ON SOME LARGE GLOBAL SOLUTIONS FOR THE COMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM

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Abstract. In this paper we consider the global well-posedness of compressible magnetohydrodynamic system in $\mathbb{R}^d$ with $d \geq 2$, in the framework of the critical Besov spaces. We can show that if the initial data, the shear viscosity and the magnetic diffusion coefficient are small comparing with the volume viscosity, then compressible magnetohydrodynamic system has a unique global solution.

1. Introduction

The present paper is devoted to the equations of magnetohydrodynamics (MHD) which describe the motion of electrically conducting fluids in the presence of a magnetic field. The barotropic compressible magnetohydrodynamic system can be written as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) = b \cdot \nabla b - \frac{1}{2}\nabla(|b|^2) + \mu \Delta u & \\ \qquad +\nabla((\mu+\lambda)\mathrm{div} u), & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \partial_t b + (\mathrm{div} u) b + u \cdot \nabla b - b \cdot \nabla u - \nu \Delta b = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \\ \mathrm{div} b = 0, & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x) \in \mathbb{R}^+$ denotes the density, $u = u(t,x) \in \mathbb{R}^d$ and $b = b(t,x) \in \mathbb{R}^d$ stand for the velocity field and the magnetic field, respectively. The barotropic assumption means that the pressure $P = P(\rho)$ is given and assumed to be strictly increasing. The constant $\nu > 0$ is the resistivity acting as the magnetic diffusion coefficient of the magnetic field. The shear and volume viscosity coefficients μ and λ are constant and fulfill the standard strong parabolicity assumption:

$$\mu > 0, \qquad \kappa = \lambda + 2\mu > 0. \tag{1.2}$$

To complete the system (1.1), the initial data are supplemented by

$$(u, b, \rho)(t,x)|_{t=0} = (u_0(x), b_0(x), \rho_0(x)) \tag{1.3}$$

and also, as the space variable tends to infinity, we assume

$$\lim_{|x|\to\infty} \rho_0(x) = 1. \tag{1.4}$$

The system of MHD involves various topics such as the evolution and dynamics of astrophysical objects, thermonuclear fusion, metallurgy and semiconductor crystal growth, see for example [2, 4]. Roughly speaking, The system (1.1) is a coupling between the compressible Navier-Stokes equations with the magnetic equations (heat equations). On the other hand, notice that $b \equiv 0$, system (1.1) reduces to the usual compressible Navier-Stokes system for baratropic fluids. Due to its physical importance, complexity, rich phenomena and mathematical challenges, there have been huge literatures on the study of the compressible MHD problem (1.1) by many physicists and mathematicians, see

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for example, [2, 15, 16, 13, 17, 18, 19, 20, 21, 5, 4, 6, 22, 23, 24, 25] and the references therein. Now, we briefly recall some results concerned with the multi-dimensional compressible MHD equations in the absence of vacuum, which are more relatively with our problem. Kawashima [5] established the local and global well-posedness of the solutions to the compressible MHD equations as the initial density is strictly positive, see also Vol'pert-Khudiaev [7] and Strohmer [6] for the local existence results. To catch the scaling invariance property, Danchin first introduce in his series papers [8, 9, 10, 11, 12] the "Critical Besov Spaces" which were inspired by those efforts on the incompressible Navier-Stokes. Recently, Danchin et.al prove that the compressible Navier-Stokes system convergence to the homogeneous incompressible case for the large volume viscosity in [3]. Motivated this, our main goal of the present paper is devoted to extend the compressible Navier-Stokes system to the MHD system. That is, we will prove the global existence of strong solutions to (1.1) for a class of large initial data. We notice that if κ tends to $+\infty$, then velocity field and magnetic field will satisfy the incompressible MHD system:

$$\begin{cases} \partial_t U + U\cdot\nabla U - \mu\Delta U + \nabla\Pi - B\cdot\nabla B - \frac{1}{2}\nabla(|B|^2) = 0,\\ \partial_t B + U\cdot\nabla B - B\cdot\nabla U - \nu\Delta B = 0,\\ \mathrm{div}U = \mathrm{div}B = 0, \qquad (U,B)|_{t=0} = (U_0, B_0) \end{cases} \tag{1.5}$$

with $U_0 = \mathcal{P}u_0$ and $B_0 = b_0$. Here, the projectors $\mathcal{P}$ and $\mathcal{Q}$ are defined by

$$\mathcal{P} := \mathrm{Id} + (-\Delta)^{-1}\nabla\mathrm{div}, \qquad \mathcal{Q} := -(-\Delta)^{-1}\nabla\mathrm{div}.$$

Our main result can state be stated as follows:

Theorem 1.1. *Assume that $d\ge 2$, $u_0 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)$ and $a_0 := \rho_0 - 1 \in \dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d)\cap \dot{B}^{\frac{d}{2}}_{2,1}(\mathbb{R}^d)$. Suppose that (1.5) generates a unique global solution $(U,B)\in \mathcal{C}(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1}(\mathbb{R}^d))$ satisfying $U_0 := \mathcal{P}u_0$ and $B_0 = b_0$. Let C be a large universal constant and denote*

$$\begin{aligned} M &:= ||U,B||_{L^\infty(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B||_{L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1})},\\ D_0 &:= Ce^{C(1+\mu^{-1}+\nu^{-1})(M+1)^2}\big(||a_0,\mathcal{Q}v_0||_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \kappa||a_0||_{\dot{B}^{\frac{d}{2}}_{2,1}} + 1\big),\\ \delta_0 &:= Ce^{2C(1+\mu^{-2}+\nu^{-2})(M+1)^2}\big(\kappa^{-1}D_0^2 + \kappa^{-\frac{1}{2}}D_0\big). \end{aligned} \tag{1.6}$$

If κ is large enough and $||a_0||_{\dot{B}^{\frac{d}{2}}_{2,1}}$ is small enough such that

$$\kappa^{-1}D_0 \ll 1, \quad \delta_0(\frac{1}{\mu}+\frac{1}{\nu}+1) \le \frac{1}{2},$$

then (1.1) has a unique global-in-time solution (ρ,u,b) which satisfies

$$\begin{aligned} &u,b \in \mathcal{C}(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1(\mathbb{R}^+;\dot{B}^{\frac{d}{2}+1}_{2,1}),\\ &a := \rho - 1 \in C(\mathbb{R}^+;\dot{B}^{\frac{d}{2}-1}_{2,1}\cap\dot{B}^{\frac{d}{2}}_{2,1})\cap L^2(\mathbb{R}^+;\dot{B}^{\frac{d}{2}}_{2,1}). \end{aligned} \tag{1.7}$$

Remark 1.2. *If $d = 2$, according to Lemma 2.6, we can set*

$$M := C||U_0,B_0||_{\dot{B}^0_{2,1}}\exp\Big(C(\frac{1}{\mu^4}+\frac{1}{\nu^4})||U_0,B_0||^4_{L^2}\Big).$$

From Theorem 1.1, we deduce that the system (1.1) has a unique global-in-time solution without any smallness condition on the initial data. On the other hand, our result improves the the previous one due to Danchin et.al who considered the compressible Navier-Stokes system in [3].

