

EDGE AND FANO ON NETS OF QUADRICS

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ABSTRACT. In a number of papers by Edge, and in a related paper by Fano, several properties are discussed about the family of scrolls $R \subset \mathbb{P}^3$ of degree 8 whose plane sections are projected bicanonical models of a genus 3 curve C . This beautiful classical subject is implicitly related to the moduli of semistable rank two vector bundles on C with bicanonical determinant. In this paper such a matter is reconstructed in modern terms from the modular point of view. In particular, the stratification of the family of scrolls R by $\text{Sing } R$ is considered and the cases where R has multiplicity ≥ 3 along a curve are described.

1. INTRODUCTION

In a series of five papers, sharing the name *Notes on a Net of Quadric Surfaces*, Edge was gathering in a unique scientific work the fundamentals of the theory of nets of complex projective quadric surfaces and of their classification, from the point of view of Invariant Theory. These papers, appeared between 1937 and 1942, go far beyond the basic properties and, somehow, interestingly relate to modern theory of vector bundles on curves and their associated moduli spaces.

A standard projective invariant of a general net N of quadrics of \mathbb{P}^{d-1} is its discriminant, that is the degree d plane curve $C \subset N$ parametrizing singular quadrics. As is well known, the locus in \mathbb{P}^d of the singular points of the quadrics parametrized by C is a smooth model of C embedded by $\omega_C \otimes \theta$, where θ is a non effective theta characteristic on C . Moreover N can be uniquely reconstructed from the pair (C, θ) up to projective equivalence. This follows from a classical result of A. C. Dixon on the representations of the equation of a plane curve as a determinant of a symmetric matrix of linear forms, see [4] and [2] p. 187.

In his *Notes* Edge studies net of quadrics in \mathbb{P}^3 , therefore he deals with plane quartics C and their non effective theta characteristics. The most beautiful geometry behind a net of quadric surfaces, and its connections to the theory of rank two vector bundles on the discriminant quartic C , is specially revealed in his paper III: *The Scroll of Trisecants of the Jacobian curve*, [7]. This paper studies the embedding of C in \mathbb{P}^3 defined by the line bundle $\omega_C \otimes \theta$ and, in particular, the scroll $R_{C,\theta}$, union of the trisecant lines to C .

$R_{C,\theta}$ is a classical object of study from several points of view. For instance it is related to theta characteristics on plane quartics, to the Scorza map and to bilinear

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Cremona transformations, cfr. [2] chapters 5, 6 and 7. However, to proceed with order and motivate the title of our note, let us continue by saying that Fano proved to be an attentive reader of Edge's *Notes*. In particular he wrote *Su alcuni lavori di W. L. Edge* where, not without some criticism, $R_{C,\theta}$ and nets of quadrics are considered as very special cases of a more general situation, [10].

His point of view is that $R_{C,\theta}$ should be considered as a special member of a wider family of octic scrolls R in \mathbb{P}^3 , which deserves to be investigated. This is the family of the scrolls R in \mathbb{P}^3 whose smooth minimal model is the projectivization of a rank two vector bundle E over C with bicanonical determinant.

In these pages we discuss the content of Fano's paper and its special connections to nets of quadrics and Edge's work. Of course never the word 'vector bundle' appears in the writings of both Edge and Fano. However Fano's point of view and their study of octic scrolls R should be considered as a picture 'ante litteram' of the moduli space of semistable vector bundles of rank two and bicanonical determinant on a smooth non hyperelliptic curve C of genus 3. Actually this picture admits several generalizations nowadays in the literature, cfr. [1, 12, 16, 18].

What makes the difference between a general R and $R_{C,\theta}$? How to describe the specializations of $\text{Sing } R$? And how to recover from $R_{C,\theta}$ a net quadrics? We are going to follow closely the geometric ideas of Edge and Fano, in order to describe their beautiful answers in modern terms. We partially rebuild and prove by modern tools Fano's comments and results as follows.

Let E be a globally generated, stable rank two vector bundle with bicanonical determinant on a non hyperelliptic curve C of genus 3 and let \overline{E} be its dual span, that is the dual of the Kernel of the evaluation $H^0(E) \otimes \mathcal{O}_C \rightarrow E$. Let $R \subset \mathbb{P}^3$ be the scroll defined by the tautological map of E :

Theorem 1.1. *Let the Jacobian of C be of Picard number one. Then a component of $\text{Sing } R$ has multiplicity $m \geq 3$ iff one of the following conditions holds.*

$m = 3$ *One has $\overline{E} \simeq E \simeq T_{\mathbb{P}^2|C} \otimes \theta$, θ is a non effective theta characteristic. R is the scroll of trisecant lines to $\text{Sing } R$ and this is C embedded in \mathbb{P}^3 by $|\omega_C \otimes \theta|$.*

$m = 4$ *\overline{E} is strictly semistable. $\text{Sing } R$ is a rational normal cubic in \mathbb{P}^3 . Moreover each line in R is bisecant to $\text{Sing } R$.*

