COLLAPSING K3 SURFACES AND MODULI COMPACTIFICATION

YUJI ODAKA AND YOSHIKI OSHIMA

ABSTRACT. This note is a summary of our work [OO], which provides an explicit and global moduli-theoretic framework for the collapsing of Ricci-flat Kähler metrics and we use it to study especially the K3 surfaces case. For instance, it allows us to discuss their Gromov-Hausdorff limits along any sequences, which are even not necessarily "maximally degenerating". Our results also give a proof of Kontsevich-Soibelman [KS04, Conjecture 1] (cf., [GW00, Conjecture 6.2]) in the case of K3 surfaces as a byproduct.

1. Introduction

Our paper [OO] is a sequel to a series by the first author [Od14, Od16, which compactified both the moduli space of compact Riemann surfaces $M_q(g \ge 2)$ and that of principally polarized abelian varieties A_a . In each case, as we actually expect an analogue for any moduli of general polarized Kähler-Einstein varieties with non-positive scalar curvatures, we introduce and study two similar (non-variety) compactifications of the moduli space \mathcal{M} , which we denote by $\overline{\mathcal{M}}^{GH}$ and $\overline{\mathcal{M}}^{T}$. The former $\overline{\mathcal{M}}^{\mathrm{GH}}$ is the Gromov-Hausdorff compactification with respect to rescaled Kähler-Einstein metrics of fixed diameters and the latter "tropical geometric compactification" $\overline{\mathcal{M}}^{T}$ should dominate the former $\overline{\mathcal{M}}^{\text{GH}}$ as its boundary $\partial \overline{\mathcal{M}}^{\text{T}}$ encodes more structure of the Gromov-Hausdorff limits (collapses) rather than just distance structure. For a precise definition of $\overline{\mathcal{M}}^{GH}$ we employ the same definition as [Od14, $\S2.3$, [Od16, $\S2.2$]. For $\overline{\mathcal{M}}^{\mathrm{T}}$, we have a case by case definition for only particular classes of varieties. Here, we recall the structure theorem of $\overline{A_q}^{\text{GH}}$ from [Od16, Theorems 2.1 2.3 and Corollary 2.5].

Theorem 1.1 ([Od16]). A_g can be explicitly compactified as $\overline{A_g}^{GH}$ whose boundary parametrizes all flat (real) tori $\mathbb{R}^i/\mathbb{Z}^i$ of diameter 1 where $1 \leq i \leq g$. Once we attach the rescaled flat Kähler metric in

¹However, its compactness is unknown at least to the authors in higher dimensional negative scalar curvature case.

the principal polarization with diameter 1 to each abelian variety, the parametrization of metric spaces on whole $\overline{A_g}^{\text{GH}}$ is continuous with respect to the Gromov-Hausdorff distance.

In the above case, we simply set $\overline{A_g}^{\rm T} := \overline{A_g}^{\rm GH}$. On the other hand, in the analogue for M_g [Od14], we distinguish $\overline{M_g}^{\rm GH}$ and $\overline{M_g}^{\rm T}$, where the boundaries of $\overline{M_g}^{\rm GH}$ (resp., $\overline{M_g}^{\rm T}$) parametrize metrized graphs (resp., metrized graphs with integer weights on the vertices). We refer the details to [Od14].

Our [OO] contains the followings.

- (i) We first apply the Morgan-Shalen type compactification for general Hermitian locally symmetric spaces and identify it with one of the Satake compactifications ([Sat60a], [Sat60b]).
- (ii) We partially prove that the boundary of the Satake compactification of the type which appears in (i) parametrizes collapses of abelian varieties and Ricci-flat K3 surfaces. This gives a generalisation of some results in [GW00], [Tos10], [GTZ13], [GTZ16], [TZ17] for the K3 surface case. For instance, a proof of the conjecture of Kontsevich-Soibelman [KS04, Conjecture 1] (see also Gross-Wilson [GW00, Conjecture 6.2]), which is related to the Strominger-Yau-Zaslow mirror symmetry [SYZ96], for the case of K3 surfaces directly follows from our description of collapsing. We also give a conjecture for higher dimensional hyperKähler varieties.

Now we move on to a more detailed description.

