# FUNCTIONAL CALCULUS OF OPERATORS WITH HEAT KERNEL BOUNDS ON NON-DOUBLING MANIFOLDS WITH ENDS

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ABSTRACT. Let  $\Delta$  be the Laplace–Beltrami operator acting on a non-doubling manifold with two ends  $\mathbb{R}^m \sharp \mathcal{R}^n$  with  $m > n \geq 3$ . Let  $\mathfrak{h}_t(x, y)$  be the kernels of the semigroup  $e^{-t\Delta}$ generated by  $\Delta$ . We say that a non-negative self-adjoint operator L on  $L^2(\mathbb{R}^m \sharp \mathcal{R}^n)$  has a heat kernel with upper bound of Gaussian type if the kernel  $h_t(x, y)$  of the semigroup  $e^{-tL}$ satisfies  $h_t(x, y) \leq C\mathfrak{h}_{\alpha t}(x, y)$  for some constants C and  $\alpha$ . This class of operators includes the Schrödinger operator  $L = \Delta + V$  where V is an arbitrary non-negative potential. We then obtain upper bounds of the Poisson semigroup kernel of L together with its time derivatives and use them to show the weak type (1, 1) estimate for the holomorphic functional calculus  $\mathfrak{M}(\sqrt{L})$  where  $\mathfrak{M}(z)$  is a function of Laplace transform type. Our result covers the purely imaginary powers  $L^{is}, s \in \mathbb{R}$ , as a special case and serves as a model case for weak type (1, 1) estimates of singular integrals with non-smooth kernels on non-doubling spaces.

## 1. INTRODUCTION

In the last fifty years, the theory of Calderón-Zygmund singular integrals has been a central part and success story of modern harmonic analysis. This theory has had extensive influence on other fields of mathematics such as complex analysis and partial differential equations.

Assume that T is a bounded operator on the space  $L^2(X)$  where X is a metric space with a distance d and a measure  $\mu$ . Also assume that T has an associated kernel k(x, y) in the sense

(1.1) 
$$Tf(x) = \int_X k(x,y)f(y)d\mu(y)$$

for any continuous function f with compact support and for x not in the support of f.

The theory of Calderón-Zygmund singular integrals established sufficient conditions on the space X and the associated kernel k(x, y) for such an operator T to be bounded on  $L^p(X)$  for  $p \neq 2$ . There are 2 key conditions:

• Doubling condition: a measure  $\mu$  on the metric (or quasi-metric) space X is said to be doubling if there exists some positive constant C such that

(1.2) 
$$0 < \mu(B(x,2r)) \le C\mu(B(x,r)) < +\infty$$

for all  $x \in X$  and r > 0, where B(x, r) denotes the ball centered at x and with radius r > 0.

• Hörmander condition: the associated kernel k(x, y) is said to satisfy the (almost  $L^1$ ) Hörmander condition if there exist positive constants c and C such that

(1.3) 
$$\int_{d(x,y_1) \ge cd(y_1,y_2)} |k(x,y_1) - k(x,y_2)| d\mu(x) \le C$$

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uniformly of  $y_1, y_2$ .

Under the doubling condition (1.2) and the Hörmander condition (1.3), it is well known that T is of weak type (1,1). By Marcinkiewicz interpolation, T is bounded on  $L^p(X)$  for 1 . If the Hörmander condition (1.3) is satisfied with x and y swapped, then T is $bounded on <math>L^p(X)$  for  $2 \leq p < \infty$ .

While the theory of Calderón-Zygmund singular integrals has been a great success, there are still many important singular integral operators which do not belong to this class. Within the last twenty years, there were two main directions of development which study operators beyond this Calderón-Zygmund class.

• Singular integrals on non-homogeneous spaces: substantial progress has been made by F. Nazarov, S. Treil, A. Volberg, X. Tolsa, T. Hytönen and others in showing that many features of the classical Calderón-Zygmund theory still hold without assuming the doubling property. More specifically, the doubling condition on  $\mathbb{R}^d$  can be replaced by the polynomial growth condition: for some fixed positive constants C and  $n \in (0, d]$ , one has

(1.4) 
$$\mu(B(x,r)) \le Cr^n \quad \text{for all } x \in \mathbb{R}^d, r > 0.$$

If the measure  $\mu$  satisfies the condition (1.4), then the space  $(\mathbb{R}^d, \mu)$  is called a nonhomogeneous space. Calderón–Zygmund theory has been developed on such non-homogeneous spaces; see for example [22, 23, 24, 29]. For the BMO and  $H^1$  function space, the Littlewood– Paley theory, and weighted norm inequalities on such non-homogeneous spaces, see [2, 25, 28, 30]; for Morrey spaces, Besov spaces and Triebel-Lizorkin spaces in this setting, see [9, 15, 26]. See also [2, 18, 19, 20] for recent work in this direction which studies a more general setting for non-homogeneous analysis on metric spaces  $(X, d, \mu)$ , where (X, d) is said to be geometrically doubling.

However, to obtain boundedness of singular integrals in this setting, one needs certain strong regularity on the associated kernels in terms of the upper doubling measure, i.e.,  $r^n$  as in (1.4) rather than  $\mu(B(x,r))$ . For example Hölder continuity on the space variables of the kernels is needed for weak type (1, 1) estimate.

• Singular integrals with non-smooth kernels: A lot of work has been carried out to study singular integrals whose associated kernels are not smooth enough to satisfy the Hörmander condition. Substantial progress has been made by X. Duong, A. McIntosh, S. Hofmann, L. Yan, J. Martell, P. Auscher, T. Coulhon and others. The Hörmander condition was replaced by a weaker one to obtain the weak type (1, 1) estimates and to study function spaces associated with operators. See for example [1, 4, 10, 12, 13, 17]. The achievements in this direction are mostly obtained for operators acting on doubling spaces.

A natural question arises: How about singular integrals with non-smooth kernels on nondoubling spaces? This is a difficult and interesting problem when both the key conditions of Calderón-Zygmund theory are missing. In this paper we study certain singular integrals with non-smooth kernels acting on non-doubling spaces. Our model here is the holomorphic functional calculus of Laplace transform type for operators with suitable heat kernel upper bounds such as the Schrödinger operator on a non-doubling manifold with two ends.

Let us recall manifolds with ends as in [14]. Let M be a complete non-compact Riemannian manifold and  $K \subset M$  be a compact set with non-empty interior and smooth boundary such that  $M \setminus K$  has k connected components  $E_1, \ldots, E_k$ . We call K the central part and for simplicity consider the case of two ends, i.e. k = 2 so that  $E_1$  is isometric to  $\mathbb{R}^m$  and  $E_2$  is isometric to  $\mathcal{R}^n := \mathbb{R}^n \times \mathbb{S}^{m-n}$  where  $m > n \ge 3$  and  $\mathbb{S}^{m-n}$  is the unit sphere in  $\mathbb{R}^{m-n}$ . We denote the non-doubling manifold with two ends as  $M = \mathbb{R}^m \sharp \mathcal{R}^n$  with distance  $|x| = \sup_{z \in K} d(x, z)$  where d(x, z) is the geodesic distance in M. One can see that |x| is separated from zero in M and  $|x| \approx 1 + d(x, K)$  where  $d(x, K) = \inf\{d(x, y : y \in K\}$ . It is easy to check that  $\mathbb{R}^m \sharp \mathcal{R}^n$  is non-doubling since

(1.5) 
$$V(x,r) \approx \begin{cases} r^m \text{ for all } x \in M, \text{ when } r \leq 1\\ r^n \text{ for } B(x,r) \subset \mathcal{R}^n, \text{ when } r > 1; \text{ and}\\ r^m \text{ for } x \in \mathcal{R}^n \backslash K, \ r > 2|x|, \text{ or } x \in \mathbb{R}^m, r > 1 \end{cases}$$

where V(x, r) is the measure of the ball B(x, r).

In [14], Grigor'yan and L. Saloff-Coste studied the kernels of the semigroup  $e^{-t\Delta}$  generated by the Laplace-Beltrami operator  $\Delta$  on  $\mathbb{R}^m \sharp \mathcal{R}^n$  and obtained by probabilistic methods the upper and lower bounds for the kernels  $\mathfrak{h}_t(x, y)$  of  $e^{-t\Delta}$ . However, no further information on  $\mathfrak{h}_t(x, y)$  are known, for example we do not know if some (good) pointwise estimates on the time derivatives and space derivatives of  $\mathfrak{h}_t(x, y)$  exist. Indeed, the standard method of extending the Gaussian upper bound on the heat kernel with t > 0 to complex z = t + is in the case of doubling space like  $\mathbb{R}^n$  or spaces of homogeneous type does not give a sharp upper bound for the complex heat kernel of the Laplace-Beltrami operator due to the missing of certain (sharp) global estimate such as the  $L^1 - L^\infty$  estimate of heat semigroup in the setting of  $\mathbb{R}^m \sharp \mathcal{R}^n$ .

We view the heat kernel of the Laplace-Beltrami operator on  $\mathbb{R}^m \sharp \mathcal{R}^n$  as the standard behaviour of heat diffusion which plays the important role of the Gaussian kernels on doubling spaces. We introduce the concept of heat kernels with upper bounds of Gaussian type as follows.

**Definition 1.1.** Let  $\Delta$  be the Laplace-Beltrami operator and L be a non-negative self-adjoint operator on  $L^2(\mathbb{R}^m \sharp \mathcal{R}^n)$ . We say that the heat kernel of L has an upper bound of Gaussian type if the kernel  $h_t(x, y)$  of  $e^{-tL}$  satisfies  $h_t(x, y) \leq C\mathfrak{h}_{\alpha t}(x, y)$  for some constants C and  $\alpha$ where  $\mathfrak{h}_t(x, y)$  is the kernel of  $e^{-t\Delta}$ .

**Remark 1.2.** (a) See Theorem A, Section 2 for the (sharp) upper bound and lower bound for the heat kernels  $\mathfrak{h}_t(x, y)$ .

(b) The operators which have heat kernels with upper bounds of Gaussian type include the Schrödinger operator  $L = \Delta + V$  where V is a non-negative potential. Indeed it follows from the Trotter formula that

$$e^{-t(\Delta+V)}|f(x)| \le e^{-t\Delta}|f(x)|$$

for  $f \in L^2(M)$ . Hence an upper bound for the kernel  $\mathfrak{h}_t(x, y)$  of  $e^{-t\Delta}$  is also an upper bound for the kernel  $h_t(x, y)$  of  $e^{-t(\Delta+V)}$ . However, a lower bound for  $\mathfrak{h}_t(x, y)$  might not be a lower bound for  $h_t(x, y)$ .

(c) There is no assumption on the smoothness of the heat kernel in the definition of upper bound of Gaussian type. In the specific case of the Schrödinger operator, due to the effect of the non-negative potential V, it is possible that the kernel  $h_t(x, y)$  of  $e^{-t(\Delta+V)}$  is discontinuous hence regularity estimates such as Hölder continuity are false for  $h_t(x, y)$  in general. We note that in [11], the authors obtained the weak type (1, 1) estimates for the maximal operator  $T(f) = \sup_{t>0} |e^{-t\Delta}f|$  by using the upper bounds of the heat kernels  $\mathfrak{h}_t(x, y)$ . The proof was a direct consequence of the sharp upper bounds on heat kernels in [14]. In [5], the authors obtained some estimates which showed that spectral multipliers for a function  $\theta(z)$  with compact support are bounded on  $L^p$  spaces on a space X which includes the case of non-doubling manifolds with ends. While the result in [5] is applicable to large class of underlying spaces X, the condition that the function  $\theta(z) = z^{is}$ , s real, which gives rise to the purely imaginary power  $\Delta^{is}$  is not covered by the result of [5]. It came to our attention recently that the Riesz transform  $\nabla \Delta^{-1/2}$  of the Laplace Beltrami operator  $\Delta$  on  $\mathbb{R}^m \sharp \mathcal{R}^n$  was proved by Carron [3] to be bounded on  $L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$  for  $p_0 for some <math>p_0 > 1$ , and after the first version of this article was completed, the  $L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$  boundedness of the Riesz transform  $\nabla \Delta^{-1/2}$  for 1 , together with the weak type (1, 1) estimate, were obtainedby Hassell and Sikora [16].

The following theorem is our main result.