2. QUADRICS THROUGH THE BICANONICAL CURVE C IN \mathbb{P}^5

Fano's point of view is well represented by the title of this section. To describe it let us consider the bicanonical embedding $C \subset \mathbb{P}^5$ of a smooth plane quartic and its ideal sheaf $\mathcal{I}_C|_{\mathbb{P}^5}$. Then, as is well known, we have a unique embedding

$$C \subset V \subset \mathbb{P}^5$$

where V is a Veronese surface. In particular the net of conics of V cuts on C the canonical linear system $|\omega_C|$. Furthermore the ideal of C is generated by quadrics and C is a quadratic section of V . We fix the notation \mathcal{B} for the moduli space

of semistable rank two vector bundles $E \rightarrow C$ of bicanonical determinant and $[E]$ for the moduli point of E . Let $\mathbb{P}^6 := |\mathcal{I}_C|_{\mathbb{P}^5}(2)|$, a natural rational map

$$\gamma : \mathcal{B} \rightarrow \mathbb{P}^6,$$

generically finite of degree two, is provided by the next construction, see [1]. Let $[E] \in \mathcal{B}$ be a stable and general point, then E is globally generated and $h^0(E) = 4$. Moreover the evaluation map ev for global sections defines the exact sequence

$$0 \rightarrow \overline{E}^* \rightarrow H^0(E) \otimes \mathcal{O}_C \xrightarrow{ev} E \rightarrow 0$$

so that \overline{E} is stable. Let $f : \mathbb{P} \rightarrow \mathbb{P}^3 := \mathbb{P}H^0(E)^*$ be the tautological map of the projectivization \mathbb{P} of E and let $R := f(\mathbb{P})$. It is also known that, for a general point $[E]$, the rational map $f : \mathbb{P} \rightarrow R$ is a birational morphism. Notice that

$$R = \bigcup_{x \in C} R_x$$

where $R_x := \mathbb{P}E_x^*$ and $E_x^* \subset H^0(E)^*$ is the linear embedding induced by ev_x . Then the next property is a standard conclusion for a general $[E]$.

Lemma 2.1. *R is an octic scroll in \mathbb{P}^3 whose general plane section is a linear projection of the bicanonical model of C . Moreover $h^0(\mathcal{O}_R(1)) = h^0(E) = 4$.*

Furthermore the determinant map $det : \wedge^2 H^0(E) \rightarrow H^0(\omega_C^{\otimes 2})$ is surjective for such an $[E]$. On the other hand let us consider the Plücker embedding

$$G \subset \mathbb{P}^5 := \mathbb{P}(\wedge^2 H^0(E)^*)$$

of the Grassmannian of lines of $\mathbb{P}^3 = \mathbb{P}H^0(E)^*$. The next lemma also follows.

Lemma 2.2. *The bicanonical embedding of C factors through the map sending x to the parameter point of the line R_x and the Plücker embedding of G .*

Let \mathcal{E}^* be the universal bundle of G and let

$$0 \rightarrow \overline{\mathcal{E}}^* \rightarrow H^0(\mathcal{E}) \otimes \mathcal{O}_G \rightarrow \mathcal{E} \rightarrow 0$$

be the exact sequence of the universal and quotient bundles. These vector bundles are uniquely defined, via Serre's construction, by the two connected components of the Hilbert scheme of planes in the quadric G . Moreover one has $\mathcal{E} \otimes \mathcal{O}_C \simeq E$ and $\overline{\mathcal{E}} \otimes \mathcal{O}_C \simeq \overline{E}$. Using this we can finally define the above mentioned map γ .

Definition 2.3. *$\gamma : \mathcal{B} \rightarrow \mathbb{P}^6$ is the map sending $[E] \in \mathcal{B}$ to the quadric G .*

The fibre of γ at G consists of two points, labeled by the spinor bundles \mathcal{E} and $\overline{\mathcal{E}}$ of G . Fano was of course aware of the relations between quadrics G through the bicanonical model of C and the corresponding moduli space of octic scrolls R in \mathbb{P}^3 whose minimal desingularization is the projectivization of a *stable* E . Edge is aware as well but, in his investigations on net of quadrics, he is only considering those special scrolls which are associated to nets of quadrics. Fano has a list of three cases of scrolls R with special properties: (a), (b), (c). Cases (a) and (b) concern special features of $\text{Sing } R$ and (a) involves nets of quadrics. Case (c) is

of different nature, though related to (a) and (b), and concerns Lüroth quartics. Since our principal interest is here $\text{Sing } R$ we treat only cases (a) and (b).

In order to describe special properties of R it is natural to observe that $\text{Sing } R$ is non empty. One expects a curve of double points having finitely many triple points, triple for R as well. This is in some sense Fano's starting point. A simple vocabulary translates these properties of R in properties of G with respect to the varieties of bisecant lines and of trisecant planes to C . This is useful to understand the special cases and intimately related to the modern description of the theta map for the moduli space \mathcal{B} . We address this matter in the next sections.