2. General Hermitian Symmetric Domain

Let \mathbb{G} be a reductive algebraic group over \mathbb{Q} , $G = \mathbb{G}(\mathbb{R})$, K (one of) its maximal compact subgroup, and D := G/K, which we suppose to have a Hermitian symmetric domain structure. We moreover assume D is irreducible so that G is simple as a Lie group. Suppose that Γ is an arithmetic subgroup of $\mathbb{G}(\mathbb{Q})$, which acts on D. Hence we can discuss Hermitian locally symmetric space $\Gamma \backslash D$.

Satake [Sat60a], [Sat60b] constructed compactifications of Riemannian locally symmetric spaces G/K associated to irreducible projective representations $\tau\colon G\to PGL(\mathbb{C})$ satisfying certain conditions. They are stratified as:

$$\overline{\Gamma \backslash D}^{\mathrm{Sat},\tau} = \Gamma \backslash D \sqcup \bigsqcup_{P} (\Gamma \cap Q(P)) \backslash M_P / (K \cap M_P).$$

Here, P runs over all the $\mu(\tau)$ -connected rational parabolic subgroups, $P = N_P A_P M_P$ denotes the Langlands decomposition, and Q(P) is the $\mu(\tau)$ -saturation of P. We are particularly interested in the case when τ is the adjoint representation $\tau_{\rm ad}$.

On the other hand, given any toroidal compactification [AMRT75] for $\Gamma \backslash D$, we can apply the Morgan-Shalen type compactification to it as [Od16, Appendix] (following [MS84, BJo17]). The Morgan-Shalen type compactification $\overline{\Gamma \backslash D}^{\text{MSBJ}}$ obtained in this way is independent of the cone decomposition for the toroidal compactification [Od16, A.13, A.14].

We now compare these two compactifications.

Theorem 2.1. Let $\Gamma \backslash D$ be a locally Hermitian symmetric space. Consider its toroidal compactification and the associated (generalised) Morgan-Shalen compactification $\overline{\Gamma \backslash D}^{\mathrm{MSBJ}}$. Then this is homeomorphic to the Satake compactification $\overline{(\Gamma \backslash D)}^{\mathrm{Sat},\tau_{\mathrm{ad}}}$ for the adjoint representation τ_{ad} of G.

In the following we make an "elementary" but important observation on a rationality phenomenon of the limits along one parameter holomorphic family, which we expect to fit well with the recent approach to extend the theta functions in [GS12] etc.

Proposition 2.2. Suppose $U \subset \overline{U}^{\text{hyb}}(\mathcal{X})$ is a Morgan-Shalen-Boucksom-Jonsson compactification associated to an arbitrary dlt stacky pair $(\mathcal{X}, \mathcal{D})$ of boundary coefficients 1 ([Od16]) with $\mathcal{U} := \mathcal{X} \setminus \mathcal{D}$, its coarse moduli space $\mathcal{U} \to \mathcal{U}$. Then for any holomorphic morphism $\Delta^* := \{z \in \mathbb{C} \mid 0 < |z| < 1\} \to \mathcal{U}$ which extend to $\Delta := \{z \in \mathbb{C} \mid |z| < 1\} \to \mathcal{X}$, it induces a continuous map $\Delta \to \overline{U}^{\text{hyb}}(\mathcal{X})$, i.e., the limit exists. Furthermore, such possible limits in $\Delta(\mathcal{D})$ are characterized as points with rational coordinates.

Corollary 2.3 (corollary to Theorem 2.1 and Proposition 2.2). Take an arbitrary holomorphic map $f: \Delta^* \to \Gamma \backslash D$, which extends to a map to a toroidal compactification of $\Gamma \backslash D$. Then f also extends to a map $\Delta \to \overline{\Gamma \backslash D}^{\operatorname{Sat},\tau_{\operatorname{ad}}}$ where 0 is sent to a point with rational coordinates, i.e., a point in the dense subset $(C(F) \cap U(F) \otimes \mathbb{Q})/\mathbb{Q}_{>0} \subset C(F)/\mathbb{R}_{>0}$.

This is partially proved in the case of A_g in [Od16] by using degeneration data in [FC90].

Remark 2.4. Although we assume that G is simple in this section, our Morgan-Shalen type compactification construction [Od16, Appendix] still works for non-simple G. Thus, our construction also gives a

new Satake-type compactification for non-simple G, e.g., of the Hilbert modular varieties.

3. Abelian varieties case

We identify our tropical geometric compactification $\overline{A_g}^{\rm T}$ ([Od16]) of A_g with the adjoint type Satake compactification.