**Theorem 1.3.** Let L be an operator which has heat kernel with upper bounds of Gaussian type. Let  $\mathfrak{M}(\sqrt{L})$  be the holomorphic functional calculus of Laplace transform type of  $\sqrt{L}$  defined by

$$\mathfrak{M}(\sqrt{L})f = \int_0^\infty [\sqrt{L}\exp(-t\sqrt{L})f]\tilde{m}(t)dt$$

in which  $\tilde{m}(t)$  is a bounded function on  $[0, \infty)$ , i.e.  $|\tilde{m}(t)| \leq C_0$ , where  $C_0$  is a constant. Then  $\mathfrak{M}(\sqrt{L})$  is of weak type (1,1). Hence by interpolation and duality, the operator  $\mathfrak{M}(\sqrt{L})$  is bounded on  $L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$  for 1 .

**Remark 1.4.** (a) In Theorem 1.3 we prove the weak type (1,1) estimate for  $\mathfrak{M}(\sqrt{L})$  for a function  $\mathfrak{M}$  of Laplace transform type. While the  $L^p$  boundedness of  $\mathfrak{M}(\sqrt{\Delta})$  for 1 can be obtained by the Littlewood–Paley theory [27] or transference method [7], the end-point weak <math>(1,1) estimate of  $\mathfrak{M}(L)$  is new even for the case when  $L = \Delta$ . Our main result includes the operators  $L^{is}$ , s real, as a special case and it is a good example for singular integrals acting on non-doubling spaces whose kernels do not satisfy the Hörmander condition (1.3).

(b) By using the same approach and similar techniques in the proof of our main result, Theorem 1.3, we can also obtain the weak type (1,1) estimate for the Littlewood–Paley square function defined via the Poisson semigroup generated by L as follows:

$$g(f)(x) = \left(\int_0^\infty \left| (t\sqrt{L})^\kappa e^{-t\sqrt{L}}(f)(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \quad \kappa \in \mathbb{N}, \kappa \ge 1.$$

(c) In addition to standard techniques of harmonic analysis of real variables, there are two key elements in our method of proofs in this paper.

(i) Since the pointwise estimates on space and time derivatives of the semigroup  $e^{-tL}$  are not known, we overcome this problem by using the subordination formula to obtain upper bounds on the time derivatives of the kernel of Poisson semigroup  $e^{-t\sqrt{L}}$  via the known upper bound for the kernel of the heat semigroup  $e^{-tL}$ . Then we approach the holomorphic functional calculus of Laplace transform type  $\mathfrak{M}(\sqrt{L})$  through the Poisson semigroup  $e^{-t\sqrt{L}}$ .

(ii) The standard Calderón–Zygmund decomposition on non-homogeneous spaces (such as [23, 29]) are not applicable to the proof for the weak type (1, 1) estimate of our singular integral  $\mathfrak{M}(\sqrt{L})$  because of lack of smoothness of its kernel. To overcome this problem, we use the technique of generalised approximation to the identity in [10] to handle the local doubling part, then carry out a number of subtle decomposition and meticulous estimates to handle the case of non-doubling balls. It turns out that the sharp upper bounds of the Poisson semigroup are sufficient for us to handle the blowing up of non-doubling volumes of balls and obtain the desired weak type (1, 1) estimate.

The method in this paper relies only on good upper bounds on the Poisson semigroup kernel and its time derivatives which can be derived from the heat semigroup kernels. It does not require, for example the contraction property of the semigroup on  $L^p$  spaces, hence can be applied to other differential operators. We believe that our method can be developed further to study boundedness of singular integrals with non-smooth kernels acting on non-doubling spaces in other settings.

# 2. Poisson semigroup and its time-derivatives

Let  $\Delta$  be the Laplace Beltrami operator acting on the manifold  $\mathbb{R}^m \sharp \mathcal{R}^n$  and  $\exp(-t\Delta)$  the heat propagator corresponding to  $\Delta$ . Here and throughout the whole paper, we use  $\mathbb{R}^m \setminus K$ to denote the large end of the manifold  $\mathbb{R}^m \sharp \mathcal{R}^n$ ,  $\mathcal{R}^n \setminus K$  to denote the small end, and K to denote the centre part of the manifold.

**Theorem A** ([14]). The kernel  $\mathfrak{h}_t(x, y)$  of  $\exp(-t\Delta)$  satisfies the following estimates: 1. For  $t \leq 1$  and all  $x, y \in \mathbb{R}^m \sharp \mathcal{R}^n$ ,

$$\mathfrak{h}_t(x,y) \approx \frac{1}{V(x,\sqrt{t})} \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

2. For t > 1 and all  $x, y \in K$ ,

$$\mathfrak{h}_t(x,y) \approx \frac{1}{t^{n/2}} \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

3. For t > 1 and  $x \in \mathbb{R}^m \setminus K$ ,  $y \in K$ ,

$$\mathfrak{h}_t(x,y) \approx \left(\frac{1}{t^{n/2}|x|^{m-2}} + \frac{1}{t^{m/2}}\right) \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

4. For t > 1 and  $x \in \mathbb{R}^n \setminus K$ ,  $y \in K$ ,

$$\mathfrak{h}_t(x,y) \approx \left(\frac{1}{t^{n/2}|x|^{n-2}} + \frac{1}{t^{n/2}}\right) \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

5. For t > 1 and  $x \in \mathbb{R}^m \setminus K$ ,  $y \in \mathcal{R}^n \setminus K$ ,

$$\mathfrak{h}_t(x,y) \approx \left(\frac{1}{t^{n/2}|x|^{m-2}} + \frac{1}{t^{m/2}|y|^{n-2}}\right) \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

6. For t > 1 and  $x, y \in \mathbb{R}^m \setminus K$ ,

$$\mathfrak{h}_t(x,y) \approx \frac{1}{t^{n/2} |x|^{m-2} |y|^{m-2}} \exp\left(-c_0 \frac{|x|^2 + |y|^2}{t}\right) + \frac{1}{t^{m/2}} \exp\left(-c_0 \frac{d(x,y)^2}{t}\right);$$

7. For t > 1 and  $x, y \in \mathcal{R}^n \setminus K$ ,

$$\mathfrak{h}_t(x,y) \approx \frac{1}{t^{n/2} |x|^{n-2} |y|^{n-2}} \exp\left(-c_0 \frac{|x|^2 + |y|^2}{t}\right) + \frac{1}{t^{n/2}} \exp\left(-c_0 \frac{d(x,y)^2}{t}\right).$$

We now recall the following result.

**Theorem B** ([11]). Let T be the maximal operator defined by  $T(f)(x) := \sup_{t>0} |\exp(-t\Delta)f(x)|$ . Then T is weak type (1,1) and for any function  $f \in L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$ , 1 , the following estimates hold

 $||Tf||_{L^p(\mathbb{R}^m \sharp \mathcal{R}^n)} \le C ||f||_{L^p(\mathbb{R}^m \sharp \mathcal{R}^n)}.$ 

For the rest of the article, let L be a non-negative self-adjoint operator whose heat kernels satisfy upper bounds of Gaussian type. We first have the following corollary.

**Corollary 2.1.** Theorem B holds for the maximal operator via the heat semigroup generated by L, i.e.,  $T_L(f)(x) := \sup_{t>0} |\exp(-tL)f(x)|$  is of weak type (1,1) and bounded on  $L^{\infty}(\mathbb{R}^m \sharp \mathbb{R}^n)$ , and hence it is bounded on  $L^p(\mathbb{R}^m \sharp \mathbb{R}^n)$  for all 1 .

Proof: This follows directly from the inequality

$$\exp(-tL)|f|(x) \le C \exp(-\alpha t\Delta)|f|(x)|$$

and Theorem B.

Next, we study the properties of the Poisson semigroup generated by L. Let  $k \in \mathbb{N}$ , we denote by  $P_{t,k}(x,y)$  the kernel of  $(t\sqrt{L})^k e^{-t\sqrt{L}}$ . For k = 0, we write  $P_t(x,y)$  instead of  $P_{t,0}(x,y)$ .

**Theorem 2.2.** For  $k \in \mathbb{N}$ , set  $k \vee 1 = \max\{k, 1\}$ . Then the exists a constant C (which depends on k) such that the kernel  $P_{t,k}(x, y)$  satisfies the following estimates:

1. For  $x, y \in K$ ,

$$|P_{t,k}(x,y)| \le \frac{C}{t^m} \Big(\frac{t}{t+d(x,y)}\Big)^{m+k\vee 1} + \frac{C}{t^n} \Big(\frac{t}{t+d(x,y)}\Big)^{n+k\vee 1};$$

2. For  $x \in \mathbb{R}^m \setminus K$ ,  $y \in K$ ,  $|P_{t,k}(x,y)| \le \frac{C}{t^m} \left(\frac{t}{t+d(x,y)}\right)^{m+k\vee 1} + \frac{C}{t^n |x|^{m-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k\vee 1}$ ;

3. For  $x \in \mathcal{R}^n \setminus K$ ,  $y \in K$ ,

$$|P_{t,k}(x,y)| \le \frac{C}{t^m} \left(\frac{t}{t+d(x,y)}\right)^{m+k\vee 1} + \frac{C}{t^n} \left(\frac{t}{t+d(x,y)}\right)^{n+k\vee 1};$$

4. For 
$$x \in \mathbb{R}^m \setminus K$$
,  $y \in \mathcal{R}^n \setminus K$ ,  
 $|P_{t,k}(x,y)| \le \frac{C}{t^m} \left(\frac{t}{t+d(x,y)}\right)^{m+k\vee 1} + \frac{C}{t^n |x|^{m-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k\vee 1} + \frac{C}{t^m |y|^{n-2}} \left(\frac{t}{t+d(x,y)}\right)^{m+k\vee 1};$ 

- 5. For  $x, y \in \mathbb{R}^m \setminus K$ ,  $|P_{t,k}(x,y)| \le \frac{C}{t^m} \left(\frac{t}{t+d(x,y)}\right)^{m+k\vee 1} + \frac{C}{t^n |x|^{m-2} |y|^{m-2}} \left(\frac{t}{t+|x|+|y|}\right)^{n+k\vee 1};$
- 6. For  $x, y \in \mathcal{R}^n \setminus K$ ,

$$|P_{t,k}(x,y)| \le \frac{C}{t^m} \Big(\frac{t}{t+d(x,y)}\Big)^{m+k\vee 1} + \frac{C}{t^n} \Big(\frac{t}{t+d(x,y)}\Big)^{n+k\vee 1}$$

*Proof.* By the subordination formula we have

(2.1) 
$$e^{-t\sqrt{L}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{t e^{-\frac{t^2}{4v}}}{\sqrt{v}} e^{-vL} \frac{dv}{v},$$

and hence

$$(t\sqrt{L})^{k}e^{-t\sqrt{L}} = (-1)^{k}\frac{t^{k}}{2\sqrt{\pi}}\int_{0}^{\infty}\partial_{t}^{k}(te^{-\frac{t^{2}}{4v}})e^{-vL}\frac{du}{v^{3/2}}$$
$$= (-1)^{k}\frac{t^{k}}{\sqrt{\pi}}\int_{0}^{\infty}\partial_{t}^{k+1}(e^{-\frac{t^{2}}{4v}})e^{-vL}\frac{dv}{\sqrt{v}}.$$

This yields that

(2.2) 
$$P_{t,k}(x,y) = (-1)^k \frac{t^k}{\sqrt{\pi}} \int_0^\infty \partial_t^{k+1} (e^{-\frac{t^2}{4v}}) H_v(x,y) \frac{dv}{\sqrt{v}}$$

where  $H_v(x, y)$  is the kernel of  $e^{-vL}$ .

