Piecewise linear sheaves

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Abstract

On a finite-dimensional real vector space, we give a microlocal characterization of (derived) piecewise linear sheaves (PL sheaves) and prove that the triangulated category of such sheaves is generated by sheaves associated with convex polyhedra. We then give a similar theorem for PL γ-sheaves, that is, PL sheaves associated with the γ-topology, for a closed convex polyhedral proper cone γ. Our motivation is that convex polyhedra may be considered as building blocks for higher dimensional barcodes.

Contents

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Introduction

Persistent homology is an essential tool of Topological Data Analysis appearing in numerous papers. To our opinion it may be interpreted as follows (see [KS18]). One has some data on a manifold X which define a constructible sheaf F on X, one has a function $f\colon X \to \mathbb{R}$ (playing the role of a Morse function) and one can calculate the direct image by f of the data, that is, the derived direct image $\mathrm{R}f_*F$. Assuming that f is proper on the support of F, one gets a constructible (derived) sheaf on $\mathbb{R}$ and a variant of a theorem of Crawley-Boevey [CB14] (see also [Gui16] and, for the non compact case, [KS18, Th. 2.17]) asserts that such an object is nothing but a graded barcode. Moreover, in practice, the data on X are associated with an order and it follows that the barcodes are half-closed intervals (e.g., closed on the left and open on the right). In the langage of sheaves, this means that one gets a γ-sheaf represented by a γ-barcode, where γ is the cone $\{t \in \mathbb{R}; t \leq 0\}$.

However it is natural in many problems to replace the ordered set $(\mathbb{R}, \leq)$ with an ordered finite dimensional vector space $\mathbb{V}$ and the order may be deduced from the data of a closed convex proper cone γ with non–empty interior. Then it is natural to endow $\mathbb{V}$ with the so-called γ-topology $\mathbb{V}_\gamma$ introduced in [KS90, Ch. III § 5]. We are lead to the study of (derived) constructible γ-sheaves (that is, sheaves on $\mathbb{V}_\gamma$) and the category of such sheaves is no more equivalent to any natural category of barcodes. The aim of this paper, a kind of continuation of [KS18], is to find a substitute to this non existing equivalence.

First, we replace constructible sheaves with PL sheaves (PL for piecewise linear) which are much easier to manipulate. A convex polyhedron is the intersection of a finite family of open or closed affine half-spaces and a constructible sheaf F is PL if there is a finite covering of $\mathbb{V}$ by convex polyhedra on which it is constant. It has been proved in loc. cit. that constructible sheaves may be approximated, for a kind of derived bottleneck distance, by PL sheaves and similarly for constructible γ-sheaves. A natural higher dimensional analogue to the category of γ-barcodes is given by the additive category of finite direct sums of constant sheaves on convex γ-locally closed polyhedra and the main result of this paper asserts that the triangulated category of PL γ-sheaves is generated by the additive category of such γ-barcodes.

This paper also contains a systematic study of PL-sheaves. We show in particular that a sheaf is PL if and only if its microsupport is a PL Lagrangian variety or, equivalently, is contained in such a Lagrangian variety. We note that the six Grothendieck operations hold for PL sheaves and that PL sheaves on $\mathbb{V}$ may be considered as the restriction to $\mathbb{V}$ of PL sheaves on its projective compactification $\mathbb{P}$. In the course of the paper, we recall several results of loc. cit. that we shall need and also give a new application of the stability theorem of the bottleneck distance.

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Remark 0.1. Some notations and conventions in this paper differ from those of [KS18].

(a) A polyhedron was called a polytope in loc. cit.

(b) A PL set or a PL sheaf in loc. cit. is called here an LPL set or an LPL sheaf.

(c) The microsupport of a sheaf F is denoted here by $\mathrm{SS}(F)$ as in [KS90], instead of $\mu\mathrm{supp}(F)$ in [KS18].

1 PL geometry

1.1 PL sets and PL stratifications

Let $\mathbb{V}$ be a real finite-dimensional vector space.

Definition 1.1. (a) A *convex polyhedron* P in $\mathbb{V}$ is the intersection of a finite family of open or closed affine half-spaces.

(b) A PL set is a finite union of convex polyhedra.

(c) A locally PL set (an LPL set for short) is a locally finite union of convex polyhedra.

Note that an LPL set is subanalytic.

The next result is obvious.

Lemma 1.2. (i) *The family of* PL *sets in* $\mathbb{V}$ *is stable by finite unions and finite intersections.*

(ii) *If* Z *is* PL*, then its closure* $\overline{Z}$*, its interior* $\mathrm{Int}(Z)$ *and its complementary set* $\mathbb{V}\setminus Z$ *are* PL*.*

(iii) *Any connected component of a* PL *set is* PL*.*

(iv) *Let* $u\colon \mathbb{V}\to\mathbb{W}$ *be a linear map.*

  (a) *If* $S\subset\mathbb{V}$ *is* PL*, then* $u(S)\subset\mathbb{W}$ *is* PL*.*

  (b) *If* $Z\subset\mathbb{W}$ *is* PL*, then* $u^{-1}(Z)\subset\mathbb{V}$ *is* PL*.*

(v) *The preceding results still hold when when replacing* PL *with* LPL*, except* (iva) *in which case one has to assume that* u *is proper on* $\overline{S}$*.*

For a locally closed submanifold $Z\subset\mathbb{V}$, one sets for short

$T^*_Z\mathbb{V} := T^*_Z U$ where U is an open subset of $\mathbb{V}$ containing Z as a closed subset.

Definition 1.3. A PL-stratification of a set S of $\mathbb{V}$ is a finite family $Z=\{Z_a\}_{a\in A}$ of non-empty convex polyhedra such that

(i) $S=\bigcup_{a\in A} Z_a$,

(ii) each Z_a is a locally closed submanifold,

(iii) $Z_a\cap Z_b=\varnothing$ for $a\neq b$,

(iv) $Z_a \cap \overline{Z}_b \neq \varnothing$ implies $Z_a \subset \overline{Z}_b$.

Replacing "a finite family" with "a locally finite family" we get the notion of an LPL-stratification.