3. THE THETA MAP $\gamma : \mathcal{B} \rightarrow \mathbb{P}^6$ AND THE COBLE QUARTIC

As is well known one has $\text{Pic } \mathcal{B} \simeq \mathbb{Z}[\mathcal{L}]$, where \mathcal{L} is ample. In the case of a smooth plane quartic we are considering, the linear system $|\mathcal{L}|$ defines an embedding

$$\mathcal{B} \subset \mathbb{P}^7.$$

\mathcal{B} is a quartic hypersurface which is well known as a *Coble quartic*, [16]. The map defining this embedding is known as the *theta map* of \mathcal{B} . It is useful for our purposes to describe it as follows. Let $\Theta := C^{[2]}$ be the 2-symmetric product of C , embedded in $J^2 := \text{Pic}^2 C$ by the Abel map, sending $d \in C^{[2]}$ to $\mathcal{O}_C(d)$. In other words Θ is just the natural Theta divisor

$$\{L \in J^2 \mid h^0(L) = 1\}.$$

In particular Θ is a *symmetric* Theta divisor, which is invariant by the involution $\iota : J^2 \rightarrow J^2$ sending L to $\omega_C \otimes L^{-1}$. As is well known a natural identification

$$\mathbb{P}^7 := |2\Theta|$$

can be fixed, so that the embedding of \mathcal{B} in $|2\Theta|$ can be described as follows. Let $[E] \in \mathcal{B}$ then there exists a unique divisor $\Theta_E \in |2\Theta|$ such that

$$\text{Supp } \Theta_E := \{L \in J^2 \mid h^0(E \otimes L^{-1}) \geq 1\},$$

we will say that Θ_E is the theta divisor of E . It is well known that

$$\mathcal{B} = \{\Theta_E \mid [E] \in \mathcal{B}\}.$$

In particular let $[E_o] := [\omega_C \oplus \omega_C]$. Then $[E_o]$ is a semistable point of \mathcal{B} and $\Theta_{E_o} = 2\Theta$. Now consider the standard exact sequence

$$0 \rightarrow \mathcal{O}_{J^2}(\Theta) \rightarrow \mathcal{O}_{J^2}(2\Theta) \rightarrow \mathcal{O}_\Theta(2\Theta) \rightarrow 0.$$

Passing to the associated long exact sequence, it follows that the restriction map

$$\rho : |2\Theta| \rightarrow \mathbb{P}^6 := \mathbb{P}H^0(\mathcal{O}_\Theta(2\Theta))$$

is the linear projection of center $[E_o]$. We fix the notation $E[k]$ for the standard vector bundle of rank k , over the k -symmetric product $C^{[k]}$, whose fibre at $d \in C^{[k]}$ is $E_d := E \otimes \mathcal{O}_d$. Let us recall that C is assumed to be non hyperelliptic and that

$C^{[2]}$ is identified to the theta divisor $\Theta \subset J^2$ via the Abel map. Then let us consider the map of vector bundles

$$ev^{[2]} : H^0(E) \otimes \mathcal{O}_{C^{[2]}} \rightarrow E[2]$$

defined by the evaluation of global sections. It is well known that the degeneracy scheme of $ev^{[2]}$ is the restriction of Θ_E to Θ . Moreover such a restriction is a curve if $[E] \neq [E_o]$. Assuming this we fix the following

Definition 3.1. *The degeneracy curve B_E of E is the restriction of Θ_E to Θ .*

Putting for simplicity $B_E = B$ we finally consider the correspondence

$$I := \{(d, z) \in \Theta \times \mathbb{P}^5 \mid z \in \ell_d\},$$

where ℓ_d is the line spanned by the degree 2 effective divisor $d \subset C \subset \mathbb{P}^5$. Let

$$\Theta \xleftarrow{\alpha} I \xrightarrow{\beta} \mathbb{P}^5$$

be the projection maps of I and let G be the smooth quadric defined as above by the vector bundle E . Then the next property is satisfied, [1] 4.22.

Lemma 3.2. *G is the unique quadric of \mathbb{P}^5 whose strict transform by β is α^*B .*

More geometrically the strict transform by β of the linear system of quadrics $|\mathcal{I}_{C|\mathbb{P}^5}(2)|$ coincides with the pull back of $|\mathcal{O}_\Theta(2\Theta)|$ by α . Then, after the natural identification of these two linear systems, the next theorem follows.

Theorem 3.3. *The map $\gamma : \mathcal{B} \rightarrow \mathbb{P}^6$ factors through the embedding of \mathcal{B} in $|2\Theta|$ via the theta map and the linear projection $\rho : |2\Theta| \rightarrow \mathbb{P}^6$.*

See [1, 16]. It is now useful to summarize more informations from the literature on the embedding $\mathcal{B} \subset |2\Theta|$ and the linear projection $\gamma : \mathcal{B} \rightarrow \mathbb{P}^6$ of center $[E_o]$. Let us briefly recall that:

- \mathcal{B} is embedded as the unique quartic hypersurface, the Coble quartic of C , which is singular along the Kummer variety of the curve C

$$K(C) := \{[E] \in \mathcal{B} \mid [E] = [\omega_C(e) \oplus \omega_C(-e)], \mathcal{O}_C(e) \in \text{Pic}^0(C)\}.$$

- $K(C)$ is $\text{Sing } \mathcal{B}$, moreover it is the locus of strictly semistable points of \mathcal{B} . Each point of $K(C)$ has multiplicity two for \mathcal{B} .