Let s > 0 and  $k \in \mathbb{N}$ . By Faà di Bruno's formula, we can write

$$\partial_t^{k+1} e^{-\frac{t^2}{s}} = \sum \frac{(-1)^{m_1+m_2}}{2 \times m_1! m_2!} e^{-t^2/s} \left(\frac{t}{s}\right)^{m_1} \left(\frac{1}{s}\right)^{m_2},$$

where the sum is taken over all pairs  $(m_1, m_2)$  of nonnegative integers satisfying  $m_1 + 2m_2 = k + 1$ . For such a pair  $(m_1, m_2)$ , there exists C > 0 so that

(2.3) 
$$e^{-\frac{t^2}{s}} \left(\frac{t}{s}\right)^{m_1} \left(\frac{1}{s}\right)^{m_2} = e^{-\frac{t^2}{s}} \left(\frac{t}{\sqrt{s}}\right)^{m_1} \left(\frac{1}{s}\right)^{m_1/2+m_2} \\ \leq C e^{-\frac{2t^2}{3s}} s^{-(k+1)/2} \max\left\{1, \left(\frac{t}{\sqrt{s}}\right)^{k+1}\right\}$$

This implies that

(2.4) 
$$|\partial_t^{k+1} e^{-\frac{t^2}{s}}| \le C e^{-\frac{t^2}{2s}} s^{-(k+1)/2}.$$

From (2.2) and (2.4) we deduce that

(2.5) 
$$|P_{t,k}(x,y)| \le \frac{C}{4\sqrt{\pi}} \int_0^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k |H_v(x,y)| \frac{dv}{v}$$

We now give the estimates for  $P_{t,k}(x, y)$  with  $k \ge 1$  only, since the remaining case k = 0 can be done similarly.

We have

$$|P_{t,k}(x,y)| \le \frac{C}{4\sqrt{\pi}} \int_0^1 e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k |H_v(x,y)| \frac{dv}{v} + \frac{C}{4\sqrt{\pi}} \int_1^\infty e^{-\frac{t^2}{8v}} \left(\frac{t}{\sqrt{v}}\right)^k |H_v(x,y)| \frac{dv}{v} =: \mathbb{J}_1(x,y) + \mathbb{J}_2(x,y).$$

Applying the upper bound in point 1 in Theorem A for  $|H_v(x, y)|$ , we have

$$\begin{split} \mathbb{J}_{1}(x,y) &\leq C \int_{0}^{1} e^{-\frac{t^{2}}{8v}} \Big(\frac{t}{\sqrt{v}}\Big)^{k} \frac{1}{v^{m/2}} \exp\Big(-\frac{d(x,y)^{2}}{cv}\Big) \frac{dv}{v} \\ &\leq C \int_{0}^{\infty} \Big(\frac{t}{\sqrt{v}}\Big)^{k} \frac{1}{v^{m/2}} \exp\Big(-\frac{d(x,y)^{2}+t^{2}}{cv}\Big) \frac{dv}{v} \\ &\leq C \left(\int_{0}^{d(x,y)^{2}+t^{2}} + \int_{d(x,y)^{2}+t^{2}}^{\infty}\right) \Big(\frac{t}{\sqrt{v}}\Big)^{k} \frac{1}{v^{m/2}} \exp\Big(-\frac{d(x,y)^{2}+t^{2}}{cv}\Big) \frac{dv}{v} \\ &\leq \frac{C}{t^{m}} \Big(\frac{t}{t+d(x,y)}\Big)^{m+k}. \end{split}$$

For the term  $\mathbb{J}_2(x, y)$ , we consider the following 6 cases: Case 1:  $x, y \in K$ .

Applying the upper bound in point 2 in Theorem A for  $|H_v(x, y)|$ , we have

$$|p_v(x,y)| \le \frac{C}{v^{n/2}} \exp\left(-c_0 \frac{d(x,y)^2}{v}\right).$$

Arguing similarly to the estimate of  $\mathbb{J}_1(x, y)$  we obtain

$$\mathbb{J}_2(x,y) \le \frac{C}{t^n} \Big(\frac{t}{t+d(x,y)}\Big)^{n+k}.$$

Case 2:  $x \in \mathbb{R}^m \backslash K, y \in K$ .

Applying the upper bound in point 3 in Theorem A for  $|H_v(x, y)|$ , we get that

$$|H_v(x,y)| \le C\left(\frac{1}{v^{n/2}|x|^{m-2}} + \frac{1}{v^{m/2}}\right) \exp\left(-c_0 \frac{d(x,y)^2}{v}\right).$$

Hence, we get

$$\mathbb{J}_{2}(x,y) \leq C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \left(\frac{1}{v^{n/2}|x|^{m-2}} + \frac{1}{v^{m/2}}\right) \exp\left(-c_{0}\frac{d(x,y)^{2}}{v}\right) \frac{dv}{v} \\
\leq \frac{C}{t^{n}|x|^{m-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} + \frac{C}{t^{m}} \left(\frac{t}{t+d(x,y)}\right)^{m+k}.$$

Case 3:  $x \in \mathcal{R}^n \setminus K, y \in K$ .

Applying the upper bound in point 4 in Theorem A for  $|H_v(x, y)|$ , we have

$$\mathbb{J}_{2}(x,y) \leq C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \left(\frac{1}{v^{n/2}|x|^{n-2}} + \frac{1}{v^{n/2}}\right) \exp\left(-c_{0}\frac{d(x,y)^{2}}{v}\right) \frac{dv}{v} \\
\leq \frac{C}{t^{n}|x|^{n-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} + \frac{C}{t^{n}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} \\
\leq \frac{C}{t^{n}} \left(\frac{t}{t+d(x,y)}\right)^{n+k}.$$

Case 4:  $x \in \mathbb{R}^m \setminus K, y \in \mathcal{R}^n \setminus K$ .

Applying the upper bound in point 5 in Theorem A for  $|H_v(x,y)|$ , we get that

$$\mathbb{J}_{2}(x,y) \leq C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \left(\frac{1}{v^{n/2}|x|^{m-2}} + \frac{1}{v^{m/2}|y|^{n-2}}\right) \exp\left(-c_{0}\frac{d(x,y)^{2}}{v}\right) \frac{dv}{v} \\
\leq \frac{C}{t^{n}|x|^{m-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} + \frac{C}{t^{m}|y|^{n-2}} \left(\frac{t}{t+d(x,y)}\right)^{m+k}.$$

Case 5:  $x, y \in \mathbb{R}^m \backslash K$ .

Applying the upper bound in point 6 in Theorem A for  $|H_v(x, y)|$ , we find that

$$\mathbb{J}_{2}(x,y) \leq C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \frac{1}{v^{n/2}|x|^{m-2}|y|^{m-2}} \exp\left(-c_{0}\frac{|x|^{2}+|y|^{2}}{v}\right) \frac{dv}{v} \\
+ C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \frac{1}{v^{m/2}} \exp\left(-c_{0}\frac{d(x,y)^{2}}{v}\right) \frac{dv}{v} \\
\leq \frac{C}{t^{n}|x|^{m-2}|y|^{m-2}} \left(\frac{t}{t+|x|+|y|}\right)^{n+k} + \frac{C}{t^{m}} \left(\frac{t}{t+d(x,y)}\right)^{m+k}.$$

Case 6:  $x, y \in \mathcal{R}^n \setminus K$ .

Applying the upper bound in point 7 in Theorem A for  $|H_v(x,y)|$ , we obtain that

$$\begin{aligned} \mathbb{J}_{2}(x,y) &\leq C \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \frac{1}{v^{n/2} |x|^{n-2} |y|^{n-2}} \exp\left(-c_{0} \frac{|x|^{2} + |y|^{2}}{v}\right) \frac{dv}{v} \\ &+ \int_{1}^{\infty} e^{-\frac{t^{2}}{8v}} \left(\frac{t}{\sqrt{v}}\right)^{k} \frac{1}{v^{n/2}} \exp\left(-c_{0} \frac{d(x,y)^{2}}{v}\right) \frac{dv}{v} \\ &\leq \frac{C}{t^{n} |x|^{n-2} |y|^{n-2}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} + \frac{C}{t^{n}} \left(\frac{t}{t+d(x,y)}\right)^{n+k} \\ &\leq \frac{C}{t^{n}} \left(\frac{t}{t+d(x,y)}\right)^{n+k}. \end{aligned}$$

We observe that the proof of Theorem 2.2 can be extended to obtain the estimates for the complex Poisson semigroup and its time derivatives  $(z\sqrt{L})^k \exp(-z\sqrt{L})$ . Indeed, we have the following result.

**Theorem 2.3.** Fix  $0 < \mu < \frac{\pi}{4}$ , let  $S^0_{\mu} = \{z \in \mathbb{C} : |\arg z| < \mu\}$  and choose  $z \in S^0_{\mu}$ . The complex Poisson semigroup and its time derivatives  $(z\sqrt{L})^k \exp(-z\sqrt{L})$  exist and satisfy the upper bounds as in Theorem 2.2 with t to be replaced by |z|.

*Proof.* Fix  $0 < \mu < \frac{\pi}{4}$ . For  $z \in S^0_{\mu}$ , define

(2.6) 
$$e^{-z\sqrt{L}} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{ze^{-\frac{z^2}{4v}}}{\sqrt{v}} e^{-vL} \frac{dv}{v}.$$

For  $0 < |\arg z| < \mu < \frac{\pi}{4}$  we have  $\Re z^2 > 0$ , hence the integral in (2.6) converges. When z is real, formula (2.6) coincides with the Poisson semigroup (2.1), hence formula (2.6) defines the

complex Poisson semigroup (which is unique by analyticity). Hence

$$(z\sqrt{L})^{k}e^{-z\sqrt{L}} = (-1)^{k}\frac{z^{k}}{2\sqrt{\pi}}\int_{0}^{\infty}\partial_{z}^{k}(ze^{-\frac{z^{2}}{4v}})e^{-vL}\frac{du}{v^{3/2}}$$
$$= (-1)^{k}\frac{z^{k}}{\sqrt{\pi}}\int_{0}^{\infty}\partial_{z}^{k+1}(e^{-\frac{z^{2}}{4v}})e^{-vL}\frac{dv}{\sqrt{v}}.$$

This yields that

(2.7) 
$$P_{z,k}(x,y) = (-1)^k \frac{z^k}{\sqrt{\pi}} \int_0^\infty \partial_z^{k+1} (e^{-\frac{z^2}{4v}}) H_v(x,y) \frac{dv}{\sqrt{v}}$$

where  $H_v(x, y)$  is the kernel of  $e^{-vL}$ . The rest of the proof is similar to Theorem 2.2.

We now obtain a weak type (1, 1) estimate for a maximal operator.

**Proposition 2.4.** Fix  $0 < \mu < \frac{\pi}{4}$ . Let  $T_k$  be the operator defined by

$$T_k(f)(x) := \sup_{z \in S^0_{\mu}} |(z\sqrt{L})^k \exp(-z\sqrt{L})f(x)|$$

for an integer  $k \geq 0$  and  $f \in L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$ . Then  $T_k$  is of weak type (1,1) and bounded on  $L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$  for 1 .

*Proof.* We point out that with the upper bound of the kernel  $P_{z,k}(x,y)$  of  $(z\sqrt{L})^k \exp(-z\sqrt{L})$ . the weak type (1,1) estimate of the maximal operator  $T_k$  follows from the same idea and approach in proof of Theorem B. For more details, we refer to [11]. 

The concept of approximations to the identity plays an important role in harmonic analysis. For a family of approximations to the identity  $\phi_t * f$  in a doubling space like  $\mathbb{R}^n$ , the upper bound on  $\phi(x)$  can be taken as the Gaussian bound with exponential decay or Poisson bound with polynomial decay. However, in a non-doubling space like a manifold with ends  $\mathbb{R}^m \sharp \mathcal{R}^n$ , it is not obvious which type of bound is deemed natural. Here we suggest to use the Poisson kernels in the definition of an approximation to the identity in this setting. We note that in the case of  $\mathbb{R}^n$ , the term  $\frac{1}{t^n}$  in the Poisson kernel  $\frac{1}{t^n} \times \frac{c_n}{(1+|x-y|^2)^{\frac{n+1}{2}}}$  is independent of xand y, whereas the corresponding term in the case of  $\mathbb{R}^m \sharp \mathcal{R}^n$  might depend on x and y.

**Definition 2.5.** A family of kernels  $\phi_t(x, y), t > 0$ , is said to be a generalised approximation to the identity if  $|\phi_t(x,y)|$  has the same upper bound as  $CP_{\alpha t,k}(x,y)$  in Theorem 2.2 for some positive constants C, k and  $\alpha$ .

We note that in the proof of our main result, Theorem 1.3, we use  $e^{-t\sqrt{L}}$  as a generalised approximation to the identity. While it is true that  $e^{-t\sqrt{L}}$  tends to the Identity as t tends to 0 in  $L^2$  sense, we do not need this property in our proof.

The following result is similar to the basic result in  $\mathbb{R}^n$  that the operator  $\sup_{t>0} |\phi_t * f|$  is bounded on  $L^p$ ,  $1 for a suitable family of kernels <math>\phi_t$ .

