RECONSTRUCTION OF GENERAL ELLIPTIC K3 SURFACES FROM THEIR GROMOV–HAUSDORFF LIMITS

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Abstract. We show that a general elliptic K3 surface with a section is determined uniquely by its discriminant, which is a configuration of 24 points on the projective line. It follows that a general elliptic K3 surface with a section can be reconstructed from its Gromov–Hausdorff limit as the volume of the fiber goes to zero.

1. Introduction

Let $f, g \in \mathbb{C}[z, w]$ be homogeneous polynomials of degree 8 and 12 respectively, and define a polynomial h of degree 24 by

$$h = f^3 + g^2. \tag{1.1}$$

We show the following in this paper:

Theorem 1.1. *If f and g are general, then the decomposition of h into the sum of a cube and a square is unique, up to the obvious ambiguity of multiplication by a cubic and a square root of unity.*

Note that the corresponding problem in number theory has non-unique solutions in general. There are three solutions

$$1^3 + 4^2 = 2^3 + 3^2, \qquad 1^3 + 8^2 = 4^3 + 1^2, \qquad 2^3 + 9^2 = 4^3 + 5^2 \tag{1.2}$$

for positive integers less than 100, and

$$1^3 + 32^2 = 4^3 + 31^2 = 5^3 + 30^2 = 10^3 + 5^2 \tag{1.3}$$

is the smallest integer which can be written as the sum of a cube and a square in more than two ways. An elliptic curve of the form

$$y^2 = x^3 + n \tag{1.4}$$

for a non-zero integer n is known as a *Mordell curve*, and has been studied for many years.

The decomposition over the function field is not unique if f and g are not general. For example, for homogeneous polynomials u and v of degree 4, one has

$$4(uv)^3 + (uv(u - v))^2 = 0^3 + (uv(u + v))^2. \tag{1.5}$$

An elliptic K3 surface with a section has a Weierstrass model

$$y^2 = 4x^3 - g_8(z, w)x - g_{12}(z, w) \tag{1.6}$$

in $\mathbb{P}(4, 6, 1, 1)$, where $g_8(z, w)$ and $g_{12}(z, w)$ are homogeneous polynomials in z and w of degree 8 and 12 respectively. The discriminant is given by

$$\Delta = g_8^3 - 27g_{12}^2, \tag{1.7}$$

which is a homogeneous polynomial of degree 24. The decomposition problem asks if g_8 and g_{12} can be reconstructed from Δ. Our interest in this problem comes from mirror symmetry. It is shown in [GW00, GTZ16] that a sequence of Kähler–Einstein metrics on an elliptic K3 surface with a fixed diameter converges in the Gromov–Hausdorff topology to a sphere with a Monge–Ampère structure with singularities, if the volume of the fiber goes to zero. Since the complement of the discriminant of the elliptic K3 surface as a punctured Riemann sphere can be reconstructed as the conformal class of the smooth part of the limit metric, Theorem 1.1 allows the reconstruction of (the Jacobian of) a general elliptic K3 surface from the Gromov–Hausdorff limit.

This paper is organized as follows: In Section 2, we associate an auxiliary elliptic surface X of general type with a polynomial $h = f^3 + g^2 \in \mathbb{C}[z,w]$ of degree 24. In Section 3, we show that the Picard number of X is 4 if f and g are very general. In Section 4, we give a proof of Theorem 1.1. In Section 5, we discuss the relationship with mirror symmetry for K3 surfaces.

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2. An elliptic surface

Let $h \in \mathbb{C}[x,y]$ be the homogeneous polynomial of degree 24 defined by homogeneous polynomials $f, g \in \mathbb{C}[z,w]$ of degree 8 and 12 respectively as in (1.1). Let further $\overline{X}$ be the hypersurface of degree 24 defined by

$$y^2 = -x^3 + h \tag{2.1}$$

in the weighted projective space $\mathbb{P}(8,12,1,1) = \operatorname{Proj} \mathbb{C}[x,y,z,w]$, where the variables x, y, z, and w are of degree 8, 12, 1, and 1 respectively. The variety $\overline{X}$ has a quotient singularity of type $\mathbb{C}^2 \big/ \big\langle \frac{1}{4}(1,1) \big\rangle$ at $[x:y:z:w] = [-1:1:0:0]$ coming from the ambient space. The exceptional divisor σ_0 of the minimal resolution $X \to \overline{X}$ is a (-4)-curve. The zw-projection $\overline{\pi} \colon \overline{X} \dashrightarrow \mathbb{P}^1$ induces an elliptic fibration $\pi \colon X \to \mathbb{P}^1$ such that σ_0 is a section.

Let e be a smooth fiber of π, which is the total transform of a hyperplane section of $\overline{X}$. The sublattice U of $\operatorname{Pic} X$ generated by e and σ_0 is the hyperbolic unimodular lattice of rank 2.

