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ON PROPERTIES OF THE SOLUTIONS TO THE α-HARMONIC EQUATION

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ABSTRACT. The aim of this paper is to establish properties of the solutions to the α-harmonic equations: $\Delta_\alpha(f(z)) = \partial z[(1-|z|^2)^{-\alpha}\overline{\partial}zf](z) = g(z)$, where $g : \overline{\mathbb{D}} \to \mathbb{C}$ is a continuous function and $\overline{\mathbb{D}}$ denotes the closure of the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. We obtain Schwarz type and Schwarz-Pick type inequalities for the solutions to the α-harmonic equation. In particular, for $g \equiv 0$, the solutions to the above equation are called α-harmonic functions. We determine the necessary and sufficient conditions for an analytic function ψ to have the property that $f \circ \psi$ is α-harmonic function for any α-harmonic function f. Furthermore, we discuss the Bergman-type spaces on α-harmonic functions.

1. Introduction

Let $\mathbb{C}$ denote the complex plane. For $a \in \mathbb{C}$, let $\mathbb{D}(a,r) = \{z : |z-a| < r\}$ $(r > 0)$ and $\mathbb{D}(0,r) = \mathbb{D}_r$, $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{T} = \partial\mathbb{D}$, the boundary of $\mathbb{D}$, and $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{T}$, the closure of $\mathbb{D}$. Furthermore, we denote by $\mathcal{C}^m(\Omega)$ the set of all complex-valued m-times continuously differentiable functions from Ω into $\mathbb{C}$, where Ω stands for a subset of $\mathbb{C}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In particular, $\mathcal{C}(\Omega) := \mathcal{C}^0(\Omega)$ denotes the set of all continuous functions in Ω. We use $d(z)$ to denote the Euclidean distance from z to $\mathbb{T}$.

1.1. **Distributions.** Let Ω be an open set in $\mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ be an infinitely many times differentiable function. We may write its partial derivatives in the form

$$\partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a *multi-index* and $\partial_j = \partial/\partial x_j$, $j = 1, 2, \dots, n$. We denote by $C_0^\infty(\Omega)$ the space of functions which are infinitely many times differentiable and have a compact support in Ω.

A distribution f in Ω is a linear form on $C_0^\infty(\Omega)$ such that for every compact set $K \subset \Omega$ there exist constants C and k such that

$$|f(\phi)| \le C \sum_{|\alpha| \le k} \sup |\partial^\alpha \phi|,$$

where $\phi \in C_0^\infty(K)$. The set of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$ (cf. [16]).

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For locally integrable u and v are on Ω and

$$-\int_{\Omega} u\partial_j\varphi\, dx = \int_{\Omega} v\varphi\, dx,$$

for all $\varphi \in C_0^\infty(\Omega)$, we shall say that $\partial_j u = v$ *in the distribution sense* (cf. [14]).

1.2. **The α-harmonic equation.** We denote by Δ_α the weighted Laplace operator corresponding to the so-called standard weight $w_\alpha = (1-|z|^2)^\alpha$, that is,

$$\Delta_\alpha(f(z)) = \partial z[(1-|z|^2)^{-\alpha}\overline{\partial} z f](z) \tag{1.1}$$

in $\mathbb{D}$, where $\alpha > -1$ (see [20, Proposition 1.5] for the reason for this constraint),

$$\partial z = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \text{ and } \overline{\partial} z = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

The weighted Laplacians of the form (1.1) were first systematically studied by Garabedian in [11]. In [20], Olofsson and Wittsten introduced the operator Δ_α and gave a counterpart of the classical Poisson integral formula for it.

Let $g \in \mathcal{C}(\mathbb{D})$ and $f \in \mathcal{C}^2(\mathbb{D})$. Of particular interest to us is the following *inhomogeneous α-harmonic equation* in $\mathbb{D}$:

$$\Delta_\alpha(f) = g. \tag{1.2}$$

We also consider the associated *Dirichlet boundary value problem* of functions f, satisfying the equation (1.2),

$$\begin{cases} \Delta_\alpha(f) = g & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \mathbb{T}. \end{cases} \tag{1.3}$$

Here the boundary data f^* is a distribution on $\mathbb{T}$, i.e. $f^* \in \mathcal{D}'(\mathbb{T})$, and the boundary condition in (1.3) is understood as $f_r \to f^* \in \mathcal{D}'(\mathbb{T})$ as $r \to 1^-$, where

$$f_r(e^{i\theta}) = f(re^{i\theta})$$

for $e^{i\theta} \in \mathbb{T}$ and $r \in [0, 1)$.

If $g \equiv 0$ in (1.2), the solutions to (1.2) are said to be *α-harmonic functions.* Obviously, α-harmonicity coincides with harmonicity when $\alpha = 0$. See [8] and the references therein for the properties of harmonic mappings.

In [20], Olofsson and Wittsten showed that if an α-harmonic function f satisfies

$$\lim_{r\to 1^-} f_r = f^* \in \mathcal{D}'(\mathbb{T}) \ \ (\alpha > -1),$$

then it has the form of a *Poisson type integral*

$$f(z) = \mathcal{P}_\alpha[f^*](z) = \frac{1}{2\pi}\int_0^{2\pi} P_\alpha(ze^{-i\theta})f^*(e^{i\theta})d\theta \tag{1.4}$$

in $\mathbb{D}$, where

$$P_\alpha(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\overline{z})^{\alpha+1}}$$

is the *α-harmonic Poisson kernel* in $\mathbb{D}$. See [17] and [19] for related discussions.