**Proposition 2.6.** Assume that  $\phi_t(x, y)$ , t > 0, is a generalised approximation to the identity on  $\mathbb{R}^m \sharp \mathcal{R}^n$ . Define the family of operators  $D_t$  by

$$D_t f(x) = \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \phi_t(x, y) f(y) \ d\mu(y)$$

for  $f \in L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$ ,  $1 . Then the operator <math>T(f)(x) := \sup_{t>0} |D_t f(x)|$  is bounded on  $L^p(\mathbb{R}^m \sharp \mathcal{R}^n)$ , 1 and is of weak type <math>(1, 1).

Sketch of proof: We note that by Definition 2.5,  $D_t|f|(x)$  has the same upper bound as  $C(\alpha t \sqrt{L})^k e^{-\alpha t \sqrt{L}}|f|(x)$ , hence the proof follows the same line as that of Proposition 2.4.

**Remark 2.7.** Theorem 2.2, Propositions 2.4 and 2.6 are of independent interest as they are useful for the study of harmonic analysis on the setting of non-doubling manifolds with ends.

### 3. Proof of main result: Theorem 1.3

To begin with, we first recall the standard definition of the maximal function and its properties. For any  $p \in [1, \infty]$  and any function  $f \in L^p$  we set

$$\mathcal{M}f(x) = \sup\left\{\frac{1}{|B(y,r)|} \int_{B(y,r)} |f(z)| dz \colon x \in B(y,r)\right\}.$$

**Theorem C** ([11]). The maximal function operator is of weak type (1,1) and bounded on all  $L^p$  spaces for 1 .

Now to prove Theorem 1.3, it suffices to show that there exists a positive constant C such that for  $f \in L^1(\mathbb{R}^m \sharp \mathcal{R}^n)$  and for every  $\lambda > 0$ ,

(3.1) 
$$|\{x \in \mathbb{R}^m \, \sharp \, \mathcal{R}^n : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}| \le C \frac{\|f\|_{L^1(\mathbb{R}^m \, \sharp \, \mathcal{R}^n)}}{\lambda}.$$

Then, to prove (3.1), it suffices to verify the following three inequalities:

(3.2) 
$$|\{x \in \mathbb{R}^m \setminus K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}| \le C \frac{\|f\|_{L^1(\mathbb{R}^m \,\sharp\,\mathbb{R}^n)}}{\lambda}$$

(3.3) 
$$|\{x \in \mathcal{R}^n \setminus K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}| \le C \frac{\|f\|_{L^1(\mathbb{R}^m \,\sharp\,\mathbb{R}^n)}}{\lambda},$$

and

(3.4) 
$$|\{x \in K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}| \le C \frac{\|f\|_{L^1(\mathbb{R}^m \,\sharp\,\mathbb{R}^n)}}{\lambda}.$$

We now set

$$f_1(x) := f(x)\chi_{\mathbb{R}^m \setminus K}, f_2(x) := f(x)\chi_{\mathcal{R}^n \setminus K}, \text{ and } f_3(x) := f(x)\chi_K.$$

Thus, f can be written as

$$f = f_1 + f_2 + f_3.$$

Since  $\mathfrak{M}(\sqrt{L})$  is a linear operator, the measure in the left-hand side of (3.2) satisfies

$$\begin{aligned} \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda \right\} \right| &\leq \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L})f_1(x)| > \frac{\lambda}{3} \right\} \right| \\ &+ \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L})f_2(x)| > \frac{\lambda}{3} \right\} \right| \\ &+ \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L})f_3(x)| > \frac{\lambda}{3} \right\} \right| \end{aligned}$$

$$=: I_1 + I_2 + I_3.$$

Similarly, we decompose and obtain the measure of the set  $\{x \in \mathcal{R}^n \setminus K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}$  is bounded by the sum of three parts  $II_1 + II_2 + II_3$  and the measure of  $\{x \in K : |\mathfrak{M}(\sqrt{L})f(x)| > \lambda\}$  is bounded by the sum  $III_1 + III_2 + III_3$  which correspond to the components  $f_1, f_2$  and  $f_3$  respectively.

It then suffices to prove that each of the terms above has an upper estimate of the form  $C\frac{\|f\|_{L^1(\mathbb{R}^m \notin \mathcal{R}^n)}}{\lambda}$ .

3.1. Estimate of  $I_1$ . In this case, since x is in  $\mathbb{R}^m \setminus K$  and the function  $f_1$  is also supported in  $\mathbb{R}^m \setminus K$ , we can restrict to the setting  $\mathbb{R}^m \setminus K$ , where the measure now becomes the standard Lebesgue measure on  $\mathbb{R}^m \setminus K$  which is doubling. However, the non-homogeneous property shows up in the kernel estimate in this case. The Poisson kernel here is not bounded by the classical upper bound and the main difficulty comes from the term

$$\frac{C}{t^{n}|x|^{m-2}|y|^{m-2}} \left(\frac{t}{t+|x|+|y|}\right)^{n+k\vee 1}$$

in the upper bound of the Poisson kernel, where the power of the time scaling is n which can be much smaller than the space dimension m while we have extra decay from the terms  $|x|^{m-2}$ and  $|y|^{m-2}$ . Hence, the new method here is to have a refined classification of the dyadic cubes (see  $\mathcal{I}_1$  and  $\mathcal{I}_2$  below in the proof) such that for most of the cubes (see the term  $I_{122}$  below), the terms  $|x|^{m-2}$  and  $|y|^{m-2}$  can provide suitable decay that makes a compensation of the lack of power of the time scaling t and that for the rest of the cubes (see the term  $I_{123}$  below), the kernel of  $m(\sqrt{L})$  itself has proper decay which enables the weak type estimate holds.

To begin the proof, we now restrict the setting to  $\mathbb{R}^m \setminus K$ , and  $f_1$  is in  $L^1(\mathbb{R}^m \setminus K)$ . Extend  $f_1$  to the whole of  $\mathbb{R}^m$  by zero extension, i.e., define  $f_1(x) := 0$  when  $x \in K$ .

We now consider the standard Calderón–Zygmund decomposition as follows. Recall that the standard dyadic cubes in  $\mathbb{R}^m$  are of the form

$$[2^k a_1, 2^k (a_1+1)) \times \cdots \times [2^k a_m, 2^k (a_m+1)),$$

where  $k, a_1, \ldots, a_m$  are integers. Decompose  $\mathbb{R}^m$  into a mesh of equal size disjoint dyadic cubes so that

$$|Q| \ge \frac{1}{\lambda} \|f_1\|_{L^1(\mathbb{R}^m)}$$

for every cube in the mesh. Subdivide each cube in the mesh into  $2^m$  congruent cubes by bisecting each of its sides. We now have a new mesh of dyadic cubes. Select a cube in the new mesh if

(3.5) 
$$\frac{1}{|Q|} \int_{Q} |f_1(x)| dx > \lambda.$$

Let  $\mathcal{S}$  be the set of all these selected cubes. Now subdividing each non-selected cube into  $2^m$  congruent subcubes by bisecting each side as before. Then select one of these new cubes if (3.5) holds. Put all these selected cubes of this generation into the set  $\mathcal{S}$ . Repeat this procedure indefinitely.

Then we have  $\mathcal{S} = \bigcup_j Q_j$ , where all these  $Q'_j s$  are disjoint, and we further have

$$|S| = \sum_{j} |Q_{j}| \le \frac{1}{\lambda} \sum_{j} \int_{Q_{j}} |f_{1}(x)| dx \le \frac{1}{\lambda} ||f_{1}||_{L^{1}(X)}.$$

Define

$$b_j(x) := \left( f_1(x) - \frac{1}{|Q_j|} \int_{Q_j} f_1(y) dy \right) \chi_{Q_j}(x)$$

and

$$b(x) := \sum_{j} b_j(x), \quad g(x) := f_1(x) - b(x).$$

For a selected  $Q_j$ , there exists a unique non-selected dyadic cube Q' with twice its side length that contains  $Q_j$ . Since Q' is not selected, we get that

$$\frac{1}{|Q'|} \int_{Q'} |f_1(y)| dy \le \lambda,$$

which implies that

$$\frac{1}{|Q_j|} \int_{Q_j} |f_1(y)| dy \le \frac{2^m}{|Q'|} \int_{Q'} |f(y)| dy \le 2^m \lambda.$$

For the good part g(x), since b = 0 on  $F := \mathbb{R}^m \setminus S$ , we have

$$g(x) = f_1(x)$$
 on  $F$ , and  $g(x) = \frac{1}{|Q_j|} \int_{Q_j} f_1(x) dx$  on  $Q_j$ .

Then it is easy to verify that

$$||g||_{L^1(\mathbb{R}^m)} \le ||f||_{L^1(\mathbb{R}^m)}$$
 and  $||g||_{L^\infty(\mathbb{R}^m)} \le C\lambda$ .

We now have

$$I_{1} \leq \left| \left\{ x \in \mathbb{R}^{m} \setminus K : |\mathfrak{M}(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right|$$
  
+ 
$$\left| \left\{ x \in (\mathbb{R}^{m} \setminus K) \setminus \bigcup_{i} 8Q_{i} : |\mathfrak{M}(\sqrt{L}) \left(\sum_{j} b_{j}\right)(x)| > \frac{\lambda}{6} \right\} \right|$$
  
+ 
$$|\bigcup_{i} 8Q_{i}|$$
  
=: 
$$I_{11} + I_{12} + I_{13}.$$

As for  $I_{11}$ , by using the  $L^2$  boundedness of  $\mathfrak{M}(\sqrt{L})$ , we obtain that

$$I_{11} = \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right|$$
  
$$\leq \frac{C}{\lambda^2} ||g||^2_{L^2(\mathbb{R}^m \backslash K)}$$
  
$$\leq \frac{C}{\lambda} ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)},$$

where we use the fact that  $|g(x)| \leq C\lambda$ .

As for  $I_{13}$ , note that we have the doubling condition in this case. So we get that

$$I_{13} \le C \sum_{i} |Q_i| \le C \frac{\|f\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}$$

As for  $I_{12}$ , we now split all the  $Q_i$ 's in the set S into two groups:

 $\mathcal{I}_1 := \{i : \text{none of the corners of } Q_i \text{ is the origin}\},\$ 

and

 $\mathcal{I}_2 = \{i : \text{one of the corners of } Q_i \text{ is the origin}\}.$ 

Write

$$\mathfrak{M}(\sqrt{L}) \Big(\sum_{i} b_{i}\Big)(x) = \sum_{i \in \mathcal{I}_{1}} \mathfrak{M}(\sqrt{L}) b_{i}(x) + \sum_{i \in \mathcal{I}_{2}} \mathfrak{M}(\sqrt{L}) b_{i}(x).$$

For each  $i \in \mathcal{I}_1$ , we further decompose

$$\mathfrak{M}(\sqrt{L})b_i(x) = \mathfrak{M}(\sqrt{L})e^{-t_i\sqrt{L}}b_i(x) + \mathfrak{M}(\sqrt{L})(I - e^{-t_i\sqrt{L}})b_i(x),$$

where  $\{e^{-t\sqrt{L}}\}_{t>0}$  is the Poisson semigroup of L as studied in Section 2, and for each  $i, t_i$  is the side length of the cube  $Q_i$ .

