}$ with the seminorms family $(p_\epsilon)_{\epsilon>0}$.*

Let γ in $\mathcal{S}$ and let H_γ be the operator on polynomials with a double root at zero, defined by:

$$H_\gamma\left(\sum_{n>1} a_n z^n\right) = \sum_{n>1} \gamma(n) a_n z^n.$$

Let E be an bounded subset of $\mathbb{C}$, star shaped with respect to 0. Let U be a neighborhood of $\bar{E}$. Then there exists $C>0$ such that for all polynomials f with a double root at 0:

$$|H_\gamma(f)|_{L^\infty(E)} \leq C|f|_{L^\infty(U)}. \tag{B.1}$$

Moreover, the constant C above can be chosen continuously in $\gamma \in \mathcal{S}$.

Note that according to the estimate of the previous Theorem B.1, and assuming U is star-shaped with respect to 0, the operators H_γ extend by density to every holomorphic function[14] on U (with a double zero at 0). So, we will apply this estimate (B.1) on entire functions (with a double zero at 0).

Proof of Proposition 5.3. The proof is made by writing $\tilde{u}_{\tilde{\xi}}(v)$ as the power series $\tilde{u}_{\tilde{\xi}}(v) = \sum \tilde{u}_{\tilde{\xi},2n} v^{2n}$, and showing that the coefficients $\tilde{u}_{\tilde{\xi},n}$ of this power series are of the form $\tilde{u}_{\tilde{\xi},2n} = \tilde{\rho}_{\tilde{\xi}}\gamma_{\tilde{\xi}}(n)\tilde{\xi}^n/n!$ for $n \geq 1$, with $\tilde{\rho}_{\tilde{\xi}}$ defined at the beginning of subsection 5.4, so that with the notation of Theorem B.1:

$$\tilde{u}_{\tilde{\xi}}(v) = 1 + \tilde{\rho}_{\tilde{\xi}} H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2} - 1)(v) \tag{B.2}$$

Then, Theorem B.1 will allow us to conclude.

Let us write $\tilde{u}_{\tilde{\xi}}(v) = \sum_{n=0}^{+\infty} \tilde{u}_{\tilde{\xi},n} v^n$. Since $\tilde{u}_{\tilde{\xi}}$ satisfies the Cauchy problem $-\tilde{u}_{\tilde{\xi}}'' + 2\tilde{\xi}v\tilde{u}_{\tilde{\xi}}' - \tilde{\rho}_{\tilde{\xi}}\tilde{u}_{\tilde{\xi}} = 0$ with initial conditions[15] $\tilde{u}_{\tilde{\xi}}(0) = 1$, $\tilde{u}_{\tilde{\xi}}'(0) = 0$, we have $\tilde{u}_{\tilde{\xi},0} = 1$, $\tilde{u}_{\tilde{\xi},2n+1} = 0$ and

$$\tilde{u}_{\tilde{\xi},n+2} = \frac{2n\tilde{\xi} - \tilde{\rho}_{\tilde{\xi}}}{(n+1)(n+2)}\tilde{u}_{\tilde{\xi},n} \tag{B.3}$$

[13] In the reference, the Theorem is stated with an open (bounded star-shaped) domain U instead of a arbitrary (bounded star-shaped) subset E of $\mathbb{C}$, but we can set $U = E^\delta$, and apply the Theorem as stated in the reference to get $|H_\gamma(f)|_{L^\infty(E)} \leq C_\delta |f|_{L^\infty(E^{2\delta})}$.

[14] According to Runge's theorem [31, Theorem 13.9], the polynomials are dense in the space of holomorphic functions on U with the topology of the convergence on every compact.