In [3], Behm found the Green function for $-\overline{\Delta_\alpha}$. The *$\alpha$-harmonic Green function* G_α is given in $\mathbb{D}$ by

$$G_\alpha(z,w) = -(1-z\overline{w})^\alpha h \circ \varphi(z,w), \quad z \neq w, \tag{1.5}$$

where

$$h(s) = \int_0^s \frac{t^\alpha}{1-t}dt = \sum_{n=0}^{\infty} \frac{s^{\alpha+1+n}}{\alpha+1+n}, \quad 0 \le s < 1, \tag{1.6}$$

and

$$\varphi(z,w) = 1 - \left|\frac{z-w}{1-\overline{z}w}\right|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}.$$

For convenience, we let

$$\mathcal{G}[g](z) = \int_{\mathbb{D}} G_\alpha(z,w)g(w)\,dA(w)$$

and

$$\mathcal{P}[f^*](z) = \frac{1}{2\pi}\int_0^{2\pi} P(ze^{-i\theta})f^*(e^{i\theta})d\theta,$$

where $dA(w) = (1/\pi)\,dx\,dy$ denotes the normalized area measure in $\mathbb{D}$ and

$$P(z) = \frac{1-|z|^2}{|1-z|^2}$$

is the *Poisson kernel* in $\mathbb{D}$.

By [20, Theorem 5.3] and [3, Theorem 2], we see that all solutions to the α-harmonic equation (1.3) are given by

$$f(z) = \mathcal{P}_\alpha[f^*](z) + \mathcal{G}[g](z). \tag{1.7}$$

2. Main results

2.1. **A Schwarz type lemma.**

The classical Schwarz lemma states that an analytic function f from $\mathbb{D}$ into itself, with $f(0)=0$, satisfies $|f(z)| \le |z|$ for all $z \in \mathbb{D}$. This result is a crucial theme in many branches of research for more than a hundred years.

Heinz [15] proved the following Schwarz lemma of complex-valued harmonic functions: If f is a complex-valued harmonic function from $\mathbb{D}$ into itself with $f(0)=0$, then, for $z \in \mathbb{D}$,

$$|f(z)| \le \frac{4}{\pi}\arctan|z|. \tag{2.1}$$

The first purpose of this paper is to consider the results of the above type for the solutions to the α-harmonic equation. Our result is the following:

Theorem 2.1. *Suppose that $g \in \mathcal{C}(\overline{\mathbb{D}})$ and $f^* \in \mathcal{C}^1(\mathbb{T})$. If $f \in \mathcal{C}^2(\mathbb{D})$ satisfying (1.3) with $\alpha \ge 0$ and $\mathcal{P}_\alpha[f^*](0) = 0$, then for $z \in \overline{\mathbb{D}}$,*

$$|f(z)| \le 2^\alpha \left[\frac{4}{\pi}\|f^*\|_\infty \arctan|z| + \|g\|_\infty (1-|z|^2)^{\alpha+1}\right], \tag{2.2}$$

where $\|f^*\|_\infty = \sup_{z\in\mathbb{T}}\{|f^*(z)|\}$, and $\|g\|_\infty = \sup_{z\in\mathbb{D}}\{|g(z)|\}$.

Moreover, if we take $\alpha = 0$, $g(z) \equiv -C$, *where* C *is a positive constant, and*

$$f(z) = C(1 - |z|^2),$$

then we see that the inequality (2.2) *is sharp.*

Clearly, if $\alpha = 0$, $g \equiv 0$ and f maps $\mathbb{D}$ into itself, then (2.2) coincides with (2.1).

2.2. **A Schwarz-Pick type lemma.** Suppose $f = u + iv$ is in $C^1(\Omega)$, where Ω is a domain in $\mathbb{C}$ and u, v are real functions. The Jacobian matrix of f at z is denoted by

$$D_f(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Then

$$\|D_f(z)\| = \sup\{|D_f(z)\varsigma| : |\varsigma| = 1\} = |f_z(z)| + |f_{\overline{z}}(z)| \tag{2.3}$$

and

$$l(D_f(z)) = \inf\{|D_f(z)\varsigma| : |\varsigma| = 1\} = \big||f_z(z)| - |f_{\overline{z}}(z)|\big|. \tag{2.4}$$

Colonna [7] obtained a sharp Schwarz-Pick type lemma for complex-valued harmonic functions, which is as follows: If f is a complex-valued harmonic function from $\mathbb{D}$ into itself, then, for $z \in \mathbb{D}$,

$$\|D_f(z)\| \leq \frac{4}{\pi}\frac{1}{1 - |z|^2}. \tag{2.5}$$

The following result establishes a Schwarz-Pick type lemma for the solutions to the α-harmonic equation.

Theorem 2.2. *Suppose that* $g \in \mathcal{C}(\overline{\mathbb{D}})$, $f \in \mathcal{C}^2(\mathbb{D})$ *satisfies* (1.3) *with* $\alpha \geq 0$ *and that* $f^* \in C(\mathbb{T})$. *Then for* $z \in \mathbb{D}$,

$$\|D_f(z)\| \leq (\alpha + 1)2^{\alpha+1}\|f^*\|_\infty \frac{1}{1 - |z|^2} + (\alpha + \frac{4}{3})2^{\alpha+1}\|g\|_\infty,$$

where $\|f^*\|_\infty$ *and* $\|g\|_\infty$ *are as in Theorem 2.1.*

In particular, if f *maps* $\mathbb{D}$ *into* $\mathbb{D}$ *and* $g \equiv 0$, *then*

$$\|D_f(z)\| \leq (\alpha + 1)2^{\alpha+1}\frac{1}{1 - |z|^2}. \tag{2.6}$$

We remark that the estimate of (2.6) is sharper than the estimate given by [17, Theorem 1.1].

2.3. **Compositions of α-harmonic functions.** Although the composition $f \circ \phi$, where f is harmonic and ϕ is analytic, is known to be harmonic, an α-harmonic function ($\alpha \neq 0$) precomposed with an analytic function may not be α-harmonic. This can be seen by the following example.