Note that $K_X = \mathcal{O}_X(2e)$. Any section σ of π has self-intersection -4 since

$$\mathcal{N}_{\sigma/X} = \mathcal{O}_\sigma\left(K_\sigma - K_X|_\sigma\right) = \mathcal{O}_\sigma(-2-2) = \mathcal{O}_\sigma(-4). \tag{2.2}$$

If σ does not intersect σ_0, then one has $\sigma^2 = -4$, $\sigma \cdot \sigma_0 = 0$, and $\sigma \cdot e = 1$, so that

$$\tau(\sigma) := \sigma - \sigma_0 - 4e \tag{2.3}$$

satisfies $\tau(\sigma) \perp U$ and $\tau(\sigma)^2 = -8$.

The elliptic fibration π has six sections

$$\begin{aligned} \sigma_1 &: (x,y) = (f,g), \\ \sigma_2 &: (x,y) = (\zeta_3 f, g), \\ \sigma_3 &: (x,y) = (\zeta_3^2 f, g), \\ \sigma_4 &: (x,y) = (f, -g), \\ \sigma_5 &: (x,y) = (\zeta_3 f, -g), \\ \sigma_6 &: (x,y) = (\zeta_3^2 f, -g) \end{aligned} \tag{2.4}$$

disjoint from σ_0, where $\zeta_k := \exp\left(2\pi\sqrt{-1}/k\right)$ for a positive integer k. Let M be the sublattice of $\operatorname{Pic} X$ generated by $\tau_i := \tau(\sigma_i)$ for $i = 1, \ldots, 6$.

Lemma 2.1. *The pair (τ_1, τ_2) is an ordered basis of M with the Gram matrix $\begin{pmatrix} -8 & 4 \\ 4 & -8 \end{pmatrix}$.*

Proof. It follows from

$$\sigma_1 \cdot \sigma_4 = \deg g = 12 \tag{2.5}$$

that

$$(\tau_1 + \tau_4)^2 = 0 \tag{2.6}$$

and hence

$$\tau_1 + \tau_4 = 0 \tag{2.7}$$

since $U^\perp \subset \operatorname{Pic} X$ is negative definite by the Hodge index theorem. Similarly, it follows from

$$\sigma_1 \cdot \sigma_2 = \sigma_2 \cdot \sigma_3 = \sigma_1 \cdot \sigma_3 = \deg f = 8 \tag{2.8}$$

that

$$(\tau_1 + \tau_2 + \tau_3)^2 = 0 \tag{2.9}$$

and hence

$$\tau_1 + \tau_2 + \tau_3 = 0. \tag{2.10}$$

One also has $\tau_2 + \tau_5 = \tau_3 + \tau_6 = \tau_4 + \tau_5 + \tau_6 = 0$, so that $\{\tau_1, \tau_2\}$ is a basis of M. It also follows from (2.8) that

$$\tau_1 \cdot \tau_2 = 4, \tag{2.11}$$

and Lemma 2.1 is proved. □

3. The Picard number of X

Let $\rho(X)$ be the Picard number of X. We prove the following in this section:

Proposition 3.1. *For very general f and g, one has $\rho(X) = 4$.*

Proof. Let $\mathcal{Y}$ be the family of elliptic surfaces over $\operatorname{Spec} \mathbb{C}[a,b]$ obtained as the simultaneous minimal resolution of the quotient singularity coming from the ambient space of the family of hypersurfaces of $\mathbb{P}(8,12,1,1)$ defined by (1.1), (2.1), and

$$f = aw^8, \quad g = w\left(z^{11} + bw^{11}\right) \tag{3.1}$$

as in the beginning of Section 2. One has

$$h = f^3 + g^2 = w^2\left(z^{11} - a'w^{11}\right)\left(z^{11} - b'w^{11}\right), \tag{3.2}$$

where $a', b' \in \mathbb{C}$ are defined by

$$a' + b' = -2b, \quad a'b' = a^3 + b^2. \tag{3.3}$$

The discriminant of the elliptic fibration is given by h^2, which is a configuration of (not necessarily distinct) 24 double points. The family $\mathcal{Y}$ is not isotrivial, since the configuration depends on the parameter, even after quotienting out the $\mathrm{PGL}(2, \mathbb{C})$-action. A general member Y of this family has a singular fiber of Kodaira type IV at $[z : w] = [1 : 0]$, consisting of three lines meeting at one point, so that one has

$$\rho(Y) \geq 4 + 2 = 6. \tag{3.4}$$

Since the discriminant is of degree 48, the topological Euler number of Y is 48, which implies that the second Betti number of Y is given by $48 - 2 = 46$. Since Y is a weighted projective hypersurface, the Griffiths–Dwork method (see e.g. [Dol82]) shows