Recall the operation $\widehat{+}$ and the notion of a μ-stratification of [KS90, Def. 6.2.4, 8.3.19].

Proposition 1.4. *Let $Z = \{Z_a\}_{a\in A}$ be an* LPL *stratification. Then*

(i) *$\{Z_a\}_{a\in A}$ is a μ-stratification, that is, $Z_a \subset \overline{Z}_b$ implies $(T^*_{Z_a}\mathbb{V} \widehat{+} T^*_{Z_b}\mathbb{V}) \cap \pi^{-1}(Z_a) \subset T^*_{Z_a}\mathbb{V}$.*

(ii) *Set $\Lambda = \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$. Then $\Lambda \widehat{+} \Lambda = \Lambda$.*

Proof. We shall prove both statements together.

Assume that $Z_a \subset \overline{Z}_b \cap \overline{Z}_c$. We may assume (in a neighborhood of a point of Z_a) that $\mathbb{V} = \mathbb{W} \oplus \mathbb{W}'$ for two linear spaces $\mathbb{W}$ and $\mathbb{W}'$ and $Z_a = \mathbb{W}$. Then $Z_b = \mathbb{W} \times S$ and $Z_c = \mathbb{W} \times L$ where S is open in some linear subspace $\mathbb{W}''$ of $\mathbb{W}'$ and similarly for L. Then one immediately checks that $(T^*_{Z_b}\mathbb{V} \widehat{+} T^*_{Z_c}\mathbb{V}) \cap \pi^{-1}(Z_a) \subset T^*_{Z_a}\mathbb{V}$.

This proves (ii). Choosing $c = a$ we get (i). □

Proposition 1.5. *Consider a finite family $\{P_b\}_{b\in B}$ of convex polyhedra. Then there exists a* PL*-stratification $\mathbb{V} = \bigsqcup_{a\in A} Z_a$ such that each P_b is a union of strata.*

In the sequel, an interval of $\mathbb{R}$ means a convex subset of $\mathbb{R}$.

Proof. There exists a finite family $\{f_1, \dots, f_l\}$ of linear forms and a finite family $\{I_c\}_{c\in C}$ such that each I_c is either an open interval or a point, $\mathbb{R} = \bigsqcup_{c\in C} I_c$ and for all $b \in B$,

$$P_b = \bigcap_{1\le j\le l} f_j^{-1}(J_{j,b}), \text{ where } J_{j,b} \text{ is a union of some } I_c,\ c \in C.$$

For any family $d = \{c_1, \dots, c_l\} \in C^l$, set

$$Z_d = \bigcap_{j=1}^{l} f_j^{-1}(I_{c_j}).$$

Then the family $\{Z_d\}_{d\in C^l}$ is a PL-stratification of $\mathbb{V}$ finer that the family $\{P_b\}_{b\in B}$. □

1.2 PL Lagrangian subvarieties

Recall that the notions of co-isotropic, isotropic and Lagrangian subanalytic subvarieties are given in [KS90, Def. 6.5.1, 8.3.9].

Proposition 1.6. *Let Λ be a locally closed conic* LPL *isotropic subset of $T^*\mathbb{V}$. Then for any $p \in \Lambda_{\mathrm{reg}}$ there exists a linear affine subspace $L \subset \mathbb{V}$ with $\Lambda \subset T^*_L\mathbb{V}$ in a neighborhood of p.*

Proof. If λ is a linear affine isotropic subspace of $T^*\mathbb{V}$, then there exists a linear affine subspace L of $\mathbb{V}$ such that $\lambda \subset T^*_L\mathbb{V}$. □

Lemma 1.7. *Let $\{L_a\}_{a\in A}$ be a finite family of affine linear subspaces in a vector space $\mathbb{W}$. Set $X = \bigcup_{a\in A} L_a$ and let S be a closed subset of X. Assume that $S \cap X_{\rm reg}$ is open in $X_{\rm reg}$ and S is the closure of $S \cap X_{\rm reg}$. Then S is* PL.

Proof. Indeed $S \cap X_{\rm reg}$ is a locally finite union of connected components of $X_{\rm reg}$. ☐

Theorem 1.8. (a) *Let Λ be a locally closed conic* PL *isotropic subset of $T^*\mathbb{V}$. Then there exists a* PL *stratification $\{P_a\}_{a\in A}$ of $\mathbb{V}$ such that $\Lambda \subset \bigsqcup_{a\in A} T^*_{P_a}\mathbb{V}$.*

(b) *Let Λ be a locally closed conic subanalytic Lagrangian subset of $T^*\mathbb{V}$ and assume that Λ is contained in a closed conic* PL *isotropic subset. Then Λ is* PL.

Proof. (a) Let $\{\Omega_i\}_{i\in I}$ be the family of connected components of $\Lambda_{\rm reg}$. Note that the Ω_i's are PL. Then there exists an affine linear subspace L_i such that $\Omega_i \subset T^*_{L_i}\mathbb{V}$ by Proposition 1.6. Choose a PL stratification $\{P_a\}_{a\in A}$ finer than the family $\{L_i\}_{i\in I}$. Then $\Lambda_{\rm reg} \subset \bigsqcup T^*_{P_a}\mathbb{V}$ and Proposition 1.4 (ii) implies that this last set is closed, hence contains Λ.

(b) follows from Lemma 1.7 with $\mathbb{W} = T^*\mathbb{V}$. ☐

Remark 1.9. In Lemma 1.7 and Theorem 1.8, all statements remain true when replacing everywhere "finite" with "locally finite" and "PL" with "LPL".

2 PL sheaves

2.1 Review on sheaves

Let us recall some definitions extracted from [KS90] and a few notations.

- Throughout this paper, $\mathbf{k}$ denotes a field. We denote by $\mathrm{Mod}(\mathbf{k})$ the abelian category of $\mathbf{k}$-vector spaces.

- For an abelian category $\mathscr{C}$, we denote by $\mathrm{D^b}(\mathscr{C})$ its bounded derived category. However, we write $\mathrm{D^b}(\mathbf{k})$ instead of $\mathrm{D^b}(\mathrm{Mod}(\mathbf{k}))$.