Theorem 3.1. There are canonical homeomorphisms between the three compactifications

$$\overline{A_g}^{\mathrm{T}} \cong \overline{A_g}^{\mathrm{Sat},\tau_{\mathrm{ad}}} \cong \overline{A_g}^{\mathrm{MSBJ}},$$

extending the identity on A_q .

The second canonical homeomorphism is a special case of Theorem 2.1 and the first is essentially reduced to matrix computations.

In [OO], we also give a purely moduli-theoritic reexplanation of the structure theory of one parameter degenerations of abelian varieties in [Mum72], [FC90], after the above Theorem 3.1 as follows.

Theorem 3.2. Take a holomorphic maximally degenerating family of principally polarized abelian varieties $\pi: (\mathcal{X}, \mathcal{L}) \to \Delta$. Consider the rescaled Gromov-Hausdorff limit $B(\mathcal{X}, \mathcal{L})$ of diameter 1 as in Theorem 1.1 ([Od16]) and its discrete Legendre transform $\check{B}(\mathcal{X}, \mathcal{L})$ ([GS11], [KS04]).

Then we can enhance the underlying integral affine structure of $\check{B}(\mathcal{X}, \mathcal{L})$ as K-affine structure (in the sense of [KS04, §7.1]) naturally via the data of π . Furthermore, such K-affine structure recovers π up to an equivalence relation generated by base change (replace t by t^a with $a \in \mathbb{Q}_{>0}$).

4. Moduli of Algebraic K3 surfaces

4.1. **Satake compactification.** Let \mathcal{F}_{2d} be the moduli space of polarized K3 surfaces of degree 2d possibly with ADE singularities. Its structure is known as follows. Let $\Lambda_{K3} := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ be the K3 lattice and fix a primitive vector λ_{2d} with $(\lambda_{2d}, \lambda_{2d}) = 2d$ and $\Lambda_{2d} := \lambda_{2d}^{\perp}$. The complex manifold

$$\Omega(\Lambda_{2d}) := \{ [w] \in \mathbb{P}(\Lambda_{2d} \otimes \mathbb{C}) \mid (w, w) = 0, \ (w, \bar{w}) > 0 \}.$$

has two connected components. We choose one component and denote by $\mathcal{D}_{\Lambda_{2d}}$. Let $O(\Lambda_{K3})$ denote the isomorphism group of the lattice Λ_{K3} preserving the bilinear form and set

$$\tilde{O}(\Lambda_{2d}) := \{ g |_{\Lambda_{2d}} : g \in O(\Lambda_{K3}), \ g(\lambda_{2d}) = \lambda_{2d} \}.$$

The group $\tilde{O}(\Lambda_{2d})$ naturally acts on $\Omega(\Lambda_{2d})$. We define $\tilde{O}^+(\Lambda_{2d})$ to be the index two subgroup of $\tilde{O}(\Lambda_{2d})$ consisting of the elements preserving each connected component of $\Omega(\Lambda_{2d})$. Then it is well-known that

$$\mathcal{F}_{2d} \simeq \tilde{O}^+(\Lambda_{2d}) \backslash \mathcal{D}_{\Lambda_{2d}} \simeq \tilde{O}(\Lambda_{2d}) \backslash \Omega(\Lambda_{2d}).$$

Let $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat},\tau_{\mathrm{ad}}}$ (or simply $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}}$ in our papers) be the Satake compactification of \mathcal{F}_{2d} corresponding to the adjoint representation of O(2,19). It decomposes as

$$\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} = \mathcal{F}_{2d} \sqcup \bigcup_{l} \mathcal{F}_{2d}(l) \sqcup \bigcup_{p} \mathcal{F}_{2d}(p),$$

where l runs over one-dimensional isotropic subspaces of $\Lambda_{2d} \otimes \mathbb{Q}$, and p runs over two-dimensional isotropic subspaces of $\Lambda_{2d} \otimes \mathbb{Q}$. Also, we simply define the tropical geometric compactification of \mathcal{F}_{2d} as this $\overline{\mathcal{F}_{2d}}^{\text{Sat}}$. The boundary component $\mathcal{F}_{2d}(l)$ is given as

$$\mathcal{F}_{2d}(l) = \{ v \in (l^{\perp}/l) \otimes \mathbb{R} \mid (v, v) > 0 \} / \sim.$$