2. Littlewood-Paley analysis

In this section, we recall the Littlewood-Paley theory, the definition of homogeneous Besov spaces and some useful properties. First, let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $\tilde{\mathcal{C}} = \{\xi \in \mathbb{R}^d, \frac{3}{4} \leq \xi \leq \frac{8}{3}\}$ such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \text{for all } \xi \neq 0.$$

The frequency localization operator $\dot{\Delta}_j$ and $\dot{S}_j$ are defined by

$$\dot{\Delta}_j f = \varphi(2^{-j}D)f = \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), \quad \dot{S}_j f = \sum_{k\leq j-1} \dot{\Delta}_k f \quad \text{for} \quad j \in \mathbb{Z}.$$

With a suitable choice of φ, one can easily verify that

$$\dot{\Delta}_j \dot{\Delta}_k f = 0 \quad \text{if} \quad |j-k| \geq 2, \quad \dot{\Delta}_j(\dot{S}_{k-1} f \dot{\Delta}_k f) = 0 \quad \text{if} \quad |j-k| \geq 5.$$

Now, we will introduce the definition of the homogeneous Besov space. We denote the space $\mathcal{Z}'(\mathbb{R}^d)$ by the dual space of $\mathcal{Z}(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d); D^\alpha \hat{f}(0) = 0; \ \forall \alpha \in \mathbb{N}^d\}$, which can be identified by the quotient space of $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ with the polynomials space $\mathcal{P}$. The formal equality $f = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f$ holds true for $f \in \mathcal{Z}'(\mathbb{R}^d)$ and is called the homogenous Littlewood-Paley decomposition.

The operators $\dot{\Delta}_j$ help us recall the definition of the homogenous Besov space (see [1])

Definition 2.1. *Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by*

$$\dot{B}^s_{p,r} = \{f \in \mathcal{Z}'(\mathbb{R}^d) : ||f||_{\dot{B}^s_{p,r}} < +\infty\},$$

where

$$||f||_{\dot{B}^s_{p,r}} \triangleq \Big|\Big|(2^{ks}||\dot{\Delta}_k f||_{L^p})_k\Big|\Big|_{\ell^r}.$$

Remark 2.2. *Let $\mathcal{C}'$ be an annulus and $(u_j)_{j\in\mathbb{Z}}$ be a sequence of functions such that*

$$Supp\ \hat{u}_j \subset 2^j\mathcal{C}' \quad and \quad ||(2^{js}||u_j||_{L^p})_{j\in\mathbb{Z}}||_{\ell^r} < \infty.$$

There exists a constant C_s depending on s such that

$$||u||_{\dot{B}^s_{p,r}} \leq C_s||(2^{js}||u_j||_{L^p})_{j\in\mathbb{Z}}||_{\ell^r}.$$

Next, we give the important product acts on homogenous Besov spaces by collecting some useful lemmas from [1].

Lemma 2.3. *Let $s_1, s_2 \leq \frac{d}{2}$, $s_1 + s_2 > 0$ and $(f, g) \in \dot{B}^{s_1}_{2,1}(\mathbb{R}^d) \times \dot{B}^{s_2}_{2,1}(\mathbb{R}^d)$. Then we have*

$$||fg||_{\dot{B}^{s_1+s_2-\frac{d}{2}}_{2,1}} \leq C||f||_{\dot{B}^{s_1}_{2,1}}||g||_{\dot{B}^{s_2}_{2,1}}.$$

Lemma 2.4. *Assume that $F \in W^{[\sigma]+2,\infty}_{loc}(\mathbb{R})$ with $F(0) = 0$. Then for any $f \in L^\infty(\mathbb{R}^d) \cap \dot{B}^s_{2,1}(\mathbb{R}^d)$, we have*

$$||F(f)||_{\dot{B}^s_{2,1}} \leq C(||f||_{L^\infty})||f||_{\dot{B}^s_{2,1}}.$$

Lemma 2.5. *For $(p, r_1, r_2, r) \in [1,\infty]^4$, $s_1 \neq s_2$ and $\theta \in (0, 1)$, the following interpolation inequality holds*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} \leq C\|u\|^{\theta}_{\dot{B}^{s_1}_{p,r_1}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r_2}}.$$

Proposition 2.6. *Let $U_0, B_0 \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ with $\mathrm{div}U_0 = \mathrm{div}B_0 = 0$. Then there exists a unique solution to (1.5) such that*

$$U, B \in \mathcal{C}(\mathbb{R}^+; \dot{B}^0_{2,1}(\mathbb{R}^2)) \cap L^1(\mathbb{R}^+; \dot{B}^2_{2,1}(\mathbb{R}^2)).$$

Furthermore, there exists some universal constant C, one has for all $T \geq 0$,

$$\begin{aligned}&||U, B||_{L^\infty_T(\dot{B}^0_{2,1})} + ||U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B||_{L^1_T(\dot{B}^0_{2,1})}\\ &\leq C||U_0, B_0||_{\dot{B}^0_{2,1}} \exp\Big(C(\frac{1}{\mu^4} + \frac{1}{\nu^4})||U_0, B_0||^4_{L^2}\Big).\end{aligned}$$

Proof. For any $t \in [0, T]$, the standard energy balance reads:

$$||U(t)||^2_{L^2} + ||B(t)||^2_{L^2} + 2\mu\int_0^t ||\nabla U||^2_{L^2} + 2\nu\int_0^t ||\nabla B||^2_{L^2} = ||U_0||^2_{L^2} + ||B_0||^2_{L^2},$$

which implies for all $T \geq 0$,

$$\mu^{\frac{1}{4}}||U||_{L^4_T(\dot{B}^{\frac{1}{2}}_{2,1})} + \nu^{\frac{1}{4}}||B||_{L^4_T(\dot{B}^{\frac{1}{2}}_{2,1})} \leq C||U_0, B_0||_{L^2}. \tag{2.1}$$

From the estimates of the Stokes system in homogeneous Besov spaces, we have

$$\begin{aligned}&||U, B||_{L^\infty_T(\dot{B}^0_{2,1})} + ||U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B||_{L^1_T(\dot{B}^0_{2,1})}\\ &\leq C\big(||U_0, B_0||_{\dot{B}^0_{2,1}} + ||U\cdot\nabla U - B\cdot\nabla B||_{L^1_T(\dot{B}^0_{2,1})} + ||B\cdot\nabla U - U\cdot\nabla B||_{L^1_T(\dot{B}^0_{2,1})}\big).\end{aligned} \tag{2.2}$$

