The p-adic Analysis of Stirling Numbers via

Higher Order Bernoulli Numbers

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Abstract

In this paper, we use our previous study of the higher order Bernoulli numbers $B_n^{(l)}$ to investigate *p*-adic properties of Stirling numbers of the second kind S(n,k). For example we give a new greatly simplified proof of the formula $\nu_2(S(2^h,k)) = d_2(k) - 1$ if $1 \le k \le 2^h$, and generalize this result to arbitrary primes *p*. We also consider the Stirling numbers of the first kind s(n,k), with new results analogous to those for the Stirling numbers of the second kind. New mod *p* congruences for Stirling numbers of both kinds are also given.

Keywords: Stirling numbers, higher order Bernoulli numbers and polynomials, *p*-adic analysis, congruences. *MSC[2010]*: 11A07, 11B68, 11B73, 11S05.

1 Introduction

The starting point of our investigation was the remarkable formula conjectured by T. Lengyel [12] that

$$\nu_2(S(2^h, k)) = \sigma_2(k) - 1 \tag{1.1}$$

if $1 \le k \le 2^h$, where S(n,k) = Stirling number of the second kind, $\nu_2 =$ 2-adic valuation, and $\sigma_2 =$ base 2 digit sum = number of base 2 digits.

This formula was conjectured by Lengyel in 1994 and proven by S. De Wannemacker [7] in 2005. De Wannemacker's proof is quite involved. Furthermore the proof appears to be only suitable for the prime p = 2.

We were surprised to observe that De Wannemacker's Theorem is an immediate consequence of our previous study of higher order Bernoulli numbers and polynomials, primarily of the pole structure, which we developed in a series of papers in the nineties [1,2,3,4]. The machinery of these papers is valid for all primes p, and enables us to extend De Wannemacker's Theorem to odd primes p without additional effort. We also get a significant improvement of this theorem, which is new even for p = 2.

Although the connection between higher order Bernoulli numbers and Stirling numbers

$$S(n,k) = \binom{n}{k} B_{n-k}^{(-k)} \text{ and } s(n,k) = \binom{n-1}{k-1} B_{n-k}^{(n)}$$
(1.2)

is well known and has been noted in [2, 3], we have not previously pursued this application in any depth.

Subsequent to De Wannemacker's proof, Lengyel used the same methods to strengthen his original conjecture to prove [13] that

$$\nu_2(S(c2^h, k)) = \sigma_2(k) - 1 \tag{1.3}$$

 $\text{ if } c \geq 1 \text{ and } 1 \leq k \leq 2^h.$

We will not prove this stronger result in this paper, but we will prove it in a subsequent paper, along with its generalization to arbitrary primes. If $\sigma_2(c) > 1$, these Stirling numbers do not have the "minimum zero property," which is the main focus of this paper.

In addition, we have found greatly simplified proofs for other important results on Stirling numbers of the second kind, e.g. we have a nice proof of the theorem proven by O-Y. Chan and D. Manna [6, Th. 2.4] that the central Stirling number S(2k, k) is odd if and only if k is Fibbinary (i.e., the base 2 representation of k has no consecutive ones). We also present a generalization valid for all primes p, namely we determine when $p \nmid S(pk, k)$, using a simple analog of the Fibbinary property. We also give a new mod p congruence for S(pk, k), which contains additional information if $p \neq 2$.

We have abstracted the role of 2^h in De Wannemacker's Theorem to the "minimum zero property," and have used this concept to strengthen the result of T. Amdeberhan et al [5], conjectured in 2008 and proven by S. Hong et al [11, Th. 3.2] in 2012, that

$$\nu_2(S(2^h+1,k+1)) = \sigma_2(k) - 1 \quad \text{if } 1 \le k \le 2^h.$$
(1.4)

This is also generalized to all primes, as well as to all "minimum zero cases."

In all instances where we have been able to exactly determine $\nu_p(S(n,k))$, we have also been able to find simple explicit mod p congruences for $\epsilon_p(S(n,k)) = p^{-\nu_p(S(n,k))}S(n,k)$, which is the part of S(n,k) prime to p.

In our subsequent paper, we will also consider some cases which are not "minimum zero cases." We have tried to incorporate enough material in our background section to facilitate this extension.