- For a vector bundle $E \to M$, we denote by $a\colon E \to E$ the antipodal map, $a(x, y) = (x, -y)$. For a subset $Z \subset E$, we simply denote by Z^a its image by the antipodal map. In particular, for a cone γ in E, we denote by $\gamma^a = -\gamma$ the opposite cone. For such a cone, we denote by γ° the polar cone (or dual cone) in the dual vector bundle E^*:

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v\rangle \geq 0 \text{ for all } v \in \gamma_x\}. \tag{2.1}$$

- Let M be a real manifold of dimension $\dim M$. We shall use freely the classical notions of microlocal sheaf theory, referring to [KS90]. We denote by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathrm{D^b}(\mathbf{k}_M)$ its bounded derived category. For short, an object of $\mathrm{D^b}(\mathbf{k}_M)$ is called a "sheaf" on M.

- For a locally closed subset $Z \subset M$, one denotes by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z extended by 0 on $M \setminus Z$. One defines similarly the sheaf L_Z for $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k})$.
- For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ we denote by $\mathrm{SS}(F)$ its singular support, or microsupport, a closed conic co-isotropic subset of T^*M.

Constructible sheaves

We refer the reader to [KS90] for terminologies not explained here.

Definition 2.1. Let M be a real analytic manifold and let $F \in \mathrm{Mod}(\mathbf{k}_M)$. One says that F is weakly $\mathbb{R}$-constructible if there exists a subanalytic stratification $M = \bigsqcup_{a \in A} M_a$ such that for each stratum M_a, the restriction $F|_{M_a}$ is locally constant. If moreover, the stalk F_x is of finite rank for all $x \in M$, then one says that F is $\mathbb{R}$-constructible.

Notation 2.2. (i) One denotes by $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves, a thick abelian subcategory of $\mathrm{Mod}(\mathbf{k}_M)$.
(ii) One denotes by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves with $\mathbb{R}$-constructible cohomology and by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c,c}}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ consisting of sheaves with compact support.

Recall that the natural functor $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_M)$ is an equivalence of categories.

2.2 Microlocal characterization of PL sheaves

Recall Remark 0.1.

Definition 2.3. One says that $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ is PL if there exists a finite family $\{P_a\}_{a \in A}$ of convex polyhedra such that $\mathbb{V} = \bigcup_{a \in A} P_a$ and $F|_{P_a}$ is constant of finite rank for any $a \in A$.

Replacing the finite family $\{P_a\}_{a \in A}$ with a locally finite family, we get the notion of an LPL sheaf.

By this definition

(2.2) $\quad F$ is PL if and only if $H^j(F)$ is PL for all $j \in \mathbb{Z}$.

One sets

$$\begin{cases} \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}});\ F \text{ is PL}\}, \\ \mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}). \end{cases} \tag{2.3}$$

Of course, $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ and $\mathrm{Mod}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is a subcategory of $\mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$.

Proposition 2.4. *The natural functor* $\mathrm{D^b}(\mathrm{Mod_{PL}}(\mathbf{k}_{\mathbb{V}})) \to \mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}})$ *is an equivalence.*

Proof. There exists a triangulation $\mathbb{S} = (S, \Delta)$ and a homeomorphism $f\colon |\mathbb{S}| \to \mathbb{V}$ such that its restriction to $|\sigma|$ is linear for any $\sigma \in \Delta$. Then the result follow from [KS90, Th. 8.1.10]. ☐

Theorem 2.5. *Let* $F \in \mathrm{D^b_{\mathbb{R}c}}(\mathbf{k}_{\mathbb{V}})$*. Then the conditions below are equivalent.*

(a) $F \in \mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}})$,

(b) $\mathrm{SS}(F)$ *is a closed conic* PL *Lagrangian subset of* $T^*\mathbb{V}$,

(c) $\mathrm{SS}(F)$ *is contained in a closed conic* PL *isotropic subset of* $T^*\mathbb{V}$.

Proof. (a)⇒(c) Consider a finite covering $\{P_b\}_{b\in B}$ by convex polyhedra such that $F|_{P_b}$ is constant and choose a finer PL stratification $\mathbb{V} = \bigsqcup_{a\in A} Z_a$. This is a μ-stratification and this implies $\mathrm{SS}(F) \subset \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$ by [KS90, Prop. 8.4.1].
(b)⇒(a) By Theorem 1.8 (a), there exists a PL stratification $\mathbb{V} = \bigsqcup_{a\in A} Z_a$ such that $\mathrm{SS}(F) \subset \bigsqcup_{a\in A} T^*_{Z_a}\mathbb{V}$. Then $F|_{Z_a}$ is locally constant for each $a \in A$ by [KS90, Prop. 8.4.1].
(b)⇔(c) in view of Theorem 1.8 (b). ☐

The next result immediately follows from Definition 2.3. It can also easily be deduced from [KS90], Theorem 1.8 and Theorem 2.5.

Corollary 2.6. (i) *The category* $\mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}})$ *is a full triangulated subcategory of the category* $\mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ *and the category* $\mathrm{Mod_{PL}}(\mathbf{k}_{\mathbb{V}})$ *is a full thick abelian subcategory of the category* $\mathrm{Mod}(\mathbf{k}_{\mathbb{V}})$.

(ii) *If* F_1 *and* F_2 *are* PL, *then so are* $F_1 \otimes F_2$ *and* $\mathrm{R}\mathscr{H}om(F_1, F_2)$.

(iii) *Let* $f\colon \mathbb{V} \to \mathbb{W}$ *be a linear map.*

  (a) *If* G *is a PL sheaf on* $\mathbb{W}$, *then* $f^{-1}G$ *and* $f^!G$ *are* PL *sheaves on* $\mathbb{V}$.

  (b) *If* F *is a* PL *sheaf on* $\mathbb{V}$ *then* $\mathrm{R}f_*F$ *and* $\mathrm{R}f_!F$ *are* PL *sheaves on* $\mathbb{W}$.

Recall that the convolution functor $\star\colon \mathrm{D^b}(\mathbf{k}_{\mathbb{V}}) \times \mathrm{D^b}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D^b}(\mathbf{k}_{\mathbb{V}})$ is defined by the formula:

$$F \star G := \mathrm{R}s_!(F \boxtimes G),$$

where $s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is the map $(x, y) \mapsto x + y$.