Then we have

$$I_{12} \leq \left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_{i \in \mathcal{I}_1} e^{-t_i \sqrt{L}} b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right|$$
  
+ 
$$\left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_{i \in \mathcal{I}_1} \left( I - e^{-t_i \sqrt{L}} \right) b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right|$$
  
+ 
$$\left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_{i \in \mathcal{I}_2} b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right|$$
  
=: 
$$I_{121} + I_{122} + I_{123}.$$

We first estimate  $I_{121}$ . To see this, we claim that

(3.6) 
$$\left\|\sum_{i\in\mathcal{I}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathbb{R}^m\sharp\mathcal{R}^n)} \le C\lambda^{\frac{1}{2}}\|f_1\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}$$

To verify this claim, it suffices to show the following 3 cases:

(3.7) 
$$\left\|\sum_{i\in\mathcal{I}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathbb{R}^m\setminus K)} \le C\lambda^{\frac{1}{2}}\|f_1\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}$$

(3.8) 
$$\left\|\sum_{i\in\mathcal{I}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathcal{R}^n\setminus K)} \le C\lambda^{\frac{1}{2}}\|f_1\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}$$

and

(3.9) 
$$\left\| \sum_{i \in \mathcal{I}_1} e^{-t_i \sqrt{L}} b_i \right\|_{L^2(K)} \le C \lambda^{\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}$$

Hence, combining the estimates of (3.7), (3.8) and (3.9), for  $I_{121}$ , we get that

$$I_{121} \leq \left\| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : |\mathfrak{M}(\sqrt{L}) \left( \sum_{j \in \mathcal{I}_1} e^{-t_j \sqrt{L}} b_j \right)(x) | > \frac{\lambda}{18} \right\} \\ \leq \frac{C}{\lambda^2} \left\| \sum_{i \in \mathcal{I}_1} e^{-t_i \sqrt{L}} b_i \right\|_{L^2(\mathbb{R}^m \sharp \mathcal{R}^n)}^2 \\ \leq \frac{C}{\lambda} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}.$$

We first estimate (3.7). Consider the function  $e^{-t_i\sqrt{L}}b_i(x)$  for  $x \in \mathbb{R}^m \setminus K$ . Since

$$e^{-t_i\sqrt{L}}b_i(x) = \int_{\mathbb{R}^m\setminus K} P_{t_i}(x,y)b_i(y)dy,$$

applying the upper bound in point 5 in Theorem 2.2 for  $|P_{t_i}(x, y)|$  we obtain that

$$\begin{split} |e^{-t_i\sqrt{L}}b_i(x)| &\leq \int_{\mathbb{R}^m\setminus K} |P_{t_i}(x,y)| \, |b_i(y)| dy \\ &\leq C \int_{\mathbb{R}^m\setminus K} \left( \frac{t_i}{|x|^{m-2}|y|^{m-2}(t_i+|x|+|y|)^{n+1}} + \frac{t_i}{(t_i+d(x,y))^{m+1}} \right) |b_i(y)| dy \\ &= C \int_{Q_i} \frac{t_i}{|x|^{m-2}|y|^{m-2}(t_i+|x|+|y|)^{n+1}} \, |b_i(y)| dy \\ &\quad + C \int_{\mathbb{R}^m \notin \mathcal{R}^n} \frac{t_i}{(t_i+d(x,y))^{m+1}} \, |b_i(y)| dy \\ &=: F_{1,i} + F_{2,i}. \end{split}$$

We turn to estimating the term  $F_{2,i}$ . In this case, we have  $x \in \mathbb{R}^m \setminus K$  and  $Q_i \subset \mathbb{R}^m \setminus K$ , dyadic, with none of the corners of  $Q_i$  being the origin. This implies that

$$\sup_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}} \le C \inf_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}}.$$

Therefore

$$F_{2,i} \leq C \sup_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}} \int_{\mathbb{R}^m \sharp \mathcal{R}^n} |b_i(y)| dy$$
$$\leq C \inf_{z \in Q_i} \frac{t_i}{(t_i + d(x, z))^{m+1}} \lambda |Q_i|$$
$$\leq C \lambda \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{t_i}{(t_i + d(x, z))^{m+1}} \chi_{Q_i}(z) dz,$$

where  $\chi_{Q_i}$  is the characteristic function of  $Q_i$ .

For any  $h \in L^2(\mathbb{R}^m \setminus K)$  with  $||h||_{L^2(\mathbb{R}^m \setminus K)} = 1$ , we get that

$$\langle F_{2,i},h\rangle = C\lambda \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \int_{\mathbb{R}^m \setminus K} \frac{t_i}{(t_i + d(x,z))^{m+1}} h(x) dx \ \chi_{Q_i}(z) dz \leq C\lambda \langle \mathcal{M}(h), \chi_{Q_i} \rangle.$$

As a consequence we obtain that

$$\left\langle \sum_{i\in\mathcal{I}_1}F_{2,i},h\right\rangle \leq C\lambda\left\langle \mathcal{M}(h),\sum_{i\in\mathcal{I}_1}\chi_{Q_i}\right\rangle,$$

which yields

$$\begin{split} \left\| \sum_{i \in \mathcal{I}_1} F_{2,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} &\leq C\lambda \left\| \sum_{i \in \mathcal{I}_1} \chi_{Q_i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \leq C\lambda \left( \sum_{i \in \mathcal{I}_1} |Q_i| \right)^{1/2} \\ &\leq C\lambda \frac{\|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} \\ &\leq C\lambda^{\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}. \end{split}$$

To handle  $F_{1,i}$ , we note that the distance of  $Q_i$  to the center K is comparable to the side length of  $Q_i$  for  $i \in \mathcal{I}_1$  since none of the corners of  $Q_i$  are the origin. Hence, we obtain that

(3.10) 
$$\sup_{z \in Q_i} |z| \approx \inf_{z \in Q_i} |z|.$$

Thus, we further obtain that

$$F_{1,i} \leq C \sup_{z \in Q_i} \frac{t_i}{|x|^{m-2} |z|^{m-2} (t_i + |x| + |z|)^{n+1}} \int_{\mathbb{R}^m \sharp \mathcal{R}^n} |b_i(y)| dy$$
  
$$\leq C \inf_{z \in Q_i} \frac{t_i}{|x|^{m-2} |z|^{m-2} (t_i + |x| + |z|)^{n+1}} \lambda |Q_i|$$
  
$$\leq C \lambda \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{t_i}{|x|^{m-2} |z|^{m-2} (t_i + |x| + |z|)^{n+1}} \chi_{Q_i}(z) dz.$$

Consequently, for any  $h \in L^2(\mathbb{R}^m \setminus K)$  with  $||h||_{L^2(\mathbb{R}^m \setminus K)} = 1$ ,

$$\begin{aligned} |\langle F_{1,i},h\rangle| &\leq C\lambda \int_{\mathbb{R}^m \setminus K} \int_{\mathbb{R}^m \setminus K} \frac{t_i}{|x|^{m-2} |z|^{m-2} (t_i + |x| + |z|)^{n+1}} |h(x)| dx \ \chi_{Q_i}(z) dz \\ &\leq C\lambda \int_{\mathbb{R}^m \setminus K} \mathcal{G}(h)(z) \chi_{Q_i}(z) dz, \end{aligned}$$

where  $\mathcal{G}$  is an operator defined as

$$\mathcal{G}(h)(z) := \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{m-2} |z|^{m-2} (|x|+|z|)^n} |h(x)| dx.$$

Next, it is direct to see that  $\mathcal{G}$  is a bounded operator on  $L^2(\mathbb{R}^m \setminus K)$ :

$$\begin{split} \|\mathcal{G}(h)\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2} &\leq \int_{\mathbb{R}^{m}\setminus K} \int_{\mathbb{R}^{m}\setminus K} \frac{1}{|x|^{2m-4}|z|^{2m-4}(|x|+|z|)^{2n}} dx \|h\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2} dz \\ &\leq \|h\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2} \int_{\mathbb{R}^{m}\setminus K} \int_{\mathbb{R}^{m}\setminus K} \frac{1}{|x|^{2m-4}|z|^{2m-4}|x|^{n}} dx dz \\ &\leq \|h\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2} \int_{\mathbb{R}^{m}\setminus K} \frac{1}{|x|^{2m-4}|x|^{n}} dx \int_{\mathbb{R}^{m}\setminus K} \frac{1}{|z|^{2m-4}|z|^{n}} dz \\ &\leq C\|h\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2}, \end{split}$$

where in the last inequality we use the condition that m > n > 2.

As a consequence, similar to the estimates for  $\sum_{i \in \mathcal{I}_1} F_{2,i}$ , we obtain that

$$\begin{split} \Big| \sum_{i \in \mathcal{I}_1} F_{1,i} \Big\|_{L^2(\mathbb{R}^m \setminus K)} &\leq C\lambda \Big\| \sum_{i \in \mathcal{I}_1} \chi_{Q_i} \Big\|_{L^2(\mathbb{R}^m \setminus K)} \\ &\leq C\lambda \Big( \sum_i |Q_i| \Big)^{1/2} \\ &\leq C\lambda^{\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R}^n \sharp \mathbb{R}^m)}^{\frac{1}{2}}. \end{split}$$

Combining the estimates with respect to  $F_{1,i}$  and  $F_{2,i}$  above, we deduce that

$$\begin{split} \left\| \sum_{i \in \mathcal{I}_1} e^{-t_i \sqrt{L}} b_i \right\|_{L^2(\mathbb{R}^m \setminus K)} &\leq \left\| \sum_{i \in \mathcal{I}_1} F_{1,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} + \left\| \sum_{i \in \mathcal{I}_1} F_{2,i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \\ &\leq C \lambda^{\frac{1}{2}} \| f_1 \|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}, \end{split}$$

which shows that the claim (3.7) holds.

We now estimate (3.8). Consider the function  $e^{-t_i\sqrt{\Delta}}b_i(x)$  for  $x \in \mathcal{R}^n \setminus K$ . Since

$$e^{-t_i\sqrt{\Delta}}b_i(x) = \int_{\mathbb{R}^m\setminus K} P_{t_i}(x,y)b_i(y)dy,$$

applying the upper bound in point 4 in Theorem 2.2 for  $|P_{t_i}(x, y)|$  we obtain that  $|e^{-t_i\sqrt{L}}b_i(x)|$ 

$$\begin{split} &\leq \int_{\mathbb{R}^m \sharp \mathcal{R}^n} |P_{t_i}(x,y)| \, |b_i(y)| dy \\ &\leq C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \left( \frac{t_i}{(t_i + d(x,y))^{m+1}} + \frac{1}{|x|^{n-2}} \frac{t_i}{(t_i + d(x,y))^{m+1}} + \frac{1}{|y|^{m-2}} \frac{t}{(t_i + d(x,y))^{n+1}} \right) |b_i(y)| dy \\ &\leq C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{t_i}{(t_i + d(x,y))^{m+1}} |b_i(y)| dy + C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{1}{|y|^{m-2}} \frac{t_i}{(t_i + d(x,y))^{n+1}} |b_i(y)| dy \\ &=: G_{1,i} + G_{2,i} \end{split}$$

where the third inequality follows from the fact that  $|x| \ge 1$ , hence the second term in the integrand is dominated by the first term.

By the equivalence in (3.10), we have

$$G_{2,i} \leq C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{1}{|y|^{m-2}} \frac{t_i}{(t_i + d(x, y))^{n+1}} |b_i(y)| dy$$
  
$$\leq C \sup_{z \in Q_i} \frac{1}{|z|^{m-2}} \frac{t_i}{(t_i + d(x, z))^{n+1}} \int_X |b_i(y)| dy$$
  
$$\leq C \inf_{z \in Q_i} \frac{1}{|z|^{m-2}} \frac{t_i}{(t_i + d(x, z))^{n+1}} \int_X |b_i(y)| dy$$
  
$$\leq C \lambda \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{1}{|z|^{m-2}} \frac{t_i}{(t_i + d(x, z))^{n+1}} \chi_{Q_i}(z) dz.$$

So for any  $h \in L^2(\mathcal{R}^n \setminus K)$  with  $||h||_{L^2(\mathcal{R}^n \setminus K)} = 1$ ,

$$\begin{aligned} |\langle G_{2,i},h\rangle| &\leq C\lambda \int_{\mathbb{R}^m \setminus K} \int_{\mathcal{R}^n \setminus K} \frac{1}{|z|^{m-2}} \frac{t_i}{(t_i + d(x,z))^{n+1}} |h(x)| dx \ \chi_{Q_i}(z) dz \\ &\leq C\lambda \int_{\mathbb{R}^m \setminus K} \mathcal{T}(h)(z) \chi_{Q_i}(z) dz, \end{aligned}$$

where the operator  $\mathcal{T}$  is defined as

$$\mathcal{T}(h)(z) := \int_{\mathcal{R}^n \setminus K} \frac{1}{|z|^{m-2}} \frac{t_i}{(t_i + d(x, z))^{n+1}} |h(x)| dx.$$

Once again, it is direct to see that  $\mathcal{T}$  is a bounded operator on  $L^2(\mathbb{R}^m \setminus K)$ :

$$\begin{aligned} \|\mathcal{T}(h)\|_{L^{2}(\mathbb{R}^{m}\setminus K)}^{2} &\leq \|h\|_{L^{2}(\mathbb{R}^{n}\setminus K)}^{2} \int_{\mathbb{R}^{m}\setminus K} \int_{\mathcal{R}^{n}\setminus K} \frac{1}{|z|^{2m-4}} \frac{t_{i}^{2}}{(t_{i}+|x|+|z|)^{2(n+1)}} dx dz \\ &\leq \|h\|_{L^{2}(\mathbb{R}^{n}\setminus K)}^{2} \int_{\mathbb{R}^{m}\setminus K} \int_{\mathcal{R}^{n}\setminus K} \frac{1}{t^{n}} \Big(\frac{t_{i}}{t_{i}+|x|}\Big)^{n+2} dx \frac{1}{|z|^{2m+n-4}} dz \\ &\leq C \|h\|_{L^{2}(\mathbb{R}^{n}\setminus K)}^{2}, \end{aligned}$$

where in the last inequality we use the condition that m > n > 2.