We also consider the Stirling numbers of the first kind s(n, k). The "minimum zero property" now necessitates that $k \leq n < kp$ in addition to p-1 | n-k. We use this property to prove an analog of DeWannemacker's Theorem, that $\nu_2(s(n, 2^h)) = h - \sigma_2(n-1)$ if $2^h \leq n < 2^{h+1}$, and we generalize this result to arbitrary primes.

Similarly we have an analog of the Hong, Zhao and Zhao result for Stirling numbers of the first kind, that $\nu_2(s(n-1,2^h-1)) = \nu_2(s(n,2^h))$ if $2^h \leq n < 2^{h+1}$, which we generalize to all primes and to all "minimum zero cases."

We have organized this paper so that the new results on the p-adic analysis of Stirling numbers appear in the early sections, with the preliminaries and background in the later sections.

2 *p*-adic analysis of Stirling numbers of the second kind

Throughout this paper, p = arbitrary prime and $\nu_p =$ exponential *p*-adic valuation. We say that *r* has a zero of order *e* if $\nu_p(r) = e > 0$, or a pole of order *e* $\nu_p(r) = -e < 0$. If $\nu_p(r) = 0$ then r is a unit. If $r \neq 0$, then $\epsilon_p(r) = p^{-\nu_p(r)}r$ is the unit part of r.

The function $\sigma_p(n) = \text{sum of the base } p$ digits of n plays an important role in this paper. For p = 2, $\sigma_2(n) = \text{the number of base } 2$ digits in n, which is sometimes denoted by $d_2(n)$. Obviously $\sigma_p(pn) = \sigma_p(n)$.

The connection between Stirling numbers of the second kind and higher order Bernoulli numbers is given by

$$S(n,k) = \binom{n}{k} B_{n-k}^{(-k)}$$

$$(2.1)$$

Using the standard formula (4.3) for $\nu_p\binom{n}{m}$, the estimate of Lemma 5.1 for $\nu_p(B_n^{(l)})$ now translates to

Lemma 2.1. $\nu(S(n,k)) \ge \lceil (\sigma(k) - \sigma(n))/(p-1) \rceil$ if $n \ge k$.

This lemma was proven for p = 2 by De Wannemacker ([7, Th. 3]). The proof he gave is non-trivial, involving Stirling number identities and induction, and doesn't appear to extend to odd primes. Lengyel has proven an estimate for odd primes [13, Theorem 5] that is less precise and never sharp. Note that since $\nu(S(n,k)) \in \mathbb{N}$, the estimate in this lemma is equivalent to the estimate $\nu(S(n,k)) \ge (\sigma(k) - \sigma(n))/(p-1).$

We define the minimum zero case for S(n,k) as one where the general inequality noted at the end of the preceding paragraph is an equality, namely

$$S(n,k)$$
 is a minimum zero case if $\nu(S(n,k)) = (\sigma(k) - \sigma(n))/(p-1)$. (2.2)

The concept of minimum zero directly relates to the concept of maximum pole for higher order Bernoulli polynomials (5.4), which we introduced in [4]. Combining these definitions with the congruence in Proposition 5.1 for the higher order Bernoulli numbers, we get the following theorem, which establishes a simple, effective binomial coefficient criterion.

Theorem 2.1. The following are equivalent:

- (i) S(n,k) is a minimum zero case,
- (ii) $B_{n-k}^{(-k)}(x)$ has maximum pole.

(iii)
$$r = (n-k)/(p-1) \in \mathbb{N}$$
 and $p \nmid \binom{-(n+1)}{r}$, i.e., $p \nmid \binom{n+r}{r}$.

Furthermore, in the minimum zero case, we have

$$\epsilon(S(n,k)) \equiv (-1)^r \epsilon(n!/k!) \binom{n+r}{r} \mod p.$$

Remarks. Since the classical theorems are all p = 2 theorems, it is worth noting what this theorem says for p = 2. In this case, (iii) simply says $\binom{n+r}{r} = \binom{n+n-k}{n}$ is odd, i.e., that n and n-k have no common base 2 digits.

Corollary 2.1. S(n,k) is a minimum zero case if and only if S(np,kp) is a minimum zero case. Furthermore, if S(n,k) is a minimum zero case, then $\nu(S(n,k) = \nu(S(np,kp)) \text{ and } \epsilon(S(n,k)) \equiv \epsilon(S(np,kp)) \mod p.$

Corollary 2.2. With the same notations as in the theorem, if $\sigma(k) = \sigma(n)$ then

$$S(n,k) \equiv (-1)^r \epsilon (n!/k!) \binom{n+r}{r} \mod p$$

Remark. This corollary implies that if $\sigma(k) = \sigma(n)$ and r = (n-k)/(p-1), then p|S(n,k) if and only if $p|\binom{n+r}{r}$, i.e. if and only if S(n,k) is not a minimum zero case. We can now easily prove an analog of De Wannemacker's Theorem valid for all primes p. The following theorem has De Wannemacker's result as the special case for p = 2. Even for p = 2, the proof is much simpler than any proofs in the literature which we know.