Here $v \sim v'$ if $g \cdot v = cv'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$ and $c \in \mathbb{R}^{\times}$. We have $\mathcal{F}_{2d}(l) = \mathcal{F}_{2d}(l')$ if $g \cdot l = l'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$ and $\mathcal{F}_{2d}(l) \cap \mathcal{F}_{2d}(l') = \emptyset$ if otherwise. Since $(l^{\perp}/l) \otimes \mathbb{R}$ has signature (1, 18), there is an isomorphism

$$\{v \in (l^{\perp}/l) \otimes \mathbb{R} \mid (v,v) > 0\}/\mathbb{R}^{\times}$$

$$\simeq O(1,18)/O(1) \times O(18)$$

and hence $\mathcal{F}_{2d}(l)$ is an arithmetic quotient of $O(1,18)/O(1) \times O(18)$. The other component $\mathcal{F}_{2d}(p)$ is a point and $\mathcal{F}_{2d}(p) = \mathcal{F}_{2d}(p')$ if and only if $g \cdot p = p'$ for some $g \in \tilde{O}^+(\Lambda_{2d})$. Therefore, if we take representatives of l and p from each equivalence class, we get a finite decomposition:

$$\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} = \mathcal{F}_{2d} \sqcup \bigsqcup_{l} \mathcal{F}_{2d}(l) \sqcup \bigsqcup_{p} \mathcal{F}_{2d}(p).$$

4.2. **Tropical K3 surfaces.** In our paper, what we mean by *tropical polarized K3 surface* is a topological space B homeomorphic to the sphere S^2 , with an affine structure away from certain finite points $\operatorname{Sing}(B)$, with a metric which is Mongé-Ampere metric g with respect to the affine structure on $B \setminus \operatorname{Sing}(B)$. Studies of such object as tropical version of K3 surfaces are pioneered in well-known papers of Gross-Wilson [GW00] and Kontsevich-Soibelman [KS04].

Here we assign such tropical K3 surface to each point in the boundary component $\mathcal{F}_{2d}(l)$ as follows. Let l be an oriented one-dimensional isotropic subspace of $\Lambda_{2d} \otimes \mathbb{Q}$. Write e for the primitive element of l such

that $\mathbb{R}_{>0}e$ agrees with the orientation of l. Take a vector $v \in (l^{\perp}/l) \otimes \mathbb{R}$ such that (v, v) > 0. Write [e, v] for the corresponding point in $\mathcal{F}_{2d}(l)$. Then there exists a (not necessarily projective) K3 surface X and a marking $\alpha_X \colon H^2(X,\mathbb{Z}) \to \Lambda$ with

- $\alpha_X(H^{2,0}) \subset \mathbb{R}\lambda + \sqrt{-1}\mathbb{R}v$, $\alpha_X^{-1}(e)$ is in the closure of Kähler cone.

The pair (X, α_X) is unique up to isomorphisms.

Let L be a line bundle on X such that $\alpha_X([L]) = e$. Then we get an elliptic fibration $f: X \to B(\simeq \mathbb{P}^1)$. Take a holomorphic volume form Ω on X such that $\alpha_X([\operatorname{Re}\Omega]) = \lambda$. The map f is a Lagrangian fibration with respect to the symplectic form Re Ω . Hence it gives an affine manifold structure on $B \setminus \Delta$, where Δ denotes the finite set of singular points. Similarly, the imaginary part $\operatorname{Im} \Omega$ gives another affine manifold structure on $B \setminus \Delta$.

We endow the base space B with the McLean metric on the base B ([ML98]), where we regard f as special Lagrangian fibration after hyperKähler rotation. A straightforward calculation shows that this coincides with the "special Kähler metric" g_{sp} introduced and studied in [DW96, Hit96, Freed99] and appears as the metric on \mathbb{P}^1 in [GTZ16]. We rescale the metric to make its diameter 1 and denote this obtained tropical K3 surface by $\Phi_{\text{alg}}([e, v])$.