Corollary 2.7. *The convolution functor induces a functor* $\star\colon \mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}}) \times \mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D^b_{PL}}(\mathbf{k}_{\mathbb{V}})$.

Proposition 2.8. *Let* Z *be a locally closed subset of* $\mathbb{V}$. *Then* Z *is* PL *if and only if* $\mathrm{SS}(\mathbf{k}_Z)$ *is* PL.

Proof. (i) Assume that Z is PL. Then the sheaf $\mathbf{k}_Z$ is PL and its microsupport is PL by Theorem 2.5.

(ii) Conversely, assume that $\mathrm{SS}(\mathbf{k}_Z)$ is PL. Set $\partial Z = \overline{Z} \setminus Z$. Since Z is locally closed, ∂Z is closed.

(ii)–(a) First, notice that $\overline{Z} = \pi(\mathrm{SS}(\mathbf{k}_Z))$ is PL.

(ii)–(b) Now consider the exact sequence of sheaves $0 \to \mathbf{k}_Z \to \mathbf{k}_{\overline{Z}} \to \mathbf{k}_{\partial Z} \to 0$. Since $\mathbf{k}_Z$ and $\mathbf{k}_{\overline{Z}}$ are PL sheaves, the sheaf $\mathbf{k}_{\partial Z}$ is PL. Therefore, ∂Z is PL and it follows that Z is PL. ☐

2.3 PL sheaves on the projective space

Let $\mathbb{V} \hookrightarrow \mathbb{P}$ be the projective compactification of $\mathbb{V}$. Hence, setting $\mathbb{W} = \mathbb{V} \times \mathbb{R}$,

$$\mathbb{P} \simeq (\mathbb{W} \setminus \{0\})/\mathbb{R}^\times,$$

where $\mathbb{R}^\times$ is the multiplicative group $\mathbb{R} \setminus \{0\}$. Denote by $\pi\colon \mathbb{W} \setminus \{0\} \to \mathbb{P}$ the projection and by $\iota\colon \mathbb{W} \setminus \{0\} \hookrightarrow \mathbb{W}$ the embedding.

We shall say that a subset A of $\mathbb{P}$ is PL if $\iota(\pi^{-1}(A))$ is PL in $\mathbb{W}$.

Similarly, one defines the category of PL sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_\mathbb{P})$ as the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_\mathbb{P})$ consisting of objects F such that $\iota_!\pi^{-1}F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_\mathbb{W})$.

Denote by $j\colon \mathbb{V} \hookrightarrow \mathbb{P}$ the open embedding and by $H_\infty = \mathbb{P} \setminus j(\mathbb{V})$ the hyperplane at infinity. The next result is easy and its proof is left to the reader.

Proposition 2.9. *The functor $j_!\colon \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_\mathbb{V}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_\mathbb{P})$ is well-defined, is fully faithful and its essential image consists of objects F such that $F|_{H_\infty} \simeq 0$.*

2.4 Distance, approximation and stability

The bottleneck distance for persistent modules is a classical subject that we shall not review here, referring to [CSEH07] and [CCSG+09, CdSGO16, Cur13, EH08, Ghr08, Les15]. Here, we use a convolution distance for sheaves similar to that of [KS18] and slightly different from the classical ones since it is defined in the derived setting. Note that this "derived" distance has recently been studied with great details in [BG18] in case of dimension one.

Assume that the vector space $\mathbb{V}$ is endowed with a norm $\|\cdot\|$ (see Remark 2.10 below). We define a family of sheaves $\{K_a\}_{a\in\mathbb{R}}$ as follows:

$$K_a \simeq \begin{cases} \mathbf{k}_{\{\|x\|\leq a\}} & \text{for } a \geq 0, \\ \mathbf{k}_{\{\|-x\|<-a\}}[\dim \mathbb{V}] & \text{for } a < 0. \end{cases} \tag{2.4}$$

There are natural morphisms and isomorphisms

$$\begin{aligned} &\chi_{a,b}\colon K_b \to K_a \text{ for } a \leq b \in \mathbb{R}, \\ &K_a \star K_b \simeq K_{a+b} \text{ for } a, b \in \mathbb{R} \end{aligned} \tag{2.5}$$

such that $\chi_{a,b} \circ \chi_{b,c} = \chi_{a,c}$ for $a \leq b \leq c$.

Remark 2.10. In [KS18] we have used the Euclidean norm, but the argument works for any norm, since (2.5) remains true. Here a norm $\|\cdot\|\colon \mathbb{V} \to \mathbb{R}_{\geq 0}$ satisfies (1) $\|x+y\| \leq \|x\| + \|y\|$, (2) $\|ax\| = a\|x\|$ for $a \geq 0$, and (3) $\|x\| = 0$ implies $x = 0$. Hence norms correspond bijectively with open relatively compact convex neighborhood of 0. Note that we do not ask $\|x\| = \|-x\|$ and that is why we define K_a for $a < 0$ as above, in order that $K_a \star K_{-a} \simeq K_0$.

Definition 2.11. ([KS18, Def. 3.2]) Let $F, G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$ and let $a \geq 0$. One says that F and G are *a-isomorphic* if there are morphisms $f\colon K_a \star F \to G$ and $g\colon K_a \star G \to F$ which satisfies the following compatibility conditions: the composition $K_{2a} \star F \xrightarrow{K_a \star f} K_a \star G \xrightarrow{g} F$ coincides with the natural morphism $\chi_{0,2a} \star F\colon K_{2a} \star F \to F$ and the composition $K_{2a} \star G \xrightarrow{K_a \star g} K_a \star F \xrightarrow{f} G$ coincides with the natural morphism $\chi_{0,2a} \star G\colon K_{2a} \star G \to G$.

One sets

$$\mathrm{dist}(F, G) = \inf\Big(\{+\infty\} \cup \{a \in \mathbb{R}_{\geq 0}\,;\, F \text{ and } G \text{ are } a\text{-isomorphic}\}\Big)$$

and calls $\mathrm{dist}(\,\bullet\,,\,\bullet\,)$ the *convolution distance*.