- $[E] = [\omega_C(e) \oplus \omega_C(-e)]$ if and only if E fits in an exact sequence

$$0 \rightarrow \omega_C(e) \rightarrow E \rightarrow \omega_C(-e) \rightarrow 0.$$

In order to describe γ notice that a stable point $[E]$ satisfies $h^1(E) = 0$ and hence $h^0(E) = 4$. Indeed we have $H^1(E) = \text{Hom}(E, \omega_C)$ by Serre duality, therefore $h^1(E) \geq 1$ implies that $[E]$ is strictly semistable. Notice that a point $[E] \in K(C)$ satisfies $h^0(E) \geq 5$ iff $[E] = [E_o]$, with $E_o = \omega_C \oplus \omega_C$. As is well known the divisor Θ_{E_o} is just 2Θ . In particular the map $\gamma : \mathcal{B} \rightarrow \mathbb{P}^6$ of the Coble quartic is the linear projection from its double point $[E_o]$ and a rational double covering. Let

$$\iota : \mathcal{B} \rightarrow \mathcal{B}$$

be its associated birational involution, since now we assume $[E] \neq [E_o]$. Let ℓ be the line joining $[E_o]$ to $[E]$, then exactly one of the following cases is possible:

- (1) $\ell \subset \mathcal{B}$,
- (2) $\ell \cdot \mathcal{B} = 3[E_o] + [E]$ and $\iota([E]) = [E_o]$,
- (3) $\ell \cdot \mathcal{B} = 2[E_o] + [E] + [\iota(E)]$ and $\iota([E]) \neq [E_o]$.

Now assume that E is globally generated, then the usual sequence

$$0 \rightarrow \overline{E}^* \rightarrow H^0(E) \otimes \mathcal{O}_C \xrightarrow{ev} E \rightarrow 0$$

is exact and either \overline{E} is stable or $[\overline{E}] = [E_o]$. Hence ι is regular at $[E]$ and $[\overline{E}] = \iota([E])$. Let \mathbb{P} and $\overline{\mathbb{P}}$ be the projectivizations of E and \overline{E} , then we have their tautological models $R \subset \mathbb{P}^3$ and $\overline{R} \subset \mathbb{P}^{3*}$. It is easy to see that, since the determinant of E is bicanonical, these are embedded as dual surfaces:

$$\overline{R} = \bigcup_{x \in C} \overline{R}_x,$$

where $\overline{R}_x \subset \mathbb{P}^{3*}$ denotes the pencil of planes through the line R_x of R . Let $G = \gamma([E])$, since $[E] \neq [E_o]$ we have $h^0(E) = 4$ and G can be constructed from $H^0(E)$ as in [1] section 2.

We recall this construction for a general $[E]$: for such an $[E]$ the determinant map $\wedge^2 H^0(E) \rightarrow H^0(\omega_C^{\otimes 2})$ is an isomorphism. Consider the natural pairing

$$\wedge^2 H^0(E) \times \wedge^2 H^0(E) \rightarrow \wedge^4 H^0(E)$$

and the quadratic form q_E induced by it on the space $\mathbb{P}^5 = \mathbb{P}H^0(\omega_C^{\otimes 2})^*$. By definition G is the zero locus of q_E . Notice also that G is the Plücker embedding of the Grassmannian of lines of $\mathbb{P}H^0(E)^*$. Moreover C is embedded in G so that E^* is the restriction to C of the universal bundle.

Let $\sigma_{2,0}$ be the family of planes of G which are dual of the planes in \mathbb{P}^3 , then each $P \in \sigma_{2,0}$ corresponds to some point $[s] \in \mathbb{P}^3$ so that $P \cdot C$ is the scheme of zeroes of $s \in H^0(E)$. In the same way let $\sigma_{1,1}$ be the family of planes of G which are parametrizing the lines through a point of \mathbb{P}^3 , then each $\overline{P} \in \sigma_{1,1}$ corresponds to a point $[\overline{s}] \in \mathbb{P}H^0(\overline{E})$ and $\overline{P} \cdot C$ is the scheme of zeroes of $\overline{s} \in H^0(\overline{E})$.

Let us also recall that $C \subseteq V \cap G$, where V is the unique Veronese surface containing C . The next theorem characterizes the locus of points $[E]$ such that E is globally generated and $\iota([E]) = [E_o]$.

Theorem 3.4. *Let E be stable and globally generated, then V is contained in G if and only if $\iota([E]) = [E_o]$.*

Proof. We have $[\overline{E}] = \iota([E])$. Let $Sec V$ be the cubic hypersurface equal to the union of the planes Π spanned by the conics of V . Assume $V \subset G$, it is easily seen that $T = G \cap Sec V$ is union of planes Π . Now $Z := \Pi \cdot C$ is a canonical divisor. Since $\deg Z = 4$ and E is stable it follows that Z is the scheme of zeroes of some $\overline{s} \in H^0(\overline{E})$. Hence \overline{E} is strictly semistable and it follows $[\overline{E}] = [E_o]$. Conversely let $[\overline{E}] = [E_o]$, then there exists a plane Π as above in G so that $Z = \Pi \cdot C$ is a

canonical divisor. But then we have $Z \subset B \subset \Pi \subset G$, where B is a conic of V . Then G contains $C \cup B$ and hence contains V by counting the degrees. \square

Theorem 3.5. *Let E be stable and globally generated then R is an octic scroll.*

Proof. Since E is globally generated, the tautological map $\tau : \mathbb{P} \rightarrow R$ is a morphism. Assume $\deg R \geq 2$, then either R is a quartic scroll or a quadric and its minimal desingularization is the projectivization of a vector bundle $F \rightarrow A$, where A is a smooth integral curve of genus 1 or 0.

Notice that $E = \pi^*F$, where $\pi : C \rightarrow A$ is a $2 : 1$ cover if $\deg R = 4$ or a $4 : 1$ cover if $\deg R = 2$. We claim that F is not stable. But then E is not stable and this contradiction implies the statement. The claim is clear if R is a quadric. If R is a quartic then R has a singular curve and hence F admits linear subbundles of degree two. Then F is not stable. \square

We assume from now the property that E is stable and globally generated, which is the case to be considered.

