

An analytical closed-form solution for free vibration of stepped circular/annular Mindlin functionally graded plate

M. Derakhshani ^{a*}, Sh. Hosseini-Hashemi ^b, M. Fadaee ^c

^a *M.Sc. Student of Mechanical Engineering, School of Mechanical Engineering, Iran University of Science and Technology, Tehran, Iran.*

^b *Professor of Mechanical Engineering, School of Mechanical Engineering, Iran University of Science and Technology, Tehran, Iran.*

^c *Assistant Professor of Mechanical Engineering, School of Mechanical Engineering, Qom University of Technology, Qom, Iran.*

* *m.derakhshany87@gmail.com*

Abstract

An exact solution based on a unique procedure is presented for free vibration of stepped circular and annular functionally graded (FG) plates via first-order shear deformation plate theory of Mindlin. A power-law distribution of the volume fraction of the components is considered for the Young's Modulus and Poisson's ratio of the studied FG plate. Free vibration of the plate is solved by introducing some new potential functions and the use of separation of variables method. Finally, several comparisons of the developed model were presented with the FEA analysis, to demonstrate the accuracy of the proposed exact procedure. The effect of the geometrical parameters such as step thickness ratios and step locations on the natural frequencies of FG plates is also investigated.

Keywords: Free vibration; Stepped circular plate; Functionally graded material; Mindlin theory.

1. Introduction

Numerous studies have been performed on free vibration of circular and annular plates with variable thickness. A summary of works on free vibration of thin circular and annular plates with

variable thickness was done by Leissa, most of which were developed based on the classical plate theory (CPT). One of the problems associated with using this theory is that it is not able to model the expected behavior for moderately thick plates. To resolve such problem, some models have been developed by researchers based on the theory of moderately thick and thick plates including the first shear deformation theory (FSDT), the third order shear deformation theory (TSDT) and the 3D elasticity, in which the effects of transverse shear deformation and rotary inertia are considered [1-4].

In recent years, functionally graded materials (FGM) have gained so many attentions as special composites with material properties that vary continuously through their thickness. This continuous change in material properties results in eliminating discontinuities of stresses, high resistance to temperature gradients, reduction in residual and thermal stresses, high wear resistance, and an increase in strength to weight ratio. Circular and annular FG plates have been reviewed by many researchers due to the multiple applications they can be used for. Efraim and Eisenberger, Tajeddini, Hosseini-Hashemi, and Gupta are the most famous researchers who have studied the vibration of FG circular/annular plates [5-8]. It has been found that there are very limited studies performed on the circular/annular FG plates with variables thickness and to the best of the authors' knowledge, there is no study for exact closed-form solutions of vibration analysis of stepped thickness circular/annular Mindlin FG plate.

In this study, an exact solution is presented for free vibration of stepped thickness moderately thick circular and annular FG plates. By use of some new auxiliary and potential functions along with a unique theoretical approach [7], the equations of motion are exactly solved. The accuracy of the current analytical approach is validated by comparing the obtained results of the proposed model with finite element analysis (FEA). Furthermore, the effect of the various parameters such as step thickness ratios, and step locations on the natural frequencies of FG plates is evaluated. More details about the current study could be found in [9].

2. Mathematical formulation

Consider an annular functionally graded plate of radius r_n and thickness h_n consists of n steps in which the radius and thickness of i^{th} annular segment is r_i and h_i respectively, as shown in **Figure 1**. The plate geometry and dimensions are defined in an orthogonal cylindrical coordinate system (r, θ, z) . In this study, the properties of the plate are assumed to vary through the plate thickness with a power-law distribution of the volume fractions of two materials. The top surface of the first segment ($z = h_1 / 2$), which assumed here the thickest part of the plate, is metal-rich while

the bottom surface of the same segment ($z = -h_1/2$) is ceramic-rich. By considering this assumption, Young's modulus and mass density are vary through the plate thickness by the following relations

$$E(z) = (E_m - E_c)V_f(z) + E_c \quad \rho(z) = (\rho_m - \rho_c)V_f(z) + \rho_c \quad (1a-b)$$

where

$$V_f(z) = \left(\frac{z}{h_1} + \frac{1}{2}\right)^g \quad (2)$$

in which the subscripts m and c represent the metallic and ceramic constituents, respectively. V_f shows the volume fraction and g is the power-law index which takes only non-negative values. Poisson's ratio ν is taken as 0.3 throughout the analysis.