In view of the interpolation inequality and Young inequality, we deduce that

$$\begin{aligned}||U\cdot\nabla U||_{L^1_T(\dot{B}^0_{2,1})} &\leq C\int_0^T ||U||_{\dot{B}^{\frac{1}{2}}_{2,1}}||\nabla U||_{\dot{B}^{\frac{1}{2}}_{2,1}}\mathrm{d}t\\ &\leq \int_0^T ||U||_{\dot{B}^{\frac{1}{2}}_{2,1}}||\nabla U||^{\frac{1}{4}}_{\dot{B}^{-1}_{2,1}}||\nabla U||^{\frac{3}{4}}_{\dot{B}^1_{2,1}}\mathrm{d}t\\ &\leq \frac{C}{\varepsilon^3\mu^3}\int_0^T ||U||^4_{\dot{B}^{\frac{1}{2}}_{2,1}}||U||_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\mu||\nabla^2 U||_{L^1_T(\dot{B}^0_{2,1})}.\end{aligned} \tag{2.3}$$

Similar argument as in (2.3), we obtain

$$\begin{aligned}||B\cdot\nabla B||_{L^1_T(\dot{B}^0_{2,1})} &\leq \frac{C}{\varepsilon^3\nu^3}\int_0^T ||B||^4_{\dot{B}^{\frac{1}{2}}_{2,1}}||B||_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\nu||\nabla^2 B||_{L^1_T(\dot{B}^0_{2,1})},\\ ||U\cdot\nabla B||_{L^1_T(\dot{B}^0_{2,1})} &\leq \frac{C}{\varepsilon^3\nu^3}\int_0^T ||U||^4_{\dot{B}^{\frac{1}{2}}_{2,1}}||B||_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\nu||\nabla^2 B||_{L^1_T(\dot{B}^0_{2,1})},\\ ||B\cdot\nabla U||_{L^1_T(\dot{B}^0_{2,1})} &\leq \frac{C}{\varepsilon^3\mu^3}\int_0^T ||B||^4_{\dot{B}^{\frac{1}{2}}_{2,1}}||U||_{\dot{B}^0_{2,1}}\mathrm{d}t + \varepsilon\mu||\nabla^2 U||_{L^1_T(\dot{B}^0_{2,1})}.\end{aligned} \tag{2.4}$$

Therefore, combing (2.2)-(2.4) and choosing ε small enough, we find that

$$\begin{aligned}&||U, B||_{L^\infty_T(\dot{B}^0_{2,1})} + ||U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B||_{L^1_T(\dot{B}^0_{2,1})}\\ &\leq C\Big((\frac{1}{\mu^3} + \frac{1}{\nu^3})\int_0^T (||U||^4_{\dot{B}^{\frac{1}{2}}_{2,1}} + ||B||^4_{\dot{B}^{\frac{1}{2}}_{2,1}})(||U||_{\dot{B}^0_{2,1}} + ||B||_{\dot{B}^0_{2,1}})\mathrm{d}t + ||U_0, B_0||_{\dot{B}^0_{2,1}}\Big).\end{aligned}$$

It follows from the Gronwall inequality and (2.1) that the desired result of this lemma. □

3. The proof of the main results

In this section, we shall give the main details for the proof of Theorem 1.1. Our main idea basically follows from the recent work in [3]

Setting $a = \rho - 1$, we infer from (1.1) that

$$\begin{cases} \partial_t a + \operatorname{div}(au) + \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + P'(1+a)\nabla a - b \cdot \nabla b + \frac{1}{2}\nabla(|b|^2) - \mu\Delta u - \nabla((\mu+\lambda)\operatorname{div} u) \\ \qquad = -a(u_t + u \cdot \nabla u), \\ \partial_t b + (\operatorname{div} u) b + u \cdot \nabla b - b \cdot \nabla u - \nu \dot{\Delta} b = 0, \\ \operatorname{div} b = 0. \end{cases} \tag{3.1}$$

Before continue on, we recall the following local well-posedness of the system (3.1).

Theorem 3.1. [14] *Assume that the initial data $(a_0 := \rho - 1, u_0, b_0)$ satisfy $\operatorname{div} b_0 = 0$ and*

$$(a_0, u_0, b_0) \in \dot{B}^{\frac{d}{2}}_{2,1} \times \dot{B}^{\frac{d}{2}-1}_{2,1} \times \dot{B}^{\frac{d}{2}-1}_{2,1}.$$

In addition, $\inf_{x\in\mathbb{R}^d} a_0(x) > -1$, then there exists some time $T > 0$ such that the system (3.1) has a local unique solution (a, u, b) on $[0,T] \times \mathbb{R}^d$ which belongs to the function space

$$E_T := \widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{d}{2}}_{2,1}) \times (\widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1_T \dot{B}^{\frac{d}{2}+1}_{2,1})^{2d},$$

where $\widetilde{\mathcal{C}}([0,T]; \dot{B}^s_{q,1}) := \mathcal{C}([0,T]; \dot{B}^s_{q,1}) \cap \widetilde{L}^\infty([0,T]; \dot{B}^s_{q,1})$ with $s \in \mathbb{R}$ and $1 \le q \le \infty$.

We set

$$v = u - U \quad \text{and} \quad c = b - B.$$

From the very beginning, the potential $\mathcal{Q}v$ and divergence-free $\mathcal{P}v$ parts of v are treated separately. Applying $\mathcal{Q}$ to the velocity equation of (3.1) and noticing that $\mathcal{Q}v = \mathcal{Q}u$ yield

$$\partial_t(\mathcal{Q}v) + \mathcal{Q}((v+U)\cdot\nabla\mathcal{Q}v) - \kappa\dot{\Delta}\mathcal{Q}v + \nabla a = -\mathcal{Q}(aU_t + av_t) - \mathcal{Q}R_1, \tag{3.2}$$

where, denoting $k(a) = P'(1+a) - 1$

$$\begin{aligned} R_1 = &(1+a)(v+U)\cdot\nabla\mathcal{P}v + (1+a)(v+U)\cdot\nabla U + a(v+U)\cdot\nabla\mathcal{Q}v \\ &+ k(a)\nabla a - (B+c)\cdot\nabla(B+c) + \frac{1}{2}\nabla(|B+c|^2). \end{aligned} \tag{3.3}$$

In view of the density equation of (3.1) and using $u = \mathcal{Q}v + \mathcal{P}v + U$, we find that a satisfies

$$\partial_t a + (v+U)\cdot\nabla a + \operatorname{div}\mathcal{Q}v = -a\operatorname{div}\mathcal{Q}v. \tag{3.4}$$