4. THETA DIVISOR OF E AND SINGULARITIES OF R

Now we want to study the degeneracy curve $B = \Theta \cdot \Theta_E$ and to see how B is related to the singular locus of the scroll R defined by E . Notice that we have

$$\mathcal{O}_\Theta(B) \simeq \mathcal{O}_\Theta(\sum (x_i + C) - \delta) \in \text{Pic } \Theta$$

cfr. [1] 4.7, where $\sum x_i$ is a bicanonical divisor and our notation is fixed as follows. We set $(x + C) := \{d \in \Theta \mid x \in \text{Supp } d\}$. Moreover 2δ is the class of the diagonal $\Delta := \{2x \in \Theta, x \in C\}$ and δ defines a natural $2 : 1$ cover $C \times C \rightarrow \Theta$.

Lemma 4.1. *B is a smooth, integral curve of genus 19 intersecting transversally a general curve $(x + C)$ at six points.*

Proof. Since the restriction $\rho : |2\Theta| \rightarrow |\mathcal{O}_\Theta(2\Theta)|$ is surjective, a general $[E] \in \mathcal{B}$ defines a general curve $B = \Theta \cdot \Theta_E \in |\mathcal{O}_\Theta(2\Theta)|$. Since this is ample and base point free, B is smooth and integral. Finally notice that $|\mathcal{O}_{J^2}(2\Theta)|$ is base point free and that its restriction to $C^{[2]}$ is $|B|$. Hence the restriction to $x + C$ of $|B|$ is also base point free, which implies that a general B is transversal to $x + C$. \square

We denote by \mathbb{P}_d the fibre of $\mathbb{P} \rightarrow C$ at $d \in \Theta$, then we have

$$\mathbb{P}_d \subset \mathbb{P}E_d^*.$$

If $d = x + y$ and $x \neq y$ then \mathbb{P}_d is the union of the skew lines $\mathbb{P}_x, \mathbb{P}_y$ in the 3-dimensional space $\mathbb{P}E_d^*$. In what follows we assume that B is smooth, then $h^0(E(-d)) = 1$ for any $d \in B$. Note that the evaluation $e_d : H^0(E) \rightarrow E_d$ defines a linear map of 3-dimensional spaces $p_d : \mathbb{P}E_d^* \rightarrow \mathbb{P}^3$. Let $d \in B$, since $h^0(E(-d)) = 1$ it follows that the tautological map $\tau|_{\mathbb{P}_d} : \mathbb{P}_d \rightarrow \mathbb{P}^3$ fails to be an embedding exactly along a 0-dimensional subscheme

$$s_d \subset \mathbb{P}_d$$

which is contracted by τ to a singular point of R . This defines a morphism

$$\sigma : B \rightarrow \text{Sing } R$$

sending $d \in B$ to the point $p_d(s_d)$. If $d = x + y$ with $x \neq y$ then $\sigma(d) = R_x \cap R_y$.

A more delicate question concerns the stratification of the family of scrolls R according the properties of σ and, in particular, according to its possible degrees. In [10] Fano considers this problem and its special cases. Let us discuss something useful about. Let $n : \mathbb{P} \rightarrow R$ be the normalization and $p : \mathbb{P} \rightarrow C$ the projection map. Assume $o \in \text{Sing } R$ and consider the divisor $p_*n^*o = x_1 + \cdots + x_k$. As usual let $[\overline{E}] = \iota([E])$, according to our convention we have

$$P_o \cdot C = x_1 + \cdots + x_k,$$

where P_o is the plane in G parametrizing all the lines through the point o .

Lemma 4.2. *For every divisor $d \subset x_1 + \cdots + x_k$ one has $h^0(F(-d)) \geq 1$.*

The proof follows immediately from the above remarks. Assume now that σ is not birational, then we have $k \geq 3$ distinct lines of the scroll R , say

$$R_{x_1} \dots R_{x_k},$$

passing through a general point $o \in \text{Sing } R$. Let $P_o \subset G$ be the plane parametrizing the lines through o , then $x_1 \dots x_k$ are in P_o . Since $\mathcal{I}_{C|\mathbb{P}^5}$ is generated by quadrics no three of these points are collinear. Since E is stable and globally generated, we know that either \overline{E} is stable or $[\overline{E}] = [E_o]$. Therefore we have $k \leq 4$ and the equality holds iff $[\overline{E}] = [E_o]$. The next statement is known more in general. We introduce a proof for completeness and to study its exceptions.

