

SEQUENTIAL PERIODS OF THE CRYSTALLINE FROBENIUS

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ABSTRACT. There is a notion of p -adic period coming from the crystalline Frobenius automorphism of the de Rham cohomology of an algebraic variety. In this paper, we consider sequences of p -adic periods, one for each prime. We study the sequences using motivic periods, and we formulate an analogue of the Grothendieck period conjecture.

1. INTRODUCTION

Let X be a smooth algebraic variety over \mathbb{Q} . The algebraic de Rham cohomology $H_{dR}(X) = \bigoplus_n H_{dR}^n(X)$ is a finite-dimensional vector space over \mathbb{Q} . For every sufficiently large prime p , there is a distinguished \mathbb{Q}_p -linear automorphism

$$F_{p,X} : H_{dR}(X) \otimes \mathbb{Q}_p \xrightarrow{\sim} H_{dR}(X) \otimes \mathbb{Q}_p,$$

the *crystalline Frobenius map*, coming from the absolute Frobenius endomorphism of the reduction of an integral model of X modulo p (see [Ked09]). If we choose a \mathbb{Q} -basis for $H_{dR}(X)$, we can represent $F_{p,X}$ as a square matrix with entries in \mathbb{Q}_p , and \mathbb{Q} -linear combinations of matrix entries are called *p -adic periods*¹ of X . Note that the \mathbb{Q} -span of matrix entries is independent of the choice of basis.

It makes sense to talk about “the same” p -adic period for different p . We consider sequences of p -adic periods, one for each sufficiently large p , living in the ring

$$\mathcal{R} := \frac{\prod_p \mathbb{Q}_p}{\bigoplus_p \mathbb{Q}_p}.$$

An element of \mathcal{R} is a prime-indexed sequence $(a_p)_p$, with $a_p \in \mathbb{Q}_p$, and two sequences are equal in \mathcal{R} if they agree for all sufficiently large p .

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¹This is one of several different meanings of the term “ p -adic period” that have appeared in the literature.

The maps $F_{p,X}$ (for all large p at once) assemble to give an \mathcal{R} -linear automorphism

$$F_{\mathcal{R},X} : H_{dR}(X) \otimes \mathcal{R} \xrightarrow{\sim} H_{dR}(X) \otimes \mathcal{R},$$

which we call the *sequential Frobenius map*. Like before, if we choose a \mathbb{Q} -basis for $H_{dR}(X)$, we can represent $F_{\mathcal{R},X}$ as a square matrix with entries in \mathcal{R} . The following definition is new.

Definition 1.1. A *sequential period* (or *\mathcal{R} -valued period*) of X is a \mathbb{Q} -linear combination of matrix entries for $F_{\mathcal{R},X}$. We denote the set of sequential periods of X by $\mathcal{P}_{\mathcal{R}}(X) \subset \mathcal{R}$.

We compute several examples of sequential periods in Sec. 2.

Remark 1.2. Sequential periods are analogous to the usual periods of X , which are complex numbers arising from the comparison between de Rham and Betti cohomology. The analogy is explained in 3.

1.1. Results. In this paper, we develop a motivic theory of sequential periods, and we formulate an analogue of the Grothendieck period conjecture for sequential periods. Our main results are several consequences of the conjecture. The conjecture has a weak form (Conjecture 1) and a strong form (Conjecture 2). Conjecture 1 predicts an answer to the question: When are two sequential periods equal? Conjecture 2 predicts an answer to the question: When are two sequential periods congruent modulo p^n for all sufficiently large p ?

Our conjectures are assertions about a category of (mixed) motives. Suppose \mathcal{M} is a Tannakian category of motives over \mathbb{Q} (the precise properties we need are given in Definition 3.1), equipped with a fibre functor ω_{dR} , the de Rham realization. Write G_{dR} for the affine group-scheme $\underline{Aut}^{\otimes}(\omega_{dR})$. The coordinate ring of G_{dR} is a commutative \mathbb{Q} -algebra \mathcal{P}^{dr} , called the *ring of de Rham periods* of \mathcal{M} (see [Bro14], §2). If \mathcal{M} satisfies the conditions of Definition 3.1, the crystalline Frobenius (for all sufficiently large p at once) gives a distinguished functorial automorphism $F_{\mathcal{R},M}$ of $\omega_{dR}(M) \otimes \mathcal{R}$ for $M \in \mathcal{M}$. These automorphisms determine a ring homomorphism

$$\text{per}_{\mathcal{R}} : \mathcal{P}^{dr} \rightarrow \mathcal{R},$$

which we call the *sequential period map*. The image of $\text{per}_{\mathcal{R}}$ is the \mathbb{Q} -span of all matrix entries for $F_{\mathcal{R},M}$ (for all $M \in \mathcal{M}$).

The weak form of our conjecture is the following assertion about \mathcal{M} .

Conjecture 1 (Sequential period conjecture). *The map $\text{per}_{\mathcal{R}}$ is injective.*

By contrast, a similar p -adic period map $\mathcal{P}^{\text{dr}} \rightarrow \mathbb{Q}_p$ is typically not injective (see Sec. 3.3).

For X smooth and projective, it is known that the restriction of $F_{p,X}$ to the n -th level of the Hodge filtration on $H_{dR}(X)$ has matrix entries divisible by p^n when p is sufficiently large. The strong form of our conjecture is a converse to this divisibility property. It asserts that if a sequential period is divisible by p^n for all sufficiently large p , then that period is motivically equivalent (in a precise sense) to something coming from the n -th level of the Hodge filtration.

To formulate this conjecture precisely, we use the decreasing filtration Fil^\bullet on \mathcal{R} given by

$$\text{Fil}^n \mathcal{R} := \{(a_p)_p : \liminf_p v_p(a_p) \geq n\}.$$

There is also a decreasing filtration Fil^\bullet on \mathcal{P}^{dr} , coming from the Hodge filtration on algebraic de Rham cohomology. The strong form of our conjecture is the following assertion about \mathcal{M} .

Conjecture 2 (Strong sequential period conjecture). *The sequential period map $\text{per}_{\mathcal{R}}$ is an embedding of filtered algebras, i.e. for every integer n ,*

$$\text{per}_{\mathcal{R}}^{-1}(\text{Fil}^n \mathcal{R}) = \text{Fil}^n \mathcal{P}^{\text{dr}}.$$

We give several consequences of Conjectures 1 and 2. We list some of them here. The first two results are consequences of Conjecture 1.