Theorem 2.2. Let $n = ap^h$ with $1 \le a \le p-1$ and assume that $1 \le k \le n$ and p-1|n-k. Then S(n,k) is a minimum zero case and

$$\nu(S(n,k)) = \frac{\sigma(k) - \sigma(n)}{p-1} = \frac{\sigma(k) - a}{p-1}.$$

Proof. If r = (n-k)/(p-1) then $r < p^h$, so $p \nmid \binom{n+r}{r}$ by the Lucas Theorem, and so we have the minimum zero case by the preceding theorem, giving the equations of Theorem 2.2.

Corollary 2.3. With the same assumptions, we have

$$\epsilon(S(n,k)) \equiv (-1)^{r+ah} a! / \epsilon(k!) \mod p.$$

Proof. We have the minimum zero case by Theorem 2.2, and $\binom{n+r}{r} \equiv 1 \mod p$ since r and n have disjoint base p representations. Finally, the standard Lemma 4.1 congruence $\epsilon((ap^h)!) \equiv (-1)^{ah}a! \mod p$ and the congruence in Theorem 2.1 give the desired result.

The next theorem shows that the minimum zero Stirling numbers of the second kind have certain invariance properties.

Theorem 2.3. Let S(n,k) be a minimum zero case and $0 \le b < \min\{p^{\nu(k)}, p^{\nu(n)}\}$. Let n' = n + b and k' = k + b. Then S(n',k') is a minimum zero case and

(i)
$$\nu(S(n',k')) = \nu(S(n,k)).$$

(*ii*)
$$\epsilon(S(n',k')) \equiv \epsilon(S(n,k)) \mod p$$
.

Proof. First observe that b is a common bottom segment of the base p representations of n' and k', and n and k are the respective top segments. We have n' - k' = n - k, so r = (n' - k')/(p - 1) = (n - k)/(p - 1). Since the base p representations of n, k, and n + r are all disjoint from the representation of b, we have $\binom{n'+r}{r} \equiv \binom{n+r}{r} \mod p$ by the Lucas congruence. Hence, S(n', k') is also a minimum zero case. Since $\sigma(k') - \sigma(n') = \sigma(k) - \sigma(n)$, part (i) is now established.

For part (ii) consider

$$(n'!/k'!)/(n!/k!) = \frac{n'!}{n!b!} / \frac{k'!}{k!b!} = {n' \choose n} / {k' \choose k}.$$

But now the disjointness of n and b implies that $\binom{n'}{n} \equiv 1 \mod p$, and similarly the disjointness of k and b implies that $\binom{k'}{k} \equiv 1 \mod p$. Hence $\epsilon(n'!/k'!) \equiv \epsilon(n!/k!) \mod p$, so by the congruence in Theorem 2.1, we have $\epsilon(S(n',k')) \equiv \epsilon(S(n,k)) \mod p$.

The following corollary, which strengthens DeWannemacker's Theorem, is a special case of Theorem 2.3.

Corollary 2.4. Let $n = ap^h$ with $1 \le a \le p - 1$, and assume that $1 \le k \le n$ and p - 1|n - k. Let n' = n + b and k' = k + b, where $0 \le b < p^{\nu(k)}$. Then S(n',k') is a minimum zero case and

(i)
$$\nu(S(n,k)) = \nu(S(n',k')) = (\sigma(k) - a)/(p-1).$$

(*ii*) $\epsilon(S(n,k)) \equiv \epsilon(S(n',k')) \mod p$.

Next we consider the central Stirling numbers S(2k, k), which are close relatives of the Catalan numbers, and are significant for combinatorics. In [6, Th. 2.4], O-Y Chan and D. Manna showed in a non-trivial way that S(2k, k) is odd if and only if k is Fibbinary, i.e., if the base 2 representation of k has no consecutive ones. We give a short proof of this theorem, generalized to all primes p. The proof given by Chan and Manna for p = 2, considers many parity cases.