Remark 4.1. Recall the concepts of the class of metric (metric class) and the radiance obstruction of Mongé-Ampére manifolds B with singularities. They are introduced in [KS04] and discussed in [GS06] in more details. We denote them by $k(B) \in H^1(B, i_* \tilde{\Lambda}^{\vee} \otimes \mathbb{R})$ and $c(B) \in H^1(B, i_*\Lambda)$, respectively. Here, Λ is the affine structure as a $\mathbb{Z}^{dim(B)}$ -local system in tangent bundle $T(B \setminus \Delta)$, $-^{\vee}$ denotes -'s dual local system, $\tilde{\Lambda}^{\vee}$ is local system of affine functions. In particular, we naturally have a morphism of local systems $f : \tilde{\Lambda}^{\vee} \to \Lambda^{\vee}$ which induces $f_*: H^1(B, i_*\tilde{\Lambda}^{\vee}) \to H^1(B, i_*\Lambda^{\vee})$. It is also easy to see that, if we slightly change the definition of the metric class, to extract its "linear" part as $f_*k(B)$. Then, it naturally recovers the data $\overline{v} \in (e^{\perp} \otimes \mathbb{R}/\mathbb{R}e)$ i.e., we have $f_*k(\Phi_{\mathrm{alg}}([e,v])) = [v]$, under the natural identification $H^1(\Phi_{\text{alg}}([e,v]), i_*\Lambda^{\vee} \otimes \mathbb{R}) \hookrightarrow (e^{\perp} \otimes \mathbb{R}/\mathbb{R}e)$ which comes from the Leray spectral sequence applied to the elliptic fibration $X \to \Phi_{\rm alg}([e,v])$ in §4.2. Our results in [Od16] and Theorem 3.1 for A_q can be re-interpretted similarly (but with weight 1).

Remark 4.2. Yuto Yamamoto [Yam] has some ongoing interesting work which seems to be related to our works, where he constructs a sphere

with an integral affine structure from the tropicalization of an anticanonical hypersurface in a toric Fano 3-fold, and computes its radiance obstruction.

4.3. Gromov-Hausdorff collapse of K3 surfaces. For a point in \mathcal{F}_{2d} we have a corresponding polarized K3 surface (X, L), equipped with a natural Ricci-flat metric. For $[e, v] \in \mathcal{F}_{2d}(l)$ we defined in a previous section $\Phi_{\text{alg}}([e, v])$. For a point in $\mathcal{F}_{2d}(p)$ we assign a (one-dimensional) segment, which we denote by $\Phi_{\text{alg}}(\mathcal{F}_{2d}(p))$. Let us normalize these metric spaces so that their diameters are one. We thus obtained a map $\Phi_{\text{alg}}: \overline{\mathcal{F}_{2d}}^{\text{Sat}} \to \{\text{compact metric spaces with diameter one}\}$. Here, we associate Gromov-Hausdorff distance to the right hand side (target space) and denote it by $CMet_1$.

Conjecture 4.3. The map

$$\Phi_{\rm alg} \colon \overline{\mathcal{F}_{2d}}^{\rm Sat} \to \mathit{CMet}_1$$

given above is continuous.

We would like to simply set the tropical geometric compactification of \mathcal{F}_{2d} as $\overline{\mathcal{F}_{2d}}^{\mathrm{T}} := \overline{\mathcal{F}_{2d}}^{\mathrm{Sat}}$. Indeed, if Conjecture 4.3 holds, we get a continuous map $\overline{\mathcal{F}_{2d}}^{\mathrm{Sat}} \to \overline{\mathcal{F}_{2d}}^{\mathrm{GH}}$ and we also observe that each $\mathcal{F}_{2d}(l)$ encodes affine structure of the limit tropical K3 surface as well. (This answers a question of Prof. B. Siebert in 2016 to the first author, regarding if one can associate tropical affine structure to limit of any collapsing sequence). So far, we have partially confirmed the conjecture. The case of $(A_1$ -singular flat) Kummer surfaces, with 3-dimensional moduli, are easily reduced to [Od16]. More generally, we have proved the following. In particular, the conjecture 4.3 holds at least away from finite points.

Theorem 4.4. The map Φ_{alg} is continuous on $\overline{\mathcal{F}_{2d}}^{\text{Sat}} \setminus (\bigcup_p \mathcal{F}_{2d}(p))$. It is continuous also when restricted to the boundary $\partial \overline{\mathcal{F}_{2d}}^{\text{Sat}} = \overline{\mathcal{F}_{2d}}^{\text{Sat}} \setminus \mathcal{F}_{2d}$.

The proof of the former half of the statements involves some symmetric space theory, hyperKähler geometry, algebraic geometry of moduli, and a priori analytic estimates. The estimates heavily depends on [Tos10, GW00, GTZ13, GTZ16, TZ17] and their extensions. One nontrivial part of the extension is, for instance, to make many of the C^2 -estimations in op.cit following methods of [Yau78] locally uniform with respect to a family of elliptic K3 surfaces even along degenerations to orbifolds.