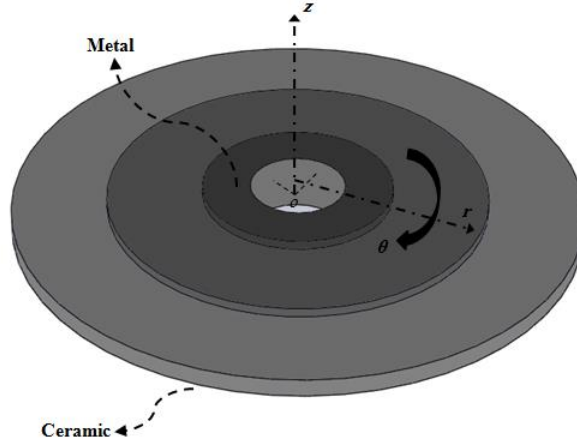


Figure 1. Geometry of a stepped annular FG plate

A stepped annular Mindlin plate with n steps can be divided to n annular plates. An annular Mindlin plate of radius r_i and thickness h_i , in which i refers to the i^{th} segment of the stepped annular FG plate as shown in **Figure 1**, is considered in this model. According to the FSDT, the displacement field is used for the i^{th} segment of the stepped annular Mindlin FG plate as follows

$$u^i(r, \theta, z, t) = u_0^i(r, \theta, t) + z\psi_r^i(r, \theta, t) \quad (3a)$$

$$v^i(r, \theta, z, t) = v_0^i(r, \theta, t) + z\psi_\theta^i(r, \theta, t) \quad (3b)$$

$$w^i(r, \theta, z, t) = w_0^i(r, \theta, t) = w^i(r, \theta, t) \quad (3c)$$

where u^i , v^i and w^i denote the displacements in r , θ and z directions, respectively. u_0^i and v_0^i denote the in-plane displacements of mid-plane in radial and circumferential directions. Moreover, ψ_r^i and ψ_θ^i show the slope rotations in r - z and θ - z planes at $z=0$ for i^{th} segment of the plate, respectively.

The exact free vibration of circular/annular Mindlin FG plate has been studied by the first author [7], recently. In this study, the same analytical approach is used to solve the free vibration of circular/annular moderately thick FG plates. For generality and convenience in deriving mathematical formulations, the following non-dimensional terms are defined:

$$R = \frac{r}{r_n}, \quad Z_i = \frac{z}{h_i}, \quad \delta_i = \frac{h_i}{r_n}, \quad \tau_i = \frac{h_i}{h_n} \quad (4a-d)$$

For harmonic motion, the displacement fields in dimensionless forms are taken as

$$\bar{u}^i(R, \theta, Z_i) = \frac{u^i(r, \theta, z, t)}{h_i} e^{-j\alpha t}, \quad \bar{v}^i(R, \theta, Z_i) = \frac{v^i(r, \theta, z, t)}{h_i} e^{-j\alpha t} \quad (5a, b)$$

$$\bar{w}^i(R, \theta) = \frac{w^i(r, \theta, t)}{r_n} e^{-j\alpha t}, \quad \bar{\psi}_r^i = \psi_r^i e^{-j\alpha t}, \quad \bar{\psi}_\theta^i = \psi_\theta^i e^{-j\alpha t} \quad (5c-e)$$

where

$$\bar{u}^i(R, \theta, Z_i) = \bar{u}_0^i + Z \bar{\psi}_r^i, \quad \bar{v}^i(R, \theta, Z_i) = \bar{v}_0^i + Z \bar{\psi}_\theta^i, \quad \bar{w}^i(R, \theta) = \bar{w}^i \quad (6a-c)$$

By substituting the stress-strain relations in polar coordinate [4] into stress resultants, which are obtained by the first author in [7] based on the FSDT, stress resultants in dimensionless forms for i^{th} segment of the stepped circular FG plate are obtained as

$$\bar{N}_k^i = \frac{N_k^i}{E_c h_i} e^{-j\alpha t}, \quad \bar{M}_k^i = \frac{M_k^i}{E_c h_i^2} e^{-j\alpha t}, \quad \bar{Q}_k^i = \frac{Q_k^i}{E_c h_i} e^{-j\alpha t} \quad k = r, \theta, r\theta \quad (7a-c)$$

in which

$$(\bar{N}_r^i, \bar{N}_\theta^i, \bar{N}_{r\theta}^i) = \int_{-h_i/2}^{h_i/2} (\sigma_{rr}^i, \sigma_{\theta\theta}^i, \sigma_{r\theta}^i) dz \quad (8a)$$

$$(\bar{M}_r^i, \bar{M}_\theta^i, \bar{M}_{r\theta}^i) = \int_{-h_i/2}^{h_i/2} (\sigma_{rr}^i, \sigma_{\theta\theta}^i, \sigma_{r\theta}^i) z dz \quad (8b)$$

$$(\bar{Q}_r^i, \bar{Q}_\theta^i) = \int_{-h_i/2}^{h_i/2} (\sigma_{rz}^i, \sigma_{\theta z}^i) dz \quad (8c)$$

By substituting Eqs. (7a-c) into equations of motion obtained in [7], the final dimensionless form of equations of motion are obtained as follows