[15] Here we use the fact that $\tilde{u}_{\tilde{\xi}}$ is even when $\tilde{\xi}$ is real positive, which is well-known from Sturm-Liouville's theory.

so, by induction, for $n \geq 1$

$$\tilde{u}_{\tilde{\xi},2n} = -\frac{\tilde{\rho}_{\tilde{\xi}}}{2}\frac{(4\tilde{\xi})^{n-1}(n-1)!}{(2n)!}\prod_{k=1}^{n-1}\left(1-\frac{\tilde{\rho}_{\tilde{\xi}}}{4\tilde{\xi}k}\right). \tag{B.4}$$

So, by defining

$$\gamma_{\tilde{\xi}}(n) = -\frac{1}{8\tilde{\xi}n}\times\frac{4^n(n!)^2}{(2n)!}\times\prod_{k=1}^{n-1}\left(1-\frac{\tilde{\rho}_{\tilde{\xi}}}{4\tilde{\xi}k}\right) \tag{B.5}$$

we have $\tilde{u}_{\tilde{\xi},2n} = \tilde{\rho}_{\tilde{\xi}}\gamma_{\tilde{\xi}}(n)\tilde{\xi}^n/n!$. So, $\tilde{u}_{\tilde{\xi}}(v) = 1+\tilde{\rho}_{\tilde{\xi}}\sum_{n\geq 1}\gamma_{\tilde{\xi}}(n)\frac{1}{n!}(v^2\tilde{\xi})^n$. Assuming that $\gamma_{\tilde{\xi}}$ is in $\mathcal{S}$, this is exactly the equation (B.2) we were claiming.

Well, let us actually prove that $\gamma_{\tilde{\xi}}$ is in the space $\mathcal{S}$ defined in Theorem B.1, i.e. that we can extend $n \mapsto \gamma_{\tilde{\xi}}(n)$ to a holomorphic function on $\Omega = \{\Re(z) > 1\}$ with subexponential growth. This is obvious for the term $-1/(8\tilde{\xi}n)$. The term $4^n(n!)^2/(2n)!$ can be extended to Ω with Euler's Gamma function, and Stirling's approximation gives us the subexponential growth (actually an equivalent in $\sqrt{\pi z}$). The product term is a tiny bit more tricky to extend to non-integer values. We define it with the following formula, which is inspired by [29], and where we have set $\alpha = -\tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$:

$$\delta_{\tilde{\xi}}(z) = \prod_{k=1}^{+\infty}\frac{1+\frac{\alpha}{k}}{1+\frac{\alpha}{k+z-1}}. \tag{B.6}$$

This product converges if $|\alpha| < 1/2$ and $\Re(z) > 1$. And if n is integer, $\delta_{\tilde{\xi}}(n)$ is a telescopic product, and we have $\delta_{\tilde{\xi}}(n) = \prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right)$. Moreover, $\delta_{\tilde{\xi}}$ is holomorphic on Ω. We also claim that there exists $c, C > 0$ such that if $|\alpha| < 1/2$ and $\Re(z) > 1$, $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$. The proof of this claim is just a few basic computations, and we postpone it after the end of the proof at hand.

Since $\alpha = \tilde{\rho}_{\tilde{\xi}}/4\tilde{\xi}$, according to Theorem 5.1, $|\alpha| < 1/2$ as soon as $|\arg(\tilde{\xi})| < \theta$ and $|\tilde{\xi}|$ is large enough, say $|\tilde{\xi}| > M$ (depending on θ). Then, according to the claim, the term $\delta_{\tilde{\xi}}(z)$ has subexponential growth in Ω, and since it is holomorphic, it is in $\mathcal{S}$. Moreover, this estimate also proves that $(\delta_{\tilde{\xi}})_{|\alpha|<1/2}$ is a bounded family of $\mathcal{S}$.

