

HYPERELLIPTIC THREEFOLDS WITH GROUP D_4 , THE DIHEDRAL GROUP OF ORDER 8

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ABSTRACT. We give a simple construction for the hyperelliptic threefolds with group D_4 .

INTRODUCTION

A Generalized Hyperelliptic Manifold is the quotient $X = T/G$ of a complex torus T by the free action of a finite group G which contains no translations. We say that we have a Generalized Hyperelliptic Variety if moreover the torus T is projective, i.e., it is an Abelian variety A .

Recently D. Kotschick observed that the classification of Generalized Hyperelliptic Manifolds of complex dimension three was not complete, since the case where G is the dihedral group D_4 of order 8 was excluded (by H. Lange in [La01]) but it does indeed occur. Indeed F.E.A. Johnson in the preprint [Jo18] showed that a construction due to Dekimpe, Hałenda and Szczepański of a flat manifold M of real dimension 6 with holonomy equal to D_4 (see [DHS08]) would give the desired Manifold (which is projective, as remarked by Kotschick¹, being Kähler with second Betti number = 2). We describe all such examples explicitly, following the method of Lange, which was based on the classification of automorphisms of complex tori of dimension 2 given by Fujiki in [Fu88].

The family we give is exactly the one obtained by taking all possible complex structures on the flat manifold M , and the upshot is that all these hyperelliptic complex manifolds X are quotients of the product of three elliptic curves by a translation of order 2.

1. THE EXAMPLE

Let E, E' be any two elliptic curves,

$$E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \quad E' = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau').$$

Set

$$A' := E \times E \times E', \quad A := A'/\langle \omega \rangle, \quad \text{where } \omega := (1/2, 1/2, 0).$$

Theorem 1.1. *The Abelian variety A admits a fixed point free action of the dihedral group*

$$D_4 := \langle r, s \mid r^4 = 1, s^2 = 1, (rs)^2 = 1 \rangle,$$

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such that D_4 contains no translations.

Proof. Set, for $z := (z_1, z_2, z_3) \in A'$:

$$\begin{aligned} r(z_1, z_2, z_3) &:= (z_2, -z_1, z_3 + 1/4) = R(z_1, z_2, z_3) + (0, 0, 1/4) \\ s(z_1, z_2, z_3) &:= (z_2 + b_1, z_1 + b_2, -z_3) = S(z_1, z_2, z_3) + (b_1, b_2, 0), \\ &\text{where } b_1 := 1/2 + \tau/2, \quad b_2 := \tau/2. \end{aligned}$$

Step 1. It is easy to verify that r, R have order exactly 4 on A' , and that $R(\omega) = \omega$, so that r descends to an automorphism of A , of order exactly 4. Moreover, any power r^j , $0 < j \leq 3$ acts freely on A , since the third coordinate of $r^j(z)$ equals $z_3 + j/4$.

Step 2. $s^2(z) = z + \omega$, since $b_1 + b_2 = 1/2$; moreover $S(\omega) = \omega$, hence s descends to an automorphism of A of order exactly 2.

Step 3. We have

$$rs(z) = (z_1 + b_2, -z_2 - b_1, -z_3 + 1/4),$$

hence

$$(rs)^2(z) = (z_1 + 2b_2, z_2, z_3) = z,$$

and we have an action of D_4 on A , since the respective orders of r, s, rs are precisely 4, 2, 2.

Step 4. We claim that also the symmetries in D_4 act freely on A and are not translations. Since there are exactly two conjugacy classes of symmetries, those of s and rs , it suffices to observe that these two transformations are not translations: in the next step we show that they both act freely.

Step 5. It is rather immediate to see that rs acts freely, since $rs(z) = z$ is equivalent to

$$(b_2, -2z_2 - b_1, -2z_3 + 1/4)$$

being a multiple of ω in A' . But this is absurd, since $2\omega = 0$, and $b_2 = \tau/2 \neq 0, 1/2$.

The condition that s acts freely, since $s(z) = z$ is equivalent to

$$\gamma := (z_2 - z_1 + b_1, z_1 - z_2 + b_2, -2z_3)$$

being a multiple of ω in A' .

However the sum of the first two coordinates of multiples of ω is 0, and this equation is not satisfied by γ , since $b_1 + b_2 = 1/2 \neq 0$. □

Corollary 1.1. The above family of hyperelliptic threefolds X with group D_4 forms a complete two dimensional family. The Kähler manifolds with the same fundamental group as X yield an open subspace of the Teichmüller space of X parametrized by the two halfplanes containing τ, τ' respectively.