Theorem 1 (Theorem 5.8). *Assume the standard conjectures on algebraic cycles, and suppose that Conjecture 1 holds for the category of pure numerical motives over \mathbb{Q} . Let X be a smooth projective variety over \mathbb{Q} . If $\alpha, \beta \in H_{dR}(X)$ satisfy $F_{p,X}(\alpha) = p^n \beta$ for all but finitely many p , then α is algebraic.*

Theorem 2 (Corollary 5.6). *Let \mathcal{M} be a category of motives satisfying Definition 3.1. If Conjecture 1 holds for \mathcal{M} , then the Ogus realization of \mathcal{M} is a fully faithful embedding.*

The following results give consequences of Conjecture 2 applied to the categories $\mathcal{MT}(\mathbb{Z})$ and $\mathcal{MT}(\mathbb{Q})$ of mixed Tate motives over \mathbb{Z} resp. \mathbb{Q} .

Theorem 3 (Theorem 5.1). *If Conjecture 2 holds for $\mathcal{MT}(\mathbb{Q})$, then for every rational number $r \neq 0, \pm 1$, there exist infinitely many p for which $r^{p-1} \not\equiv 1 \pmod{p^2}$.*

Theorem 4 (Theorem 5.2). *If Conjecture 2 holds for $\mathcal{MT}(\mathbb{Z})$, then for every odd $k \geq 3$, there exist infinitely many primes p for which $p \nmid B_{p-k}$.*

Theorem 5 (Theorem 8.11). *Assume that the Grothendieck period conjecture (Conjecture 0 below) holds for $\mathcal{MT}(\mathbb{Z})$. Then the truth of Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$ is equivalent to Kaneko-Zagier’s conjecture on finite multiple zeta values.*

1.2. Outline. In Section 2 we compute several examples of sequential periods. Section 3 gives the properties necessary for a category of motives in order to construct the sequential period map, and to state Conjecture 1. In Section 4, we consider divisibility properties of sequential periods and state Conjecture 2.

Sections 5, 6, and 7 are independent of one another, and can be read in any order. In Section 5, we deduce several consequences of Conjectures 1 and 2. Sections 6 and 7 describe two variations of sequential periods. The first variation involves reducing sequential periods modulo p , and this variation includes Kaneko-Zagier’s finite multiple zeta values as a special case. The second variation involves allowing uniformly convergent infinite sums of sequential periods.

In Section 8, we describe the sequential periods of the category of mixed Tate motives over \mathbb{Z} .

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2. EXAMPLES

We compute some examples of sequential periods.

2.1. The projective line. The vector space $H_{dR}^2(\mathbb{P}^1)$ is 1-dimensional, and F_p acts on $H_{dR}^2(\mathbb{P}^1)$ by multiplication by p . This shows that

$$\mathbf{p} := (p)_p \in \mathcal{R}$$

is a sequential period. The complex number $2\pi i$ is a period of $H^2(\mathbb{P}^1)$, so we view \mathbf{p} as the sequential analogue of $2\pi i$. Note that \mathbf{p} , like $2\pi i$, is transcendental over \mathbb{Q} .

2.2. Point counts. Let X be a smooth projective variety over \mathbb{Q} . Choose an integral model \tilde{X} of X , and consider the sequence

$$\left(\# \tilde{X}(\mathbb{F}_p) \right)_p \in \mathcal{R}, \tag{2.1}$$

which is independent of the choice of \tilde{X} . The Lefschetz fixed point formula expresses $\# \tilde{X}(\mathbb{F}_p)$ as an alternating sum of traces of $F_{p,X}$ on cohomology of X , so (2.1) is a sequential period of X .

2.3. The p -adic logarithm. For $r \in \mathbb{Q}_{>0}$, the logarithm $\log(r) \in \mathbb{R}$ is a period. The p -adic logarithm is defined by

$$\log_p(x) = \sum_{n \geq 1} (-1)^{n+1} \frac{(x-1)^n}{n},$$

which converges p -adically if $v_p(x-1) \geq 1$. For fixed r , the element

$$\left(\log_p(r^{p-1})\right)_p \in \mathcal{R}$$

is a sequential period (see Sec. 5.1).

2.4. The ℓ -adic Frobenius. Let X be a smooth projective variety over \mathbb{Q} , and fix an integer $n \geq 0$ and a prime ℓ . The for $p \neq \ell$, the geometric Frobenius at p acting on the ℓ -adic cohomology $H_\ell^n(X; \mathbb{Q}_\ell)$ has the same characteristic polynomial as $F_{p,X}$. This implies the sequence of characteristic polynomials of the Frobenius at p acting on H_ℓ^n , as p varies, is a polynomial with coefficients in $\mathcal{P}_{\mathcal{R}}(X)$.

2.5. Modular forms. Let $f(z) = \sum_{n \geq 0} a_n e^{2\pi i n z}$ be a normalized cusp form of weight k for $\Gamma_0(N)$, with $a_n \in \mathbb{Q}$. Then there is a variety V over \mathbb{Q} (a power of a family of elliptic curves over the modular curve $X_0(N)$) such that a_p is the trace of the Frobenius at p acting on an ℓ -adic cohomology group of V , and it follows that the sequence $(a_p)_p \in \mathcal{R}$ is a sequential period.

3. MOTIVIC PERIODS AND PERIOD MAPS

Conjectures 1 and 2 are formulated relative to a category of motives. We begin by listing the properties required of this category.

Definition 3.1. We say \mathcal{M} is a *category of motives (over \mathbb{Q})* if \mathcal{M} is a neutral Tannakian category ([Del89], Definition 2.19) over \mathbb{Q} equipped with the following structure.

- (A) There is a fibre functor ω_{dR} (the de Rham realization), $M \mapsto M_{dR}$, equipped with a decreasing, separated, exhaustive filtration Fil^\bullet , the Hodge filtration. The Hodge filtration is compatible with the tensor product, and if $f : M \rightarrow N$ is a morphism in \mathcal{M} , then the induced map $f_{dR} : M_{dR} \rightarrow N_{dR}$ satisfies $f_{dR}(\text{Fil}^n M_{dR}) = f(M_{dR}) \cap \text{Fil}^n N_{dR}$ for all n .
- (B) There is a fibre functor ω_B (the Betti realization), $M \mapsto M_B$, and a functorial \mathbb{C} -linear isomorphism

$$\text{comp}_M : M_{dR} \otimes \mathbb{C} \xrightarrow{\sim} M_B \otimes \mathbb{C} \tag{3.1}$$

compatible with the tensor product.