So $(\gamma_{\tilde{\xi}})$ is a bounded family of $\mathcal{S}$ for $|\arg(\tilde{\xi})| < \theta$ and $|\tilde{\xi}| > M$. So, according to Theorem B.1 and the following remark, for any neighborhood U of $[-1+\epsilon, 1-\epsilon]$ that is star-shaped with respect to 0, there exists $C > 0$ such that for all $|\tilde{\xi}| > M$ with $|\arg(\tilde{\xi})| < \theta$ and for every $v \in (-1+\epsilon, 1-\epsilon)$:

$$\left|H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2}-1)(v)\right| \leq C(1+|e^{\tilde{\xi}v^2}|_{L^\infty(U)}) \tag{B.7}$$

and if we choose U to be small enough, we have $|H_{\gamma_{\tilde{\xi}}}(e^{\tilde{\xi}v^2}-1)(v)| \leq C'|e^{(1-\delta)\tilde{\xi}}|$. Finally, thanks to equation (B.2) and Theorem 5.1, we have

$$|\tilde{u}_{\tilde{\xi}}(v) - 1| \leq C_\delta|\tilde{\xi}|^{3/2}|e^{-\delta\tilde{\xi}}|. \tag{B.8}$$

□

Proof of the claim that $|\delta_{\tilde{\xi}}(z)| \leq C|z|^c$. We first write

$$\delta_{\tilde{\xi}}(z) = \exp\left(\sum_{k=1}^{+\infty}\ln\left(1+\frac{\alpha}{k}\right)-\ln\left(1+\frac{\alpha}{k+z-1}\right)\right). \tag{B.9}$$

Let us also remind that we assume $|\alpha| < 1/2$ and $\Re(z) > 1$, so that for $k \in \mathbb{N}^*$ $|\alpha/k| < 1/2$ and $|\alpha/(k+z-1)| < 1/2$. We denote $k_0 = \lfloor |z| \rfloor$, and we separate the sum into two parts:

$$S_{\leq k_0} = \sum_{k=1}^{k_0} \ln\left(1 + \frac{\alpha}{k}\right) - \ln\left(1 + \frac{\alpha}{k+z-1}\right) \qquad S_{>k_0} = \sum_{k=k_0+1}^{+\infty} \ln\left(1 + \frac{\alpha}{k}\right) - \ln\left(1 + \frac{\alpha}{k+z-1}\right)$$

About the part of a sum for $k \leq k_0$, we have thanks to the triangle inequality and the fact that for $|x| < 1/2$, $|\ln(1+x)| \leq c|x|$:

$$|S_{\leq k_0}| \leq 2c|\alpha| \sum_{k=1}^{k_0} \frac{1}{k} \leq 2c|\alpha|(\ln(k_0) + C') \leq 2c|\alpha|(\ln(|z|) + C'), \tag{B.10}$$

where we used the relation between the harmonic sum and the logarithm and the fact that $k_0 = \lfloor |z| \rfloor$. About the rest of the sum, we have by writing $\ln(1+b) - \ln(1+a) = \int_a^b \frac{\mathrm{d}x}{1+x}$,

$$|S_{>k_0}| \leq \sum_{k=k_0+1}^{+\infty} \left| \int_{\alpha/k}^{\alpha/(k+z-1)} \frac{\mathrm{d}x}{1+x} \right| \leq \sum_{k=k_0+1}^{+\infty} 2 \left| \frac{\alpha}{k} - \frac{\alpha}{k+z-1} \right| \leq 2|\alpha(z-1)| \sum_{k=k_0+1}^{+\infty} \frac{1}{k^2}, \tag{B.11}$$

where we used the fact that for $x \in [\frac{\alpha}{k}, \frac{\alpha}{k+z-1}]$, $|\frac{1}{1+x}| \leq 2$. By comparing this sum with an integral,

$$|S_{>k_0}| \leq 2|\alpha(z-1)| \int_{k_0}^{+\infty} \frac{\mathrm{d}x}{x^2} \leq 2|\alpha| \frac{|z-1|}{k_0} \leq C''|\alpha|, \tag{B.12}$$

where we again used that $k_0 = \lfloor |z| \rfloor$. Summing the two inequalities (B.10) and (B.12), and plugging this into equation (B.9) proves the claim. ∎

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