In loc. cit. we have proved that any object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ can be approximated with LPL-sheaves. A similar result holds for constructible sheaves on $\mathbb{V}_\infty$.

Notation 2.12. Let us denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}})$ consisting of objects F such that there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{P}})$ with $F \simeq G|_{\mathbb{V}}$.

Theorem 2.13 (The approximation theorem)**.** *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$. For each $\varepsilon > 0$ there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ such that $\mathrm{dist}(F, G) \leq \varepsilon$ and $\mathrm{supp}(G) \subset \mathrm{supp}(F) + B_\varepsilon$, where $B_\varepsilon = \{x; \|x\| \leq \varepsilon\}$.*

The proof is the same as in [KS18] after noticing that one can choose the simplicial complex $\mathbf{S} = (S, \Delta)$ (with the notations of loc. cit.) to be finite, thanks to the fact that $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}_\infty})$.

Recall the stability theorem of [KS18, Th. 3.7], a derived version of a classical theorem (see [CSEH07]).

For a set X and a map $f\colon X \to \mathbb{V}$, one sets

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

Theorem 2.14 (The stability theorem)**.** *Let X be a locally compact space and let $f_1, f_2\colon X \to \mathbb{V}$ be two continuous maps. Then, for any $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, we have*

$$\mathrm{dist}(\mathrm{R}f_{1*}F, \mathrm{R}f_{2*}F) \leq \|f_1 - f_2\| \quad \textit{and} \quad \mathrm{dist}(\mathrm{R}f_{1!}F, \mathrm{R}f_{2!}F) \leq \|f_1 - f_2\|.$$

2.5 Generators for PL sheaves

Consider a triangulated category $\mathscr{D}$ and a family of objects $\mathscr{G}$. Consider the full subcategory $\mathscr{T}$ of $\mathscr{D}$ defined as follows. An object $F \in \mathscr{D}$ belongs to $\mathscr{T}$ if there exists a finite sequence $F_0, \dots, F_N$ in $\mathscr{D}$ with $F_0 = 0$, $F_N = F$ and distinguished triangles (d. t. for short) $F_k \to F_{k+1} \to G_k[m_k] \xrightarrow{+1}$, $0 \le k < N$ with $m_k \in \mathbb{Z}$ and $G_k \in \mathscr{G}$.

The next result is well-known from the specialists but we recall its proof for the reader's convenience.

Lemma 2.15. *The subcategory $\mathscr{T}$ of $\mathscr{D}$ is triangulated. It is the smallest triangulated subcategory of $\mathscr{D}$ which contains $\mathscr{G}$.*

Proof. We may assume from the beginning that $\mathscr{G}$ is an additive category stable by isomorphisms and by shifts. Define the additive subcategory $\mathscr{T}_n$ by induction as follows:

> $\mathscr{T}_0 = \{0\}$, $X \in \mathscr{T}_n$ if and only if there is a d. t. $X_{n-1} \to X \to X_1 \xrightarrow{+1}$ with $X_{n-1} \in \mathscr{T}_{n-1}$ and $X_1 \in \mathscr{G}$.

Clearly, $\mathscr{T}_1 = \mathscr{G}$.

Consider a d. t. $X_n \to X \xrightarrow{\alpha} X_p \xrightarrow{+1}$ with $X_i \in \mathscr{T}_i$ as above. By the definition of $\mathscr{T}_p$, there exists a d. t. $X_{p-1} \to X_p \xrightarrow{\beta} X_1 \xrightarrow{+1}$ with $X_i \in \mathscr{T}_i$, $i = p-1, 1$. Consider a d. t. $Y \to X \xrightarrow{\beta\circ\alpha} X_1 \xrightarrow{+1}$. By the octahedral axiom we get a d. t. $X_n \to Y \to X_{p-1} \xrightarrow{+1}$. By the induction hypothesis, $Y \in \mathscr{T}_{n+p-1}$. Hence $X \in \mathscr{T}_{n+p}$. ☐

In this paper, we shall say that $\mathscr{G}$ generates $\mathscr{D}$ if $\mathscr{T} = \mathscr{D}$.

Theorem 2.16. *The triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ is generated by the family $\{\mathbf{k}_P\}$ where P ranges over the family of locally closed convex polyhedra.*

Proof. (i) We denote by $\mathscr{G}$ the family of sheaves isomorphic to some $\mathbf{k}_P$, P a locally closed convex polyhedron, and denote by $\mathscr{T}$ the triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ generated by $\mathscr{G}$.

(ii) We argue by induction on $\dim \mathbb{V}$. The case where $\dim \mathbb{V}$ is 0 is clear.

(iii) Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$. By truncation, we may reduce to the case where F is concentrated in degree 0.

(iv) There exists a finite family $\{H_a\}_{a\in A}$ of affine hyperplanes such that, setting $U = \mathbb{V} \setminus \bigcup_a H_a$, the restriction of F to U is locally constant. Let $U = \bigsqcup_i U_i$ be the decomposition of U into connected component. Each U_i is an open convex polyhedron. Set $Z = \bigcup_a H_a$ and consider the exact sequence $0 \to F_U \to F \to F_Z \to 0$. The sheaf F_U is a finite direct sum of sheaves of the type $\mathbf{k}_{U_i}$. Hence $F_U \in \mathscr{T}$ and it remains to show that F_Z belongs to $\mathscr{T}$.

(v) We argue by induction on $\#A$. If $\#A = 1$, then the result follows from the induction hypothesis on the dimension of $\mathbb{V}$ since we may identify F_{H_a} with a sheaf on the affine space H_a. Let $a \in A$ and define G by the exact sequence $0 \to G \to F_Z \to F_{H_a} \to 0$. By the induction hypothesis G and F_{H_a} belong to $\mathscr{T}$ and the result follows. ☐

3 PL γ-sheaves

3.1 Review on γ-sheaves

In this subsection we shall review some definitions and results extracted from [KS90, KS18]. The so-called γ-topology has been studied with some details in [KS90, § 3.4].