Example 2.1. Let $\alpha \in (-1,0)\cup(0,+\infty)$ and let $k \geq 1$ be an integer. Denote by f the function

$$\begin{aligned} f(z) &= \Big(\sum_{n=0}^{\infty}\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}(1-|z|^2)^n \\ &\quad -(1-|z|^2)^{\alpha+1}\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha+k+1)}{\Gamma(n+\alpha+2)\Gamma(k)}(1-|z|^2)^n\Big)\overline{z}^k \end{aligned}$$

for $z \in \mathbb{D}\setminus\{0\}$, where $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ $(s > 0)$ is the Gamma function. Let $\psi(z) = z^2$. Then

(1) f is an α-harmonic function in $\mathbb{D}\setminus\{0\}$;
(2) $f\circ\psi$ is not an α-harmonic function in $\mathbb{D}\setminus\{0\}$.

To see this note that, by [20, Lemma 1.6], f is an α-harmonic function in $\mathbb{D}\setminus\{0\}$. By letting $\eta = \psi(z) = z^2$, we have

$$\frac{\partial}{\partial\overline{z}}f\circ\psi(z) = \frac{\partial}{\partial\overline{\eta}}f(\eta)\cdot\frac{\partial\overline{\eta}}{\partial\overline{z}} = 2\frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)(k-1)!}(1-|z|^2)^{\alpha}(1+|z|^2)^{\alpha}\overline{z}^{2k-1},$$

and then

$$\frac{\partial}{\partial z}\Big((1-|z|^2)^{-\alpha}\frac{\partial}{\partial\overline{z}}f\circ\psi(z)\Big) \neq 0.$$

Hence, it follows that $f\circ\psi$ is not an α-harmonic function in $\mathbb{D}\setminus\{0\}$.

Now we are ready to state our result.

Theorem 2.3. *Let ψ be an analytic function in $\mathbb{D}$. Then for any α-harmonic function f with $\alpha \in (-1,0)\cup(0,+\infty)$, $f\circ\psi$ is α-harmonic if and only if $\psi(z) = e^{it}z$, $t \in [0,2\pi]$.*

2.4. **Bergman-type spaces.** For $\nu,\mu,t \in \mathbb{R}$,

$$\mathcal{D}_f(\nu,\mu,t) = \int_{\mathbb{D}} d^{\nu}|f(z)|^{\mu}\|Df(z)\|^t dA(z) < \infty$$

is called *Dirichlet-type energy integral* of the complex-valued function f (cf. [1, 2, 4, 5, 6, 10, 12, 13, 21]). In particular, for $\nu > -1$, $0 < \mu < \infty$ and $t = 0$, we denote by $b_{\nu,\mu}(\mathbb{D})$ the *Bergman-type space*, consisting of all $f \in \mathcal{C}^0(\mathbb{D})$ with the norm

$$\|f\|_{b_{\nu,\mu}} = |f(0)| + (\mathcal{D}_f(\nu,\mu,0))^{\frac{1}{\mu}} < \infty.$$

We refer to [6, 9, 12, 13, 22] for basic characterizations of analytic or harmonic Bergman-type spaces and Dirichlet-type spaces. However, very few related studies can be found from the literature for the general complex-valued functions. The following is a characterization of α-harmonic functions in Bergman-type spaces.

Theorem 2.4. *Let $f \in \mathcal{C}^2(\mathbb{D})$ be an α-harmonic function in $\mathbb{D}$ with $\alpha > -1$, $\mathrm{Re}\,(f\overline{\Delta f}) \geq 0$ and $\sup_{z\in\overline{\mathbb{D}}}|f(z)| \leq M$, where M is a constant. Then for $p \geq 2$, $f \in b_{p-1,p}(\mathbb{D})$.*

We will give proofs of Theorem 2.1 and Theorem 2.2 in Section 3. Theorem 2.3 and Theorem 2.4 will be proved in Section 4.

3. Schwarz type and Schwarz-Pick type lemmas for the solutions of the α-harmonic equation

The aim of this section is to prove Theorems 2.1 and 2.2. First, we show Theorem 2.1. For this, we need some lemmas.

Lemma 3.1. *For $\alpha \geq 0$, the function $h(s)$ given by* (1.6) *satisfies the estimate*

$$h(s) \leq s^{\alpha} \log \frac{1}{1-s}.$$

Proof. By (1.6), we obtain that

$$h(s) = \sum_{n=0}^{\infty} \frac{s^{\alpha+1+n}}{\alpha+1+n} = \sum_{n=1}^{\infty} \frac{s^{\alpha+n}}{\alpha+n} = s^{\alpha} \sum_{n=1}^{\infty} \frac{s^{n}}{\alpha+n} \leq s^{\alpha} \sum_{n=1}^{\infty} \frac{s^{n}}{n} = s^{\alpha} \log \frac{1}{1-s},$$

and the result follows. □

Lemma 3.2. *For $\alpha \geq 0$, the function $G_\alpha(z,w)$ given by* (1.5) *satisfies the estimate*

$$|G_\alpha(z,w)| \leq 2^{\alpha}(1-|z|^2)^{\alpha} \log \left|\frac{1-\overline{z}w}{z-w}\right|^2.$$

Proof. Applying Lemma 3.1, shows that

$$\begin{aligned} |G_\alpha(z,w)| &\leq |1-z\overline{w}|^{\alpha} \frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\alpha}}{|1-z\overline{w}|^{2\alpha}} \log \left|\frac{1-\overline{z}w}{z-w}\right|^2 \\ &\leq 2^{\alpha}(1-|z|^2)^{\alpha} \log \left|\frac{1-\overline{z}w}{z-w}\right|^2, \end{aligned}$$

and the result follows. □

Proof of Theorem 2.1. For a given $g \in \mathcal{C}(\overline{\mathbb{D}})$, by (1.4), we have

$$|f(z)| \leq |\mathcal{P}_\alpha[f^*](z)| + |\mathcal{G}[g](z)|.$$

Since

$$\begin{aligned} |\mathcal{P}_\alpha[f^*](z)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2} \cdot \frac{(1-|z|^2)^{\alpha}}{(1-\overline{z}e^{i\theta})^{\alpha}} f^*(e^{i\theta})d\theta\right| \\ &\leq (1+|z|)^{\alpha} \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|1-ze^{-i\theta}|^2} |f^*(e^{i\theta})|d\theta \\ &\leq 2^{\alpha}\mathcal{P}[|f^*|](z) \end{aligned}$$

and $\mathcal{P}[|f^*|](z)$ is harmonic in $\mathbb{D}$, by (2.1), we know that, for $z \in \mathbb{D}$

$$|\mathcal{P}_\alpha[f^*](z)| \leq 2^{\alpha}\mathcal{P}[|f^*|](z) \leq 2^{\alpha} \cdot \frac{4}{\pi}\|f^*\|_\infty \arctan|z|. \tag{3.1}$$