Theorem 4.3. *$\sigma : B \rightarrow \text{Sing } R$ is birational onto its image for a general $[E]$ and $\text{Sing } R$ is an integral curve of multiplicity two for R .*

Proof. Consider the open set $U := \{t \in C^{[3]} \mid h^0(\omega_C(-t)) = 0\}$ and

$$Q := \{(t, G) \in U \times \mathbb{P}^6 \mid P_t \subset G\}.$$

Since C is generated by quadrics the projection $p : Q \rightarrow U$ is a \mathbb{P}^3 -bundle. The fibre of p at t is the linear system of all quadrics through $C \cup P_t$. Let $q : Q \rightarrow \mathbb{P}^6$ be the second projection. Since Q is irreducible of dimension 6 the fibre of q at a general t is a finite set. Hence the smooth quadric G associated to $[E]$ contains a finite set of planes P_t . Since E and \overline{E} are stable, no plane in G of the same ruling of P_t is k -secant to C for $k \geq 4$. Let $o \in \text{Sing } R$ then at most two lines R_x, R_y of R contain o . Hence $\sigma^{-1}(o) = \{x + y\}$ and $\sigma : B \rightarrow \text{Sing } R$ is birational. \square

Actually the surjectivity of $q : Q \rightarrow \mathbb{P}^6$ is well known, see e.g. [13]. Equivalently the set of planes P_t , considered above, is finite and not empty for a general $[E]$. Thi set is strictly related to the scheme of triple points of R . To study this scheme of triple points of R let us consider \overline{E} and the map of vector bundles

$$ev^{[3]} : H^0(\overline{E}) \otimes \mathcal{O}_{C^{[3]}} \rightarrow \overline{E}[3].$$

The degeneracy scheme of this map is expected to be of codimension three. It is essentially the parameter space for the locus of points of multiplicity three in R .

Definition 4.4. *The degeneracy scheme of $ev^{[3]}$ is the triple locus T_E of E .*

T_E has dimension zero for a general $[E]$. The count of the length of T_E is of course possible, both by classical and modern methods. Notice that, for a general E , the curve $\sigma(B) = \text{Sing } R$ is the curve of double points of R having degree 18 and geometric genus 19. Counting multiplicities the length of T_E is the number of triple points of R and of $\sigma(B)$ as well. This can be computed via Cayley-Zeuthen formulae, see [2] 10.4.7 and 10.4.9, as Fano does:

Lemma 4.5. *If E is general then R has eight triple points.*

As it has been already emphasized, it is not forbidden that T_E be a curve in special cases: even when E is stable. This leads to the list of specializations considered by Fano and to the beautiful interplay with nets of quadrics considered by Edge. We reconstruct this matter in the next section, via Coble quartic and modern tools from the theory of vector bundles on curves and their moduli.

5. SPECIALIZATIONS: SCORZA CORRESPONDENCE AND NETS OF QUADRICS

When the morphism $\sigma : B \rightarrow \text{Sing } R$ is not birational? In what follows we keep the assumption that E is stable and globally generated. Under this assumption the next result, already pointed out by Fano, reveals beautiful geometry.

Theorem 5.1. *Let $\sigma : B \rightarrow \text{Sing } R$ be not birational, then two cases may occur:*

(a) $\iota([E]) \neq [E_o]$: σ has degree 3 and $\text{Sing } R$ is the sextic embedding of C defined by $\omega_C \otimes \theta$, where θ is a non effective theta on C .

(b) $\iota([E]) = [E_o]$: σ has degree 4 and $\text{Sing } R$ is a rational normal cubic.

Moreover these are the only cases if the Jacobian of C has Picard number one.

Some remarks are in order. At first notice that R is not a cone: indeed this implies $h^0(E \otimes \omega_C^{-1}) \geq 1$ and E unstable. As was remarked before, ι is the birational involution of the Coble quartic \mathcal{B} , induced by the projection from $[E_o] \in \text{Sing } \mathcal{B}$.

Lemma 5.2. *Let $[F] = \iota([E])$ then $[F] \neq [E_o] \Rightarrow F$ is stable.*

Proof. Let $[F]$ be not stable, then $[F] \in \text{Sing } \mathcal{B}$ and the line containing the distinct points $[E_o], [E], [F]$ is in \mathcal{B} . Hence E is not globally generated: a contradiction. \square

Since R is not a cone, a point $o \in \text{Sing } R$ defines on C the divisor

$$p_*n^*(o) := x_1 + \cdots + x_k.$$

We can start our proofs of cases (a) and (b), adding something more, which is due in order to reveal some nice properties appearing.

◦ *Proof and more geometry on case (a)*

We assume here that F is *stable*. Then we have $k \leq 3$ for any $o \in \text{Sing } R$ and we already know that $h^0(F(p_*n^*o)) \geq 1$. Finally we consider the reduced curve

$$\Gamma_k \subset \text{Sing } R$$

which is union of the irreducible components of $\text{Sing } R$ having multiplicity k for R . We assume that σ is not birational, which implies that Γ_3 is not empty. Actually Γ_3 is the support of the triple locus T_E of E , defined in 4.4. Let

$$a : \Gamma_3 \rightarrow \text{Pic}^3(C)$$

be the map sending $t_o := p_*n^*o$ to $\mathcal{O}_C(T - o)$. Let $\tau \in \text{Pic}^2(C)$ and

$$\tau + C := \{\tau(x), x \in C\} \subset \text{Pic}^3(C).$$

Assume that $\text{Pic}^3(C)$ has Picard number one. Then the next theorem is the key to describe geometrically the case we are considering.

Theorem 5.3. *The image of a is $\tau + C$, for a non effective $\tau \in \text{Pic}^2(C)$.*

Proof. We claim that $a(\Gamma_3) \cdot (x + \Theta)$ has length $\ell \leq 3$ for x general. This implies $\ell = 3$, since the numerical class of $a(\Gamma)$ is in $\mathbb{Z}[\frac{\Theta^2}{2}]$. Then, by Matsusaka criterion [14], $a(\Gamma)$ is $\tau + C$ for some $\tau \in \text{Pic}^2 C$. Finally τ is not effective: otherwise we would have $h^0(F(-a - b - x)) \geq 1$, where $a, b \in C$ are fixed points and $\tau \simeq \mathcal{O}_C(a + b)$. Since R is not a cone it follows $R_a = R_b := L$. Let $o_x = L \cap R_x$, then we have a rational map $f : C \rightarrow L$ such that $f(x) = o_x$. Since F is stable f is injective: a contradiction.