During our work, we learnt that Kenji Hashimoto, Yuichi Nohara, Kazushi Ueda [HNU] also studied the Gromov-Hausdorff collapses along certain 2-dimensional subvariety of \mathcal{F}_{2d} , i.e., the moduli of $E_8^{\oplus 2} \oplus U(\oplus \langle -2 \rangle)$ -polarized K3 surfaces. Moereover, a result of Hashimoto and Ueda [HU] implies that the restriction of Φ_{alg} to the boundary is a generically two-to-one map. We appreciate their gentle discussion with us.

Theorem 4.4 (resp., Conjecture 4.3) combined with Proposition 2.2 determines the Gromov-Hausdorff limits of Type III (resp., Type II) one parameter family of Ricci-flat algebraic K3 surfaces, which solves a conjecture of Kontsevich-Soibelman [KS04, Conjecture 1], Todorov, and Gross-Wilson (cf., e.g., [Gross12, Conjecture 6.2]) in the K3 surfaces case.

In the next section, we discuss collapsing of general Kähler K3 surfaces, which are not necessarily algebraic.

5. Moduli of Kähler K3 surfaces

It is known (cf., [Tod80], [Looi81], [KT87]) that the moduli space of all Einstein metrics on a Kähler K3 surfaces (including orbifold-metrics) has again a structure of the locally Riemannian symmetric space:

$$O(\Lambda_{K3}) \setminus SO_0(3, 19) / (SO(3) \times SO(19)),$$

which we denote by \mathcal{M}_{K3} . An enriched version encoding also complex structures of the K3 surfaces is

$$\mathbb{R}_{>0} \times (O(\Lambda_{K3}) \setminus SO_0(3,19) / (SO(2) \times SO(19))).$$

Roughly speaking, this is a union of Kähler cones of ADE K3 surfaces with marking of the minimal resolutions.

Thus we can again compare a Satake compactification of \mathcal{M}_{K3} with the Gromov-Hausdorff compactification. Inside the Satake compactification for the adjoint representation, we consider an open locus (a partial compactification of \mathcal{M}_{K3}) $\mathcal{M}_{K3} \sqcup \mathcal{M}_{K3}(a)$, where $\mathcal{M}_{K3}(a)$ denotes the 36-dimensional boundary stratum corresponding to an isotropic rational line $l = \mathbb{Q}e$ in $\Lambda_{K3} \otimes \mathbb{Q}$, with primitive integral generator e, which are unique up to $O(\Lambda_{K3})$. Then for each point $p = \langle e, v_1, v_2 \rangle$ in strata $\mathcal{M}_{K3}(a)$, we consider the marked (possibly ADE) K3 surface X_p with period $\langle v_1, v_2 \rangle$. Then it is known that there is an elliptic K3 surface structure on X_p with the fiber class e. Then we define $\Phi(p)$ as its base biholomorphic to \mathbb{P}^1 with the McLean metric, which only depends on $\langle v_1, v_2 \rangle$. Similarly to the projective case Theorem 4.4, [OO] proves that for non-algebraic situation:

Theorem 5.1. The map

$$\Phi \colon \mathcal{M}_{\mathrm{K3}} \sqcup \mathcal{M}_{\mathrm{K3}}(a) \to CMet_1$$

given above is continuous. Here, we put the Gromov-Hausdorff topology for the right hand side.

In [OO], we further explicitly define an extension to the whole Satake compactification $\Phi \colon \overline{\mathcal{M}_{\mathrm{K3}}}^{\mathrm{Sat}} \to CMet_1$, and conjecture that this is still continuous with respect to the Gromov-Hausdorff topology. For the boundary strata other than $\mathcal{M}_{\mathrm{K3}}(a)$, we assign flat tori $\mathbb{R}^i/\mathbb{Z}^i$ (i=1,2,3) modulo (-1)-multiplication. We show that Φ restricted to the closure of the locus which parametrizes $\mathbb{R}^4/\mathbb{Z}^4$ modulo ± 1 , that includes those boundary strata, is continuous. Furthermore, we also prove the restriction of Φ to the closure of $\mathcal{M}_{\mathrm{K3}}(a)$ is continuous by using Weierstrass models.

6. Higher dimensional case

We expect that our results for K3 surfaces naturally extend to higher dimensional compact hyperKähler manifolds. Let us focus on algebraic case in this notes. We set up as follows. Fix any connected moduli M of polarized 2n-dimensional irreducible holomorphic symplectic manifolds (X, L) whose second cohomology $H^2(X, \mathbb{Z})$ is isomorphic (as a lattice) to Λ . By [Ver13, Mark11] ([GHS13, Theorem 3.7]), it is a Zariski open subset of a Hermitian locally symmetric space of orthogonal type $\Gamma \backslash \mathcal{D}_M$.