On the other hand, by Lemma 3.2, we get

$$|\mathcal{G}[g](z)| = \left|\int_{\mathbb{D}} G_\alpha(z,w)g(w)\, dA(w)\right| \leq 2^{\alpha}\|g\|_\infty(1-|z|^2)^{\alpha} J_1,$$

where

$$J_1 = \int_{\mathbb{D}} \log \left| \frac{1-\overline{z}w}{z-w} \right|^2 dA(w).$$

In order to estimate J_1, we let

$$w \mapsto \zeta = \phi(w) = \frac{z-w}{1-w\overline{z}} = re^{i\vartheta}$$

so that $\phi = \phi^{-1}$,

$$w = \frac{z-\zeta}{1-\zeta\overline{z}},\ \phi'(w) = -\frac{1-|z|^2}{(1-w\overline{z})^2},$$

and thus,

$$dA(w) = |(\phi^{-1})'(\zeta)|^2 dA(\zeta) = \frac{(1-|z|^2)^2}{|1-\zeta\overline{z}|^4} dA(\zeta).$$

Consequently, by switching to the polar coordinates and using [18, (2.3)], we obtain

$$\begin{aligned} J_1 &= \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\zeta\overline{z}|^4} \log \frac{1}{|\zeta|^2}\, dA(\zeta) \\ &= \frac{(1-|z|^2)^2}{\pi} \int_0^1 \int_0^{2\pi} \frac{r}{|1-\overline{z}re^{i\vartheta}|^4} \log \frac{1}{r^2}\, d\vartheta\, dr \\ &= 2(1-|z|^2)^2 \sum_{n=0}^{\infty} (n+1)^2 |z|^{2n} \int_0^1 r^{2n+1} \log \frac{1}{r^2}\, dr \\ &= 1-|z|^2, \end{aligned}$$

which implies that

$$|\mathcal{G}[g](z)| \leq 2^{\alpha} \|g\|_{\infty} (1-|z|^2)^{\alpha+1}. \tag{3.2}$$

Hence, it follows from (3.1) and (3.2) that (2.2) holds, and the proof of the theorem is complete.

□

In order to prove Theorem 2.2, we need some auxiliary lemmas. The following result is from [17].

Lemma A. ([17, Lemma 2.1]) *If $\alpha > -1$ and $f^* \in C(\mathbb{T})$, then*

$$\frac{\partial}{\partial z} \int_0^{2\pi} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial z} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta$$

and

$$\frac{\partial}{\partial \overline{z}} \int_0^{2\pi} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta = \int_0^{2\pi} \frac{\partial}{\partial \overline{z}} P_{\alpha}(ze^{-i\theta}) f^*(e^{i\theta}) d\theta.$$

Lemma 3.3. *Assume that $f^* \in C(\mathbb{T})$ and $\alpha \geq 0$. Then*

$$\|D_{\mathcal{P}_{\alpha}[f^*]}(z)\| \leq (\alpha+1) 2^{\alpha+1} \|f^*\|_{\infty} \frac{1}{1-|z|^2}.$$

Proof. By elementary calculations, we obtain

$$\frac{\partial}{\partial z}P(ze^{-i\theta})=\frac{e^{-i\theta}}{(1-ze^{-i\theta})^2}$$

and

$$\frac{\partial}{\partial \overline{z}}P(ze^{-i\theta})=\frac{e^{i\theta}}{(1-\overline{z}e^{i\theta})^2}.$$

Then

$$\begin{aligned}\frac{\partial}{\partial z}P_\alpha(ze^{-i\theta}) &= \frac{(1-|z|^2)^\alpha\left[e^{-i\theta}(1-|z|^2)-(\alpha+1)\overline{z}(1-ze^{-i\theta})\right]}{(1-ze^{-i\theta})^2(1-\overline{z}e^{i\theta})^{\alpha+1}}\\ &= \frac{(1-|z|^2)^\alpha}{(1-\overline{z}e^{i\theta})^{\alpha+1}}\left[1-|z|^2-(\alpha+1)\overline{z}e^{i\theta}(1-ze^{-i\theta})\right]\frac{\partial}{\partial z}P(ze^{-i\theta}) \qquad (3.3)\end{aligned}$$

and

$$\frac{\partial}{\partial \overline{z}}P_\alpha(ze^{-i\theta})=\frac{(\alpha+1)(1-|z|^2)^\alpha e^{i\theta}}{(1-\overline{z}e^{i\theta})^{\alpha+2}}=\frac{(\alpha+1)(1-|z|^2)^\alpha}{(1-\overline{z}e^{i\theta})^{\alpha}}\frac{\partial}{\partial \overline{z}}P(ze^{-i\theta}). \qquad (3.4)$$

Since

$$\begin{aligned}\left|1-|z|^2-(\alpha+1)\overline{z}e^{i\theta}(1-ze^{-i\theta})\right| &= \left|(1-ze^{-i\theta})(-\alpha\overline{z}e^{i\theta})+(1-\overline{z}e^{i\theta})\right|\\ &\leq (\alpha+1)|1-\overline{z}e^{i\theta}|,\end{aligned}$$

we see from (3.3) that

$$\left|\frac{\partial}{\partial z}P_\alpha(ze^{-i\theta})\right|\leq(\alpha+1)2^\alpha\left|\frac{\partial}{\partial z}P(ze^{-i\theta})\right|.$$