(C) For each $M \in \mathcal{M}$, there is a distinguished automorphism

$$F_{p,M} : M_{dR} \otimes \mathbb{Q}_p \xrightarrow{\sim} M_{dR} \otimes \mathbb{Q}_p,$$

the crystalline Frobenius map, defined for all sufficiently large p . The crystalline Frobenius is functorial in M (for large p) and compatible with the tensor product.

Definition 3.1 holds, for example, for categories satisfying the axioms in [Del89], §1.3.

Remark 3.2. The data of $F_{p,M}$ is equivalent to the data of an \mathcal{R} -linear automorphism $F_{\mathcal{R},M}$ of $M_{dR} \otimes \mathcal{R}$. The *Ogus category* $\mathbf{Og}(\mathbb{Q})$ is defined to be the category whose objects are pairs (V, F) , where V is a finite-dimensional vector space over \mathbb{Q} and F is an \mathcal{R} -linear automorphism of $V \otimes \mathcal{R}$. If \mathcal{M} is a category of motives, the *Ogus realization* of \mathcal{M} is the functor $\mathcal{M} \rightarrow \mathbf{Og}(\mathbb{Q})$, $M \mapsto (M_{dR}, F_{\mathcal{R},M})$ (see [And04], Sec. 7.1.5).

For the remainder of this section, we fix a category of motives \mathcal{M} .

3.1. Complex periods. Suppose $M \in \mathcal{M}$. If we choose \mathbb{Q} -bases for M_{dR} and M_B , we can represent $comp_M$ as a square matrix with entries in \mathbb{C} . A (complex) *period* of M is a \mathbb{Q} -linear combination of matrix entries for $comp_M$.

For any two fibre functors ω and η on \mathcal{M} , the functor of \mathbb{Q} -algebras $R \mapsto \mathrm{Hom}^{\otimes}(\omega \otimes R, \eta \otimes R)$ is representable by an affine pro-algebraic scheme $\underline{\mathrm{Hom}}^{\otimes}(\omega, \eta)$ ([DM82], Theorem 3.2). The scheme $\underline{\mathrm{Hom}}^{\otimes}(\omega_{dR}, \omega_B)$ is called the *torsor of periods* of \mathcal{M} , and the coordinate ring \mathcal{P}^m of $\underline{\mathrm{Hom}}^{\otimes}(\omega_{dR}, \omega_B)$ is called the *ring of motivic periods* of \mathcal{M} . The comparison isomorphism (3.1) determines an element of $\underline{\mathrm{Hom}}^{\otimes}(\omega_{dR}, \omega_B)(\mathbb{C})$, and evaluation at this point induces a ring homomorphism

$$\mathrm{per} : \mathcal{P}^m \rightarrow \mathbb{C}, \tag{3.2}$$

the *period map* (see [Bro17], §1.2).

The image of per is the set of all periods of all objects of \mathcal{M} . We write $\mathcal{P}_{\mathbb{C}}(\mathcal{M})$ for the image of per , and we call $\mathcal{P}_{\mathbb{C}}(\mathcal{M})$ the *ring of periods of \mathcal{M}* . The Grothendieck period conjecture for \mathcal{M} is the following assertion.

Conjecture 0 (Period conjecture). *The map per is injective.*

Conjecture 0 is an algebraic independence statement for periods, and is expected to be difficult to resolve.

The ring \mathcal{P}^m is a torsor² for the affine (pro-algebraic) group scheme $\underline{Aut}^{\otimes}(\omega_B)$. If Conjecture 0 holds, then the action descends to an action of $\underline{Aut}^{\otimes}(\omega_B)$ on $\mathcal{P}_{\mathbb{C}}(\mathcal{M})$, and we get a ‘‘Galois theory of periods’’. For the category of Artin motives (that is, motives of 0-dimensional varieties), Conjecture 0 is known to hold, and per maps \mathcal{P}^m isomorphically onto $\overline{\mathbb{Q}} \subset \mathbb{C}$. The group $\underline{Aut}^{\otimes}(\omega_B)$ is canonically isomorphic to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (viewed as a profinite constant group scheme over \mathbb{Q}), and thus the Galois theory of periods for Artin motives is just the classical Galois theory (see [Bro17], §5.1).

3.2. Sequential periods. The affine group scheme $G_{dR} := \underline{Aut}^{\otimes}(\omega_{dR})$ is called the *de Rham Galois group* of \mathcal{M} , and the coordinate ring \mathcal{P}^{dr} of G_{dR} is called the *ring of de Rham periods* of \mathcal{M} . The automorphisms $F_{\mathcal{R},M}$ determine an element $F_{\mathcal{R}} \in \mathcal{P}^{\text{dr}}(\mathcal{R})$. The following definition is new.

Definition 3.3. The *sequential period map* is the ring homomorphism

$$\text{per}_{\mathcal{R}} : \mathcal{P}^{\text{dr}} \rightarrow \mathcal{R}$$

induced by evaluation at $F_{\mathcal{R}}$.

Here \mathcal{R} plays the same role as \mathbb{C} plays for the complex period map (3.2). We write $\mathcal{P}_{\mathcal{R}}(\mathcal{M}) \subset \mathcal{R}$ for the image of $\text{per}_{\mathcal{R}}$, and we call $\mathcal{P}_{\mathcal{R}}(\mathcal{M})$ the *ring of sequential periods of \mathcal{M}* . Our weaker analogue of the period conjecture for sequential periods is the following assertion about \mathcal{M} .

Conjecture 1 (Sequential period conjecture). *The map $\text{per}_{\mathcal{R}}$ is injective.*

Conjecture 0 is equivalent to the statement that $\text{per}_{\mathcal{R}}$ induces an isomorphism of algebras $\mathcal{P}^{\text{dr}} \xrightarrow{\sim} \mathcal{P}_{\mathcal{R}}(\mathcal{M})$. As with Conjecture 0, the truth of Conjecture 1 would give a Galois theory of sequential periods (see Sec. 5.5). We expect that Conjecture 1 will be difficult to resolve.