$$H^{2,0}(Y) = \bigoplus_{i=0}^{2} \mathbb{C}\Omega_i, \tag{3.5}$$

where

$$\Omega_i = \mathrm{Res}_Y \left(z^i w^{2-i} \frac{8x dy \wedge dz \wedge dw - 12y dx \wedge dz \wedge dw + \cdots}{x^3 + y^2 - h} \right) \tag{3.6}$$

for $i = 0, 1, 2$. Hence the $\mathbb{Z}/33\mathbb{Z}$-action generated by

$$\alpha \colon [x : y : z : w] \mapsto [\zeta_3 x : y : z : w], \tag{3.7}$$

$$\beta \colon [x : y : z : w] \mapsto [x : y : \zeta_{11} z : w] \tag{3.8}$$

satisfies

$$\alpha^* \Omega_i = \zeta_3 \Omega_i, \quad \beta^* \Omega_i = \zeta_{11}^{i+1} \Omega_i. \tag{3.9}$$

Let V be the irreducible representation of $\mathbb{Z}/33\mathbb{Z}$ over $\mathbb{Q}$ with eigenvalues $\left\{ \zeta_{33}^j \right\}_{(j,33)=1}$. One has $\dim V = \phi(33) = 20$, where ϕ is Euler's totient function. It follows from (3.9) that $H^2(Y, \mathbb{Q})$ contains $V^{\oplus k}$ for some $k \geq 1$. If $k = 1$, then $H^{2,0}(Y)$ does not depend on the parameters a and b, which contradicts the non-isotriviality of the family $\mathcal{Y}$ and the local Torelli theorem for elliptic surfaces [Kiĭ78, Cha84, Sai83, Klo04]. Hence one has $k = 2$, which implies

$$\rho(Y) \leq 46 - \dim V^{\oplus 2} = 6, \tag{3.10}$$

so that $\rho(Y) = 6$ for very general Y.

Now let X be the elliptic surface defined by very general f and g, so that any singular fiber is of type II and $\rho(X) \leq 6$. (For example, if we put $f = z^8$, $g = w^{12}$, then we have $h = z^{24} + w^{24}$. This implies that all roots of h are pairwise distinct for general f and g.) Assume $\rho(X) = 6$ for a contradiction. Then there is a deformation of X to Y such that $H^{2,0}(X) \subset V^{\oplus 2} \otimes \mathbb{C}$ under the induced identification $H^2(X; \mathbb{Z}) = H^2(Y; \mathbb{Z})$, which implies $\mathrm{Pic}\, X = \mathrm{Pic}\, Y$. Let d be a (-2)-vector in $\mathrm{Pic}\, Y$ coming from the singular fiber of type IV. The Riemann–Roch theorem shows

$$\begin{aligned}
\chi(\mathcal{O}_X(d)) &:= h^0(\mathcal{O}_X(d)) - h^1(\mathcal{O}_X(d)) + h^2(\mathcal{O}_X(d)) \qquad (3.11)\\
&= \frac{1}{2}d.(d-K_X) + \frac{1}{12}(K_X^2 + c_2(X)) \qquad (3.12)\\
&= \frac{1}{2}d^2 + \frac{1}{12}c_2(X) \qquad (3.13)\\
&= -1 + \frac{48}{12} \qquad (3.14)\\
&= 3, \qquad (3.15)
\end{aligned}$$

so that either d or $K_X - d = -d + 2e$ is effective. Since d is orthogonal to e and every singular fiber of X is irreducible, the divisor d must be a multiple of e, which contradicts $d^2 = -2$ and $e^2 = 0$.

Next assume for a contradiction that $\rho(X) = 5$. Then $\operatorname{Pic} X \otimes \mathbb{Q}$ is generated over $\mathbb{Q}$ by U, M and δ with $\alpha^*\delta = \delta$. By the theory of elliptic surfaces [Shi90], the Mordell–Weil group of X is isomorphic to

(3.16)

$$\operatorname{Pic}(X)/(U + (\text{the lattice generated by irreducible components of singular fibers})).$$

Hence the class δ corresponds to an α-invariant section different from σ_0. Such a section is given by $x = 0$ and $y = \psi$ for a homogeneous polynomial ψ in z and w of degree 12 satisfying $\psi^2 = h$. The existence of a square root ψ of h contradicts the assumption that f and g are very general, and $\rho(X) = 4$ is proved. ☐

Remark 3.2. In the published version of this paper, the authors have cited only [Kiĭ78, Cha84, Sai83] for the local Torelli theorem for elliptic surfaces. Kloosterman pointed out that the results in these papers are not strong enough to deduce the local Torelli theorem for the family $\mathcal{Y}$, but [Klo04, Theorems 1.1 and 3.3] are. He also pointed out that Theorem 1.1 follows from the proof of [HL02, Proposition 2.1].