Hence, by combining the above with (3.4), we conclude

$$\left|\frac{\partial}{\partial z}P_\alpha(ze^{-i\theta})\right|+\left|\frac{\partial}{\partial \overline{z}}P_\alpha(ze^{-i\theta})\right|\leq(\alpha+1)2^\alpha\left(\left|\frac{\partial}{\partial z}P(ze^{-i\theta})\right|+\left|\frac{\partial}{\partial \overline{z}}P(ze^{-i\theta})\right|\right).$$

Consequently, from Lemma A and the identities

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial z}P(ze^{-i\theta})\right|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial \overline{z}}P(ze^{-i\theta})\right|d\theta=\frac{1}{1-|z|^2},$$

it follows that

$$\|D_{\mathcal{P}_\alpha[f^*]}(z)\|\leq(\alpha+1)2^{\alpha+1}\|f^*\|_\infty\frac{1}{1-|z|^2},$$

which is what is needed. □

Identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$, and denote by $L^1_{loc}(\mathbb{D})$ the space of locally integrable functions in $\mathbb{D}$. Functions $\psi\in L^1_{loc}(\mathbb{D})$ with distributional partial derivatives in $\mathbb{D}$, we have the action

$$\langle\psi_z,\varphi\rangle=-\int_{\mathbb{D}}\psi\varphi_z\,dA,\quad \varphi\in C_0^\infty(\mathbb{D}),$$

for the distributional partial derivative ψ_z have, and similarly for the distribution $\psi_{\overline{z}}$ (cf. [3, 14, 16]).

Lemma 3.4. *For $\alpha \geq 0$ and $g \in \mathcal{C}(\overline{\mathbb{D}})$, the function $G_\alpha(z,w)$, given by (1.5), satisfies the following inequality:*

(1) $\displaystyle\int_{\mathbb{D}} |G_\alpha(z,w)g(w)|\, dA(w) \leq 2^\alpha \|g\|_\infty$ *and* $\int_{\mathbb{D}} |G_\alpha(z,w)|\, dA(z) < \infty$*;*

(2) *For fixed $w \in \mathbb{D}$,*

$$\frac{\partial G_\alpha(z,w)}{\partial z} = \alpha\overline{w}(1-z\overline{w})^{\alpha-1} h\circ g(z,w) - \frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(1-\overline{z}w)^\alpha(1-z\overline{w})} \cdot \frac{1}{z-w}$$

and

$$\frac{\partial G_\alpha(z,w)}{\partial \overline{z}} = \frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(1-\overline{z}w)^{\alpha+1}(\overline{z}-\overline{w})}$$

in the sense of distributions in $\mathbb{D}$;

(3) (a) $\displaystyle\frac{\partial \mathcal{G}[g](z)}{\partial z} = \int_{\mathbb{D}} \frac{\partial G_\alpha(z,w)}{\partial z} g(w)\, dA(w)$*, and*

(b) $\displaystyle\left|\frac{\partial \mathcal{G}[g](z)}{\partial z}\right| \leq \int_{\mathbb{D}} \left|\frac{\partial G_\alpha(z,w)}{\partial z} g(w)\right| dA(w) \leq (\alpha + \frac{2}{3}) 2^{\alpha+1}\|g\|_\infty$*,*

in the sense of distributions in $\mathbb{D}$;

(4) (a) $\displaystyle\frac{\partial \mathcal{G}[g](z)}{\partial \overline{z}} = \int_{\mathbb{D}} \frac{\partial G_\alpha(z,w)}{\partial \overline{z}} g(w)\, dA(w)$*, and*

(b) $\displaystyle\left|\frac{\partial \mathcal{G}[g](z)}{\partial \overline{z}}\right| \leq \int_{\mathbb{D}} \left|\frac{\partial G_\alpha(z,w)}{\partial \overline{z}} g(w)\right| dA(w) \leq \frac{2^{\alpha+2}}{3}\|g\|_\infty$*,*

in the sense of distributions in $\mathbb{D}$.

Proof. Obviously, it follows from (3.2) and the proof of [3, Proposition 4] that (1) holds. Therefore, $G_\alpha(z,w) \in L^1(\mathbb{D})$ so that its derivative has the action

$$\left\langle \frac{\partial G_\alpha(z,w)}{\partial z}, \varphi(z) \right\rangle = -\int_{\mathbb{D}} G_\alpha(z,w)\varphi_z(z)\, dA(z), \quad \varphi \in C_0^\infty(\mathbb{D}).$$

By Lebesgue's dominated convergence theorem we get

$$\int_{\mathbb{D}} G_\alpha(z,w)\varphi_z(z)\, dA(z) = \lim_{\varepsilon\to 0} \int_{\mathbb{D}\setminus\mathbb{D}(w,\varepsilon)} G_\alpha(z,w)\varphi_z(z)\, dA(z).$$

For $\varepsilon > 0$, let $D_\varepsilon = D(w,\varepsilon)$. Partial integration gives

$$\begin{aligned}\int_{\mathbb{D}\setminus D_\varepsilon} G_\alpha(z,w)\varphi_z(z)\, dA(z) &= -\int_{\partial D_\varepsilon} G_\alpha(z,w)\varphi(z)v(z)\, ds(z) \\ &\quad -\int_{\mathbb{D}\setminus D_\varepsilon} \frac{\partial G_\alpha(z,w)}{\partial z}\varphi(z)\, dA(z),\end{aligned}$$

where v is the unit exterior normal of $\mathbb{D}\setminus D_\varepsilon$, that is, the inward directed unit normal of D_ε and ds denotes the normalized arc length measure.