To generalize to arbitrary primes p, define S(pk, k) as a p-central Stirling number and k as p-Fibbinary if the sum of any two consecutive digits of the base p representation of k is at most p - 1. These concepts clearly specialize to central Stirling number and Fibbinary number for p = 2.

Theorem 2.4. $p \nmid S(pk, k)$ if and only if k is p-Fibbinary.

Proof. Since if n = pk, then r = (n - k)/(p - 1) = k and $\sigma_p(n) = \sigma_p(k)$. Hence S(pk, k) is a minimum zero case iff $\nu(S(pk, k)) = 0$, so $p \nmid S(pk, k)$ iff $p \nmid \binom{pk+k}{k}$. But by Lucas' Theorem this is equivalent to the *p*-Fibbinary condition for *k*.

Corollary 2.5. If $k = \sum_{i} a_{i}p^{i}$ is the base *p* representation, then

$$S(pk,k) \equiv \prod_{i} {a_i + a_{i+1} \choose a_i} \mod p.$$

Proof. This follows immediately from the Lucas congruence for $\binom{pk+k}{k}$, with $\epsilon((pk)!/k!) \equiv (-1)^k \mod p$.

We now turn to a result conjectured by T. Amdeberhan et al in [5] and

proven by Hong et al ([8, Th. 3.2]) several years later, namely

$$\nu_2(S(2^h+1,k+1)) = \sigma_2(k) - 1. \tag{2.3}$$

We give a proof of this result, which is more general since it works for all primes p, and replaces the assumption that $n = 2^h$ by the weaker assumption that S(n,k) is a minimum zero case. The proof is also shorter and we believe more instructive than the one given for the special case p = 2 in [8].

Theorem 2.5. Suppose
$$S(n,k)$$
 is a minimum zero case. Then
 $\nu(S(n+1,k+1)) = \nu(S(n,k))$ and $\epsilon(S(n+1,k+1)) \equiv \epsilon(S(n,k)) \mod p$.

Proof. By the standard recursion for Stirling numbers of the second kind, we have

$$S(n+1, k+1) = S(n, k) + (k+1)S(n, k+1)$$

Hence it will suffice for the first assertion to show that $\nu((k+1)S(n, k+1)) > \nu(S(n, k))$, by a standard property of valuations. By Lemma 2.1, $\nu(S(n, k+1)) \ge \lceil (\sigma(k+1) - \sigma(n))/(p-1) \rceil$, so by Lemma 4.2 we have $\nu(k+1) + \nu(S(n, k+1)) \ge \lceil (\nu(k+1)(p-1) + \sigma(k+1) - \sigma(n))/(p-1) \rceil$

$$= \left\lceil (1 + \sigma(k) - \sigma(n))/(p-1) \right\rceil.$$

But $p-1|(\sigma(k) - \sigma(n))$ by assumption, so this number equals $1 + \nu(S(n,k))$. The proof of the congruence now follows from (4.10).

Note that simple examples show that S(n+1, k+1) may not be a minimum zero case in Theorem 2.5. For example, for p = 2, we have S(5, 3) is a minimum zero case, since n = 5 and n - k = 2 have no common base 2 digit, i.e., $2 \nmid \binom{n+r}{r}$. However, S(6, 4) is not a minimum zero case since now n + r = 6 + 2, which does have a base 2 carry.

3 *p*-adic analysis of Stirling numbers of the first kind

We give some results for Stirling numbers of the first kind s(n,k), which are analogous to the results for Stirling numbers of the second kind. We believe they are all new.

We now have the connecting formula

$$s(n,k) = {\binom{n-1}{k-1}} B_{n-k}^{(n)}$$
(3.1)

Our first result, which is analogous to Lemma 2.1, and has essentially the same proof, is the following.

Lemma 3.1. $\nu_p(s(n,k)) \ge \lceil \frac{\sigma(k-1) - \sigma(n-1)}{p-1} \rceil$.

Remarks. In [14] Lengyel gives several striking estimates for the *p*-adic values of s(n,k), including $\nu_p(s(n,k)) \to \infty$ as $n \to \infty$ for *k* fixed. Our methods do not suffice to yield these results. He also considers the case where n-k is fixed, and in this case our estimate compares well with his.

References [11, 15] extend the *p*-adic analysis of Stirling numbers of the first kind, with [11] making heavy use of the Newton polygon of the horizontal generating function $(x)_n$.