3.3. p -adic periods. Suppose p is a prime of good reduction for every object of \mathcal{M} , meaning that for every $M \in \mathcal{M}$, $F_{p,M}$ is defined and is functorial in M . In this case we get a *p -adic period map*

$$\text{per}_p : \mathcal{P}^{\text{dr}} \rightarrow \mathbb{Q}_p.$$

However, per_p is typically not injective. For example, F_p acts on the de Rham realization of the Tate motive by multiplication by p^{-1} , which is rational, so per_p fails to be injective if the Tate motive is in \mathcal{M} . It can also be shown that per_p is not injective for the category of Artin motives.

²In fact \mathcal{P}^m is a bitorsor for $\underline{Aut}^{\otimes}(\omega_B)$ and $\underline{Aut}^{\otimes}(\omega_{dR})$.

In the case of mixed Tate motives, a modified version of the p -adic period conjecture is expected to hold. For mixed Tate motives, the de Rham Galois group decomposes as a semi-direct product

$$G_{dR} = \mathbb{G}_m \ltimes U_{dR},$$

where U_{dR} is pro-unipotent. If we write A_{dR} for the coordinate ring of U_{dR} , then there is a quotient map $\mathcal{P}^{\text{dr}} \twoheadrightarrow A_{dR}$. Although the p -adic period map does not factor through A_{dR} , there is a rescaled version $\varphi_p : \mathcal{P}^{\text{dr}} \rightarrow \mathbb{Q}_p$ that does factor through A_{dR} , and it is conjectured ([Yam10], Conjecture 4) that $\varphi_p : A_{dR} \rightarrow \mathbb{Q}_p$ is injective for every p .

4. VALUATIONS OF SEQUENTIAL PERIODS

Let X be a smooth projective variety over \mathbb{Z}_p . It is known that matrix coefficients for F_p coming from the n -th level of the Hodge filtration on $H_{dR}(X)$ are divisible by p^n if $\dim(X) < p$ (see [Maz72], p. 666). We formulate a version of this statement for sequential periods. Recall that there is a decreasing filtration Fil^\bullet on \mathcal{R} given by

$$\text{Fil}^n \mathcal{R} := \left\{ (a_p)_p : \liminf_p v_p(a_p) \geq n \right\}.$$

We also denote by Fil^\bullet the Hodge filtration on $H_{dR}(X)$.

Proposition 4.1. *Suppose X is smooth and projective over \mathbb{Q} . Then for all integers $n \geq 0$, the sequential Frobenius $F_{\mathcal{R}, X}$ takes $\text{Fil}^n H_{dR}(X)$ into $H_{dR}(X) \otimes \text{Fil}^n \mathcal{R}$.*

Proof. Choose an integral model \tilde{X} of X over $\mathbb{Z}[1/N]$ for some positive integer N . For $p \nmid N$, let X_p be the reduction of \tilde{X} modulo p . There is a canonical isomorphism

$$H_{dR}(X) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H_{\text{crys}}(X_p; \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \quad (4.1)$$

Choose a basis B for $H_{dR}(X)$ with the property that $B \cap \text{Fil}^n H_{dR}(X)$ is a basis for $\text{Fil}^n H_{dR}(X)$ for all n . There is an integer $M > N$ such that for all $p > M$, (4.1) takes B to a \mathbb{Z}_p -basis of $H_{\text{crys}}(X_p; \mathbb{Z}_p)$. We choose M large enough that $\text{Fil}^M H_{dR}(X) = 0$. For $p > M$, it is known that the crystalline Frobenius on $H_{\text{crys}}(X_p; \mathbb{Z}_p)$ takes the n -th level of the Hodge filtration into $p^n H_{\text{crys}}(X_p; \mathbb{Z}_p)$. So for $p > M$, the matrix of $F_{p, X}$ with respect to our chosen basis has the property that columns corresponding to elements in $\text{Fil}^n H_{dR}(X)$ have each entry in $p^n \mathbb{Z}_p$. It follows that the corresponding columns in the matrix for $F_{\mathcal{R}, X}$ are in $\text{Fil}^n \mathcal{R}$, so that $F_{\mathcal{R}}$ takes $\text{Fil}^n H_{dR}(X)$ into $H_{dR}(X) \otimes \text{Fil}^n \mathcal{R}$. \square

The strong form of our period conjecture is a kind of converse to Proposition 4.1. Roughly speaking, the conjecture asserts that if a

sequential periods is divisible by p^n for all sufficiently large p , then that period is motivically equivalent something coming from the n -th level of the Hodge filtration.

To formulate the conjecture precisely, we fix a category \mathcal{M} of motives. The Hodge filtration on ω_{dR} induces a filtration on \mathcal{P}^{dr} . If $F_{p,M}$ takes $\text{Fil}^n M_{dR}$ into $M_{dR} \otimes \text{Fil}^n \mathcal{R}$ for all $M \in \mathcal{M}$ and $n \in \mathbb{Z}$ (Proposition 4.1 suggests we should expect this to be the case), then $\text{per}_{\mathcal{R}}$ takes $\text{Fil}^n \mathcal{P}^{\text{dr}}(M)$ into $\text{Fil}^n \mathcal{R}$. We expect that $\text{per}_{\mathcal{R}}$ is compatible with the filtrations in a stronger sense. The strong form of our period conjecture is the following assertion about \mathcal{M} .

Conjecture 2 (Strong sequential period conjecture). *For all integers n , we have*

$$\text{per}_{\mathcal{R}}^{-1}(\text{Fil}^n \mathcal{R}) = \text{Fil}^n \mathcal{P}^{\text{dr}}.$$

Conjecture 2 is equivalent to the statement that $\text{per}_{\mathcal{R}}$ induces an isomorphism of filtered algebras $\mathcal{P}^{\text{dr}} \cong \mathcal{P}_{\mathcal{R}}(\mathcal{M})$. Conjecture 2 implies Conjecture 1 because the filtration on \mathcal{P}^{dr} is separated. In some cases, it is possible to show directly that $\text{per}_{\mathcal{R}}(\text{Fil}^n \mathcal{P}^{\text{dr}}) \subset \text{Fil}^n \mathcal{R}$. This is the case for mixed Tate motives over \mathbb{Z} (see Sec. 8). The hard part of the conjecture is the converse: if $x \in \mathcal{P}^{\text{dr}}$ satisfies $\text{per}(x) \in \text{Fil}^n \mathcal{R}$, then $x \in \text{Fil}^n \mathcal{P}^{\text{dr}}$.