$$\begin{aligned} & \bar{K}_1 \left[\left(\frac{\partial^2 \bar{u}_0^i}{\partial R^2} + \frac{\partial \bar{u}_0^i}{R \partial R} - \frac{\bar{u}_0^i}{R^2} + \frac{\partial^2 \bar{v}_0^i}{R \partial R \partial \theta} - \frac{\partial \bar{v}_0^i}{R^2 \partial \theta} \right) + \frac{1-\nu}{2} \left(\frac{\partial^2 \bar{u}_0^i}{R^2 \partial \theta^2} - \frac{\partial \bar{v}_0^i}{R^2 \partial \theta} - \frac{\partial \bar{v}_0^i}{R \partial R \partial \theta} \right) \right] \\ & + \bar{K}_2 \left[\left(\frac{\partial^2 \bar{\psi}_r^i}{\partial R^2} + \frac{\partial \bar{\psi}_r^i}{R \partial R} - \frac{\bar{\psi}_r^i}{R^2} + \frac{\partial^2 \bar{\psi}_\theta^i}{R \partial R \partial \theta} - \frac{\partial \bar{\psi}_\theta^i}{R^2 \partial \theta} \right) + \frac{1-\nu}{2} \left(\frac{\partial^2 \bar{\psi}_r^i}{R^2 \partial \theta^2} - \frac{\partial \bar{\psi}_\theta^i}{R^2 \partial \theta} - \frac{\partial^2 \bar{\psi}_\theta^i}{R^2 \partial R \partial \theta} \right) \right] \\ & = -S_1^i \lambda_1^2 (\bar{I}_1^i \bar{u}_0^i + \bar{I}_2^i \bar{\psi}_r^i) \end{aligned} \quad (9a)$$

$$\begin{aligned}
& \bar{K}_1^i \left[\frac{\partial \bar{u}_0^i}{R^2 \partial \theta} + \frac{\partial^2 \bar{u}_0^i}{R \partial R \partial \theta} + \frac{\partial^2 \bar{v}_0^i}{R^2 \partial \theta^2} + \frac{1-\nu}{2} \left(-\frac{\partial^2 \bar{u}_0^i}{R \partial R \partial \theta} + \frac{\partial \bar{u}_0^i}{R^2 \partial \theta} - \frac{\bar{v}_0^i}{R^2} + \frac{\partial \bar{v}_0^i}{R \partial R} + \frac{\partial^2 \bar{v}_0^i}{\partial R^2} \right) \right] \\
& + \bar{K}_2^i \left[\frac{\partial \bar{\psi}_r^i}{R^2 \partial \theta} + \frac{\partial^2 \bar{\psi}_r^i}{R \partial R \partial \theta} + \frac{\partial^2 \bar{\psi}_\theta^i}{R^2 \partial \theta^2} + \frac{1-\nu}{2} \left(-\frac{\partial^2 \bar{\psi}_r^i}{R \partial R \partial \theta} + \frac{\partial \bar{\psi}_r^i}{R^2 \partial \theta} - \frac{\bar{\psi}_\theta^i}{R^2} + \frac{\partial \bar{\psi}_\theta^i}{R \partial R} + \frac{\partial^2 \bar{\psi}_\theta^i}{\partial R^2} \right) \right] \\
& = -S_1^i \lambda_i^2 (\bar{I}_1^i \bar{v}_0^i + \bar{I}_2^i \bar{\psi}_\theta^i)
\end{aligned} \tag{9b}$$

$$\begin{aligned}
& \bar{K}_2^i \left[\left(\frac{\partial^2 \bar{u}_0^i}{\partial R^2} + \frac{\partial \bar{u}_0^i}{R \partial R} - \frac{\bar{u}_0^i}{R^2} + \frac{\partial^2 \bar{v}_0^i}{R \partial R \partial \theta} - \frac{\partial \bar{v}_0^i}{R^2 \partial \theta} \right) + \frac{1-\nu}{2} \left(\frac{\partial^2 \bar{u}_0^i}{R^2 \partial \theta^2} - \frac{\partial \bar{v}_0^i}{R^2 \partial \theta} - \frac{\partial \bar{v}_0^i}{R \partial R \partial \theta} \right) \right] \\
& + \bar{K}_3^i \left[\left(\frac{\partial^2 \bar{\psi}_r^i}{\partial R^2} + \frac{\partial \bar{\psi}_r^i}{R \partial R} - \frac{\bar{\psi}_r^i}{R^2} + \frac{\partial^2 \bar{\psi}_\theta^i}{R \partial R \partial \theta} - \frac{\partial \bar{\psi}_\theta^i}{R^2 \partial \theta} \right) + \frac{1-\nu}{2} \left(\frac{\partial^2 \bar{\psi}_r^i}{R^2 \partial \theta^2} - \frac{\partial \bar{\psi}_\theta^i}{R^2 \partial \theta} - \frac{\partial^2 \bar{\psi}_\theta^i}{R^2 \partial R \partial \theta} \right) \right] \\
& - S_2^i \left(\bar{\psi}_r^i + \frac{\partial \bar{w}^i}{\partial R} \right) = -S_1^i \lambda_i^2 (\bar{I}_2^i \bar{u}_0^i + \bar{I}_3^i \bar{\psi}_r^i)
\end{aligned} \tag{9c}$$