Let $\mathbb{V}$ be a finite-dimensional real vector space. Recall that we denote by $a\colon \mathbb{V} \to \mathbb{V}$ the antipodal map $x \mapsto -x$ and by $s\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ the map $(x, y) \mapsto x + y$. Hence, for two subsets A, B of $\mathbb{V}$, one has $A + B = s(A \times B)$.

A subset A of $\mathbb{V}$ is called a cone if $0 \in A$ and $\mathbb{R}_{>0}A \subset A$. A convex cone A is proper if $A \cap A^a = \{0\}$.

Throughout the rest of the paper, we consider a cone $\gamma \subset \mathbb{V}$ and we assume:

(3.1) *γ is closed proper convex with non-empty interior.*

In § 3.3 we shall make the extra assumption that γ is *polyhedral*, meaning that it is a finite intersection of closed half-spaces.

We say that a subset A of $\mathbb{V}$ is *γ-invariant* if $A + \gamma = A$. Note that a subset A is γ-invariant if and only if $\mathbb{V} \setminus A$ is γ^a-invariant.

The family of γ-invariant open subsets of $\mathbb{V}$ defines a topology, which is called the *γ-topology* on $\mathbb{V}$. One denotes by $\mathbb{V}_\gamma$ the space $\mathbb{V}$ endowed with the γ-topology and one denotes by

$$\varphi_\gamma\colon \mathbb{V} \to \mathbb{V}_\gamma \tag{3.2}$$

the continuous map associated with the identity. Note that the closed sets for this topology are the γ^a-invariant closed subsets of $\mathbb{V}$.

Definition 3.1. Let A be a subset of $\mathbb{V}$.

(a) One says that A is *γ-open* (resp. *γ-closed*) if A is open (resp. closed) for the γ-topology.

(b) One says that A is *γ-locally closed* if A is the intersection of a γ-open subset and a γ-closed subset.

(c) One says that A is *γ-flat* if $A = (A + \gamma) \cap (A + \gamma^a)$.

(d) One says that a closed set A is *γ-proper* if the addition map s is proper on $A \times \gamma^a$.

Remark that a closed subset A is γ-proper if and only if $A \cap (x + \gamma)$ is compact for any $x \in \mathbb{V}$.

Proposition 3.2 ([KS18, Prop. 4.4])**.** *The set of γ-flat open subsets Ω of $\mathbb{V}$ and the set of γ-locally closed subsets Z of $\mathbb{V}$ correspond bijectively by*

$$\Omega \longmapsto (\Omega + \gamma) \cap \overline{\Omega + \gamma^a}$$
$$\mathrm{Int}(Z) \longleftarrow\!\shortmid Z.$$

In particular, γ-locally closed subsets are γ-flat.

We shall use the notations:

$$
(3.3)\qquad \begin{cases} \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \{F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}}); \mathrm{SS}(F) \subset \mathbb{V} \times \gamma^{\circ a}\}, \\ \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathbb{R}\mathrm{c},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}_{\mathbb{R}\mathrm{c}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{Mod}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}). \end{cases}
$$

We call an object of $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ *a γ-sheaf.*

It follows from [KS90, Prop. 5.4.14] that for $F, G \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ and $H \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ}}(\mathbf{k}_{\mathbb{V}})$, the sheaves $F \otimes G$ and $\mathrm{R}\mathscr{H}om(H, F)$ belong to $\mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$.

The next result is implicitly proved in [KS90] and explicitly in [KS18]. (In this statement, the hypothesis that $\mathrm{Int}(\gamma)$ is non empty is not necessary.)

Theorem 3.3. *Let γ be a closed convex proper cone in $\mathbb{V}$. The functor $\mathrm{R}\varphi_{\gamma *}\colon \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}_\gamma})$ is an equivalence of triangulated categories with quasi-inverse φ_γ^{-1}. Moreover, this equivalence preserves the natural t-structures of both categories. In particular, for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{V}})$, the condition $F \in \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$ is equivalent to the condition: $\mathrm{SS}(H^j(F)) \subset \mathbb{V} \times \gamma^{\circ a}$ for any $j \in \mathbb{Z}$.*

Thanks to this theorem, the reader may ignore microlocal sheaf theory, at least in a first reading.

Corollary 3.4 ([KS18, Cor. 2.8])**.** *Let A be a γ-locally closed subset of $\mathbb{V}$. Then $\mathrm{SS}(\mathbf{k}_A) \subset \mathbb{V} \times \gamma^{\circ a}$.*

Proposition 3.5. *Assume (3.1). Let $U = U + \gamma$ be a γ-open set and let $x_0 \in \partial U$. Then there exist a linear coordinate system $(x_1, \dots, x_n)$ on $\mathbb{V}$, an open neighborhood V of x_0, an open subset W of $\mathbb{V}$ and a bi-Lipschitz isomorphism $\varphi\colon V \xrightarrow{\sim} W$ such that $\varphi(V \cap U) = W \cap \{x \in \mathbb{V}; x_n > 0\}$.*

Proof. The proofs of [GS16, Lem. 2.36, 2.37] (which were formulated for subanalytic open subsets) extend immediately to our situation. ☐

Recall the duality functor

$$
\mathrm{D}'_M(\,\bullet\,) = \mathrm{R}\mathscr{H}om(\,\bullet\,, \mathbf{k}_M).
$$

Corollary 3.6. *Let Z be a γ-locally closed subset of $\mathbb{V}$. Then, $\mathrm{D}'_M(\mathbf{k}_Z)$ is concentrated in degree 0. Moreover, $\mathrm{D}'_M(\mathbf{k}_Z) \simeq \mathbf{k}_S$ with $\Omega = \mathrm{Int}(Z)$ and $S = \overline{\Omega + \gamma} \cap (\Omega + \gamma^a)$.*

Proof. It follows from Proposition 3.5 that $\mathrm{D}'_M(\mathbf{k}_{\Omega+\gamma}) \simeq \mathbf{k}_{\overline{\Omega+\gamma}}$ and $\mathrm{D}'_M(\mathbf{k}_{\overline{\Omega+\gamma^a}}) \simeq \mathbf{k}_{\Omega+\gamma^a}$.