5. CONSEQUENCES OF THE PERIOD CONJECTURES

In this section, we describe several consequences of Conjectures 1 and 2 applied to various categories of motives.

5.1. Wieferich primes. Fix a rational number $r \neq 0, \pm 1$. Fermat's Little Theorem asserts that, for all primes p not dividing the numerator or denominator of r ,

$$r^{p-1} \equiv 1 \pmod{p}. \tag{5.1}$$

A *Wieferich prime to the base r* is a prime which (5.1) holds modulo p^2 . A heuristic argument suggests that, for each r , the set of Wieferich primes is infinite but has density 0. Little is known about the Wieferich primes, and it is an open problem to produce an r for which there are infinitely many non-Wieferich primes. It is known [Sil88] that the truth of the ABC-conjecture would imply there are infinitely many non-Wieferich primes to the base 2.

Theorem 5.1. *If Conjecture 2 holds for the category of mixed Tate motives over \mathbb{Q} , then for every rational $r \neq 0, \pm 1$, there exist infinitely many non-Wieferich primes to the base r .*

Proof. For $r \in \mathbb{Q} \setminus \{0, 1\}$, there is a Kummer motive $K(r) \in \mathcal{MT}(\mathbb{Q})$ sitting in a short exact sequence

$$0 \rightarrow \mathbb{Q}(0) \rightarrow K(r) \rightarrow \mathbb{Q}(-1) \rightarrow 0.$$

It follows from [Del89], §2.9 that with respect to an appropriate basis, the matrix for F_p on $K(r)_{dR} \otimes \mathbb{Q}_p$ is given by

$$\begin{bmatrix} 1 & \log_p(r^{p-1}) \\ 0 & p \end{bmatrix},$$

where \log_p is the p -adic logarithm. It is not hard to check that $\log_p(r^{p-1})$ is a non-zero element of $p\mathbb{Z}_p$ for all primes p at which r is a unit, and for these p , $v_p(\log_p(r^{p-1})) \geq 2$ precisely when p is a Wieferich prime to the base r . Since $\text{Fil}^2 K(r)_{dR} = 0$, Conjecture 2 implies that every non-zero sequential period of $K(r)$ has valuation exactly 1 for infinitely many p . It follows that Conjecture 2 implies that there are infinitely many non-Wieferich primes to the base r . \square

5.1.1. *Fermat's Last Theorem.* In 1909, Wieferich proved that if the first case of Fermat's Last Theorem fails for a prime p , then p must be a Wieferich prime to the base 2. So Conjecture 2 would imply that the first case of Fermat's Last Theorem is true for infinitely many p . By comparison, the first proof that the first case of Fermat's Last Theorem is true for infinitely many p appeared in 1985 [AHB85], so this gives a lower bound on the difficulty of proving Conjecture 2 for $\mathcal{MT}(\mathbb{Q})$.

5.2. **Bernoulli numbers.** The Bernoulli numbers $B_n \in \mathbb{Q}$ are a sequence of rational numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}.$$

While the denominator of B_n is the product of those primes p for which $p-1|n$, the numerators are much more mysterious. For p prime, the Herbrand-Ribet theorem gives a connection between the set of Bernoulli numbers with numerator divisible by p and the class group of the cyclotomic field $\mathbb{Q}(\zeta_p)$ (see [Her32] and [Rib76]).

Conjecture 2 applied to the category of mixed Tate motives over \mathbb{Z} has the following consequence for the Bernoulli numbers.

Theorem 5.2. *If Conjecture 2 holds for the category of mixed Tate motives over \mathbb{Z} , then for every odd $k \geq 3$, there exist infinitely primes p for which $p \nmid B_{p-k}$.*

Proof. Let \mathcal{P}^{dr} be the ring of de Rham periods of $\mathcal{MT}(\mathbb{Z})$, which is described in Sec. 8.2. There is an element $\zeta^{\text{dr}}(k) \in \text{Fil}^k \mathcal{P}^{\text{dr}} \setminus \text{Fil}^{k+1} \mathcal{P}^{\text{dr}}$, with the property that

$$\text{per}_p(\zeta^{\text{dr}}(k)) = \zeta_p(k) \equiv p^k \frac{B_{p-k}}{k} \pmod{p^{k+1}}.$$

Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$ then implies $(p^k B_{p-k}/k) \notin \text{Fil}^{k+1} \mathcal{R}$, which is the statement that there are infinitely many p not dividing B_{p-k} . \square

Theorem 5.2 can be generalized considerably.

Theorem 5.3. *Assume the truth of Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$, and let $f \in \mathbb{Q}[x_1, \dots, x_n]$ be a non-zero polynomial. Then there are infinitely many primes p for which p does not divide the numerator of*

$$f(B_{p-3}, B_{p-5}, \dots, B_{p-2n-1}).$$

The proof is given in Sec. 8.5.

5.3. Fullness of the Ogus realization. The following definition is a sequential analogue of a definition in [Maz72].

Definition 5.4. A *sequential span* is a triple (V, W, F) , where V and W are finite-dimensional vector spaces over \mathbb{Q} and $F : V \otimes \mathcal{R} \xrightarrow{\sim} W \otimes \mathcal{R}$ is an \mathcal{R} -linear isomorphism. The collection of all sequential spans forms a neutral Tannakian category, which is denoted $\mathbf{Span}(\mathbb{Q})$

There is a functor $\mathbf{Og}(\mathbb{Q}) \rightarrow \mathbf{Span}(\mathbb{Q})$, $(V, F) \mapsto (V, V, F)$ that “forgets” that the two vector spaces are the same.