$$\begin{aligned}
& \bar{K}_2^i \left[\frac{\partial \bar{u}_0^i}{R^2 \partial \theta} + \frac{\partial^2 \bar{u}_0^i}{R \partial R \partial \theta} + \frac{\partial^2 \bar{v}_0^i}{R^2 \partial \theta^2} + \frac{1-\nu}{2} \left(-\frac{\partial^2 \bar{u}_0^i}{R \partial R \partial \theta} + \frac{\partial \bar{u}_0^i}{R^2 \partial \theta} - \frac{\bar{v}_0^i}{R^2} + \frac{\partial \bar{v}_0^i}{R \partial R} + \frac{\partial^2 \bar{v}_0^i}{\partial R^2} \right) \right] \\
& + \bar{K}_3^i \left[\frac{\partial \bar{\psi}_r^i}{R^2 \partial \theta} + \frac{\partial^2 \bar{\psi}_r^i}{R \partial R \partial \theta} + \frac{\partial^2 \bar{\psi}_\theta^i}{R^2 \partial \theta^2} + \frac{1-\nu}{2} \left(-\frac{\partial^2 \bar{\psi}_r^i}{R \partial R \partial \theta} + \frac{\partial \bar{\psi}_r^i}{R^2 \partial \theta} - \frac{\bar{\psi}_\theta^i}{R^2} + \frac{\partial \bar{\psi}_\theta^i}{R \partial R} + \frac{\partial^2 \bar{\psi}_\theta^i}{\partial R^2} \right) \right] \\
& - S_2^i \left(\bar{\psi}_\theta^i + \frac{\partial \bar{w}^i}{R \partial \theta} \right) = -S_1^i \lambda_i^2 (\bar{I}_2^i \bar{v}_0^i + \bar{I}_3^i \bar{\psi}_\theta^i)
\end{aligned} \tag{9d}$$

$$\delta_i^2 S_2^i \left[\left(\frac{\partial \bar{\psi}_r^i}{\partial R} + \frac{\bar{\psi}_r^i}{R} + \frac{\partial \bar{\psi}_\theta^i}{R \partial \theta} \right) + \bar{\Delta} \bar{w}^i \right] = -S_1^i \lambda_i^2 \bar{I}_1^i \bar{w}^i \tag{9e}$$

where

$$(\bar{I}_1^i, \bar{I}_2^i, \bar{I}_3^i) = \int_{-1/2}^{1/2} \frac{\rho(z)}{\rho_c} (1, Z_i, Z_i^2) dZ_i \tag{10}$$

$$(\bar{K}_1^i, \bar{K}_2^i, \bar{K}_3^i) = \frac{1}{E_c h_i^k} \int_{-h_i/2}^{h_i/2} \frac{E(z)}{1-\nu^2} (1, z, z^2) dz, \quad k=1,2,3 \tag{11}$$

$$\bar{\Delta} = \frac{\partial^2}{\partial R^2} + \frac{\partial}{R \partial R} + \frac{\partial^2}{R^2 \partial \theta^2} \tag{12}$$

$$S_1^i = \frac{\delta_i^2}{12(1-\nu^2)}, \quad S_2^i = \frac{\kappa^2 (1-\nu) \bar{K}_1^i}{2\delta_i^2}, \quad \lambda_i = \omega r_n^2 \sqrt{\frac{\rho_c h_i}{D_i}}, \quad D_i = \frac{E_c h_i^3}{12(1-\nu^2)} \tag{13a-d}$$

In order to solve displacement field \bar{w}^i , two auxiliary functions are defined as follows

$$\Phi_1^i = \frac{\partial \bar{u}_0^i}{\partial R} + \frac{\bar{u}_0^i}{R} + \frac{\partial \bar{v}_0^i}{R \partial \theta}, \quad \Phi_2^i = \frac{\partial \bar{\psi}_r^i}{\partial R} + \frac{\bar{\psi}_r^i}{R} + \frac{\partial \bar{\psi}_\theta^i}{R \partial \theta} \tag{14a, b}$$

Using these Eqs. (14a, b) and after some mathematical manipulation, finally a sixth order partial differential equation with constant coefficients is acquired in terms of \bar{w}^i as follows

$$A_1^i \bar{\Delta} \bar{\Delta} \bar{\Delta} \bar{w}^i + A_2^i \bar{\Delta} \bar{\Delta} \bar{w}^i + A_3^i \bar{\Delta} \bar{w}^i + A_4^i \bar{w}^i = 0 \tag{15}$$

where the coefficients A_k^i ($k=1,2,3,4$) are determined by

$$A_1^i = S_2^i \delta_i^2 (\bar{K}_1^i \bar{K}_3^i - \bar{K}_2^i{}^2) \tag{16a}$$

$$A_2^i = S_1^i \lambda_i^2 \left[\bar{I}_1^i (\bar{K}_1^i \bar{K}_3^i - \bar{K}_2^{i2}) + S_2^i \delta_i^2 (\bar{K}_1^i \bar{I}_3^i + \bar{K}_3^i \bar{I}_1^i - 2\bar{K}_2^i \bar{I}_2^i) \right] \quad (16b)$$

$$A_3^i = S_1^i \lambda_i^2 \left[S_1^i \lambda_i^2 (\bar{I}_1^i (\bar{K}_1^i \bar{I}_3^i + \bar{K}_3^i \bar{I}_1^i - 2\bar{K}_2^i \bar{I}_2^i) + S_2^i \delta_i^2 (\bar{I}_1^i \bar{I}_3^i - \bar{I}_2^{i2})) - S_2^i \bar{K}_1^i \bar{I}_1^i \right] \quad (16c)$$

$$A_4^i = S_1^{i2} \lambda_i^4 \bar{I}_1^i [S_1^i \lambda_i^2 (\bar{I}_1^i \bar{I}_3^i - \bar{I}_2^{i2}) - S_2^i \bar{I}_1^i] \quad (16d)$$