Because $\mathcal{P}U = U$ and $\mathcal{P}(\mathcal{Q}v\cdot\nabla\mathcal{Q}v) = \mathcal{P}(a\nabla a) = 0$, applying $\mathcal{P}$ to the velocity equation of (3.1), we discover that

$$\partial_t(\mathcal{P}v) + \mathcal{P}((v+U)\cdot\nabla\mathcal{P}v) - \mu\dot{\Delta}\mathcal{P}v = -\mathcal{P}(aU_t + av_t + a\nabla a) - \mathcal{P}R_2, \tag{3.5}$$

where

$$\begin{aligned} R_2 = &(1+a)(v+U)\cdot\nabla\mathcal{Q}v + (1+a)v\cdot\nabla U + a(v+U)\cdot\nabla\mathcal{P}v \\ &+ aU\cdot\nabla U - (B+c)\cdot\nabla c - c\cdot\nabla B \\ = &(1+a)\mathcal{P}v\cdot\nabla(U+\mathcal{Q}v) + (1+a)U\cdot\nabla\mathcal{Q}v + (1+a)\mathcal{Q}v\cdot\nabla U \\ &+ a(v+U)\cdot\nabla\mathcal{P}v + aU\cdot\nabla U + a\mathcal{Q}v\cdot\nabla\mathcal{Q}v - (B+c)\cdot\nabla c - c\cdot\nabla B. \end{aligned} \tag{3.6}$$

According to the magnetic equation of (3.1), we can show that c satisfies

$$\partial_t c + (v+U)\cdot\nabla c - \nu\dot{\Delta} c = -R_3, \tag{3.7}$$

where

$$R_3 = (\mathrm{div}\mathcal{Q}v)B + (\mathrm{div}\mathcal{Q}v)c + v\cdot\nabla B - (B+c)\cdot\nabla v - c\cdot\nabla U. \tag{3.8}$$

In the sequel, we denote a^ℓ and a^h the low and high frequencies parts of a as

$$a^\ell = \sum_{2^j\kappa\le 1}\dot{\Delta}_j a, \qquad a^h = \sum_{2^j\kappa>1}\dot{\Delta}_j a,$$

and set

$$\begin{aligned}
&X_d(T) = ||\mathcal{Q}v, a, \kappa\nabla a||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||\mathcal{Q}v_t + \nabla a, \kappa\nabla^2\mathcal{Q}v, \kappa\nabla^2 a^\ell, \nabla a^h||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})},\\
&Y_d(T) = Y_{d,1}(T) + Y_{d,2}(T) := ||\mathcal{P}v, c||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||\mathcal{P}v_t, c_t, \mu\nabla^2\mathcal{P}v, \nu\nabla^2 c||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})},\\
&Z_d(T) = ||U, B||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||U_t, B_t, \mu\nabla^2 U, \nu\nabla^2 B||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})},\\
&X_d(0) = ||a_0, \mathcal{Q}v_0||_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \kappa||a_0||_{\dot{B}^{\frac{d}{2}}_{2,1}}.
\end{aligned}$$

It is easy to show that

$$Z_d(T) \le M \quad \text{for all } T>0. \tag{3.9}$$

We concentrate our attention on the proof global in time a priori estimates, as the local well-posedness issue has been ensured by Theorem 3.1. We claim that if κ is large enough then one may find some (large) D and (small) δ so that there holds for all $T<T^*$,

$$\begin{cases} X_d(T)\le D, \quad Y_d(T)\le\delta, \quad \kappa^{-1}D\ll 1,\\ \delta(\frac{1}{\mu}+\frac{1}{\nu}+1)\le 1, \quad D\ge (M+1), \quad ||a(t,\cdot)||_{L^\infty}\le\frac12. \end{cases} \tag{3.10}$$

Step 1. Estimate on the terms $\mathcal{P}v$ and c.

We first consider the estimates for $\mathcal{P}v$. Applying $\dot{\Delta}_j$ to (3.5), taking the L^2 inner product with $\dot{\Delta}_j\mathcal{P}v$ then using that $\mathcal{P}^2=\mathcal{P}$, we deduce that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}||\dot{\Delta}_j\mathcal{P}v||^2_{L^2} + \mu||\nabla\dot{\Delta}_j\mathcal{P}v||^2_{L^2} = \int_{\mathbb{R}^d}([v+U,\dot{\Delta}_j]\cdot\nabla\mathcal{P}v)\cdot\dot{\Delta}_j\mathcal{P}v\mathrm{d}x \tag{3.11}$$

$$-\int_{\mathbb{R}^d}\dot{\Delta}_j(aU_t + av_t + a\nabla a + R_2)\cdot\dot{\Delta}_j\mathcal{P}v\mathrm{d}x - \frac12\int_{\mathbb{R}^d}|\dot{\Delta}_j\mathcal{P}v|^2\mathrm{div}v\mathrm{d}x. \tag{3.12}$$

According to the commutator estimates of Lemma 2.100 in [1], the commutator term may be estimated as follows:

$$2^{j(\frac{d}{2}-1)}||[v+U,\dot{\Delta}_j]\cdot\nabla\mathcal{P}v||_{L^2} \le Cc_j||\nabla(v+U)||_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}, \quad \text{with} \quad ||c_j||_{l^1}=1. \tag{3.13}$$

Now, multiplying both sides of (3.11) by $2^{j(\frac{d}{2}-1)}$ and summing up over $j\in\mathbb{Z}$, using Lemma 2.3 and (3.13), we obtain that

$$\begin{aligned}
&||\mathcal{P}v||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + \mu||\nabla^2\mathcal{P}v||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \le C\int_0^T ||\nabla(v+U)||_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t\\
&+ C||a(U_t+\mathcal{P}v_t+(\mathcal{Q}v_t+\nabla a))||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + C||R_2||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}.
\end{aligned} \tag{3.14}$$

In order to bound $||\mathcal{P}v_t||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}$, we infer from (3.5) and (3.14) that

$$\begin{aligned}
&||\mathcal{P}v||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||\mathcal{P}v_t, \mu\nabla^2\mathcal{P}v||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \leq C\int_0^T ||\nabla(v+U)||_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t \\
&\quad + C\int_0^T ||(v+U)\cdot\nabla\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t + C\int_0^T ||a(U_t+\mathcal{P}v_t+(\mathcal{Q}v_t+\nabla a))||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t \\
&\quad + C\int_0^T ||R_2||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t.
\end{aligned} \tag{3.15}$$

Next, we will estimate the Besov norm of the right-hand side for (3.15). For the second term of the right-hand side for (3.15), we can infer from Lemma 2.3 that

$$\begin{aligned}
&||(v+U)\cdot\nabla\mathcal{P}v||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
&\leq C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t + C\int_0^T ||\mathcal{Q}v+U||_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}}_{2,1}}\mathrm{d}t \\
&\leq C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t \\
&\quad + \frac{C}{\varepsilon\mu}\int_0^T ||(\mathcal{Q}v,U)||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t + C\mu\varepsilon||\nabla^2\mathcal{P}v||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
&\leq C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t + C\varepsilon Y_d(T) + \frac{C}{\varepsilon\mu}\int_0^T ||(\mathcal{Q}v,U)||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t.
\end{aligned} \tag{3.16}$$