Theorem 5.5. *Conjecture 1 holds for \mathcal{M} if and only if the functor $\mathcal{M} \rightarrow \mathbf{Span}(\mathbb{Q})$, $M \mapsto (M_{dR}, M_{dR}, F_{\mathcal{R}, M})$ is full.*

Proof. The Betti-de Rham category over \mathbb{Q} is the category whose objects are triples (V, W, c) , where V and W are finite-dimensional vector spaces over \mathbb{Q} and

$$c : V \otimes \mathbb{C} \xrightarrow{\sim} W \otimes \mathbb{C}.$$

It is well-known (see e.g. [HMS17], Proposition 13.2.8) that Conjecture 0 is equivalent to the assertion that the functor from \mathcal{M} to the Betti-de Rham category is full. The same proof works here. \square

Corollary 5.6. *Conjecture 1 implies the Ogus realization is full.*

Proof. Fullness of $\mathcal{M} \rightarrow \mathbf{Span}(\mathbb{Q})$ implies fullness of $\mathcal{M} \rightarrow \mathbf{Og}(\mathbb{Q})$ because $\mathbf{Og}(\mathbb{Q}) \rightarrow \mathbf{Span}(\mathbb{Q})$ is faithful. \square

Remark 5.7. Conjecture 1 is a priori stronger than the fullness of the Ogus realization. Conjecture 1 is equivalent to the statement that the crystalline Frobenius elements of G_{dR} are Zariski dense, whereas the fullness of the Ogus realization is equivalent to the statement that the crystalline Frobenius elements generate a Zariski dense subgroup of G_{dR} .

5.4. Existence of algebraic cycles. Here we describe a consequence of Conjecture 1 for algebraic cycles. We will need to assume the standard conjectures on algebraic cycles. Specifically, we assume that for every smooth projective variety over \mathbb{Q} ,

- (1) The Künneth projectors are algebraic.
- (2) Numerical equivalence of cycles equals homological equivalence.

It is shown in [Jan92] (Corollary 2) that condition (1) above implies that the category of pure numerical motives over \mathbb{Q} (with coefficients in \mathbb{Q}) is neutral Tannakian. Condition (2) implies that ω_{dR} and ω_B are fibre functors, and the de Rham-Betti comparison and crystalline Frobenius map come from the corresponding operations on cohomology.

Theorem 5.8. *Assume that the assertions (1) and (2) above hold, and let \mathcal{M} be the category of pure numerical motives over \mathbb{Q} . The truth of Conjecture 1 for \mathcal{M} is equivalent to the assertion that, whenever X is a smooth projective variety, $n \in \mathbb{Z}_{\geq 0}$, and $\alpha, \beta \in H_{dR}^{2n}(X)(n)$ satisfy $F_{p,X}(\alpha) = \beta$ for all but finitely many p , then α is algebraic (so in fact $\alpha = \beta$).*

Here $H_{dR}(X)(n)$ is a Tate twist, meaning $F_{p,X}$ is scaled by p^{-n} .

Proof. By Theorem 5.5, Conjecture 1 for \mathcal{M} is equivalent to the assertion that $\mathcal{M} \rightarrow \mathbf{Span}(\mathbb{Q})$ is full. For $M \in \mathcal{M}$, write $Sp(M)$ for the sequential span associated with M . Then $\mathcal{M} \rightarrow \mathbf{Span}(\mathbb{Q})$ is full if and only if, for every $M \in \mathcal{M}$,

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}(0), M) = \mathrm{Hom}_{\mathbf{Span}(\mathbb{Q})}(Sp(\mathbb{Q}(0)), Sp(M)).$$

It suffices to consider only those M of the form $H(X)(n)$. Morphisms of motives $\mathbb{Q}(0) \rightarrow H(X)(n)$ are in natural bijection with algebraic classes in $H_{dR}^{2n}(X)$. Morphisms of sequential spans $Sp(\mathbb{Q}(0)) \rightarrow Sp(H(X)(n))$ correspond to pairs $\alpha, \beta \in H_{dR}(X)(n)$ such that $F_{\mathcal{R}}(\alpha) = \beta$, and such α and β must live in $H_{dR}^{2n}(X)(n)$ for weight reasons. This completes the proof. \square

Remark 5.9. An *Ogus cycle* is an element $\alpha \in H_{dR}^{2n}(X)(n)$ such that $F_{p,X}(\alpha) = \alpha$ for all sufficiently large p . It is conjectured that every Ogus cycle is algebraic (see [Ogu82], §4), and statement is equivalent to the fullness of the Ogus realization.

5.5. Galois theory. Let \mathcal{M} be a category of motives. As the coordinate ring of an affine group scheme, \mathcal{P}^{dr} has the structure of a commutative Hopf algebra over \mathbb{Q} . The ring of motivic periods \mathcal{P}^{m} is an algebra-comodule over \mathcal{P}^{dr} , and the corresponding group action

$$\text{Spec } \mathcal{P}^{\text{dr}} \curvearrowright \text{Spec } \mathcal{P}^{\text{m}}$$

makes $\text{Spec } \mathcal{P}^{\text{m}}$ into a torsor for $\text{Spec } \mathcal{P}^{\text{dr}}$.

If Conjecture 1 holds for \mathcal{M} , the Hopf algebra structure descends to $\mathcal{P}_{\mathcal{R}}(\mathcal{M})$. If in addition Conjecture 0 holds for \mathcal{M} , then the coaction also descends. In this case we get ring homomorphisms

$$\Delta_{\mathcal{R}} : \mathcal{P}_{\mathcal{R}}(\mathcal{M}) \rightarrow \mathcal{P}_{\mathcal{R}}(\mathcal{M}) \otimes \mathcal{P}_{\mathcal{R}}(\mathcal{M}), \quad (5.2)$$

$$\Delta_{\mathbb{C}} : \mathcal{P}_{\mathbb{C}}(\mathcal{M}) \rightarrow \mathcal{P}_{\mathcal{R}}(\mathcal{M}) \otimes \mathcal{P}_{\mathbb{C}}(\mathcal{M}), \quad (5.3)$$

making $\mathcal{P}_{\mathcal{R}}(\mathcal{M})$ into a Hopf algebra and making $\mathcal{P}_{\mathbb{C}}(\mathcal{M})$ into an algebra-comodule for $\mathcal{P}_{\mathcal{R}}(\mathcal{M})$.