The function $\bar{w}^i(R, \theta)$ can be written as

$$\bar{w}^i(R, \theta) = \hat{w}^i(R) \cos(p\theta) \quad (17)$$

in which the non-negative integer p represents the circumferential wave number of the corresponding mode shape. By substituting Eq. (17) into Eq. (15), the reduced following form of the equation is obtained as

$$(\hat{\Delta} - x_1^i)(\hat{\Delta} - x_2^i)(\hat{\Delta} - x_3^i)\hat{w}^i(R) = 0 \quad (18)$$

where $\hat{\Delta} = \frac{\partial^2}{\partial R^2} + \frac{\partial}{R \partial R} - \frac{p^2}{R^2}$ and x_1^i , x_2^i , and x_3^i are the roots of the following equation

$$A_1^i x^3 + A_2^i x^2 + A_3^i x + A_4^i = 0 \quad (19)$$

After solving the obtained third order Eq. (19), finally the general solution of Eq. (15) can be expressed as the summation of three Bessel functions as follows

$$\bar{w}^i(R, \theta) = \sum_{k=1}^3 [c_k^i w_{k1}^i(p, \chi_k^i R) + c_{k+3}^i w_{k2}^i(p, \chi_k^i R)] \cos(p\theta) \quad (20)$$

where $\chi_k^i = \sqrt{|x_k^i|}$, and

$$w_{k1}^i = \begin{cases} J_p(\chi_k^i R), & x_k^i < 0 \\ I_p(\chi_k^i R), & x_k^i > 0 \end{cases}, \quad w_{k2}^i = \begin{cases} Y_p(\chi_k^i R), & x_k^i < 0 \\ K_p(\chi_k^i R), & x_k^i > 0 \end{cases} \quad k = 1, 2, 3 \quad (21a,b)$$

c_k^i are unknown coefficients and J_p and Y_p are the Bessel functions of the first and second kind, respectively, whereas I_p and K_p are the modified Bessel functions of the first and second kind, respectively.

In order to solve displacement fields $\bar{u}_0^i, \bar{v}_0^i, \bar{\psi}_r^i$ and $\bar{\psi}_\theta^i$, four auxiliary functions f_1^i, f_2^i, f_3^i and f_4^i [7] are introduced as follows

$$\begin{aligned} f_1^i &= \bar{K}_1^i \bar{u}_0^i + \bar{k}_2^i \bar{\psi}_r^i & f_2^i &= \bar{K}_1^i \bar{v}_0^i + \bar{K}_2^i \bar{\psi}_\theta^i \\ f_3^i &= \bar{K}_2^i \bar{u}_0^i + \bar{k}_3^i \bar{\psi}_r^i & f_4^i &= \bar{K}_2^i \bar{v}_0^i + \bar{K}_3^i \bar{\psi}_\theta^i \end{aligned} \quad (22a-d)$$

In order to determine f_1^i, f_2^i, f_3^i and f_4^i , the following forms of solution are considered

$$f_1^i = a_1^i \frac{\partial \bar{w}_1^i}{\partial R} + a_2^i \frac{\partial \bar{w}_2^i}{\partial R} + a_3^i \frac{\partial \bar{w}_3^i}{\partial R} + a_4^i \frac{\partial \bar{w}_4^i}{R \partial \theta} + a_5^i \frac{\partial \bar{w}_5^i}{R \partial \theta} \quad (23a)$$

$$f_2^i = b_1^i \frac{\partial \bar{w}_1^i}{R \partial \theta} + b_2^i \frac{\partial \bar{w}_2^i}{R \partial \theta} + b_3^i \frac{\partial \bar{w}_3^i}{R \partial \theta} + b_4^i \frac{\partial \bar{w}_4^i}{\partial R} + b_5^i \frac{\partial \bar{w}_5^i}{\partial R} \quad (23b)$$

$$f_3^i = a_6^i \frac{\partial \bar{w}_1^i}{\partial R} + a_7^i \frac{\partial \bar{w}_2^i}{\partial R} + a_8^i \frac{\partial \bar{w}_3^i}{\partial R} + a_9^i \frac{\partial \bar{w}_4^i}{R \partial \theta} + a_{10}^i \frac{\partial \bar{w}_5^i}{R \partial \theta} \quad (23c)$$

$$f_4^i = b_6^i \frac{\partial \bar{w}_1^i}{R \partial \theta} + b_7^i \frac{\partial \bar{w}_2^i}{R \partial \theta} + b_8^i \frac{\partial \bar{w}_3^i}{R \partial \theta} + b_9^i \frac{\partial \bar{w}_4^i}{\partial R} + b_{10}^i \frac{\partial \bar{w}_5^i}{\partial R} \quad (23d)$$

in which a_k^i and b_k^i are unknown coefficients. Also, \bar{w}_4^i and \bar{w}_5^i are unknown functions. By substituting Eqs. (22a-d) into Eqs. (9a-e) and using the solutions (23a-d), the coefficients a_k^i and b_k^i as well as the functions \bar{w}_4^i and \bar{w}_5^i can be determined as follows