Then (a rough version of) our conjecture for algebraic case (in [OO]) is as follows:

Conjecture 6.1. There is a continuous map Ψ (call "geometric realization map") from the Satake compactification $(M \subset) \overline{\Gamma \backslash \mathcal{D}_M}^{\operatorname{Sat},\tau_{\operatorname{ad}}}$ with respect to the adjoint representation to the Gromov-Hausdorff compactification of M, extending the identity map on M. The $(b_2(X)-4)$ -dimensional boundary strata of $\overline{\Gamma \backslash \mathcal{D}_M}^{\operatorname{Sat},\tau_{\operatorname{ad}}}$ parametrize via Ψ the projective space \mathbb{P}^n with special Kähler metrics in the sense of [Freed99] and the metric space parametrized by 0-dimensional cusps are all homeomorphic to the closed ball of dimension n.

At the moment of writing this notes, the authors have only succeeded in proving that $(M \subset)\Gamma\backslash\mathcal{D}_M$ is the moduli of polarized symplectic varieties with continuous (non-collapsing) weak Ricci-flat Kähler metrics, and making some progress on the necessary algebro-geometric preparations in particular for the case of K3^[n]-type. Remark 6.2 (Calabi-Yau case). In [OO], we also propose an extension of Conjecture 4.3 for general Calabi-Yau varieties under some technical conditions, although there are much fewer evidences in that case.

Acknowledgement We appreciate for giving us the chances to talk on [OO] in various countries and cities. The first was at a talk by the first author at a Clay conference held at Oxford in September 2016, when Theorems 4.4 and 5.1 were only partially proved and claimed, whose confirmation in the form of this notes has taken long time. In particular, we appreciate Kenji Hashimoto, Shouhei Honda, Radu Laza, Daisuke Matsushita, Shigeru Mukai, Yoshinori Namikawa, Bernd Siebert, Cristiano Spotti, Song Sun, Yuichi Nohara, Kazushi Ueda, Ken-ichi Yoshikawa for helpful discussions. There are plans of some lecture series by the first author on this topic during the next fall semester in Nagoya, Tokyo. The first author is partially supported by JSPS Grant-in-Aid (S), No. 16H06335, Grand-in-Aid for Early-Career Scientists No. 18K13389. The second author is partially supported by JSPS KAKENHI Grant No. 16K17562.

References

- [AMRT75] A. Ash, D. Mumford, M. Rapoport, Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, Cambridge Mathematical Library (1st edition: 1975, 2nd edition: 2010).
- [BJo17] S. Boucksom, M. Jonsson, Tropical and non-Archimedean limits of degenerating families of volume forms, J. Éc. polytech. Math. 4 (2017), 87–139.
- [DW96] R. Donagi, E. Witten, Supersymmetric Yang-Mills theory and integrable systems, Nuclear Phys. B **460** (1996), no. 2, 299–334.
- [FC90] G. Faltings, C.-L. Chai, Degeneration of abelian varieties, Springer-Verlag, 1990.
- [Freed99] D. Freed, Special Kähler manifolds, Comm. Math. Phys. **203** (1999), no. 1, 31–52.
- [GHS13] V. Gritsenko, K. Hulek, G.K. Sankaran, Moduli of K3 surfaces and irreducible symplectic manifolds, Handbook of moduli. Vol. I, Adv. Lect. Math. (ALM), 24 (2013), 459–526.
- [Gross12] M. Gross, Mirror Symmetry and the Strominger-Yau-Zaslow conjecture, *Current Developments in Mathematics 2012*, 133–191, Int. Press, Somerville, MA, 2013.
- [GS06] M. Gross, B. Siebert, Mirror symmetry via logarithmic degeneration data. I, J. Differential Geom. 72 (2006), no. 2, 169–338.
- [GS11] M. Gross, B. Siebert, From real affine geometry to complex geometry, Ann. of Math. 174 (2011), no. 3, 1301–1428.
- [GS12] M. Gross, B. Siebert, Theta functions and mirror symmetry, Surveys in differential geometry 2016. Advances in geometry and mathematical physics, 95–138, Surv. Differ. Geom., 21, Int. Press, Somerville, MA, 2016.
- [GTZ13] M. Gross, V. Tosatti, Y. Zhang, Collapsing of abelian fibered Calabi-Yau manifolds, Duke Math. J. **162** (2013), no. 3, 517–551.