Set $A = \Omega + \gamma$ and $B = \overline{\Omega + \gamma^a}$. Then $\mathbf{k}_A$ and $\mathbf{k}_B$ are cohomologically constructible. By Corollary 3.4, $\mathrm{SS}(\mathbf{k}_A) \cap \mathrm{SS}(\mathbf{k}_B) \subset T^*_{\mathbb{V}}\mathbb{V}$. Then $\mathrm{D}'_M(\mathbf{k}_A \otimes \mathbf{k}_B) \simeq \mathrm{D}'_M(\mathbf{k}_A) \otimes \mathrm{D}'_M(\mathbf{k}_B)$ by [KS90, Cor. 6.4.3]. ☐

3.2 An application of the stability theorem

Here we give an application of the stability theorem (Theorem 2.14) which did not appear in [KS18]. Of course, this application is well-known for the classical (non-derived) distance.

Let (M, d) be a subanalytic metric space and let K_1 and K_2 be two subanalytic compact subsets. Define for $i = 1, 2$

$$\begin{aligned} &f_i(\cdot) = d(\cdot, K_i), \\ &\Gamma_i = \{(x,t) \in M \times \mathbb{R}; d(x, K_i) = t\}, \quad G_i = \mathbf{k}_{\Gamma_i}, \\ &Z_i = \{(x,t) \in M \times \mathbb{R}; d(x, K_i) \leq t\}, \quad F_i = \mathbf{k}_{Z_i}. \end{aligned}$$

Hence, Γ_i is the graph of f_i and Z_i is the epigraph of f_i.

Lemma 3.7. *One has* $d(K_1, K_2) = \|f_1 - f_2\|$.

Lemma 3.8. *One has* $\mathrm{dist}(\mathrm{R}f_{1*}\mathbf{k}_M, \mathrm{R}f_{2*}\mathbf{k}_M) \leq \|f_1 - f_2\|$.

Proof. This is a particular case of Theorem 2.14. ☐

Lemma 3.9. *Denote by γ the cone $\{t \leq 0\}$ in $\mathbb{R}$ and still denote by φ_γ the map $M \times \mathbb{R} \to M \times \mathbb{R}_\gamma$. Then $F_i \simeq \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_i$.*

Let p denote the projection $M \times \mathbb{R} \to \mathbb{R}$.

Lemma 3.10. *One has* $\mathrm{dist}(\mathrm{R}p_*\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_1, \mathrm{R}p_*\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}G_2) \leq \mathrm{dist}(\mathrm{R}p_*G_1, \mathrm{R}p_*G_2)$.

Proof. The two functors $\mathrm{R}p_*$ and $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma_*}$ commute (with obvious notations). Then the result follows from [KS18, Prop. 3.6]. ☐

Applying these lemmas, one gets a derived version of a result of [CSEH07].

Theorem 3.11. *One has* $\mathrm{dist}(\mathrm{R}p_*F_1, \mathrm{R}p_*F_2) \leq d(K_1, K_2)$.

Proof. We have

$$\mathrm{R}p_*G_i \simeq \mathrm{R}f_{i*}\mathbf{k}_M.$$

Applying Lemma 3.8, we get

$$\mathrm{dist}(\mathrm{R}p_*G_1, \mathrm{R}p_*G_2) \leq \|f_1 - f_2\|.$$

To conclude, apply Lemma 3.9, 3.10 and 3.7.

☐

3.3 Review on PL γ-sheaves

From now on, we shall assume that the cone γ satisfies:

(3.4) *γ is a closed proper convex polyhedral cone with non-empty interior.*

Definition 3.12. Assume (3.4).

(a) A PL γ-*barcode* (A, Z) in $\mathbb{V}$ is the data of a *finite* set of indices A and a family $Z = \{Z_a\}_{a \in A}$ of non-empty, γ-locally closed, convex polyhedra of $\mathbb{V}$.

(b) A disjoint γ-barcode is a γ-barcode (A, Z) such that $Z_a \cap Z_b = \varnothing$ for $a \neq b$.

(c) The support of a γ-barcode (A, Z), denoted by $\operatorname{supp}(A, Z)$, is the set $\bigcup_{a \in A} \overline{Z_a}$.

Remark 3.13. In [KS18], we defined a PL γ-stratification of a closed set S as a barcode (A, Z) such that $\operatorname{supp}(A, Z) = S$ and $Z_a \cap Z_b = \varnothing$ for $a \neq b$. However, since a PL γ-stratification is not a PL stratification (see Definition 1.3), we prefer here to avoid this terminology and use the notion of a disjoint γ-barcode.

We shall use the notations:

$$
\begin{cases} \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}}), \\ \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) := \mathrm{Mod}(\mathbf{k}_{\mathbb{V}}) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}). \end{cases} \tag{3.5}
$$

Note that, in view of (2.2) and Theorem 3.3:

$$
F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) \Leftrightarrow H^j(F) \in \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}) \text{ for all } j \in \mathbb{Z}. \tag{3.6}
$$

Proposition 3.14. *Assume (3.4). If $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ then $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$.*

Proof. By Theorem 3.3, it remains to prove that $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F$ is PL. Denote by $Z(\gamma)$ the set $\{(x, y) \in \mathbb{V} \times \mathbb{V}; y - x \in \gamma\}$ and denote by q_1 and q_2 the first and second projections defined on $\mathbb{V} \times \mathbb{V}$. Then (see [KS90, Prop. 3.5.4]):

$$
\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F \simeq \mathrm{R}q_{1*}(\mathbf{k}_{Z(\gamma)} \otimes q_2^{-1}F)
$$

and the result follows from Corollary 2.6. □

Recall Notation 2.12.

Corollary 3.15. *Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}c,\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}_\infty})$. For each $\varepsilon > 0$ there exists $G \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ such that* $\operatorname{dist}(F, G) \leq \varepsilon$.

Proof. We shall apply Theorem 2.13. There exists $G' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL}}(\mathbf{k}_{\mathbb{V}})$ such that $\operatorname{dist}(F, G') \leq \varepsilon$. Then, $G := \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}G' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ by Proposition 3.14. Moreover, by [KS18, Prop. 3.6]:

$$
\operatorname{dist}\big(\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F, \varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}G'\big) \leq \operatorname{dist}(F, G') \leq \varepsilon.
$$

Since F is a γ-sheaf, one has $\varphi_\gamma^{-1}\mathrm{R}\varphi_{\gamma *}F \simeq F$. □

The three theorems below are the main results of [KS18]. We recall them for the reader's convenience.