As a consequence we obtain that

$$\begin{split} \left\| \sum_{i \in \mathcal{I}_1} G_{2,i} \right\|_{L^2(\mathcal{R}^n \setminus K)} &\leq C\lambda \left\| \sum_{i \in \mathcal{I}_1} \chi_{Q_i} \right\|_{L^2(\mathbb{R}^m \setminus K)} \\ &\leq C\lambda \left( \sum_i |Q_i| \right)^{1/2} \\ &\leq C\lambda^{\frac{1}{2}} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}. \end{split}$$

Similar to the estimates for the terms  $F_{2,i}$ , we also obtain

$$\left\|\sum_{i\in\mathcal{I}_1}G_{1,i}\right\|_{L^2(\mathcal{R}^n\setminus K)}\leq C\lambda^{\frac{1}{2}}\|f_1\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}.$$

Combining the estimates for  $G_{1,i}$  and  $G_{2,i}$  above, we obtain that (3.8) holds.

We now verify (3.9). Consider the function  $e^{-t_i\sqrt{L}}b_i(x)$  for  $x \in K$ . Since

$$e^{-t_i\sqrt{L}}b_i(x) = \int_{\mathbb{R}^m\setminus K} P_{t_i}(x,y)b_i(y)dy,$$

applying the upper bound in point 5 in Theorem 2.2 for  $|P_{t_i}(x, y)|$  we obtain that

$$\begin{aligned} |e^{-t_i\sqrt{L}}b_i(x)| &\leq \int_{\mathbb{R}^m \sharp \mathcal{R}^n} |P_{t_i}(x,y)| \, |b_i(y)| dy \\ &\leq C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{t_i}{(t_i + d(x,y))^{m+1}} |b_i(y)| dy + C \int_{\mathbb{R}^m \sharp \mathcal{R}^n} \frac{1}{|y|^{m-2}} \frac{t}{(t_i + d(x,y))^{n+1}} |b_i(y)| dy \\ &=: H_{1,i} + H_{2,i}. \end{aligned}$$

Arguing similarly to the estimates for the terms  $G_{1,i}$  and  $G_{2,i}$ , we get the same estimates for the terms  $H_{1,i}$  and  $H_{2,i}$ , respectively. This implies that (3.9) holds.

We now consider the term  $I_{122}$ . Note that

$$I_{122} \leq \left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_j \left( I - e^{-t_j \sqrt{L}} \right) b_j \right)(x) \right| > \frac{\lambda}{12} \right\} \right|$$
$$\leq \frac{C}{\lambda} \sum_j \int_{(8Q_j)^c} |\mathfrak{M}(\sqrt{L}) \left( I - e^{-t_j \sqrt{L}} \right) b_j(x)| dx.$$

Note that for each j, we get that

(3.11)  
$$\int_{(8Q_j)^c} |\mathfrak{M}(\sqrt{L}) (I - e^{-t_j \sqrt{L}}) b_j(x)| dx$$
$$\leq \int_{(8Q_j)^c} \int_{Q_j} |k_{t_j}(x, y)| |b_j(y)| dy dx$$
$$= \int_{Q_j} \int_{(8Q_j)^c} |k_{t_j}(x, y)| dx \ |b_j(y)| dy$$

where we use  $k_{t_j}(x, y)$  to denote the kernel of the operator  $\mathfrak{M}(\sqrt{L})(I - e^{-t_j\sqrt{L}})$ . By definition, we have

$$\mathfrak{M}(\sqrt{L})\left(I - e^{-t_j\sqrt{L}}\right) = \int_0^\infty \sqrt{L}e^{-s\sqrt{L}}m(s)ds \int_0^{t_j} -\frac{d}{dt}e^{-t\sqrt{L}}dt$$
$$= \int_0^\infty \sqrt{L}e^{-s\sqrt{L}}m(s)ds \int_0^{t_j} \sqrt{L}e^{-t\sqrt{L}}dt$$
$$= \int_0^{t_j} \int_0^\infty (\sqrt{L})^2 e^{-(s+t)\sqrt{L}} m(s) \, dsdt$$
$$= \int_0^{t_j} \int_0^\infty (s+t)^2 (\sqrt{L})^2 e^{-(s+t)\sqrt{L}} \frac{m(s)}{(s+t)^2} \, dsdt.$$

Hence, we obtain that

$$k_{t_j}(x,y) = \int_0^{t_j} \int_0^\infty P_{s+t,2}(x,y) \,\frac{m(s)}{(s+t)^2} \, ds dt.$$

We now claim that there exists an absolute positive constant C such that

(3.12) 
$$\int_{(8Q_j)^c} |k_{t_j}(x,y)| dx \le C.$$

To see this, applying the kernel expression above and Case 5 in Theorem 2.2 for  $P_{t,2}(x,y)$ , we get that

$$\begin{split} &\int_{(8Q_j)^c} |k_{t_j}(x,y)| dx \\ &\leq \int_{(8Q_j)^c} \int_0^{t_j} \int_0^{\infty} |P_{s+t,2}(x,y)| \, \frac{|m(s)|}{(s+t)^2} \, ds dt dx \\ &\leq \int_{(8Q_j)^c} \int_0^{t_j} \int_0^{\infty} \frac{C}{(s+t)^m} \Big(\frac{s+t}{s+t+d(x,y)}\Big)^{m+2} \frac{1}{(s+t)^2} \, ds dt dx \\ &\quad + \int_{(8Q_j)^c} \int_0^{t_j} \int_0^{\infty} \frac{C}{(s+t)^n |x|^{m-2} |y|^{m-2}} \Big(\frac{s+t}{s+t+|x|+|y|}\Big)^{n+2} \frac{1}{(s+t)^2} \, ds dt dx \\ &=: E_1 + E_2. \end{split}$$

We first consider the term  $E_1$ . Note that

$$\begin{split} E_1 &\leq C \int_0^{t_j} \int_0^\infty \int_{d(x,y) \geq 2t_j} \frac{1}{(s+t)^m} \Big(\frac{s+t}{s+t+d(x,y)}\Big)^{m+2} \frac{1}{(s+t)^2} \, dx \, ds dt \\ &\leq C \int_0^{t_j} \int_0^{t_j} \int_{d(x,y) \geq 2t_j} \Big(\frac{1}{s+t+d(x,y)}\Big)^{m+2} \, dx \, ds dt \\ &+ C \int_0^{t_j} \int_{t_j}^\infty \int_{d(x,y) \geq 2t_j} \frac{1}{(s+t)^m} \Big(\frac{s+t}{s+t+d(x,y)}\Big)^{m+2} \, dx \frac{1}{(s+t)^2} \, ds dt \\ &\leq C \int_0^{t_j} \int_0^{t_j} \int_{t_j}^\infty \frac{1}{r^{m+2}} r^{m-1} \, dr \, ds dt + C \int_0^{t_j} \int_{t_j}^\infty \frac{1}{(s+t)^2} \, ds dt \\ &\leq C, \end{split}$$

where in the last inequality, we use polar coordinates to estimate the first term and we use the following fact for the second term

$$\int_{d(x,y) \ge 2t_j} \frac{1}{(s+t)^m} \Big(\frac{s+t}{s+t+d(x,y)}\Big)^{m+2} \, dx \le C.$$

Next we consider the term  $E_2$ . Note that

$$E_{2} \leq C \int_{0}^{t_{j}} \int_{0}^{t_{j}} \int_{d(x,y)\geq 2t_{j}}^{t_{j}} \frac{1}{|x|^{m-2}|y|^{m-2}} \left(\frac{1}{|x|+|y|}\right)^{n+2} dx \, ds dt \\ + C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \int_{d(x,y)\geq 2t_{j}} \frac{1}{(s+t)^{n}|x|^{m-2}|y|^{m-2}} \left(\frac{s+t}{s+t+|x|+|y|}\right)^{n+2} dx \, \frac{1}{(s+t)^{2}} \, ds dt \\ =: E_{21} + E_{22}.$$

As for  $E_{21}$ , we first suppose  $t_j \ge 1$ . Then by noting that  $d(x, y) \le |x| + |y|$  and that  $|x| \ge 1$  and  $|y| \ge 1$ , we have

$$\frac{1}{|x|^{m-2}} \le \frac{1}{d(x,y)^{m-2}} \quad \text{or} \quad \frac{1}{|y|^{m-2}} \le \frac{1}{d(x,y)^{m-2}},$$

which implies that

$$\frac{1}{|x|^{m-2}|y|^{m-2}} \le \frac{1}{d(x,y)^{m-2}}.$$

As a consequence,

$$E_{21} \leq C \int_{0}^{t_j} \int_{0}^{t_j} \int_{d(x,y)\geq 2t_j}^{t_j} \frac{1}{d(x,y)^{m-2}} \left(\frac{1}{d(x,y)}\right)^{n+2} dx \, ds dt$$
$$\leq C \int_{0}^{t_j} \int_{0}^{t_j} \int_{t_j}^{\infty} \frac{1}{r^{m+n}} r^{m-1} \, dr ds dt \leq C \frac{t_j^2}{t_j^n} \leq C.$$

We now suppose  $t_j < 1$ . Then it is direct that

$$E_{21} \leq C \int_0^1 \int_0^1 \int_{d(x,y) \geq 2t_j} \frac{1}{|x|^{m-2} |y|^{m-2}} \left(\frac{1}{|x| + |y|}\right)^{n+2} dx \, ds dt$$
$$\leq C \int_{\mathbb{R}^m \setminus K} \frac{1}{|x|^{m-2}} \left(\frac{1}{|x|}\right)^{n+2} dx \leq C.$$

As for  $E_{22}$ , again, noting that  $d(x, y) \leq |x| + |y|$ , we have

$$E_{22} \le C \int_0^{t_j} \int_{t_j}^\infty \int_{d(x,y)\ge 2t_j} \frac{1}{(s+t)^n |x|^{m-2} |y|^{m-2}} \left(\frac{s+t}{s+t+|x|+|y|}\right)^{n+2} dx \frac{1}{(s+t)^2} \, ds dt$$

$$\begin{split} &\leq C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \int_{d(x,y) \geq 2t_{j}} \frac{1}{(s+t)^{n} |x|^{m-2} |y|^{m-2}} \Big( \frac{s+t}{s+t+|x|+|y|} \Big)^{n} \, dx \, \frac{1}{(s+t)^{2}} \, ds dt \\ &\leq C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \int_{d(x,y) \geq 2t_{j}} \frac{1}{|x|^{m-2} |y|^{m-2}} \Big( \frac{1}{|x|+|y|} \Big)^{n} \, dx \, \frac{1}{(s+t)^{2}} \, ds dt \\ &\leq C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \int_{\mathbb{R}^{m} \setminus K} \frac{1}{|x|^{m-2}} \Big( \frac{1}{|x|} \Big)^{n} \, dx \, \frac{1}{(s+t)^{2}} \, ds dt \\ &\leq C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \frac{1}{(s+t)^{2}} \, ds dt \\ &\leq C \int_{0}^{t_{j}} \int_{t_{j}}^{\infty} \frac{1}{(s+t)^{2}} \, ds dt \\ &\leq C. \end{split}$$

Combining the estimates of  $E_{22}$ ,  $E_{12}$  and  $E_1$ , we obtain that the claim (3.12) holds. As a consequence, from (3.11) we obtain that for each j,

$$\int_{(8Q_j)^c} |\mathfrak{M}(\sqrt{L})(I - e^{-t_j\sqrt{L}})b_j(x)|dx \le \int_{Q_j} |b_j(y)|dy \le C\lambda |Q_j|,$$

which implies that

$$I_{122} \leq \frac{C}{\lambda} \sum_{j} \int_{(8Q_j)^c} |\mathfrak{M}(\sqrt{L}) \left( I - e^{-t_j \sqrt{L}} \right) b_j(x)| dx$$
$$\leq C \sum_{j} |Q_j| \leq \frac{C}{\lambda} ||f_1||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}.$$