For the third term of the right-hand side for (3.15), we can infer from that

$$\begin{aligned}
&\int_0^T ||a(U_t+\mathcal{P}v_t+(\mathcal{Q}v_t+\nabla a))||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t \\
&\leq C||a||_{L^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})}(||U_t||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||\mathcal{P}v_t||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||\mathcal{Q}v_t+\nabla a||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}) \\
&\leq C\kappa^{-1}X_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big).
\end{aligned} \tag{3.17}$$

For the last term of the right-hand side for (3.15), in view of Lemma 2.3, we can estimate them into the following parts:

$$\begin{aligned}
&||(1+a)\mathcal{P}v\cdot\nabla(U+\mathcal{Q}v)||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
&\leq C\int_0^T (1+||a||_{\dot{B}^{\frac{d}{2}}_{2,1}})||U+\mathcal{Q}v||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&||(1+a)(\mathcal{Q}v\cdot\nabla U+U\cdot\nabla\mathcal{Q}v)||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \\
&\leq C(1+||a||_{L^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})})||\mathcal{Q}v||_{L^2_T(\dot{B}^{\frac{d}{2}}_{2,1})}||U||_{L^2_T(\dot{B}^{\frac{d}{2}}_{2,1})} \\
&\leq C(1+\kappa^{-1}X_d(T))\kappa^{-\frac{1}{2}}\mu^{-\frac{1}{2}}X_d(T)Z_d(T),
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
&\|a(v+U)\cdot\nabla\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}\|v+U\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\\
&\le C\kappa^{-1}\mu^{-1}X_d(T)Y_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big),
\end{aligned}\tag{3.20}$$

$$\begin{aligned}
&\|aU\cdot\nabla U+a\mathcal{Q}v\cdot\nabla\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C\|a\|_{L^\infty_T(\dot B^{\frac d2}_{2,1})}(\|U\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|U\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}+\|\mathcal{Q}v\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}\|\mathcal{Q}v\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})})\\
&\le C\kappa^{-1}X_d(T)\Big(\mu^{-1}Z_d^2(T)+\kappa^{-1}X_d^2(T)\Big),
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
&\|(B+c)\cdot\nabla c+c\cdot\nabla B\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C\int_0^T\|c\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+C\int_0^T\|B\|_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t.\\
&\le C\int_0^T\|c\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+C\varepsilon\nu\|c\|_{L^1_T(\dot B^{\frac d2+1}_{2,1})}+\frac{C}{\varepsilon\nu}\int_0^T\|B\|^2_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\
&\le C\int_0^T\|c\|_{\dot B^{\frac d2-1}_{2,1}}\|c\|_{\dot B^{\frac d2+1}_{2,1}}\mathrm{d}t+C\varepsilon Y_d(T)+\frac{C}{\varepsilon\nu}\int_0^T\|B\|^2_{\dot B^{\frac d2}_{2,1}}\|c\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t.
\end{aligned}\tag{3.22}$$

Therefore, summing up (3.15)-(3.22), we obtain

$$\begin{aligned}
&\|\mathcal{P}v\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}+\|\mathcal{P}v_t,\mu\nabla^2\mathcal{P}v\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C\varepsilon Y_d(T)+C\kappa^{-1}X_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big)\\
&\quad+C(1+\kappa^{-1}X_d(T))\kappa^{-\frac12}\mu^{-\frac12}X_d(T)Z_d(T)+C\kappa^{-1}X_d(T)\Big(\mu^{-1}Z_d^2(T)+\kappa^{-1}X_d^2(T)\Big)\\
&\quad+C\kappa^{-1}\mu^{-1}X_d(T)Y_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big)\\
&\quad+C\int_0^T\Big((1+\|a\|_{\dot B^{\frac d2}_{2,1}})\|(\mathcal{P}v,\mathcal{Q}v,U,c)\|_{\dot B^{\frac d2+1}_{2,1}}+\frac{1}{\varepsilon\mu}\|(\mathcal{Q}v,U)\|^2_{B^{\frac d2}_{2,1}}+\frac{1}{\varepsilon\nu}\|B\|^2_{B^{\frac d2}_{2,1}}\Big)Y_{d,1}\mathrm{d}t.
\end{aligned}\tag{3.23}$$

Now, we estimate the term for c. Similar argument as in (3.15) and (3.16), we infer from (3.7) that

$$\begin{aligned}
&\|c\|_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}+\|(c_t,\nu\nabla^2c)\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le\|b_0-B_0\|_{\dot B^{\frac d2-1}_{2,1}}+C\int_0^T\|(\mathcal{P}v,\mathcal{Q}v,U)\|_{\dot B^{\frac d2+1}_{2,1}}\|c\|_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\
&\quad+C\|(v+U)\cdot\nabla c\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}+C\|R_3\|_{L^1_T(\dot B^{\frac d2-1}_{2,1})}.
\end{aligned}\tag{3.24}$$

For the last two terms of the right-hand side for (3.24), according to Lemma 2.3, we can tackle with them as follows:

$$\begin{aligned}
&||(v+U)\cdot\nabla c||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\
\le\, & C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t + C\int_0^T ||\mathcal{Q}v+U||_{\dot{B}^{\frac{d}{2}}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t\\
\le\, & C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t + \frac{C}{\varepsilon\nu}\int_0^T ||(\mathcal{Q}v,U)||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t + C\nu\varepsilon||c||_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
\le\, & C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}\mathrm{d}t + C\varepsilon Y_d(T) + \frac{C}{\varepsilon\nu}\int_0^T ||(\mathcal{Q}v,U)||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t.
\end{aligned} \tag{3.25}$$

$$||(\mathrm{div}\mathcal{Q}v)c - c\cdot\nabla v - c\cdot\nabla U||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} \le C\int_0^T ||(U,\mathcal{P}v,\mathcal{Q}v)||_{\dot{B}^{\frac{d}{2}+1}_{2,1}}||c||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\mathrm{d}t, \tag{3.26}$$

$$\begin{aligned}
&||(\mathrm{div}\mathcal{Q}v)B + v\cdot\nabla B - B\cdot\nabla v||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\
\le\, & C\int_0^T ||\mathcal{Q}v||_{\dot{B}^{\frac{d}{2}}_{2,1}}||B||_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t + C\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}}_{2,1}}||B||_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t\\
\le\, & C\int_0^T ||\mathcal{Q}v||_{\dot{B}^{\frac{d}{2}}_{2,1}}||B||_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t + \frac{C}{\varepsilon\nu}\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||B||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t + C\varepsilon\nu||\mathcal{P}v||_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})}\\
\le\, & C\kappa^{-\frac12}\nu^{-\frac12}X_d(T)Z_d(T) + C\varepsilon Y_d(T) + \frac{C}{\varepsilon\nu}\int_0^T ||\mathcal{P}v||_{\dot{B}^{\frac{d}{2}-1}_{2,1}}||B||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\,\mathrm{d}t.
\end{aligned} \tag{3.27}$$