It follows from Lemma 3.2 that

$$|G_\alpha(z,w)| \leq 2^\alpha(1-|z|^2)^\alpha \log\left|\frac{1-\overline{z}w}{z-w}\right|^2.$$

Then

$$\begin{aligned}\left|\int_{\partial D_\varepsilon} G_\alpha(z,w)\varphi(z)v(z)\,ds(z)\right| &\le C_\varphi \int_{\partial D_\varepsilon} |G_\alpha(z,w)|\,ds(z)\\ &\le 2^\alpha C_\varphi \varepsilon \log\frac{4}{\varepsilon^2} \to 0,\end{aligned}$$

as $\varepsilon \to 0$, where C_φ is a constant depending only on $\sup\varphi$, and the supremum is taken over all functions φ. A straightforward computation shows that for $z \neq w$ we have

$$\frac{\partial G_\alpha(z,w)}{\partial z} = H(z,w) = \alpha\overline{w}(1-z\overline{w})^{\alpha-1} h\circ g(z,w) + \frac{(1-|z|^2)^\alpha(1-|w|^2)^\alpha}{(1-\overline{z}w)^\alpha(1-z\overline{w})}\cdot\frac{1}{z-w}.$$

It follows from Lemma 3.1 that

$$\begin{aligned}\left|\frac{\partial G_\alpha(z,w)}{\partial z}\right| &\le \alpha\cdot 2^{\alpha+1}(1-|z|^2)^{\alpha-1}\log\left|\frac{1-\overline{z}w}{z-w}\right|^2\\ &\quad + 2^\alpha(1-|z|^2)^\alpha\frac{1-|w|^2}{|1-z\overline{w}|\cdot|z-w|}.\end{aligned}\tag{3.5}$$

Therefore

$$\begin{aligned}\left\langle\frac{\partial G_\alpha(z,w)}{\partial z},\varphi(z)\right\rangle &= \lim_{\varepsilon\to 0}\int_{\mathbb{D}\setminus D_\varepsilon}\frac{\partial G_\alpha(z,w)}{\partial z}\varphi(z)\,dA(z)\\ &= \int_{\mathbb{D}} H(z,w)\varphi(z)\,dA(z) = \langle H(z,w),\varphi(z)\rangle.\end{aligned}$$

By a similar reasoning as above, one obtains that

$$\frac{\partial G_\alpha(z,w)}{\partial\overline{z}} = \frac{(1-|z|^2)^\alpha(1-|w|^2)^{\alpha+1}}{(1-\overline{z}w)^{\alpha+1}(\overline{z}-\overline{w})}$$

in the sense of distributions in $\mathbb{D}$. Hence, the assertion (2) of the lemma holds.

To prove the assertion (3), it follows from (3.5) that

$$\begin{aligned}\int_{\mathbb{D}}\left|\frac{\partial G_\alpha(z,w)}{\partial z}\right| dA(w) &\le \alpha 2^{\alpha+1}(1-|z|^2)^{\alpha-1}\int_{\mathbb{D}}\log\left|\frac{1-\overline{w}z}{z-w}\right|^2 dA(w)\\ &\quad + 2^\alpha(1-|z|^2)^\alpha\int_{\mathbb{D}}\frac{1-|w|^2}{|1-z\overline{w}|\cdot|z-w|}\,dA(w).\end{aligned}$$

Moreover, as before, by using the transformation

$$w\mapsto\zeta=\phi(w)=\frac{z-w}{1-w\overline{z}}=re^{i\vartheta}$$

we obtain after computation that

$$\begin{aligned}\int_{\mathbb{D}} \frac{1-|w|^2}{|z-w|\cdot|1-\overline{w}z|}\,dA(w) &= \int_{\mathbb{D}} \frac{(1-|z|^2)(1-|\zeta|^2)}{|\zeta|\cdot|1-\zeta\overline{z}|^4}\,dA(\zeta)\\ &= \frac{(1-|z|^2)}{\pi}\int_0^1\int_0^{2\pi}\frac{1-r^2}{|1-\overline{z}re^{i\vartheta}|^4}\,d\vartheta\,dr\\ &= 2(1-|z|^2)\sum_{n=0}^{\infty}(n+1)^2|z|^{2n}\int_0^1 r^{2n}(1-r^2)\,dr\\ &= 4(1-|z|^2)\sum_{n=0}^{\infty}\frac{(n+1)^2}{(2n+1)(2n+3)}|z|^{2n}\\ &\le \frac{4(1-|z|^2)}{3}\sum_{n=0}^{\infty}|z|^{2n} = \frac{4}{3}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}\int_{\mathbb{D}}\left|\frac{\partial G_\alpha(z,w)}{\partial z}\right| dA(w) &\le \alpha 2^{\alpha+1}(1-|z|^2)^{\alpha-1}J_1 + 2^\alpha\frac{4}{3}(1-|z|^2)^\alpha\\ &= \alpha 2^{\alpha+1}(1-|z|^2)^{\alpha} + 2^\alpha\frac{4}{3}(1-|z|^2)^\alpha\\ &\le (\alpha+\frac{2}{3})2^{\alpha+1}.\end{aligned}$$

Hence

$$\int_{\mathbb{D}}\left|\frac{\partial G_\alpha(z,w)}{\partial z}g(w)\right| dA(w) \le (\alpha+\frac{2}{3})2^{\alpha+1}\|g\|_\infty. \tag{3.6}$$