We can define the minimum zero case for s(n, k) by

$$\nu_p(s(n,k)) = (\sigma(k-1) - \sigma(n-1)))/(p-1).$$
(3.2)

Since $\nu_p(s(n,k)) \in \mathbb{N}$, this is equivalent to sharpness of the estimate in Lemma 3.1 and to the maximum pole case for $B_{n-k}^{(n)}(x)$, i.e. to $\nu(B_{n-k}^{(n)}) = -\sigma(n-k)/(p-1)$. It is also equivalent to $p \nmid {k-1 \choose r}$, where $r = (n-k)/(p-1) \in \mathbb{N}$.

This last formula points to an essential difference between the Stirling numbers of the first and second kinds, namely the minimum zero case here requires that $r \leq k - 1$ since $p \nmid \binom{k-1}{r}$, so $k \leq n < kp$ is a necessary condition for the Stirling number s(n,k) to be a minimum zero case. There is nothing comparable for Stirling numbers of the second kind.

We get the following theorem, essentially by definition.

Theorem 3.1. If r = (n - k)/(p - 1), then in the minimum zero case

$$\nu(s(n,k)) = (\sigma(k-1) - \sigma(n-1))/(p-1)$$

and

$$\epsilon(s(n,k)) \equiv \epsilon((n-1)!/(k-1)!)\binom{k-1}{r} \mod p.$$

Corollary 3.1. s(n,k) is a minimum zero case if and only if s(np,kp) is a minimum zero case. Furthermore, if s(n,k) is a minimum zero case, then $\epsilon(s(n,k)) \equiv \epsilon(s(np,kp)) \mod p$.

We have a theorem for Stirling numbers of the first kind analogous to De Wannemacker's Theorem, generalized to arbitrary primes. **Theorem 3.2.** Let k have a single base p digit, i.e. $k = ap^h$ with $1 \le a \le p-1$. Then the minimum zero case holds for all s(n,k) with $k \le n < kp$ such that p-1|n-k.

Proof. If r = (n-k)/(p-1) then clearly $r \le k-1$ since n-k < k(p-1)which implies by Lucas's Theorem that $p \nmid \binom{k-1}{r}$, since $k-1 = (a-1)p^h + (p-1)p^{h-1} + \dots + (p-1)$.

Corollary 3.2. With the same assumptions and notations

$$\nu(s(n, ap^{h})) = \frac{a - 1 - \sigma(n - 1)}{p - 1} + h.$$

and

$$\epsilon(s(n,ap^h)) \equiv (-1)^{ah+r-r_h} \frac{\epsilon((n-1)!)}{(a-1)!} \binom{a-1}{r_h} \mod p$$

where r_h is the coefficient of p^h in the base p representation of r.

Proof. $\sigma(k-1)$ is given in the above proof, namely $\sigma(k-1) = a - 1 + h(p-1)$, which gives the first part. For the congruence part, use Lemma 4.1 applied to $\epsilon(k!)$ with (k-1)! = k!/k, together with the Lucas congruence with the last line of the preceding proof, and the fact that $\binom{p-1}{r_i} \equiv \binom{-1}{r_i} = (-1)^{r_i} \mod p$, for each digit r_i of r, together with $\sigma(r) \equiv r \mod p - 1$, so $\sigma(r)$ and r have the same parity if $p \neq 2$.

Remark. The presence of h in $\nu(s(n, ap^h))$ is different from the situation for $\nu(S(ap^h, k))$, and illustrates that the Stirling numbers of the first and second kind have different character.

The special case for p = 2 is particularly simple and is worth noting.

Corollary 3.3. Let $k = 2^h$. Then if $2^h \le n < 2^{h+1}$, we have

$$\nu_2(s(n,k)) = h - \sigma_2(n-1).$$

We have an invariance property for Stirling numbers of the first kind analogous to the Stirling numbers of the second kind. The proof is essentially similar, and we will omit it.

Theorem 3.3. Let s(n,k) be a minimum zero case. Assume that $p^{\nu(t)} > n$. Let n' = t + n and k' = t + k. Then s(n',k') is a minimum zero case and

- (i) $\nu(s(n',k')) = \nu(s(n,k)).$
- (*ii*) $\epsilon(s(n', k')) \equiv \epsilon(s(n, k)) \mod p$.

In this case t is the common top segment of n' and k', and n and k are the respective bottom segments.