$$a_k^i = b_k^i = \begin{cases} \frac{G_2^i G_5^i}{x_k^{i2} - (G_1^i + G_4^i) x_k^i + G_1^i G_4^i - G_2^i G_3^i}, & k = 1, 2, 3 \\ \frac{(x_{k-5}^i - G_1^i) a_{k-5}^i}{G_2^i}, & k = 6, 7, 8 \end{cases} \quad (24)$$

where

$$\begin{aligned} G_1^i &= \frac{S_1^i \lambda_i^2 (\bar{K}_3^i \bar{I}_1^i - \bar{K}_2^i \bar{I}_2^i)}{\bar{K}_2^{i2} - \bar{K}_1^i \bar{K}_3^i} & G_2^i &= \frac{S_1^i \lambda_i^2 (\bar{K}_1^i \bar{I}_2^i - \bar{K}_2^i \bar{I}_1^i)}{\bar{K}_2^{i2} - \bar{K}_1^i \bar{K}_3^i} \\ G_3^i &= \frac{S_1^i \lambda_i^2 (\bar{K}_3^i \bar{I}_2^i - \bar{K}_2^i \bar{I}_3^i) + S_2^i \bar{K}_2^i}{\bar{K}_2^{i2} - \bar{K}_1^i \bar{K}_3^i} & G_4^i &= \frac{S_1^i \lambda_i^2 (\bar{K}_1^i \bar{I}_3^i - \bar{K}_2^i \bar{I}_2^i) - S_2^i \bar{K}_1^i}{\bar{K}_2^{i2} - \bar{K}_1^i \bar{K}_3^i} \\ G_5^i &= S_2^i & G_6^i &= \frac{S_2^i \delta_i^2 \bar{K}_2^i}{\bar{K}_2^i - \bar{K}_1^i \bar{K}_3^i} \\ G_7^i &= \frac{-S_2^i \delta_i^2 \bar{K}_1^i}{\bar{K}_2^i - \bar{K}_1^i \bar{K}_3^i} & G_8^i &= -S_1^i \lambda_i^2 \bar{I}_1^i \end{aligned} \quad (25a-h)$$

$$\bar{w}_k^i = \hat{w}_k^i \sin(p\theta), \quad k = 4, 5 \quad (26)$$

$$\hat{w}_4^i = c_7^i w_{41}^i(p, \chi_4^i R) + c_8^i w_{42}^i(p, \chi_4^i R) \quad (27a)$$

$$\hat{w}_5^i = c_9^i w_{51}^i(p, \chi_5^i R) + c_{10}^i w_{52}^i(p, \chi_5^i R) \quad (27b)$$

in which $\chi_k^i = \sqrt{|x_k^i|}$, and

$$x_4^i = \frac{\xi_1^i + \sqrt{\xi_1^{i2} - 4\xi_2^i}}{2}, \quad x_5^i = \frac{\xi_1^i - \sqrt{\xi_1^{i2} - 4\xi_2^i}}{2} \quad (28a,b)$$

$$\xi_1^i = \frac{2(G_1^i + G_4^i)}{1-\nu}, \quad \xi_2^i = \frac{4(G_1^i G_4^i - G_2^i G_3^i)}{(1-\nu^2)} \quad (29a,b)$$

$$w_{k1}^i = \begin{cases} J_p(\chi_k^i R), & x_k^i < 0 \\ I_p(\chi_k^i R), & x_k^i > 0 \end{cases}, \quad w_{k2}^i = \begin{cases} Y_p(\chi_k^i R), & x_k^i < 0 \\ K_p(\chi_k^i R), & x_k^i > 0 \end{cases} \quad k = 4, 5 \quad (30a,b)$$

$$a_k^i = -b_k^i = \begin{cases} \frac{G_2^i}{\left(\frac{1-\nu}{2}\right)x_k^i - G_1^i}, & k = 4,5 \\ 1, & k = 9,10 \end{cases} \quad (31)$$

Finally, the exact solutions for \bar{u}_0^i , \bar{v}_0^i , $\bar{\psi}_r^i$ and $\bar{\psi}_\theta^i$ according to Mindlin's theory, are obtained by using Eqs. (22a-d). In order to satisfy the continuity conditions, the dimensional forms of the displacement components and stress resultants must be used instead of the dimensionless ones. This is because the annular segments of the stepped plate have different values of the thicknesses h_i . Based on the FSDT, ten continuity conditions should be satisfied for each step location, which can be written as follows

$$\begin{aligned} w^i &= w^{i+1}, \quad u_0^i = u_0^{i+1}, \quad v_0^i = v_0^{i+1}, \quad \psi_r^i = \psi_r^{i+1}, \quad \psi_\theta^i = \psi_\theta^{i+1} \\ Q_r^i &= Q_r^{i+1}, \quad N_r^i = N_r^{i+1}, \quad N_\theta^i = N_\theta^{i+1}, \quad M_r^i = M_r^{i+1}, \quad M_\theta^i = M_\theta^{i+1} \end{aligned} \quad (32)$$

Both inner and outer edges of the stepped circular/annular plate can take any combinations of classical boundary conditions, including free, soft simply supported, hard simply supported and clamped. It should be noted that, for stepped circular FG plates, second type of Bessel function becomes singular at $r=0$. Hence, the unknown coefficients of them ($c_4^i, c_5^i, c_6^i, c_8^i, c_{10}^i$) should be equal to zero. The satisfaction of the continuity and boundary conditions lead to a coefficient matrix. For a nontrivial solution, the determinant of the coefficient matrix must be set to zero for each p and solving this obtained eigenvalue equations yields the frequency parameters β .