In practice, (5.2) and (5.3) can be computed explicitly. The computation depends a priori on some choices, and Conjecture 0 and 1 imply that the result is independent of the choices. Concretely, suppose $M \in \mathcal{M}$. Given $\omega \in M_{dR}$ and $\eta \in M_{dR}^*$ (where M_{dR}^* is the \mathbb{Q} -linear dual), we get a sequential period of M

$$\langle \omega, \eta \rangle_{\mathcal{R}} := \left(\langle F_{p,M} \omega, \eta \rangle \right)_p \in \mathcal{R},$$

and $\mathcal{P}_{\mathcal{R}}(\mathcal{M})$ is spanned over \mathbb{Q} by elements of this form. To compute the comultiplication (5.2), we choose a basis $\{v\}$ of M_{dR} , with dual basis $\{v^*\}$, and we have

$$\Delta_{\mathcal{R}} \left(\langle \omega, \eta \rangle_{\mathcal{R}} \right) = \sum_v \langle \omega, v^* \rangle_{\mathcal{R}} \otimes \langle v, \eta \rangle_{\mathcal{R}}. \quad (5.4)$$

Similarly, for $\omega \in M_{dR}$ and $\gamma \in M_B^*$, we get a complex period of M

$$\langle \omega, \gamma \rangle_{\mathbb{C}} := \langle \text{comp}(\omega), \eta \rangle \in \mathbb{C}.$$

The coaction (5.3) is given by

$$\Delta_{\mathbb{C}} \left(\langle \omega, \gamma \rangle_{\mathbb{C}} \right) = \sum_v \langle \omega, v^* \rangle_{\mathcal{R}} \otimes \langle v, \gamma \rangle_{\mathbb{C}}. \quad (5.5)$$

Remark 5.10. The formulas (5.4) and (5.5) for the comultiplication and coaction make sense for varieties, even without an underlying category of motives.

6. \mathcal{A} -VALUED PERIODS

Kaneko and Zagier (unpublished) observe that certain elements of the ring

$$\mathcal{A} := \frac{\prod_p \mathbb{Z}/p\mathbb{Z}}{\bigoplus_p \mathbb{Z}/p\mathbb{Z}} \cong \frac{\mathrm{Fil}^0 \mathcal{R}}{\mathrm{Fil}^1 \mathcal{R}}$$

are analogous to periods. In this section we construct a version of sequential periods living in \mathcal{A} . Applied to the category of mixed Tate motives over \mathbb{Z} , our construction recovers the finite multiple zeta values of Kaneko-Zagier (see Sec. 8.5).

We first give a concrete definition for varieties. Suppose X is a smooth projective variety over \mathbb{Q} . By Proposition 4.1, $\mathbf{p}^{-n} F_{\mathcal{R}, X}$ takes $\mathrm{Fil}^n H_{dR}(X)$ into $H_{dR}(X) \otimes \mathrm{Fil}^0 \mathcal{R}$, and we write

$$F_{\mathcal{A}, X}^{(n)} : \mathrm{Fil}^n H_{dR}(X) \rightarrow H_{dR}(X) \otimes \mathcal{A}$$

for the composition of $\mathbf{p}^{-n} F_{\mathcal{R}, X} : \mathrm{Fil}^n H_{dR} \rightarrow H_{dR}(X) \otimes \mathrm{Fil}^0 \mathcal{R}$ with the projection $\mathrm{Fil}^0 \mathcal{R} \rightarrow \mathcal{A}$. If we choose a \mathbb{Q} -basis for $\mathrm{Fil}^n H_{dR}(X)$, and extend it to a basis of $H_{dR}(X)$, we can represent $F_{\mathcal{A}, X}^{(n)}$ by a non-square matrix with entries in \mathcal{A} .

Definition 6.1. An \mathcal{A} -valued period of X is a \mathbb{Q} -linear combination of matrix coefficients for $F_{\mathcal{A}, X}^{(n)}$ (as n varies). We denote the set of \mathcal{A} -valued periods of X by $\mathcal{P}_{\mathcal{A}}(X) \subset \mathcal{A}$.

Concretely, the matrix for the sequential Frobenius on $H_{dR}(X)$ has columns that are divisible by various powers of p , according to the Hodge filtration on $H_{dR}(X)$, and we get an \mathcal{A} -valued period by taking a matrix entry whose column is in the n -th level of the Hodge filtration, dividing by p^n , and reducing modulo p .

To understand \mathcal{A} -valued periods in terms of motivic periods, we need a category \mathcal{M} of motives with Ogus realization that is known to satisfy the easier direction of Conjecture 2. We make the following definition, which is the sequential version of a definition in [Maz72], p. 665.

Definition 6.2. Let \mathcal{M} be a category of motives. We say the Ogus realization on \mathcal{M} is *divisible* if $\mathrm{per}_{\mathcal{R}}(\mathrm{Fil}^n \mathcal{P}^{\mathrm{or}}) \subset \mathrm{Fil}^n \mathcal{R}$ for all n .

For example, it is known that the category mixed Tate motives over \mathbb{Z} has divisible Ogus realization (see Sec. 8). The truth of the standard conjectures on algebraic cycles would imply that the Ogus realization for pure numerical motives is divisible.

For the remainder of this section, we assume \mathcal{M} is a category of motives equipped with a divisible Ogus realization. Consider

$$\mathcal{P}_0^{\text{dr}} := \frac{\text{Fil}^0 \mathcal{P}^{\text{dr}}}{\text{Fil}^1 \mathcal{P}^{\text{dr}}},$$

which is the degree 0 part of the associated graded ring of \mathcal{P}^{dr} . The following definition is new.

Definition 6.3. The \mathcal{A} -valued period map is the ring homomorphism

$$\text{per}_{\mathcal{A}} : \mathcal{P}_0^{\text{dr}} \rightarrow \mathcal{A}$$

induced by $\text{per}_{\mathcal{R}}$. We write $\mathcal{P}_{\mathcal{A}}$ for the image of $\text{per}_{\mathcal{A}}$, and we call $\mathcal{P}_{\mathcal{A}}$ the *ring of \mathcal{A} -valued periods* of \mathcal{M} .

We expect that the map $\text{per}_{\mathcal{A}}$ is injective (this statement could be called the *\mathcal{A} -valued period conjecture*). Injectivity of $\text{per}_{\mathcal{A}}$ is essentially equivalent to Conjecture 2.

Theorem 6.4. *Conjecture 2 implies that $\text{per}_{\mathcal{A}}$ is injective. The converse is true if \mathcal{M} admits Tate twists.*

Proof. It is immediate that Conjecture 2 implies $\text{per}_{\mathcal{R}}$ is injective. Conversely, suppose that $\text{per}_{\mathcal{R}}$ is injective, and suppose $\alpha \in \mathcal{P}^{\text{dr}}$ satisfies $\text{per}_{\mathcal{R}}(\alpha) \in \text{Fil}^n \mathcal{R}$. Let $m = \max\{M \leq n : \alpha \in \text{Fil}^M \mathcal{P}^{\text{dr}}\}$. We will show that $m = n$.