To prove the claim consider the normalization $n : \mathbb{P} \rightarrow R$ and the projection $p : \mathbb{P} \rightarrow C$. Let $a_x : C \rightarrow \text{Pic}^2(C)$ be the Abel map sending y to $\mathcal{O}_C(x + y)$. Since $\deg \mathcal{O}_{x+C}(B) = 6$ the length of $b_x := a_x^*B$ is 6. Let $B_x := p^*b_x \cdot n^*\text{Sing } R$ and $\Gamma_x = R_x \cdot \text{Sing } R$, we have the diagram

$$\Gamma_x \xleftarrow{n} B_x \xrightarrow{p} b_x$$

Since x is general and σ birational, we can assume that $p : B_x \rightarrow b_x$ is biregular. Otherwise we would have a length two divisor d embedded in $B_x \cdot \mathbb{P}_y$ for some $y \in \text{Supp } b_x$. Since σ is an embedding on d , this implies $\sigma(R_x) = \sigma(R_y)$ and finally $R_x \subset \text{Sing } R$, which is impossible for x . Now consider $n : B_x \rightarrow \Gamma_x$. Since F is stable, n contracts a scheme $d \subset B_x$ to a point iff $h^0(F(-x - d)) \geq 1$ and $\deg d = 2$ that is iff $n(d) \in \Gamma_3$. Since $\deg b_x = 6$ it follows that n contracts at most 3 disjoint schemes d . Hence $a(\Gamma) \cdot (x + \Theta)$ has length $\ell \leq 3$. \square

Once established $a(\Gamma_3) = \tau + C$ we can reconstruct $\Theta_E = \Theta_F$ from $\tau + C$ and hence the curve B . It is well known that the cohomology class of the divisor

$$T_\tau := \{\tau(x - y), (x, y) \in C \times C\} \subset \text{Pic}^2(C),$$

is twice the class of the theta divisor. The same is true for the difference divisor

$$C - C := \{N \in \text{Pic}^0(C), |N \simeq \mathcal{O}_C(x - y), (x, y) \in C \times C\}.$$

This is actually the theta divisor in $\text{Pic}^0(C)$ of a special rank two vector bundle on the quartic model $C \subset \mathbb{P}^2$, namely $T_{\mathbb{P}^2|C}(-1)$. Indeed the Euler sequence

$$0 \rightarrow \mathcal{O}_C(-1) \rightarrow H^0(\omega_C) \otimes \mathcal{O}_C \rightarrow T_{\mathbb{P}^2|C}(-1) \rightarrow 0$$

implies, tensoring by $\mathcal{O}_C(x-y)$ and passing to the long exact sequence, that

$$C - C = \{N \in \text{Pic}^0(C) \mid h^0(T_{\mathbb{P}^2|C} \otimes N^{-1}) \geq 1\}.$$

Let \mathcal{B}_J be the moduli space of semistable vector bundles on C of degree 8 and $\mathbb{P}_J \rightarrow \text{Pic}^0(C)$ the \mathbb{P}^6 -bundle with fibre at e the linear system $|\mathcal{O}_\Theta(2(\Theta+e))|$. Then the map $\gamma : \mathcal{B} \rightarrow |2\Theta|$ immediately extends to the rational double covering

$$\gamma_J : \mathcal{B}_J \rightarrow \mathbb{P}_J^6,$$

where, for a general $[U] \in \mathcal{B}_J$, the element $\gamma_J([U])$ is the curve $\Theta_U \cdot \Theta$ and Θ_U is the theta divisor in $\text{Pic}^2(C)$ defined by the vector bundle U . The involution $\iota_J : \mathcal{B}_J \rightarrow \mathcal{B}_J$, induced by γ_J , is defined again by the standard exact sequence

$$0 \rightarrow \overline{U}^* \rightarrow H^0(U) \otimes \mathcal{O}_C \xrightarrow{ev} U \rightarrow 0.$$

Let $E_\tau := T_{\mathbb{P}^2|C} \otimes \tau^{-1}$, then E_τ is stable. Notice that T_τ is the 2-Theta divisor associated to E_τ . This just follows because the condition $N \simeq \tau(x-y)$ is equivalent to $h^0(E_\tau \otimes N^{-1}) \geq 1$. In particular it follows that $\gamma_J([E_\tau]) = T_\tau$.