3. Results and Discussion

This section contains two parts; firstly, the authors try to validate the present solution with finite element analysis. After verification of results, the effects of the geometrical properties of stepped thickness circular/annular plates on the frequency parameters will be discussed. It should be noted that in this paper Poisson's ratio is assumed to be 0.3 and the shear correction factor has been taken to be 5/6. The numbers in parentheses (m, n) show that the vibrating mode has m nodal diameters and vibrates in the n th mode for the given m value. The type of FG plate which is used in this study has the Young's modulus and mass density $E_m = 70GPa$, $\rho_m = 2700kg/m^3$ and $E_c = 380GPa$, $\rho_c = 3800kg/m^3$ for metallic and ceramic constituents, respectively and power-law index g sets to be 1 in all calculations. Also, in this paper, the dimensionless frequency parameter $\beta = \omega r_n^2 \sqrt{\rho_c h_n / D_n}$ is defined in order to evaluate the effects of geometrical properties of stepped thickness circular plates on the natural frequencies.

A comparative study for evaluation of first ten natural frequencies (Hz) of stepped FG circular/annular plate between the present exact solution and the finite element analysis is carried out in

Table 1. Numerical results have been calculated for free, soft-simply supported and clamped FG circular Mindlin plates with one step variation. The step locations and thicknesses are selected as $h_1 = 0.2(m)$, $r_1 = 1(m)$, $h_2 = 0.1(m)$, and $r_2 = 2(m)$. To demonstrate further the high accuracy of the present exact solution, in **Table 1**, ANSYS software package of version 12 and ABAQUS software package of version 6.10 are used to model stepped FG circular plate. The plate is analysed with a shell element of type Shell 281 in ANSYS and a solid element in ABAQUS, created on the basis of the FSDT and the 3D elasticity, respectively. A mesh sensitivity analysis was carried out to ensure independency of finite element (FE) results from the number of elements. Results of **Table 1** reveal that very good agreement is achieved for the circular FG plate.

Table 1. Comparison study of first ten natural frequencies (Hz) for FG circular plate with one step variation

B. C's	Mode number (m,n)	(2,1)	(0,1)	(3,1)	(4,1)	(1,1)	(5,1)	(2,2)	(6,1)	(0,2)	(7,1)
Free	Present	83.543	128.289	146.398	227.583	230.416	331.664	395.639	457.797	477.222	604.337
	FEM (FSDT)	83.578	128.340	146.460	227.670	230.510	331.800	395.790	458.000	477.400	604.660
	FEM (3D)	82.773	125.658	145.156	225.139	226.525	331.018	392.833	457.637	477.737	604.696
	Mode number (m,n)	(0,1)	(1,1)	(2,1)	(0,2)	(3,1)	(1,2)	(4,1)	(5,1)	(2,2)	(0,3)
Soft-Simply Supported	Present	58.084	144.063	297.991	382.925	468.489	606.805	628.476	790.895	796.988	867.957
	FEM (FSDT)	58.108	144.120	298.110	383.070	468.650	607.000	628.690	791.170	797.230	868.220
	FEM (3D)	57.317	142.386	297.923	383.924	469.695	603.836	630.583	790.894	794.823	860.899
	Mode number (m,n)	(0,1)	(1,1)	(2,1)	(0,2)	(3,1)	(1,2)	(4,1)	(5,1)	(2,2)	(0,3)
Clamped	Present	110.629	223.279	392.735	493.537	594.290	767.116	782.061	958.938	978.960	1044.083
	FEM (FSDT)	110.670	223.360	392.870	493.710	594.480	767.330	782.290	959.220	979.210	1044.300
	FEM (3D)	109.880	220.465	392.132	495.484	596.716	767.374	785.848	963.867	972.748	1033.770

Figure 2 shows the variations of the first three frequency parameters β versus the step location $\mathfrak{R} = r_1 / r_2$ for free circular FG Mindlin plate ($g=1$) with one step variation. The step thickness ratio h_1 / h_2 and plate thickness ratio h_2 / r_2 are set to be $3/2$ and 0.1 , respectively. **Figure 2** shows that as the increase of the step location \mathfrak{R} , the first three frequency parameters of free stepped circular FG plate increase slowly to their maximum values and then decrease to the values corresponding to a free circular FG plate without step variation. The maximum values of the first, second and third modes are around 0.8 , 0.85 and 0.9 , respectively.