Hence, collecting the estimates (3.24)-(3.27), we get

$$\begin{aligned}
&||c||_{L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})} + ||(c_t,\nu\nabla^2 c)||_{L^1_T(\dot{B}^{\frac{d}{2}-1}_{2,1})}\\
\le\, & ||b_0 - B_0||_{\dot{B}^{\frac{d}{2}-1}_{2,1}} + \frac{C}{\nu}Y_d^2(T) + C\varepsilon Y_d(T) + C\kappa^{-\frac12}\nu^{-\frac12}X_d(T)Z_d(T)\\
&+ C\int_0^T \Big(||(\mathcal{P}v,\mathcal{Q}v,U,c)||_{\dot{B}^{\frac{d}{2}+1}_{2,1}} + \frac{1}{\varepsilon\nu}||(\mathcal{Q}v,U,B)||^2_{\dot{B}^{\frac{d}{2}}_{2,1}}\Big)Y_d\mathrm{d}t.
\end{aligned} \tag{3.28}$$

Then, combining (3.23) and (3.28) and choosing ε small enough, we can conclude from Gronwall's inequality that

$$\begin{aligned}
Y_d(T) \le\, & Ce^{C||\mathcal{P}v,\mathcal{Q}v,U,c||_{L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})} + (\frac{C}{\mu}+\frac{C}{\nu})||(\mathcal{Q}v,U,B)||^2_{L^1_T(\dot{B}^{\frac{d}{2}}_{2,1})}}\Big\{\kappa^{-1}X_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big)\\
&+ (1+\kappa^{-1}X_d(T))\kappa^{-\frac12}\mu^{-\frac12}X_d(T)Z_d(T) + \kappa^{-1}X_d(T)\Big(\mu^{-1}Z_d^2(T)+\kappa^{-1}X_d^2(T)\Big)\\
&+ \kappa^{-1}\mu^{-1}X_d(T)Y_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big) + \kappa^{-\frac12}\nu^{-\frac12}X_d(T)Z_d(T)\Big\}.
\end{aligned} \tag{3.29}$$

Step 2. Estimate on the terms $\mathcal{Q}v$ and a.
Now, applying $\dot{\Delta}_j$ to (3.1) and (3.2) yields that

$$\begin{cases}\partial_t a_j + (v+U)\cdot\nabla a_j + \mathrm{div}\mathcal{Q}v_j = g_j,\\ \partial_t\mathcal{Q}v_j + \mathcal{Q}((v+U)\cdot\nabla\mathcal{Q}v_j) - \kappa\dot{\Delta}\mathcal{Q}v_j + \nabla a_j = f_j,\end{cases} \tag{3.30}$$

where

$$\begin{aligned}&a_j=\dot{\Delta}_j a,\quad \mathcal{Q}v_j=\dot{\Delta}_j\mathcal{Q}v,\quad g_j=-\dot{\Delta}_j(a\mathrm{div}\mathcal{Q}v)-[\dot{\Delta}_j,(v+U)]\cdot\nabla a,\\&f_j=-\dot{\Delta}_j\mathcal{Q}(aU_t+av_t)-\dot{\Delta}_j\mathcal{Q}R_1-\mathcal{Q}[\dot{\Delta}_j,(v+U)]\cdot\nabla\mathcal{Q}v.\end{aligned}\tag{3.31}$$

We take the L^2 inner product for the first equation of (3.30) with a_j and the second equation of (3.30) with $\mathcal{Q}v_j$ to obtain

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||a_j||_{L^2}^2+(a_j,\mathrm{div}\mathcal{Q}v_j)=\frac{1}{2}(\mathrm{div}v,a_j^2)+(g_j,a_j),\\ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\mathcal{Q}v_j||_{L^2}^2+\kappa||\nabla\mathcal{Q}v_j||_{L^2}^2-(a_j,\mathrm{div}\mathcal{Q}v_j)=\frac{1}{2}(\mathrm{div}v,|\mathcal{Q}v_j|^2)+(f_j,\mathcal{Q}v_j),\end{cases}\tag{3.32}$$

We next want to estimate for $||\nabla a_j||_{L^2}^2$. From the first equation of (3.30), we have

$$\partial_t\nabla a_j+(v+U)\cdot\nabla\nabla a_j+\nabla\mathrm{div}\mathcal{Q}v_j=\nabla g_j-\nabla(v+U)\cdot\nabla a_j.\tag{3.33}$$

Following (3.33) and second equation of (3.30) and taking the L^2 inner product, we obtain

$$\begin{cases}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\nabla a_j||_{L^2}^2+((v+U)\cdot\nabla\nabla a_j,\nabla a_j)+(\nabla\mathrm{div}\mathcal{Q}v_j,\nabla a_j)\\ \qquad=(\nabla g_j-\nabla(v+U)\cdot\nabla a_j,\nabla a_j),\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{Q}v_j,\nabla a_j)+(v+U,\nabla(\mathcal{Q}v_j\cdot\nabla a_j))-\kappa(\dot{\Delta}\mathcal{Q}v_j,\nabla a_j)+||\nabla a_j||_{L^2}^2\\ \qquad+(\nabla\mathrm{div}\mathcal{Q}v_j,\mathcal{Q}v_j)=(\nabla g_j-\nabla(v+U)\cdot\nabla a_j,\mathcal{Q}v_j)+(f_j,\nabla a_j).\end{cases}\tag{3.34}$$

Noticing that $(\nabla\mathrm{div}\mathcal{Q}v_j,\nabla a_j)=(\dot{\Delta}\mathcal{Q}v_j,\nabla a_j)$ and $\dot{\Delta}\mathcal{Q}v_j=\nabla\mathrm{div}\mathcal{Q}v_j$, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\kappa||\nabla a_j||_{L^2}^2+2(\mathcal{Q}v_j\cdot\nabla a_j))+(||\nabla a_j||_{L^2}^2-||\nabla\mathcal{Q}v_j||_{L^2}^2)\\&=(\frac{1}{2}\kappa|\nabla a_j|^2+\mathcal{Q}v_j\cdot\nabla a_j,\mathrm{div}v)+\kappa(\nabla g_j-\nabla(v+U)\cdot\nabla a_j,\nabla a_j)\\&\quad+(\nabla g_j-\nabla(v+U)\cdot\nabla a_j,\mathcal{Q}v_j)+(f_j,\nabla a_j).\end{aligned}\tag{3.35}$$