By the assertion (1) and (3.6), we see that

$$\begin{aligned}\left\langle \frac{\partial}{\partial z}\int_{\mathbb{D}} G_\alpha(z,w)g(w)dA(w),\varphi(z)\right\rangle &= -\int_{\mathbb{D}}\left(\int_{\mathbb{D}} G_\alpha(z,w)g(w)dA(w)\right)\varphi_z(z)\,dA(z)\\ &= -\int_{\mathbb{D}}\left(\int_{\mathbb{D}} G_\alpha(z,w)g(w)\varphi_z(z)dA(z)\right)dA(w)\\ &= \int_{\mathbb{D}}\left\langle\frac{\partial}{\partial z}G_\alpha(z,w)g(w),\varphi(z)\right\rangle dA(w)\\ &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{\partial}{\partial z}G_\alpha(z,w)g(w)\varphi(z)dA(z)\right)dA(w)\\ &= \int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{\partial}{\partial z}G_\alpha(z,w)g(w)dA(w)\right)\varphi(z)\,dA(z)\\ &= \left\langle\int_{\mathbb{D}}\frac{\partial}{\partial z}G_\alpha(z,w)g(w)dA(w),\varphi(z)\right\rangle\end{aligned}$$

Hence, we conclude that

$$\frac{\partial\mathcal{G}[g](z)}{\partial z} = \int_{\mathbb{D}}\frac{\partial G_\alpha(z,w)}{\partial z}g(w)\,dA(w),$$

and then the assertion (3) holds.

For the assertion (4), it follows from the assertion (2) that

$$\left|\frac{\partial G_\alpha(z,w)}{\partial \overline{z}}\right| \leq 2^\alpha (1-|z|^2)^\alpha \frac{1-|w|^2}{|1-\overline{z}w||z-w|},$$

and then

$$\begin{aligned}\int_{\mathbb{D}} \left|\frac{\partial G_\alpha(z,w)}{\partial \overline{z}} g(w)\right| dA(w) &\leq 2^\alpha (1-|z|^2)^\alpha \|g\|_\infty \int_{\mathbb{D}} \frac{1-|w|^2}{|z-w|\cdot|1-\overline{w}z|}\, dA(w) \\ &\leq \frac{2^{\alpha+2}}{3}\|g\|_\infty.\end{aligned}$$

The proof of the remaining assertion (4) is similar to the proof of the assertion (3). Hence the proof of the lemma is complete. ☐

Proof of Theorem 2.2. The result follows from Lemma 3.3 and Lemma 3.4 together with (2.3). ☐

4. Some properties of α-harmonic functions

In this section, we will prove Theorem 2.3 and Theorem 2.4.

In [20], the authors obtained the following homogeneous expansion of α-harmonic functions (see [20, Theorem 1.2]):

A function f in $\mathbb{D}$ is α-harmonic if and only if it has the following convergent power series expansion:

$$f(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|z|^2)\overline{z}^k, \tag{4.1}$$

where

$$P_{\alpha,k}(x) = \int_0^1 t^{k-1}(1-tx)^\alpha dt, \tag{4.2}$$

$-1 < x < 1$, and $\{c_k\}_{k=-\infty}^{\infty}$ denotes a sequence of complex numbers with

$$\lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1.$$

Now, we are ready to prove Theorem 2.3.

Proof of Theorem 2.3. The sufficiency is obvious, and now we prove the necessity. Suppose that $f\circ\psi$ is α-harmonic. By (4.1), we get

$$f\circ\psi(z) = \sum_{k=0}^{\infty} c_k \psi(z)^k + \sum_{k=1}^{\infty} c_{-k} P_{\alpha,k}(|\psi(z)|^2)\overline{\psi(z)}^k.$$

Take the derivative of (4.2), we obtain

$$P'_{\alpha,k}(x) = -\int_0^1 t^k \alpha (1-tx)^{\alpha-1} dt,$$

and an integration by parts gives

$$xP'_{\alpha,k}(x)+kP_{\alpha,k}(x)=(1-x)^{\alpha}. \tag{4.3}$$

Let $\tau(z)=P_{\alpha,k}(|\psi(z)|^2)\overline{\psi(z)}^k$. Then, by (4.3), we obtain

$$\begin{aligned}\tau_{\overline{z}}(z) &= \overline{\psi'(z)}\cdot\overline{\psi(z)}^{k-1}\Big(|\psi(z)|^2P'_{\alpha,k}(|\psi(z)|^2)+kP_{\alpha,k}(|\psi(z)|^2)\Big)\\ &= \overline{\psi'(z)}\cdot\overline{\psi(z)}^{k-1}(1-|\psi(z)|^2)^{\alpha}.\end{aligned}$$

This together with the fact that $f\circ\psi$ is α-harmonic, shows that

$$\frac{\partial}{\partial z}\Big[(1-|z|^2)^{-\alpha}(1-|\psi(z)|^2)^{\alpha}\Big]=0,$$

and then

$$z\psi'(z)\overline{\psi(z)}=\frac{1-|\psi(z)|^2}{1-|z|^2}|z|^2 \tag{4.4}$$

is real valued. This implies that $\psi(z)=a_kz^k$, $k=1,2,\ldots$, where a_k are constants. Therefore, it follows from (4.4) that, for $z\in\mathbb{D}\backslash\{0\}$,

$$|a_k|^2=\frac{1}{k|z|^{2k-2}-(k-1)|z|^{2k}}.$$

Hence, for $k\geq 2$, a_k are not constants. This contradicts with the assumption that a_k are constants. Therefore, $\psi(z)=a_1z$. By (4.4), we have that

$$|a_1|^2|z|^2=\frac{1-|a_1|^2|z|^2}{1-|z|^2}|z|^2.$$

Therefore, we conclude that $|a_1|^2=1$, which completes the proof. □

Proof of Theorem 2.4. By [6, Theorem 3], we only need to prove

$$\int_{\mathbb{D}}(1-|z|^2)^{p+1}\Delta(|f(z)|^p)\,dA(z)<\infty. \tag{4.5}$$

From Theorem 2.2 we know that

$$\|D_f(z)\|\leq\frac{C_1}{1-|z|^2}, \tag{4.6}$$

where $C_1=(\alpha+1)2^{\alpha+1}$. Since f is α-harmonic, the function $(1-|z|^2)^{-\alpha}f_{\overline{z}}$ is antianalytic. Therefore it has a power series expansion of the form

$$(1-|z|^2)^{-\alpha}f_{\overline{z}}(z)=\sum_{k=0}^{\infty}a_k\overline{z}^k,\quad z\in\mathbb{D}.$$