We now consider the term  $I_{123}$ . Note that for each  $i \in \mathcal{I}_2$  we have  $t_i \geq 1/2$ . Fix  $i \in \mathcal{I}_2$ . Denote by  $k_{\mathfrak{M}(\sqrt{L})}(x, y)$  the associated kernel of  $\mathfrak{M}(\sqrt{L})$ . For  $x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i$  and  $y \in Q_i$ , by point 5 in Theorem 2.2 we have

$$\begin{aligned} |k_{\mathfrak{M}(\sqrt{L})}(x,y)| &\leq \int_{0}^{\infty} |P_{t}(x,y)| \frac{dt}{t} \\ &\leq \int_{0}^{\infty} \frac{C}{t^{m+1}} \Big(\frac{t}{t+d(x,y)}\Big)^{m+1} dt + \int_{0}^{\infty} \frac{C}{t^{n+1}|x|^{m-2}|y|^{m-2}} \Big(\frac{t}{t+|x|+|y|}\Big)^{n+1} dt \\ &=: K_{1}(x,y) + K_{2}(x,y). \end{aligned}$$

Since  $d(x, y) \sim d(x, x_{Q_i})$ , we have

$$K_{1}(x,y) \leq \int_{0}^{\infty} \frac{C}{t^{m+1}} \left(\frac{t}{t+d(x,x_{Q_{i}})}\right)^{m+1} dt$$
$$\leq \int_{0}^{d(x,x_{Q_{i}})} \frac{C}{d(x,x_{Q_{i}})^{m+1}} dt + \int_{d(x,x_{Q_{i}})}^{\infty} \frac{C}{t^{m+1}} dt$$
$$\leq \frac{C}{d(x,x_{Q_{i}})^{m}}.$$

Using the fact that  $|x||y| \gtrsim |x| + |y| \gtrsim d(x, y) \gtrsim d(x, x_{Q_i})$  we have

$$K_{2}(x,y) \leq \int_{0}^{\infty} \frac{C}{t^{n+1}d(x,x_{Q_{i}})^{m-2}} \left(\frac{t}{t+d(x,x_{Q_{i}})}\right)^{n+1} dt$$
  
$$\leq \int_{0}^{d(x,x_{Q_{i}})} \frac{C}{d(x,x_{Q_{i}})^{m+n-1}} dt + \int_{d(x,x_{Q_{i}})}^{\infty} \frac{C}{t^{n+1}d(x,x_{Q_{i}})^{m-2}} dt$$
  
$$\leq \frac{C}{d(x,x_{Q_{i}})^{m+n-2}}$$
  
$$\leq \frac{C}{d(x,x_{Q_{i}})^{m}}$$

where in the last inequality we used  $d(x, x_{Q_i}) \ge 2t_i \ge 1$ .

From the estimates of  $K_1(x, y)$  and  $K_2(x, y)$ , for each  $i \in \mathcal{I}_2$  and  $x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i$  we have

$$\sup_{y \in Q_i} |k_{\mathfrak{M}(\sqrt{L})}(x, y)| \le \frac{C}{d(x, x_{Q_i})^m}.$$

Moreover, observe that since  $i \in \mathcal{I}_2$  and  $x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i$  we have

$$\frac{1}{d(x, x_{Q_i})^m} \sim \frac{1}{|x|^m}.$$

As a consequence, for each  $i \in \mathcal{I}_2$  and  $x \in (\mathbb{R}^m \setminus K) \setminus 8Q_i$  we have

$$\sup_{y \in Q_i} |k_{\mathfrak{M}(\sqrt{L})}(x,y)| \le \frac{C}{|x|^m}.$$

This implies that for each  $x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i$ , we have

$$\left|\sum_{i\in\mathcal{I}_2}\mathfrak{M}(\sqrt{L})b_i(x)\right| \le C\frac{\sum_{i\in\mathcal{I}_2}\|b_i\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}}{|x|^m}.$$

Therefore,

$$I_{123} \leq \left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : \left| \sum_{i \in \mathcal{I}_2} \mathfrak{M}(\sqrt{L}) b_i(x) \right| > \frac{\lambda}{18} \right\} \right|$$
  
$$\leq \left| \left\{ x \in (\mathbb{R}^m \setminus K) \setminus \bigcup_i 8Q_i : C|x|^{-m} \left( \sum_{i \in \mathcal{I}_2} \|b_i\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)} \right) > \frac{\lambda}{18} \right\} \right|$$
  
$$\leq C \frac{\sum_{i \in \mathcal{I}_2} \|b_i\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}$$
  
$$\leq C \frac{\|f\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}.$$

Combining all cases of  $I_{11}$ ,  $I_{13}$ ,  $I_{121}$ ,  $I_{122}$  and  $I_{123}$ , we obtain that

$$I_1 \le \frac{C}{\lambda} \|f_1\|_{L^1(X)}.$$

3.2. Estimate of  $I_2$ . We now consider the term  $I_2$ . Note that in the case, x is in the large end  $\mathbb{R}^m \setminus K$  and the function  $f_2$  is supported in the small end  $\mathcal{R}^n \setminus K$ , and hence the measure will become non-doubling since if we enlarge a ball contained in  $\mathcal{R}^n \setminus K$ , then the enlargement can be partially contained in  $\mathbb{R}^m \setminus K$ . The standard Calderón–Zygmund decomposition on non-homogeneous space such as in [23, 29] does not apply since in that decomposition, we only know the existence of a

sequence of Calderón–Zygmund cubes but we do not know where they are exactly. And the Poisson kernel upper bound depends heavily on the position of the variables x and y in different ends.

Thus, to deal with this case, we use a Whitney type decomposition of the level set  $\Omega$  below and then we make clever use of the Poisson kernel upper bound in this case to handle the weak type estimate, without enlarging those cubes, which avoids the case of non-doubling measure. The genesis of this approach is an adaptation of an idea from [23].

Note that  $f_2$  is supported in  $\mathcal{R}^n \setminus K$ . We now split  $\mathcal{R}^n \setminus K$  into two parts according to  $f_2$ . Define

$$F := \{ x \in \mathcal{R}^n \backslash K : M_2(f_2) \le \lambda \}$$

and

$$\Omega := \{ x \in \mathcal{R}^n \backslash K : M_2(f_2) > \lambda \},\$$

where  $M_2$  is the Hardy–Littlewood maximal function defined on  $\mathcal{R}^n \setminus K$ .

Then we define

$$f_{2,\lambda}(x) := f_2(x)\chi_F(x)$$
 and  $f_2^{\lambda}(x) := f_2(x)\chi_{\Omega}(x).$ 

Then we have

$$I_{2} \leq \left| \left\{ x \in \mathbb{R}^{m} \setminus K : |\mathfrak{M}(\sqrt{L})f_{2,\lambda}(x)| > \frac{\lambda}{6} \right\} \right| \\ + \left| \left\{ x \in \mathbb{R}^{m} \setminus K : |\mathfrak{M}(\sqrt{L})f_{2}^{\lambda}(x)| > \frac{\lambda}{6} \right\} \right| \\ =: I_{21} + I_{22}.$$

As for  $I_{21}$ , by using the  $L^2$  boundedness of  $m(\sqrt{L})$ , we obtain that

$$I_{21} = |\{x \in \mathbb{R}^m \setminus K : |\mathfrak{M}(\sqrt{L})f_{2,\lambda}(x)| > \frac{\lambda}{6}\}|$$
  
$$\leq \frac{C}{\lambda^2} ||f_{2,\lambda}||^2_{L^2(\mathcal{R}^n \setminus K)}$$
  
$$\leq \frac{C}{\lambda} ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)},$$

where we use the fact that  $|f_{2,\lambda}(x)| = |f_2(x)|\chi_F(x) \le |M_2(f_2)(x)|\chi_F(x) \le \lambda$ .

As for  $I_{22}$ , we consider the function  $f_2^{\lambda}$ . We now apply a covering lemma in [6] (see also [8, Lemma 5.5]) for the set  $\Omega$  in the homogeneous space  $\mathcal{R}^n$  to obtain a collection of balls  $\{Q_i := B(x_i, r_i) : x_i \in \Omega, r_i = d(x_i, \Omega^c)/2, i = 1, ...\}$  so that

- (i)  $\Omega = \bigcup_i Q_i;$
- (ii)  $\{B(x_i, r_i/5)\}_{i=1}^{\infty}$  are disjoint;

(iii) there exists a universal constant C so that  $\sum_k \chi_{Q_k}(x) \leq C$  for all  $x \in \Omega$ .

Hence, we can further decompose

$$f_2^{\lambda}(x) = \sum_i f_{2,i}^{\lambda}(x),$$

where  $f_{2,i}^{\lambda}(x) = \frac{\chi_{Q_i}(x)}{\sum_k \chi_{Q_k}(x)} f_2^{\lambda}(x).$ 

Next, note that for  $x \in \mathbb{R}^m \setminus K$ ,

$$\begin{aligned} |\mathfrak{M}(\sqrt{L})(f_{2,i}^{\lambda})(x)| &= \left| \int_{0}^{\infty} t\sqrt{L} \exp(-t\sqrt{L})(f_{2,i}^{\lambda})(x)\tilde{m}(t)\frac{dt}{t} \right| \\ &\leq \int_{0}^{\infty} \int_{Q_{i}} |P_{t,1}(x,y)| \left|\tilde{m}(t)\right| \left|f_{2,i}^{\lambda}(y)\right| dy\frac{dt}{t} \end{aligned}$$

Applying the upper bound in point 4 in Theorem 2.2 for  $|P_{t,1}(x,y)|$  we obtain that

$$(3.13) \qquad |\mathfrak{M}(\sqrt{L})(f_{2,i}^{\lambda})(x)| \leq C \int_{0}^{\infty} \int_{Q_{i}} \left( \frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|x|^{m-2}} \frac{t}{(t+d(x,y))^{n+1}} + \frac{1}{|y|^{n-2}} \frac{t}{(t+d(x,y))^{m+1}} \right) |f_{2,i}^{\lambda}(y)| \, dy \frac{dt}{t}.$$

Note that  $d(x,y) \approx |x| + |y|$  since  $x \in \mathbb{R}^m \setminus K$  and  $y \in \mathcal{R}^n \setminus K$ . Hence,

$$\begin{split} \int_0^\infty \left( \frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|x|^{m-2}} \frac{t}{(t+d(x,y))^{n+1}} + \frac{1}{|y|^{n-2}} \frac{t}{(t+d(x,y))^{m+1}} \right) \frac{dt}{t} \\ &\leq C \int_0^\infty \left( \frac{t}{(t+|x|)^{m+1}} + \frac{1}{|x|^{m-2}} \frac{t}{(t+|x|)^{n+1}} \right) \frac{dt}{t}. \end{split}$$

Splitting this into two integrals over (0, |x|) and  $(|x|, \infty)$ , by a straightforward calculation we have

$$(3.14) \qquad \int_0^\infty \left(\frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|x|^{m-2}}\frac{t}{(t+d(x,y))^{n+1}} + \frac{1}{|y|^{n-2}}\frac{t}{(t+d(x,y))^{m+1}}\right)\frac{dt}{t} \le \frac{C}{|x|^m}.$$

Inserting into (3.13), we have

$$|\mathfrak{M}(\sqrt{L})(f_{2,i}^{\lambda})(x)| \leq \frac{C}{|x|^m} \int_{Q_i} |f_{2,i}^{\lambda}(y)| dy,$$

which implies that

$$\begin{split} |\mathfrak{M}(\sqrt{L})(f_2^{\lambda})(x)| &\leq \sum_i |\mathfrak{M}(\sqrt{L})(f_{2,i}^{\lambda})(x)| \\ &\leq C \frac{1}{|x|^m} \sum_i \int_{Q_i} |f_{2,i}^{\lambda}(y)| dy \\ &\leq C \frac{1}{|x|^m} \|f_2\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}. \end{split}$$

Hence, we obtain that

$$I_{22} = \left| \left\{ x \in \mathbb{R}^m \backslash K : |\mathfrak{M}(\sqrt{L}) f_2^{\lambda}(x)| > \frac{\lambda}{6} \right\} \right|$$
  
$$\leq \left| \left\{ x \in \mathbb{R}^m \backslash K : C \frac{1}{|x|^m} ||f_2||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)} > \frac{\lambda}{6} \right\} \right|$$
  
$$= \left| \left\{ x \in \mathbb{R}^m \backslash K : |x|^m < \frac{6C ||f_2||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda} \right\} \right|$$
  
$$\leq \frac{C ||f_2||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}$$
  
$$\leq \frac{C ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}.$$

3.3. Estimate of  $I_3$ . For the term  $I_3$ , we point out that we can handle this case by using the same approach as in the estimates for the term  $I_1$  with minor modifications and hence we omit the details.