Multiplying (3.35) by κ and adding up twice (3.32) yield

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_j^2+\kappa(||\nabla\mathcal{Q}v_j||_{L^2}^2+||\nabla a_j||_{L^2}^2)\\&=\int_{\mathbb{R}^d}(2g_ja_j+2f_j\cdot\mathcal{Q}v_j+\kappa^2\nabla g_j\cdot\nabla a_j+\kappa\nabla g_j\cdot\mathcal{Q}v_j+\kappa f_j\cdot\nabla a_j)\mathrm{d}x\\&\quad+\frac{1}{2}\int_{\mathbb{R}^d}\mathcal{L}_j^2\mathrm{div}v\mathrm{d}x-\kappa\int_{\mathbb{R}^d}(\nabla(v+U)\cdot\nabla a_j)\cdot(\kappa\nabla a_j+\mathcal{Q}v_j)\mathrm{d}x,\end{aligned}\tag{3.36}$$

with

$$\begin{aligned}\mathcal{L}_j^2&=\int_{\mathbb{R}^d}(2a_j^2+2|\mathcal{Q}v_j|^2+2\kappa\mathcal{Q}v_j\cdot\nabla a_j+|\kappa\nabla a_j|^2)\mathrm{d}x\\&=\int_{\mathbb{R}^d}(2a_j^2+|\mathcal{Q}v_j|^2+|\mathcal{Q}v_j+\kappa\nabla a_j|^2)\mathrm{d}x\approx||(\mathcal{Q}v_j,a_j,\kappa\nabla a_j)||_{L^2}^2.\end{aligned}\tag{3.37}$$

By (3.37), we obtain

$$\kappa(||\nabla\mathcal{Q}v_j||_{L^2}^2+||\nabla a_j||_{L^2}^2)\geq c\min(\kappa 2^{2j},\kappa^{-1})\mathcal{L}_j^2,$$

which along with (3.36) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_j^2+c\min(\kappa 2^{2j},\kappa^{-1})\mathcal{L}_j^2\leq&(\frac{1}{2}||\mathrm{div}v||_{L^\infty}+||\nabla(v+U)||_{L^\infty})\mathcal{L}_j^2\\&+C||(g_j,f_j,\kappa\nabla g_j)||_{L^2}\mathcal{L}_j.\end{aligned}\tag{3.38}$$

Multiplying both sides of (3.38) by $2^{j(\frac{d}{2}-1)}$ and then summing up over $j\in\mathbb{Z}$, we infer from Remark 2.2 that

$$\begin{aligned}
&||(a,\kappa\nabla a,\mathcal{Q}v)||_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}+||(\kappa\nabla^2\mathcal{Q}v,\kappa\nabla^2a^\ell,\nabla a^h)||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C||(a,\kappa\nabla a,\mathcal{Q}v)(0)||_{\dot B^{\frac d2-1}_{2,1}}+C\int_0^T||(v,U)||_{\dot B^{\frac d2+1}_{2,1}}||(a,\kappa\nabla a,\mathcal{Q}v)||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\
&\quad+C\int_0^T\sum_{j\in\mathbb{Z}}2^{j(\frac d2-1)}||(g_j,f_j,\kappa\nabla g_j)||_{L^2}\mathrm{d}t.
\end{aligned}\tag{3.39}$$

Combining the estimates

$$||a\mathrm{div}\mathcal{Q}v,\kappa\nabla(a\mathrm{div}\mathcal{Q}v)||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\le C\int_0^T||\mathrm{div}\mathcal{Q}v||_{\dot B^{\frac d2}_{2,1}}||a,\kappa\nabla a||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,$$

and

$$\begin{aligned}
&\int_0^T\sum_j 2^{j(\frac d2-1)}||[\dot\Delta_j,(v+U)]\nabla a,\kappa\nabla([\dot\Delta_j,(v+U)]\nabla a)||_{L^2}\mathrm{d}t\\
&\le C\int_0^T||\nabla(v+U)||_{\dot B^{\frac d2}_{2,1}}||a,\kappa\nabla a||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,
\end{aligned}$$

we have

$$\int_0^T\sum_{j\in\mathbb{Z}}2^{j(\frac d2-1)}||(g_j,\kappa\nabla g_j)||_{L^2}\mathrm{d}t\le C\int_0^T||(v,U)||_{\dot B^{\frac d2+1}_{2,1}}||(a,\kappa\nabla a)||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t.\tag{3.40}$$

Next, we will estimate the last term $\int_0^T\sum_{j\in\mathbb{Z}}2^{j(\frac d2-1)}||f_j||_{L^2}\mathrm{d}t$. According to Lemmas 2.3-2.4 and the commutator estimates of Lemma 2.100 in [1], we have

$$\begin{aligned}
&||(1+a)(v+U)\cdot\nabla(\mathcal{P}v+U)||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\
&\le C(1+||a||_{L^\infty_T(\dot B^{\frac d2}_{2,1})})||(\mathcal{P}v,U)||_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}||(\nabla\mathcal{P}v,\nabla U)||_{L^1_T(\dot B^{\frac d2}_{2,1})}\\
&\quad+C\int_0^T(1+||a||_{\dot B^{\frac d2}_{2,1}})||(\nabla\mathcal{P}v,\nabla U)||_{\dot B^{\frac d2}_{2,1}}||\mathcal{Q}v||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t\\
&\le C\big(1+\kappa^{-1}X_d(T)\big)\mu^{-1}\big(Y_d^2(T)+Z_d^2(T)\big)\\
&\quad+C\int_0^T(1+||a||_{\dot B^{\frac d2}_{2,1}})||(\nabla\mathcal{P}v,\nabla U)||_{\dot B^{\frac d2}_{2,1}}||\mathcal{Q}v||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t,
\end{aligned}\tag{3.41}$$

$$\begin{aligned}
||a(v+U)\cdot\nabla\mathcal{Q}v||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}&\le C||a||_{L^\infty_T(\dot B^{\frac d2}_{2,1})}||(v,U)||_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}||\nabla\mathcal{Q}v||_{L^1_T(\dot B^{\frac d2}_{2,1})}\\
&\le C\kappa^{-2}X_d^2(T)\big(X_d(T)+Y_d(T)+Z_d(T)\big),
\end{aligned}\tag{3.42}$$

$$\begin{aligned}
||k(a)\nabla a||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}&\le C\int_0^T||a||^2_{\dot B^{\frac d2}_{2,1}}\mathrm{d}t\le C\int_0^T(||a^\ell||^2_{\dot B^{\frac d2}_{2,1}}+||a^h||^2_{\dot B^{\frac d2}_{2,1}})\mathrm{d}t\\
&\le C\int_0^T(||a^\ell||_{\dot B^{\frac d2-1}_{2,1}}||a^\ell||_{\dot B^{\frac d2+1}_{2,1}}+||a^h||^2_{\dot B^{\frac d2}_{2,1}})\mathrm{d}t\le C\kappa^{-1}X_d^2(T),
\end{aligned}\tag{3.43}$$