Figure 3 shows the variations of the first three frequency parameters β of simply supported circular FG plate with one step variation versus the step thickness ratio $\tau = h_1 / h_2$. The step location r_1 / r_2 and the plate thickness ratio h_2 / r_2 are set to 0.5 and 0.05 , respectively. As it is shown in **Figure 3**, For third mode of the simply supported FG plate, as the step thickness ratio τ enhances, the frequency parameter β increases, keeping all other parameters fixed. But the frequency parameters of the first and second modes of the simply supported plate increase slowly to their maximum val-

ues and then decrease, indicating that the plate stiffness corresponding to the first and second modes are maximum at the points around $\tau = 2.2$ and $\tau = 1.8$, respectively.

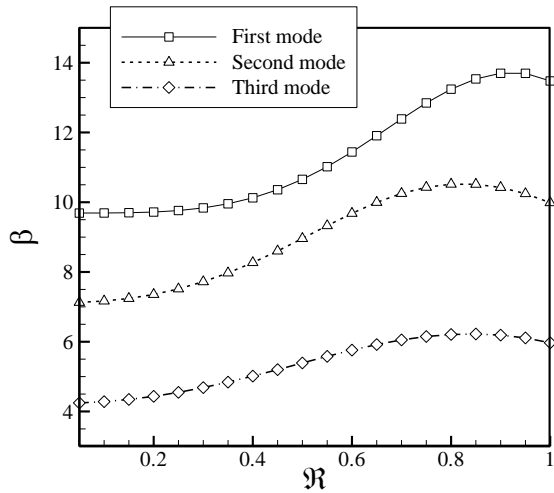


Figure 2. Variation of the first three frequency parameters β versus the step location $\mathfrak{R} = r_1 / r_2$ for free circular FG Mindlin plate with one step variation.

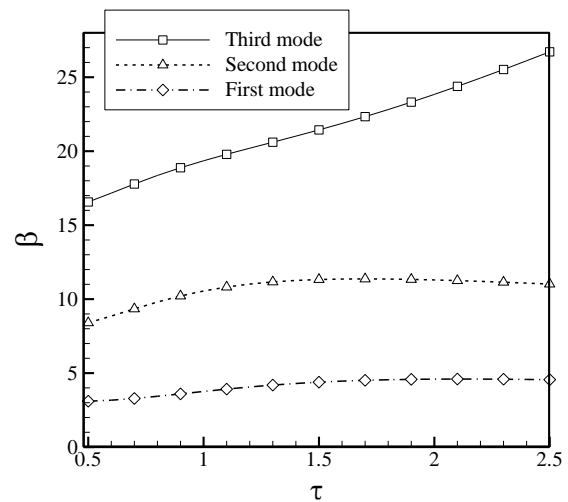


Figure 3. Variation of the first three frequency parameter β versus the step location $\tau = h_1 / h_2$ for simply supported circular FG plate with one step variation.

4. Conclusion

The main objective of this paper was to develop an analytical procedure for solving the free vibration of stepped circular FG plates. By introducing some auxiliary and potential functions, the domain decomposition method is employed to find the free dynamic response of stepped FG plate. The proposed model is validated by comparing its dynamic results with finite element analysis and finally, the influence of different parameters of the stepped FG plate such as step locations and step thickness ratios on the natural frequencies is investigated. Results show that the step parameters play a significant role in the determination of vibration behavior of the FG plate, especially for higher vibrating modes.

REFERENCES

1. J.H. Kang, A.W. Leissa, "Three-dimensional vibrations of thick, linearly tapered, annular plates", *Journal of Sound and Vibration*, 217, 927–44 (1998).
2. Y. Xiang, L. Zhang, "Free vibration analysis of stepped circular Mindlin plates", *Journal of Sound and Vibration*, 280, 633–55 (2005).
3. L. Hang, C.M. Wang, T.Y.Wu, "Exact vibration results for stepped circular plates with free edges", *International Journal of Mechanical Sciences*, 47, 1224–48 (2005).
4. Sh. Hosseini-Hashemi, M. Es'haghi, H. Rokni Damavandi Taher, M. Fadaee, "Exact closed-

- form frequency equations for thick circular plates using a third-order shear deformation theory", *Journal of Sound and Vibration*, 329, 3382–96 (2010).
5. U.S. Gupta, R. Lal, S. Sharma, "Vibration of non-homogeneous circular Mindlin plates with variable thickness", *Journal of Sound and Vibration*, 302, 1–17 (2007).
 6. E. Efraim, M. Eisenberger, "Exact vibration analysis of variable thickness thick annular isotropic and FGM plates", *Journal of Sound and Vibration*, 299, 720–38 (2007).
 7. Sh. Hosseini-Hashemi, M. Fadaee, M. Es'haghi, "A novel approach for in-plane/out-of-plane frequency analysis of functionally graded circular/annular plates", *International Journal of Mechanical Sciences*, 52, 1025–35 (2010).
 8. V. Tajeddini, A. Ohadi, A. Sadighi, "Three-dimensional free vibration of variable thickness thick circular and annular isotropic and functionally graded plates on Pasternak foundation", *International Journal of Mechanical Sciences*, 53, 300–08 (2011).
 9. M. Derakhshani, "An analytical solution for the free vibration of stepped circular/annular and sectorial Mindlin functionally graded plates", *Iran University of Science and Technology, Tehran, Iran*, (2011).