- [GTZ16] M. Gross, V. Tosatti, Y. Zhang, Gromov-Hausdorff collapsing of Calabi-Yau manifolds, Comm. Anal. Geom, 24 (2016), no. 1, 93–113.
- [GW00] M. Gross, P. M. H. Wilson, Large complex structure limits of K3 surfaces, J. Differential Geom. 55 (2000), no. 3, 475–546.
- [HU] K. Hashimoto and K. Ueda, Reconstruction of general elliptic K3 surfaces from their Gromov–Hausdorff limits, preprint.
- [HNU] K. Hashimoto, Y. Nohara, K. Ueda, private communication.
- [Hit96] N. J. Hitchin, The moduli space of complex Lagrangian submanifolds, Asian J. Math. 3 (1999), no. 1, 77–91.
- [KT87] R. Kobayashi, A. Todorov, Polarized period map for generalized K3 surfaces and the moduli of Einstein metrics, Tohoku Math. J. (2) **39** (1987), no. 3, 341–363.
- [KS04] M. Kontsevich, Y. Soibelman, Affine structures and non-Archimedean analytic spaces, The Unity of Mathematics, 321–385, Progr. Math. 244, Birkhäuser Boston, 2006.
- [KW65] A. Koranyi, J. Wolf, Generalized Cayley transformations of bounded symmetric domains, Amer. J. Math. 87 (1965), 899–939.
- [Looi81] E. Looijenga, A Torelli theorem for Kähler-Einstein K3 surfaces, 107–112, Lecture Notes in Math., 894, Springer, Berlin-New York, 1981.
- [Mark11] E. Markman, A survey of Torelli and monodromy results for holomorphicsymplectic varieties, *Complex and differential geometry*, 257–322 Springer Proc. Math., 8, Springer, Heidelberg 2011.
- [ML98] R. McLean, Deformations of calibrated submanifolds, Comm. Anal. Geom. **6** (1998), no. 4, 705–747.
- [MS84] J. Morgan, P. B. Shalen, Valuations, trees, and degenerations of hyperbolic structures. I, Ann. of Math. **120** (1984), no. 3, 401–476.
- [Mum72] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings, Compositio Math. 24 (1972), 239–272.
- [Od14] Y. Odaka, Tropical geometric compactification of Moduli, I M_g case -, arXiv:1406.7772.
- [Od16] Y. Odaka, Tropical geometric compactification of Moduli, II A_g case and holomorphic limits -, Int. Math. Res. Not. (2018).
- [OO] Y. Odaka, Y. Oshima, in preparataion.
- [PS66] I.I. Piatetsky-Shapiro, Géométrie des domaines classiques et théorie des fonctions automorphes, Travaux et Rechereches Mathématiques, No. 12 Dunod, Paris 1966.
- [Sat60a] I. Satake, On representations and compactifications of symmetric Riemannian spaces, Ann. of Math. **71** (1960), 77–110.
- [Sat60b] I. Satake, On compactifications of the quotient spaces for arithmetically defined discontinuous groups, Ann. of Math. **72** (1960), 555–580.
- [SYZ96] A. Strominger, S-T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.
- [Tod80] A. Todorov, Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces, Invent. Math. 61 (1980), no. 3, 251–265.
- [Tos10] V. Tosatti, Adiabatic limits of Ricci-flat Kähler metrics, J. Differential Geom. 84 (2010), no. 2, 427–453.
- [TZ17] V. Tosatti, Y. Zhang, Collapsing hyperKähler manifolds, arXiv:1705.03299.

[Ver13] M. Verbitsky, Mapping class group and a global Torelli theorem for hyperKähler manifolds, Appendix A by E. Markman. Duke Math. J. 162 (2013), no. 15, 2929-2986.

[Yam] Y. Yamamoto, Periods of tropical K3 hypersurfaces, in preparation.

[Yau78] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation. I, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

Contact (Yuji Odaka): yodaka@math.kyoto-u.ac.jp Department of Mathematics, Kyoto University, Kyoto 606-8285. JAPAN

Contact (Yoshiki Oshima): oshima@ist.osaka-u.ac.jp Graduate School of Information Science and Technology, Osaka University, Suita, Osaka 565-0871, JAPAN