Theorem 3.16 ([KS18, Th. 4.10]). *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$*. Then for each* $x \in \mathbb{V}$*, there exists an open neighborhood* U *of* x *such that* $F|_{(x+\gamma^a)\cap U}$ *is constant.*

Theorem 3.17 ([KS18, Th. 4.14]). *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\mathrm{c},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$*. Let* Ω *be a* γ*-flat open set and let* $Z = (\Omega + \gamma) \cap \overline{\Omega + \gamma^a}$*, a* γ*-locally closed subset. Assume that* $F|_\Omega$ *is locally constant. Then* $F|_Z$ *is locally constant.*

Theorem 3.18 ([KS18, Lem. 4.16, Th. 4.17]). *Assume* (3.4) *and let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$*. Then there exists a disjoint* γ*-barcode* (A, Z) *with* $\mathrm{supp}(A, Z) = \mathrm{supp}(F)$ *and such that* $F|_{Z_a}$ *is constant for each* $a \in A$*. Moreover,* $F_x \simeq 0$ *for* $x \notin \bigsqcup_{a\in A} Z_a$*.*

(In fact, Theorem 3.18 was proved for LPL sheaves but the proof can easily be adapted to PL sheaves.)

3.4 Generators for PL γ-sheaves

In [KS18] we have constructed a category $\mathbf{Bar}_\gamma$ whose objects are the γ-barcodes and a fully faithful functor

$$\Psi\colon \mathbf{Bar}_\gamma \to \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}}), \quad Z = \{Z_a\}_{a\in A} \mapsto \bigoplus_{a\in A} \mathbf{k}_{Z_a}. \tag{3.7}$$

However, as shown in [KS18, Ex. 2.14, 2.15], the functor Ψ is not essentially surjective as soon as $\dim \mathbb{V} > 1$.

Definition 3.19. An object of $\mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is a barcode γ-sheaf if it is in the essential image of Ψ.

In [KS18] we made the following conjecture.

Conjecture 3.20. Let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ and assume that F has compact support. Then there exists a bounded complex $F^\bullet \in \mathrm{C}^{\mathrm{b}}(\mathrm{Mod}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}_\gamma}))$ whose image in $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is isomorphic to F and such that each component F^j of $F^\bullet$ is a barcode γ-sheaf with compact support.

As usual, for an additive category $\mathscr{C}$, $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ denotes the category of bounded complexes of objects of $\mathscr{C}$.

In this subsection, we shall prove a weaker form of this conjecture, namely:

Theorem 3.21. *The triangulated category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ *is generated by the family* $\{\mathbf{k}_P\}_P$ *where* P *ranges over the family of* γ*-locally closed convex polyhedra.*

In particular, the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{PL},\gamma^{\circ \mathrm{a}}}(\mathbf{k}_{\mathbb{V}})$ is generated by the barcodes γ-sheaves.

Proof. Let $F \in \mathrm{Mod}_{\mathrm{PL},\gamma^{\circ a}}(\mathbf{k}_{\mathbb{V}})$. There exists $\{\xi_k\}_{1\le k\le l}$ in γ^{0a} and $\{c_j\}_{0\le j\le N}$ in $\mathbb{R}$ with $-\infty = c_0 < c_1 < \cdots < c_{N-1} < c_N = +\infty$ such that, setting

$$H_{k,j} = \{x \in \mathbb{V}; \langle x, \xi_k\rangle = c_j\}, \quad U := \mathbb{V} \setminus \bigsqcup_{k,j} H_{k,j},$$

the sheaf $F|_U$ is locally constant.

For $n = (n_1, \dots, n_l)$ with $0 \le n_k < N$, define

$$Z_n = \bigcap_{k=1}^{l} \{x; c_{n_k} \le \langle x, \xi_k\rangle < c_{1+n_k}\}, \quad \Omega_n = \bigcap_{k=1}^{l} \{x; c_{n_k} < \langle x, \xi_k\rangle < c_{1+n_k}\}.$$

Then $Z_n = \overline{\Omega_n + \gamma^a} \cap (\Omega_n + \gamma)$ and Z_n is γ-locally closed.

Since F_{Ω_n} is constant, F_{Z_n} is constant by Theorem 3.17.

Now we have $\mathbb{V} = \bigsqcup_{n\in[0,N-1]^l} Z_n$. Set $A = \{n \in [0, N-1]^l\,; F|_{Z_n} \not\simeq 0\}$. Then $\mathrm{supp}(F) = \bigcup_{n\in A} \overline{Z_n}$.

Lemma 3.22. *There exists $n \in A$ such that Z_n is open in* $\mathrm{supp}(F)$*.*

Proof of the lemma. We order the set of n's by $n \le n'$ if $n_j \le n'_j$ for all $j \in [1, \dots, l]$. Let n be a minimal element of A. Then Z_n is open in $\mathrm{supp}(F)$. Indeed,

$$\begin{aligned} Z_n = \mathrm{supp}(F) \cap Z_n &= \mathrm{supp}(F) \cap \bigcap_k \{x; c_{n_k} \le \langle x, \xi_k\rangle < c_{n_k+1}\} \\ &= \mathrm{supp}(F) \cap \bigcap_k \{x; \langle x, \xi_k\rangle < c_{n_k+1}\}. \end{aligned}$$

Note that the last equality is true since otherwise there exists $n' < n$ in A such that $\mathrm{supp}(F) \cap Z_{n'} \neq \varnothing$, and n would not be minimal. □

Now we can complete the proof of Theorem 3.21.

Let us take $n \in A$ such that $Z_n \subset \mathrm{supp}(F)$ is open in $\mathrm{supp}(F)$. Then we have an exact sequence

$$0 \to \mathbf{k}_{Z_n} \otimes F(Z_n) \to F \to F'' \to 0$$

and $\mathrm{supp}\, F'' \subset \mathrm{supp}(F) \setminus Z_n$. Then the proof goes by induction on $\#A$. □

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