4. Proof of the main theorem

We prove Theorem 1.1 in this section. We first prove the uniqueness of the decomposition for very general f and g. For any elements φ and ψ in $\mathbb{C}[z,w]$ of degrees 8 and 12 satisfying

$$h = \varphi^3 + \psi^2, \qquad (4.1)$$

the section σ defined by $(x,y) = (\varphi, \psi)$ does not intersect σ_0, so that $\tau(\sigma) \in U^\perp \subset \operatorname{Pic} X$. For very general f and g, one has $\rho(X) = 4$, and hence $\tau(\sigma) \in M \otimes \mathbb{Q}$. Recall that the Gram matrix of the ordered basis (τ_1, τ_2) of M is $\begin{pmatrix} -8 & 4 \\ 4 & -8 \end{pmatrix}$. By a direct computation, one can see that there are no elements $\rho \in M \otimes \mathbb{Q}$ such that $\rho.M \subset \mathbb{Z}$ and $\rho^2 = -8$ other than $\tau_1, \dots, \tau_6$, and the uniqueness of the decomposition follows from the fact that the Mordell–Weil group of X is naturally isomorphic to $\operatorname{Pic}(X)/U \cong (M \otimes \mathbb{Q}) \cap \operatorname{Pic}(X)$.

In order to prove the uniqueness of the decomposition for general f and g, let S and T be the subspaces of $\mathbb{C}[z,w]$ consisting of homogeneous polynomials of degrees 8 and 12 respectively, and define a subscheme Z of $(S \times T)^2$ by

$$Z := \left\{ ((f,g),(\varphi,\psi)) \in (S \times T)^2 \;\middle|\; f^3 + g^2 = \varphi^3 + \psi^2 \right\}. \qquad (4.2)$$

The uniqueness of the decomposition for very general f and g implies that the first projection $Z \to S \times T$, which is a morphism of schemes, is generically six-to-one. Hence the first projection is six-to-one outside of a Zariski closed subset, and Theorem 1.1 is proved.

5. Mirror symmetry

It is conjectured by Strominger, Yau, and Zaslow [SYZ96] that any Calabi–Yau manifold has a special Lagrangian torus fibration, and the mirror manifold is obtained as the dual special Lagrangian torus fibration. This picture has been refined in [GW00, KS01] to the conjecture that the Calabi–Yau metric with the diameter normalized to one converges in the Gromov–Hausdorff topology to a Monge–Ampère manifold with singularities as one approaches a large complex structure limit. Here, a *Monge–Ampère manifold with singularities* is a manifold B with a subset B^{sing} of Hausdorff codimension 2 such that $B \setminus B^{\mathrm{sing}}$ has a tropical affine structure (i.e., an atlas whose transformation functions are in $\mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$) and a Monge–Ampère metric (i.e., a Riemannian metric of Hessian form $g_{ij} = \frac{\partial^2 K}{\partial x^i \partial x^j}$ in the affine coordinate satisfying $\det\left((g_{ij})_{i,j}\right) = \text{constant}$). A Monge–Ampère manifold comes with a dual pair of affine structures, and mirror symmetry should interchange them.

In the case of a K3 surface, a special Lagrangian torus fibration can be turned into an elliptic fibration by a hyperKähler rotation. It is shown in [GW00, GTZ16] that a sequence of Calabi–Yau metrics on an elliptic K3 surface with a fixed diameter converges in the Gromov–Hausdorff topology to a sphere with a Monge–Ampère structure with singularities as the volume of the fiber goes to zero. The limit sphere B can naturally be identified with the base $\mathbb{P}^1$ of the elliptic K3 surface, and the discriminant B^{sing} of the Monge–Ampère structure can be identified with the discriminant of the elliptic K3 surface. Under this identification, the Monge–Ampère metric g on $B \setminus B^{\mathrm{sing}}$ is written as

$$g = \mathfrak{Im}\,(\overline{\tau}_1 \tau_2)\, dz \otimes d\overline{z}, \tag{5.1}$$

where z is the holomorphic local coordinate on the base $\mathbb{P}^1$ and (τ_1, τ_2) are the periods, along a symplectic basis, of the relative holomorphic one-form λ on the elliptic fibration dual to dz with respect to the holomorphic volume form of the K3 surface. It follows that the complex structure of the base $\mathbb{P}^1$ minus the discriminant can be reconstructed from the limit metric, up to the choice of an orientation. Note that the metric (5.1) depends only on the Jacobian fibration, so that one can assume that the elliptic K3 surface has a section. It follows from Theorem 1.1 that a general elliptic K3 surface with a section can be reconstructed from the limit metric up to complex conjugation.

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