By calculations, we have

$$\Delta f(z)=4f_{\overline{z}z}(z)=\frac{-4\alpha\overline{z}}{1-|z|^2}f_{\overline{z}}(z).$$

Hence, it follows from (4.6) that

$$|\Delta f(z)|\leq\frac{4\alpha}{1-|z|^2}\|D_f(z)\|\leq\frac{C_2}{(1-|z|^2)^2}, \tag{4.7}$$

where $C_2=\alpha(\alpha+1)2^{\alpha+3}$. Furthermore,

$$\begin{aligned}|f(z)| &\leq |f(0)|+\left|\int_{[0,z]} df(\xi)\right|\\ &\leq |f(0)|+\int_{[0,z]}\|D_f(\xi)\|\,|d\xi|\\ &\leq |f(0)|+\frac{C_1}{1-|z|^2},\end{aligned}$$

where $[0,z]$ denotes the line segment from 0 to z. Then, for $z\in\mathbb{D}$,

$$|f(z)|^{p-1}\leq 2^{p-2}\left(|f(0)|^{p-1}+\frac{C_1^{p-1}}{(1-|z|^2)^{p-1}}\right), \tag{4.8}$$

and

$$|f(z)|^{p-2}\leq 2^{p-2}\left(|f(0)|^{p-2}+\frac{C_1^{p-2}}{(1-|z|^2)^{p-2}}\right). \tag{4.9}$$

We divide the remaining part of the proof into two cases, namely, $p\in[4,\infty)$ and $p\in[2,4)$. For the case $p\in[4,\infty)$, a straightforward computation gives

$$\begin{aligned}\Delta(|f|^p) &= p(p-2)|f|^{p-4}|f\overline{f_z}+f_{\overline{z}}\overline{f}|^2+2p|f|^{p-2}(|f_z|^2+|f_{\overline{z}}|^2)+p|f|^{p-2}\mathrm{Re}\,(\overline{f}\Delta f)\\ &\leq p^2|f|^{p-2}\|D_f\|^2+p|f|^{p-1}|\Delta f|.\end{aligned}$$

Hence, by (4.6), (4.7), (4.8) and (4.9), we conclude that

$$\begin{aligned}&(1-|z|^2)^{p+1}\Delta(|f(z)|^p)\\ \leq\ & p^2(1-|z|^2)^{p+1}|f(z)|^{p-2}\|D_f(z)\|^2+p(1-|z|^2)^{p+1}|f(z)|^{p-1}|\Delta f(z)|\\ \leq\ & p^22^{p-2}(1-|z|^2)^{p+1}\left(|f(0)|^{p-2}+\frac{C_1^{p-2}}{(1-|z|^2)^{p-2}}\right)\frac{C_1^2}{(1-|z|^2)^2}\\ & +p2^{p-2}(1-|z|^2)^{p+1}\left(|f(0)|^{p-1}+\frac{C_1^{p-1}}{(1-|z|^2)^{p-1}}\right)\frac{C_2}{(1-|z|^2)^2}\\ \leq\ & p^22^{p-2}(|f(0)|^{p-2}+C_1^{p-2})C_1^2(1-|z|^2)+p2^{p-2}(|f(0)|^{p-1}+C_1^{p-1})C_2\\ <\ & \infty.\end{aligned}$$

In the case $p\in[2,4)$, let $F_n^p=(|f|^2+\frac{1}{n})^{\frac{p}{2}}$ for $n\in\{1,2,\ldots\}$. Then

$$\begin{aligned}\Delta(F_n^p) &= p(p-2)\Big(|f|^2+\frac{1}{n}\Big)^{\frac{p}{2}-2}|f\overline{f_z}+f_{\overline{z}}\overline{f}|^2\\ &\quad+2p\Big(|f|^2+\frac{1}{n}\Big)^{\frac{p}{2}-1}(|f_z|^2+|f_{\overline{z}}|^2)+p\Big(|f|^2+\frac{1}{n}\Big)^{\frac{p}{2}-1}\mathrm{Re}\,(\overline{f}\Delta f).\end{aligned}$$

Let

$$\begin{aligned}F &= p(p-2)|f|^{p-2}\|D_f(z)\|^2+2p\big(|f|^2+1\big)^{\frac{p}{2}-1}(|f_z|^2+|f_{\overline{z}}|^2)\\ &\quad+p\big(|f|^2+1\big)^{\frac{p}{2}-1}\mathrm{Re}\,(\overline{f}\Delta f).\end{aligned}$$

For $r \in (0,1)$, $\Delta(F_n^p)$ and F are integrable in $\mathbb{D}_r$, and $\Delta(F_n^p) \leq F$. By Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{D}_r}(1-|z|^2)^{p+1}\Delta\big(F_n^p(z)\big)\,dA(z) &= \int_{\mathbb{D}_r}(1-|z|^2)^{p+1}\lim_{n\to\infty}\Big(\Delta\big(F_n^p(z)\big)\Big)\,dA(z)\\ &\leq \int_{\mathbb{D}_r}(1-|z|^2)^{p+1}\Big[p^2|f(z)|^{p-2}\|D_f(z)\|^2\\ &\quad +p|f(z)|^{p-1}|\Delta f(z)|\Big]\,dA(z)\\ &\leq \int_{\mathbb{D}_r}\Big[p^2 2^{p-2}(|f(0)|^{p-2}+C_1^{p-2})C_1^2(1-|z|^2)\\ &\quad +p2^{p-2}(|f(0)|^{p-1}+C_1^{p-1})C_2\Big]\,dA(z)\\ &< \infty.\end{aligned}$$

Therefore, (4.5) follows from the above two estimates and so the proof of the theorem is complete. □

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