The special case when $k = ap^h$ with $1 \le a \le p-1$ and $pk > n \ge k$ and p-1|n-k, has the same invariance, which is a strengthening of the analog of DeWannemacker's Theorem for Stirling numbers of the first kind.

Finally we prove an analog of the Hong, Zhao and Zhao result for Stirling numbers of the first kind, also valid for all primes p, and generalized to minimum zero cases.

Theorem 3.4. Let s(n,k) be a minimum zero case. Then

$$\begin{split} \nu(s(n-1,k-1)) &= \nu(s(n,k)) \quad and \\ \epsilon(s(n-1,k-1)) &\equiv \epsilon(s(n,k)) \mod p. \end{split}$$

Proof. This is entirely analogous to the previous proof for the Stirling numbers of the second kind, now using the basic recursion

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k).$$

The rest of the proof is essentially the same as in Theorem 2.5, so we omit the details.

Observe that s(n-1, k-1) may not be a minimum zero case.

4 *p*-adic preliminaries

We now collect, for reference purposes, some useful standard and elementary *p*-adic results.

$$n \equiv \sigma_p(n) \mod (p-1)$$
, i.e. $p-1|(n-\sigma_p(n))$. (4.1)

This paper makes heavy use of standard results on factorials and binomial coefficients, which we now summarize:

$$\nu_p(n!) = (n - \sigma_p(n))/(p - 1). \tag{4.2}$$

$$\nu_p \binom{n}{m} = (\sigma_p(m) + \sigma_p(n-m) - \sigma_p(n))/(p-1).$$
(4.3)

Remark. From (4.2) and (4.3), it immediately follows that if p-1|n-k then $\binom{n}{k} = \epsilon (n!/k!)p^{(n-k)/(p-1)}p^{(\sigma(k)-\sigma(n))/(p-1)}/(n-k)!.$

It also follows that $\nu_p \binom{n}{m} =$ number of carries for the base p addition of mand n - m, whence we have the Lucas Theorem that

$$p \nmid \binom{n}{m}$$
 iff $n_i \ge m_i$ for all base p digits. (4.4)

In fact, the Lucas congruence says

$$\binom{n}{m} \equiv \prod_{i} \binom{n_i}{m_i} \mod p. \tag{4.5}$$

An important special case of the Lucas congruence is that if r and n have disjoint base p representations then

$$\binom{n+r}{r} \equiv 1 \mod p. \tag{4.6}$$

There is a more subtle congruence discovered by H. Anton in 1869 that if $\nu_p \binom{n}{m} = e$ and r = n - m then

$$\frac{(-1)^e}{p^e} \binom{n}{m} \equiv \prod \frac{n_i!}{m_i! r_i!} \mod p \tag{4.7}$$

where n_i, m_i, r_i are the base p digits of n, m, r respectively. This is a mod p congruence for $\epsilon_p \binom{n}{m}$, up to the sign $(-1)^e$.

Since the base p digits of np are the same as those of n shifted one place to the left, it follows immediately from the Lucas and Anton congruences that

$$\nu_p \binom{np}{mp} = \nu_p \binom{n}{m} \text{ and } \epsilon_p \binom{np}{mp} \equiv \epsilon_p \binom{n}{m} \mod p.$$
(4.8)

If p is understood by the context, we may suppress the p in our notations, i.e. use ν, σ, ϵ instead of $\nu_p, \sigma_p, \epsilon_p$ respectively.

Finally, we make frequent use of the formula

$$\binom{-a}{r} = (-1)^r \binom{a+r-1}{r}, \text{ i.e. } \binom{-(n+1)}{r} = (-1)^r \binom{n+r}{r}.$$
 (4.9)

By basic properties of valuations, it is clear that $\epsilon(ab) = \epsilon(a)\epsilon(b)$, and

if
$$\nu(a) < \nu(b)$$
, then $\nu(a+b) = \nu(a)$ and $\epsilon(a+b) \equiv \epsilon(a) \mod p$. (4.10)

Remark. It is worth noting that $c/d \equiv 1 \mod p$ if and only if $\nu(c) = \nu(d)$ and $\epsilon(c) \equiv \epsilon(d) \mod p$.

These observations lead immediately to the following lemma. We omit the proof, which is a straightforward generalization of Wilson's Theorem and proof.