Proof. Take the above family of varieties $X = A/G$, where $G = D_4$, and observe that, setting $U := \mathbb{C}^3$, $H^1(\Theta_A)^G = (U \otimes \overline{U}^\vee)^G$ can be calculated as follows. We have $U = U_1 \oplus W$, where U_1, W are real (self-conjugate) representations, U_1 is irreducible and W is a character of $G/\langle r \rangle$.

Hence $(U \otimes \overline{U}^\vee)^G = (U \otimes U^\vee)^G = \text{End}(U_1 \oplus W)^G$ has dimension 2 by Schur's lemma.

Following Theorem 1 of [CC17], and since, as we show below in proposition 1.3, there is only one possible Hodge-Type, we conclude that the open subspace of the Teichmüller space of X corresponding to Kähler manifolds is irreducible and equal to the product of two halfplanes. \square

Remark 1.2. Let

$$\mathcal{R} := \mathcal{R}_4 := \mathbb{Z}[x]/(x^2 + 1)$$

be the 4th cyclotomic ring, also called the ring of Gaussian integers.

We denote by σ the Galois involution sending $z = a + xb \mapsto \sigma(z) := a - xb$.

We define, according to Dekimpe et al. ([DHS08]) the following \mathcal{R} -module:

$$L := \mathcal{R} \oplus \mathcal{R} \oplus (\mathbb{Z}e_5 \oplus \mathbb{Z}e_6) =: L_1 \oplus L_2 \oplus L_3,$$

where the module L_3 is the trivial \mathcal{R} -module.

The real torus $T := (L \otimes \mathbb{R})/L$ admits a free action of the dihedral group D_4 , defined as follows:

$$\begin{aligned} r(z_1, z_2, z_3) &= (xz_1 + x/2, xz_2 + 1/2, z_3 - 1/4e_5), \\ s_2(z_1, z_2, z_3) &= (x\sigma(z_1), \sigma(z_2) + 1/2, -z_3). \end{aligned}$$

It is easy to see that the flat manifolds A/D_4 are the same as the flat manifolds T/D_4 .

In fact, we define $\omega_1, \omega_2 \in \mathbb{C}^3$ as the vectors

$$\omega_1 := (1/2, 1/2, 0), \omega_2 := (\tau, 0, 0).$$

Then we set L_1 to be the free \mathcal{R} -module generated by ω_1 , L_2 the free \mathcal{R} -module generated by ω_2 , and L_3 the trivial \mathcal{R} -module generated by $(0, 0, 1)$ and $(0, 0, \tau')$.

We have then that A is the quotient \mathbb{C}^3/L .

Proposition 1.3. The above \mathcal{R} -module L admits a unique Hodge-Type. More precisely, the D_4 -invariant complex structures form a 2-dimensional complex family, obtained choosing respective 1 dimensional subspaces $U(1) \subset V(1) = V_3$ and $U(i) \subset V(i)$, such that

$$(**) \quad U(1) \oplus \overline{U(1)} = V(1), \quad U(i) \oplus \overline{sU(i)} = V(i),$$

and defining $U(-i) := sU(i)$.

Proof.

Consider the complex vector space

$$V := L \otimes \mathbb{C} = (L_1 \otimes \mathbb{C}) \oplus (L_2 \otimes \mathbb{C}) \oplus (L_3 \otimes \mathbb{C}),$$

where we observe that each summand is stable by the action of D_4 .

By looking at the eigenspace decomposition of V with respect to the linear action of r , given by the diagonal matrix with entries $(x, x, 1)$ we can decompose:

$$V = (V_1(i) \oplus V_1(-i)) \oplus (V_2(i) \oplus V_2(-i)) \oplus V_3.$$

The second summand is the conjugate of the first, the fourth is the conjugate of the third. To get a free action of D_4 one must give a Hodge decomposition

$$V = U \oplus \overline{U},$$

where the holomorphic subspace U must be invariant by R , and must split as the sum of three eigenspaces for R :

$$U = U(i) \oplus U(-i) \oplus U(1).$$

Let us look at $V_1(i)$: it is spanned by $(1 - ix, 0, 0)$ and the linear part of s sends it to $-i(1 + ix)$, an element in $V_1(-i)$.

Similarly S sends $V_2(i)$ to $V_2(-i)$, hence

$$SV(i) = V(-i) = \overline{V(i)}.$$

Since S preserves the complex structure U , we obtain that

$$S(U(i)) = U(-i),$$

so all eigenspaces have dimension one.

We have already seen that it must hold $U(-i) = SU(i)$, from which follows that $S(U(i)) = U(-i)$.

The condition $V = U \oplus \overline{U}$ amounts then to the two properties (**). One can directly verify that the second holds true on some open set of the Grassmannian.

However, this also follows from the explicit description of our examples. \square

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