$$\int_0^T\sum_{j\in\mathbb{Z}}2^{j(\frac d2-1)}||[\dot\Delta_j,v+U]\nabla\mathcal{Q}v||_{L^2}\mathrm{d}t\le C\int_0^T||\nabla(v+U)||_{\dot B^{\frac d2}_{2,1}}||\mathcal{Q}v||_{\dot B^{\frac d2-1}_{2,1}}\mathrm{d}t.\tag{3.44}$$

$$\begin{aligned}&||aU_t||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}+||av_t||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\le C||(U_t,\mathcal{P}v_t,\mathcal{Q}v_t+\nabla a)||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}||a||_{L^\infty_T(\dot B^{\frac d2}_{2,1})}+C\int_0^T||a||^2_{\dot B^{\frac d2}_{2,1}}\,\mathrm{d}t\\ &\le C\kappa^{-1}X_d(T)\Big(X_d(T)+Y_d(T)+Z_d(T)\Big),\end{aligned}\tag{3.45}$$

$$\begin{aligned}&||\frac12\nabla(|B+c|^2)-(B+c)\cdot\nabla(B+c)||_{L^1_T(\dot B^{\frac d2-1}_{2,1})}\\ &\le C||(B,c)||_{L^\infty_T(\dot B^{\frac d2-1}_{2,1})}||(B,c)||_{L^1_T(\dot B^{\frac d2+1}_{2,1})}\le\frac{C}{\nu}\big(Z^2_d(T)+Y^2_d(T)\big).\end{aligned}\tag{3.46}$$

Therefore, collecting (3.39)-(3.46), we have

$$\begin{aligned}X_d(T)\le Ce^{C(1+||a||_{L^\infty_T(\dot B^{\frac d2}_{2,1})})||\mathcal{P}v,\mathcal{Q}v,U||_{L^1_T(\dot B^{\frac d2+1}_{2,1})}}\Big\{X_d(0)\\ +C\big(1+\kappa^{-1}X_d(T)\big)(\mu^{-1}+\nu^{-1})\big(Y^2_d(T)+Z^2_d(T)\big)\\ +C\big(\kappa^{-2}X^2_d(T)+\kappa^{-1}X_d(T)\big)\big(X_d(T)+Y_d(T)+Z_d(T)\big)\Big\}.\end{aligned}\tag{3.47}$$

By (3.9) and (3.10), we can deduce that

$$\begin{aligned}(1+||a||_{L^\infty_T(\dot B^{\frac d2}_{2,1})})||\mathcal{P}v,\mathcal{Q}v,U,c||_{L^1_T(\dot B^{\frac d2+1}_{2,1})}&\le(1+\kappa^{-1}D)(\kappa^{-1}D+\mu^{-1}M+\mu^{-1}\delta+\nu^{-1}\delta)\\ &\le 2(1+\mu^{-1}+\nu^{-1})(M+1),\end{aligned}\tag{3.48}$$

and

$$\begin{aligned}(\mu^{-1}+\nu^{-1})||(\mathcal{Q}v,U,B)||^2_{L^1_T(\dot B^{\frac d2}_{2,1})}&\le(\mu^{-1}+\nu^{-1})\big(\kappa^{-1}D^2+(\mu^{-1}+\nu^{-1})M^2\big)\\ &\le(1+\mu^{-2}+\nu^{-2})(M+1)^2.\end{aligned}\tag{3.49}$$

According to (3.10), (3.29) and (3.47)–(3.49), we have

$$\begin{aligned}Y_d(T)&\le Ce^{C(1+\mu^{-2}+\nu^{-2})(M+1)^2}\big(\kappa^{-1}D^2+\kappa^{-\frac12}(\mu^{-\frac12}+\nu^{-\frac12})DM+\kappa^{-1}\mu^{-1}DM^2\\ &\quad+\kappa^{-1}\mu^{-1}D^2\delta\big)\\ &\le Ce^{C(1+\mu^{-2}+\nu^{-2})(M+1)^2}\big(\kappa^{-1}D^2+\kappa^{-\frac12}D\big),\end{aligned}\tag{3.50}$$

and

$$\begin{aligned}X_d(T)&\le Ce^{C(1+\mu^{-1}+\nu^{-1})(M+1)}\big(X_d(0)+(\mu^{-1}+\nu^{-1})(1+M^2)+M+1\big)\\ &\le Ce^{C(1+\mu^{-1}+\nu^{-1})(M+1)^2}\big(X_d(0)+1\big),\end{aligned}\tag{3.51}$$

for a suitable large (universal) constant C. So it is natural to take first

$$D:=Ce^{C(1+\mu^{-2}+\nu^{-2})(M+1)^2}\big(X_d(0)+1\big),\tag{3.52}$$

and then to set

$$\delta=Ce^{2C(1+\mu^{-2}+\nu^{-2})(M+1)^2}\big(\kappa^{-1}D^2+\kappa^{-\frac12}D\big).\tag{3.53}$$

for a suitable large (universal) constant C. It is easy to prove that $||a(t,\cdot)||_{L^\infty}\le C\kappa^{-1}D$. Therefore, if we make the assumption that κ is large enough such that

$$\kappa^{-1}D\ll 1,\quad \delta(\frac1\mu+\frac1\nu+1)\le\frac12,$$

then we deduce from (3.50)-(3.53) that the desired result (3.10).

Proof of Theorem 1.1 First, Theorem 3.1 implies that there exists a unique maximal solution (a, u, b) to (3.1) which belongs to $\widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{d}{2}}_{2,1}) \times (\widetilde{\mathcal{C}}([0,T]; \dot{B}^{\frac{d}{2}-1}_{2,1}) \cap L^1_T \dot{B}^{\frac{d}{2}+1}_{2,1})^{2d}$ on some time interval $[0, T^*)$, with the global a priori estimates (3.9) and (3.10) at our hand, then one conclude that $T^* = +\infty$. In fact, let us assume (by contradiction) that $T^* < \infty$. Next, applying (3.9) and (3.10) for all $t < T^*$ yields

$$||a, u, b||_{L^\infty_{T^*}(\dot{B}^{\frac{d}{2}-1}_{2,1})} \leq C < \infty. \tag{3.54}$$

Then, for all $t_0 \in [0, T^*)$, one can solve (3.1) starting with data (a_0, u_0, b_0) at time $t = t_0$ and get a solution according to Theorem 3.1 on the interval $[t_0, T + t_0]$ with T independent of t_0. Choosing $t_0 > T^* - T$ thus shows that the solution can be continued beyond T^*, a contradiction.

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