Lemma 4.1. Assume $1 \le a \le p-1$. Then

$$\epsilon((ap^h)!) \equiv (-1)^{ah}a! \mod p.$$

It is also well-known and easy to prove that

$$\epsilon_p((pk)!) \equiv (-1)^k \epsilon_p(k!) \mod p. \tag{4.11}$$

Finally we conclude with a useful, elementary lemma.

Lemma 4.2. $\sigma_p(k+1) = \sigma_p(k) + 1 - (p-1)u$ where $u = \nu_p(k+1) = number$ of consecutive digits at the bottom of the base p representation of k which are equal to p - 1.

Proof. The effect of adding one to k is to replace the bottom u digits by zeros and increase the next digit by one.

5 Background on Stirling numbers and higher order Bernoulli numbers and polynomials

If $n \in \mathbb{N}$ and $l \in \mathbb{Z}$, the Bernoulli polynomials $B_n^{(l)}(x)$ of order l and degree n are defined by

$$\left(\frac{t}{e^t - 1}\right)^l e^{tx} = \sum_{n=0}^{\infty} B_n^{(l)}(x) \frac{t^n}{n!}.$$
(5.1)

The higher order Bernoulli numbers are the constant terms $B_n^{(l)} = B_n^{(l)}(0)$. The polynomial $B_n^{(l)}(x) \in \mathbb{Q}[x]$ is monic with degree n.

The Stirling numbers of the first kind s(n,k) can be defined by

$$(x)_n = \sum_{k=1}^{\infty} s(n,k) x^k \tag{5.2}$$

where $(x)_n = x(x-1)\cdots(x-(n-1)) = n!\binom{x}{n}$.

The s(n,k) are integers and the sign of s(n,k) is $(-1)^{n-k}$. The unsigned Stirling numbers |s(n,k)| count the number of *n*-permutations with *k* cycles.

The Stirling numbers of the second kind S(n,k) can be defined combinatorially by

$$S(n,k) =$$
 number of partitions of an *n*-set into *k* subsets. (5.3)

Remarks. We showed in [1] how to precisely locate the successively increasing order poles of the coefficients of $B_n^{(l)}(x)$, arranged from top degree down, which we call the poles of $B_n^{(l)}(x)$, and we showed that these poles have a remarkably regular pattern. The salient features of the pole pattern are that the first pole has order 1, the next bigger pole has order 2, etc., and that all these first occurrences appear in codegrees i, where p - 1|i and $p \nmid {n \choose i}$. Subsequently in [4] we interpreted these results in terms of the Newton polygon of $B_n^{(l)}(x)$ and gave a precise, algorithmic, description of the descending portion of this Newton polygon, which summarizes the pole pattern.

The following lemma was proven in [1], and by a different method, also in [3].

Lemma 5.1.

$$\nu(B_n^{(l)}) \ge -\lfloor \sigma(n)/(p-1) \rfloor.$$

We were also able to prove some general congruences for the higher order Bernoulli numbers $B_n^{(l)}$ in [3]. We will generally assume that p - 1|n (or p - 1|n - k for the applications to Stirling numbers S(n, k) and s(n, k)), since that is simplest. The following proposition is the special case of [3, Th. 1] where p - 1|n, with some notational changes.

Proposition 5.1. Suppose p - 1|n and let r = n/(p - 1). Then

$$(-1)^n p^r B_n^{(l)} / n! \equiv (-1)^r \binom{n+r-l}{r} \mod p.$$

Note that since p - 1|n, we can omit the factor $(-1)^n$ from the preceding congruence.

We introduced the concept of maximum pole in [4] for $B_n^{(l)}(x)$ by

$$\nu_p(B_n^{(l)}) = -\sigma(n)/(p-1), \tag{5.4}$$

which is the theoretical minimum value and obviously is only attainable if p-1|n. Observe that if p-1|n, this is equivalent to sharpness of the estimate in Lemma 5.1. In the maximum pole case, $B_n^{(l)}(x)$ has a pole if n > 0, which is the biggest pole for all the coefficients of $B_n^{(l)}(x)$. This occurs when the Newton polynomial of $B_n^{(l)}(x)$ is strictly decreasing. By the preceding analysis, there is a maximum pole iff

$$r = n/(p-1) \in \mathbb{N}$$
 and $p \nmid \binom{l-n-1}{r}$. (5.5)

In the maximum pole case, we have the nontrivial congruences

$$p^r B_n^{(l)}/n! \equiv (-1)^n \binom{l-n-1}{r} \equiv (-1)^r \binom{n-l+r}{r} \mod p.$$
 (5.6)

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