Theorem 5.4. *The vector bundles E_τ , E and F are isomorphic and τ is a non effective theta characteristic on C .*

Proof. For a general τ , and hence for any τ , it is easy to check that

$$T_\tau \cdot \Theta = \{d \in \Theta \mid d + y \in |\tau(x)|, (x, y) \in C^2\} = \{d \in \Theta \mid h^0(F(-d)) \geq 1\} = B.$$

Then $[E_\tau]$ is in the fibre of $\gamma_J : \mathcal{B}_J \rightarrow \mathbb{P}_J^6$ at B and it is isomorphic to E or F . This implies $\omega_C^{\otimes 3} \otimes \tau^{-2} \simeq \det E_\tau \simeq \omega_C^{\otimes 2}$ and hence that τ is a theta characteristic. Finally it is well known that $\iota([E_\tau]) = [E_\tau]$, which implies the statement. \square

More in general let $[E] \in \mathcal{B}_J$, where E is stable and globally generated and $\text{Pic}^2(C)$ has Picard number one. The same arguments used above imply that:

$$R \text{ has multiplicity } \geq 3 \text{ along a curve} \Leftrightarrow E \simeq T_{\mathbb{P}^2|C} \otimes \tau^{-1}$$

where R is tautological model of $\mathbb{P} = \mathbb{P}(T_{\mathbb{P}^2|C}^* \otimes \tau)$ and τ satisfies $h^0(\tau) = 0$. We cannot avoid to sketch very briefly the classical properties of R , cfr. [2] 5.5 and 6. Keeping our notation, we have the following description. Consider the curves:

- $T_E = \{t \in C^{[3]} \mid t \in |\tau(x)| \text{ for some } x \in C\}$,
- $B = \{d \in \Theta \mid d \subset t, \text{ for some } t \in T_E\}$.

Notice that, since $h^0(\tau) = 0$, T_E is biregular to C via the map $t \rightarrow \mathcal{O}_C(t - \tau)$.

Let $t = a + b + c \in T_E$. Since $E = F = E_\tau$, we know that $h^0(E_\tau(-t)) \geq 1$ and hence that the lines R_a, R_b, R_c intersect in one point $o_t := R_a \cap R_b \cap R_c$. Let

$$f : C \rightarrow \mathbb{P}^3$$

be the map $t \rightarrow o_t$: it turns out that f is the embedding defined by $\omega_C \otimes \tau$. Its image Γ is a curve of degree 6 and multiplicity 3 for R . On the other hand we know that, for a general E , $\text{Sing } R$ is instead a double curve of degree 18. This implies, for degree reasons, that $f(C) = \text{Sing } \Gamma$. Recall also that, for every x , the scheme $(x + C) \cdot B$ has length 6 and defines in C a divisor $b_x = \sum_{1..3} y_i + z_i$ such that, up to reindexing, $h^0(E_\tau(-x - y_i - z_i)) = 1$, $1 \leq i \leq 3$. Equivalently the line R_x of R is trisecant to the curve Γ . Under the previous assumptions we conclude that:

$$E \simeq T_{\mathbb{P}^2|C} \otimes \tau^{-1} \Leftrightarrow R \text{ is the scroll of the trisecant lines to } \Gamma$$

Of course, as very well considered by Edge in his series of papers, if τ is a non effective theta characteristic the Γ is the locus of the singular points of the net of quadrics defined by the pair (C, τ) . Notice also that the trisecant locus T_E of this case gives rise to the well known and interesting Scorza's correspondence

$$S(\theta) = \{(x, y) \in C \times C \mid h^0(\theta(x - y)) = 1\}$$

and its related topics, see [3].

Proof and geometric description of case (b)

Now we are left to discuss the case $[E_0] = \iota([E])$. This means that the line joining $[E_0]$ to $[E]$ is an inflexional tangent to \mathcal{B} at $[E_0]$ not contained in \mathcal{B} , see section 3. Let us conclude very briefly by this case, which is remarked by Fano in [10] as an additional case to Edge's analysis of nets of quadrics. Actually it representative of the net of quadrics whose base locus is a rational normal cubic. Since $[E_0] = [\omega_C^2]$, it follows that $F = \overline{E}$, fits in an exact sequence

$$0 \rightarrow \omega_C \rightarrow F \rightarrow \omega_C \rightarrow 0.$$

Moreover we know from theorem that the quadric G defines by $[E]$ has rank six, since F and E are not isomorphic. With our conventions G is the Grassmannian of lines of $\mathbb{P}^3 = \mathbb{P}H^0(E)^*$ and the ruling of the dual of the planes of \mathbb{P}^3 corresponds to the planes of class $\sigma_{2,0}$. A non zero section $s \in H^0(E)$ defines a plane P of this class and its scheme of zeroes $P \cdot C$ has always ≤ 3 . Furthermore we know from theorem 3.4 that G contains the Veronese V surface already considered. Also we know that the secant variety $\text{Sec } V$ of V cuts on G a threefold T which is union of planes $\overline{P} \subset G$. Each \overline{P} contains a conic of V and $\overline{P} \cdot C$ is a canonical divisor. Since its degree is 4, it follows that the planes \overline{P} have the opposite class $\sigma_{1,1}$. As is well known the class of V in G is then $3\sigma_{2,0} + \sigma_{1,1}$. Equivalently the point of V parametrize the bisecant lines to a skew cubic

$$\Gamma \subset \mathbb{P}^3.$$

Each point $o \in \Gamma$ defines the plane \overline{P}_o parametrizing the lines through o . The cone of vertex o containing Γ is parametrized by a conic $B \subset V$, moreover it is now clear that $b_o := B \cdot C$ is a canonical divisor of C . Counting multiplicities, b_o parametrizes four lines in the scroll R passing through the point o . This explains,

up to further details, why R has multiplicity four along Γ and completely describes this scroll.

To conclude, let us say that papers of Edge and Fano we discussed in this article, though dedicated to important and known classical topics, still are rich of arguments to be investigated. Our note is a small contribution to support this point of view.

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