3.4. Estimate of  $II_1$ . For the term  $II_1$ , we point out that we can handle this case by using similar way as in the estimates for the term  $I_2$ . We sketch the proof as follows.

Define  $F := \{x \in \mathbb{R}^m \setminus K : M_1(f_1) \leq \lambda\}$  and  $\Omega := \{x \in \mathbb{R}^m \setminus K : M_1(f_1) > \lambda\}$ , where  $M_1$  is the Hardy–Littlewood maximal function defined on  $\mathbb{R}^m \setminus K$ . Then let  $f_{1,\lambda}(x) := f_1(x)\chi_F(x)$  and  $f_1^{\lambda}(x) := f_1(x)\chi_{\Omega}(x)$ .

Then we have

$$II_{1} \leq \left| \left\{ x \in \mathcal{R}^{n} \backslash K : |\mathfrak{M}(\sqrt{L})f_{1,\lambda}(x)| > \frac{\lambda}{6} \right\} \right| + \left| \left\{ x \in \mathcal{R}^{n} \backslash K : |\mathfrak{M}(\sqrt{L})f_{1}^{\lambda}(x)| > \frac{\lambda}{6} \right\}$$
$$=: II_{11} + II_{12}.$$

Using the  $L^2$  boundedness of  $\mathfrak{M}(\sqrt{L})$  and the fact that  $|f_{1,\lambda}(x)| \leq \lambda$ , we obtain  $II_{11} \leq \frac{C}{\lambda} ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}$ .

As for  $II_{12}$ , by using the Whitney decomposition, we obtain  $\Omega = \bigcup_i Q_i$  such that  $\sum_i |Q_i| = |\Omega|$ , which gives

$$f_1^{\lambda}(x) = \sum_i f_{1,i}^{\lambda}(x),$$

where  $f_{1,i}^{\lambda}(x) = f_1^{\lambda}(x)\chi_{Q_i}(x)$ .

Next, for  $x \in \mathcal{R}^n \setminus K$ ,

$$|\mathfrak{M}(\sqrt{L})(f_{1,i}^{\lambda})(x)| \leq \int_0^\infty \int_{Q_i} |P_{t,1}(x,y)| \, |\tilde{m}(t)| \, |f_{1,i}^{\lambda}(y)| \, dy \frac{dt}{t}$$

Applying the upper bound in point 4 in Theorem 2.2 for  $|P_{t,1}(x,y)|$  we obtain that

$$\begin{split} |\mathfrak{M}(\sqrt{L})(f_{1,i}^{\lambda})(x)| &\leq C \int_{0}^{\infty} \int_{Q_{i}} \left( \frac{t}{(t+d(x,y))^{m+1}} + \frac{1}{|y|^{m-2}} \frac{t}{(t+d(x,y))^{n+1}} \right. \\ &+ \frac{1}{|x|^{n-2}} \frac{t}{(t+d(x,y))^{m+1}} \right) |f_{1,i}^{\lambda}(y)| \, dy \frac{dt}{t} \\ &\leq \frac{C}{|x|^{n}} \int_{Q_{i}} |f_{1,i}^{\lambda}(y)| \, dy, \end{split}$$

where the last inequality follows from similar estimates for (3.14). This implies that

$$|\mathfrak{M}(\sqrt{L})(f_1^{\lambda})(x)| \le \sum_i |m(\sqrt{L})(f_{1,i}^{\lambda})(x)| \le C \frac{1}{|x|^n} \sum_i \int_{Q_i} |f_{1,i}^{\lambda}(y)| dy \le C \frac{1}{|x|^n} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)} dx \le C \frac{1}{|x|^n} \|f_1\|_{L^1(\mathbb{R}^n \Bbbk \mathcal{R}^n)} dx \le C \frac{1}{|x|^n} \|f_$$

Hence, we obtain that

$$\begin{split} II_{12} &= \left| \left\{ x \in \mathcal{R}^n \backslash K : |\mathfrak{M}(\sqrt{L}) f_1^{\lambda}(x)| > \frac{\lambda}{6} \right\} \right| \le \left| \left\{ x \in \mathcal{R}^n \backslash K : C\frac{1}{|x|^n} \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)} > \frac{\lambda}{6} \right\} \right| \\ &= \left| \left\{ x \in \mathcal{R}^n \backslash K : |x|^n < \frac{6C \|f_1\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda} \right\} \right| \\ &\le \frac{C \|f\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}. \end{split}$$

3.5. Estimate of  $II_2$ . We will apply a similar approach as that in [10] and using similar estimates for the term  $I_1$  in our Section 3.1 to estimate  $II_2$ .

We restrict the setting to  $\mathcal{R}^n \setminus K$ , and  $f_2$  is in  $L^1(\mathcal{R}^n \setminus K)$ . We now extend  $f_2$  to all of  $\mathbb{R}^n$  by zero extension, i.e., define  $f_2(x) := 0$  when  $x \in K$ .

Similar to the Calderón–Zygmund decomposition in  $I_1$ , we get

$$f_2(x) = g(x) + \sum_j b_j(x)$$

with  $||g||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)} \le ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}$  and  $||g||_{L^\infty(\mathbb{R}^m \sharp \mathcal{R}^n)} \le C\lambda$  and

$$b_j(x) := \left( f_2(x) - \frac{1}{|Q_j|} \int_{Q_j} f_2(y) dy \right) \chi_{Q_j}(x).$$

Then we get

$$II_{2} \leq \left| \left\{ x \in \mathcal{R}^{n} \setminus K : |\mathfrak{M}(\sqrt{L})g(x)| > \frac{\lambda}{6} \right\} \right|$$
  
+ 
$$\left| \left\{ x \in (\mathcal{R}^{n} \setminus K) \setminus \bigcup_{i} 8Q_{i} : |\mathfrak{M}(\sqrt{L})(\sum_{i} b_{i})(x)| > \frac{\lambda}{6} \right\} \right| + |\bigcup_{i} 8Q_{i}|$$
  
=: 
$$II_{21} + II_{22} + II_{23}.$$

By using the  $L^2$  boundedness of  $\mathfrak{M}(\sqrt{L})$  and the fact that  $||g||_{L^{\infty}(X)} \leq C\lambda$ , we obtain that  $II_{21} \leq \frac{C}{\lambda} ||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}$ . Next, from the doubling condition in this case, we get that  $II_{23} \leq C \sum_i |Q_i| \leq C \frac{||f||_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}}{\lambda}$ . For the term  $II_{22}$ , we have

$$II_{22} \leq \left| \left\{ x \in (\mathcal{R}^n \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_i e^{-t_i \sqrt{L}} b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right| \\ + \left| \left\{ x \in (\mathcal{R}^n \setminus K) \setminus \bigcup_i 8Q_i : \left| \mathfrak{M}(\sqrt{L}) \left( \sum_i \left( I - e^{-t_i \sqrt{L}} \right) b_i \right)(x) \right| > \frac{\lambda}{18} \right\} \right| \\ =: II_{221} + II_{222},$$

where for each *i*,  $t_i$  is the side length of the cube  $Q_i$ . Note that the term  $II_{222}$  can be handled similarly by using the same approach as that for  $I_{122}$  and using upper bound in point 6 in Theorem 2.2 for  $|P_{t,2}(x,y)|$ , which yields that  $II_{222}$  is bounded by  $C\frac{\|f\|_{L^1(\mathbb{R}^m \sharp \mathbb{R}^n)}}{\lambda}$ .

As for  $II_{221}$ , we now split all the  $Q_i$ 's into two groups:

 $\mathcal{J}_1 := \{i : \text{none of the corners of } Q_i \text{ is the origin}\},\$ 

and

 $\mathcal{J}_2 = \{i : \text{one of the corners of } Q_i \text{ is the origin}\}.$ 

Similarly  $I_{12}$ , we need only to claim that

(3.15) 
$$\left\|\sum_{i\in\mathcal{J}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathbb{R}^m\sharp\mathcal{R}^n)} \le C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}$$

To see this claim, it suffices to show the following 3 cases:

(3.16) 
$$\left\|\sum_{i\in\mathcal{J}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathbb{R}^m\setminus K)} \le C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}},$$

(3.17) 
$$\left\|\sum_{i\in\mathcal{J}_1}e^{-t_i\sqrt{L}}b_i\right\|_{L^2(\mathcal{R}^n\setminus K)} \le C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}},$$

and

(3.18) 
$$\left\| \sum_{i \in \mathcal{J}_1} e^{-t_i \sqrt{L}} b_i \right\|_{L^2(K)} \le C \lambda^{\frac{1}{2}} \|f_2\|_{L^1(\mathbb{R}^m \sharp \mathcal{R}^n)}^{\frac{1}{2}}.$$

We now point out that (3.16) can be obtained by using similar estimates as those for (3.8) and that (3.18) can be obtained by using similar estimates as those for (3.9). We omit the details.

As for (3.17), applying the upper bound in point 6 in Theorem 2.2 for  $|P_{t_i}(x, y)|$  we obtain that

$$\begin{aligned} |e^{-t_i\sqrt{L}}b_i(x)| &\leq \int_{\mathcal{R}^n\setminus K} |P_{t_i}(x,y)| \, |b_i(y)| dy \\ &\leq C \int_{\mathcal{R}^n\setminus K} \left(\frac{t_i}{(t_i+d(x,y))^{m+1}} + \frac{t_i}{(t_i+d(x,y))^{n+1}}\right) |b_i(y)| dy \\ &=: \mathcal{F}_{1,i} + \mathcal{F}_{2,i}. \end{aligned}$$

For the term  $\mathcal{F}_{2,i}$ , by using smilar technique of the sup-inf estimate as in the estimate for  $F_{2,i}$  in Subsection 3.1, we obtain that  $\langle \sum_i \mathcal{F}_{2,i}, h \rangle \leq C \lambda \langle M_2(h), \sum_i \chi_{Q_i} \rangle$  for any h with  $\|h\|_{L^2(\mathcal{R}^n \setminus K)} = 1$ , which yields that

$$\left\|\sum_{i\in\mathcal{J}_1}F_{2,i}\right\|_{L^2(\mathbb{R}^m\setminus K)}\leq C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}.$$

For the term  $\mathcal{F}_{2,i}$ , we consider the position of  $Q_i$ , the support of  $b_i$ , as follows: if one of the corners of  $Q_i$  is origin, then  $t_i \geq \frac{1}{2}$ , since otherwise the function  $f_2$  on  $Q_i$  is zero which yields that this  $Q_i$ can not be chosen from the Calderón–Zygmund decomposition; if none of the corners of  $Q_i$  is origin, then if  $t_i < 1$ ,  $d(Q_i, 0) \geq \frac{1}{2}$ . Combining all these cases, we get that

$$\frac{t_i}{(t_i + d(x, y))^{m+1}} \le C \frac{t_i}{(t_i + d(x, y))^{n+1}},$$

which shows that

$$\left\|\sum_{i\in\mathcal{J}_1}F_{1,i}\right\|_{L^2(\mathbb{R}^m\setminus K)} \le C\left\|\sum_{i\in\mathcal{I}_1}F_{2,i}\right\|_{L^2(\mathbb{R}^m\setminus K)} \le C\lambda^{\frac{1}{2}}\|f_2\|_{L^1(\mathbb{R}^m\sharp\mathcal{R}^n)}^{\frac{1}{2}}.$$

3.6. Estimate of  $II_3$ ,  $III_1$ ,  $III_2$ ,  $III_3$ . We point out that the estimates of  $II_3$  follows from the upper bound in point 3 in Theorem 2.2 for  $|P_{t,1}(x,y)|$  and from similar estimates as for  $I_3$  in Subsection 3.3. The estimates of  $III_1$  and  $III_2$  can be obtained by using similar techniques as in  $I_3$  and  $II_3$ , respectively.  $III_3$  can also be obtained using similar approaches as in  $II_3$ . We omit the details here.

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