For the sake of contradiction, suppose $m < n$. By assumption there is a rank-1 object $\mathbb{Q}(-1) \in \mathcal{M}$ (the Lefschetz motive), whose de Rham period $\mathbb{L}^{\text{dr}} \in \mathcal{P}^{\text{dr}}$ is mapped to \mathfrak{p} by $\text{per}_{\mathcal{R}}$. We have $(\mathbb{L}^{\text{dr}})^{-m} \alpha \in \text{Fil}^0 \mathcal{P}^{\text{dr}}$, and

$$\text{per}_{\mathcal{R}}((\mathbb{L}^{\text{dr}})^{-m} \alpha) = \mathfrak{p}^{-m} \text{per}_{\mathcal{R}}(\alpha) \in \text{Fil}^{n-m} \mathcal{R} \subset \text{Fil}^1 \mathcal{R}.$$

This shows $\text{per}_{\mathcal{A}}((\mathbb{L}^{\text{dr}})^{-m} \alpha) = 0$. Because $\text{per}_{\mathcal{A}}$ is injective by assumption, we conclude $(\mathbb{L}^{\text{dr}})^{-m} \alpha \in \text{Fil}^1 \mathcal{P}^{\text{dr}}$. It follows that $\alpha \in \text{Fil}^{m+1} \mathcal{P}^{\text{dr}}$, contradicting the maximality of m . \square

The finite multiple zeta values of Kaneko-Zagier are the \mathcal{A} -valued periods of the category of mixed Tate motives over \mathbb{Z} (Theorem 8.10 below). The category $\mathcal{AM}_{\mathbb{Q}}$ of Artin motives over \mathbb{Q} has trivial Hodge filtration, and for this category the \mathcal{A} -valued period map is a map $\text{per}_{\mathcal{R}} : \mathcal{P}^{\text{dr}} \rightarrow \mathcal{A}$. The paper [Ros18b] involves an application of the \mathcal{A} -valued period map for $\mathcal{AM}_{\mathbb{Q}}$ to an analogue of the Skolem-Mahler-Lech theorem.

Remark 6.5. If one is interested in congruences modulo p^n , one can consider the ring

$$\mathcal{A}_n = \frac{\prod_p \mathbb{Z}/p^n \mathbb{Z}}{\bigoplus_p \mathbb{Z}/p^n \mathbb{Z}},$$

and there is an \mathcal{A}_n -valued period map

$$\text{per}_{\mathcal{A}_n} : \frac{\text{Fil}^0 \mathcal{P}^{\text{dr}}}{\text{Fil}^n \mathcal{P}^{\text{dr}}} \rightarrow \mathcal{A}_n.$$

If \mathcal{M} admits Tate twists, then Conjecture 2 is equivalent to the statement that $\text{per}_{\mathcal{A}_n}$ is injective for one (equivalently, every) positive integer n .

7. $\widehat{\mathcal{R}}$ -VALUED PERIODS

Some arithmetically interesting quantities can be expressed as infinite sums of p -adic periods in a manner that is uniform in p (we give some examples in Sec. 8.6). For example, for $k \geq 2$ an integer, the p -adic zeta value $\zeta_p(k)$ is a p -adic period, and a result of Washington [Was98] expresses a harmonic number in terms of $\zeta_p(k)$:

$$p^s \sum_{\substack{n=1 \\ p \nmid n}}^{pr} \frac{1}{n^s} = \sum_{k=0}^{\infty} (-1)^k \binom{r+k}{k+1} r^{k+1} \zeta_p(s+k+1), \quad (7.1)$$

This formula has a generalization due to Jarossay [Jar15b].

Here we describe a version of sequential periods for treating these infinite sums. This notion of period was considered in [Ros18a] for mixed Tate motives over \mathbb{Z} .

The filtration Fil^\bullet on \mathcal{R} is neither exhaustive nor separated. To deal with infinite sums of sequential periods, we replace \mathcal{R} with

$$\widehat{\mathcal{R}} := \frac{\bigcup_n \text{Fil}^n \mathcal{R}}{\bigcap_n \text{Fil}^n \mathcal{R}} = \frac{\left\{ (a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \text{ bounded below} \right\}}{\left\{ (a_p) \in \prod_p \mathbb{Q}_p : v_p(a_p) \rightarrow \infty \text{ as } p \rightarrow \infty \right\}}.$$

We write Fil^\bullet for the induced filtration on $\widehat{\mathcal{R}}$, which is exhaustive and separated. The ring $\widehat{\mathcal{R}}$ is complete with respect to the topology induced by Fil^\bullet .

Fix a category of motives \mathcal{M} with divisible Ogus realization (Definition 6.2). This means $\text{per}_{\mathcal{R}}$ takes $\text{Fil}^n \mathcal{P}^{\text{dr}}$ into $\text{Fil}^n \mathcal{R}$ for each n , so $\text{per}_{\mathcal{R}}$ induces a continuous homomorphism of the completions.

Definition 7.1. The *completed sequential period map* is the continuous ring homomorphism

$$\text{per}_{\widehat{\mathcal{R}}} : \widehat{\mathcal{P}}^{\text{dr}} \rightarrow \widehat{\mathcal{R}},$$

where $\widehat{\mathcal{P}}^{\text{dr}}$ is the completion of \mathcal{P}^{dr} with respect to the Hodge filtration. We write $\mathcal{P}_{\widehat{\mathcal{R}}}(\mathcal{M})$ for the image of $\text{per}_{\widehat{\mathcal{R}}}$, and we call $\mathcal{P}_{\widehat{\mathcal{R}}}(\mathcal{M})$ the *ring of $\widehat{\mathcal{R}}$ -valued periods* of \mathcal{M} .

In terms of the example (7.1), we will see in Sec. 8 that $\zeta_p(k)$ is a p -adic period of the category of mixed Tate motives over \mathbb{Z} . For this category, there are elements $\zeta^{\text{dr}}(k) \in \mathcal{P}^{\text{dr}}$ that $\text{per}_{\mathcal{R}}$ maps to $(\zeta_p(k))_p$, and (7.1) implies $\text{per}_{\mathcal{R}}$ takes the element

$$\sum_{k=0}^{\infty} (-1)^k \binom{r+k}{k+1} r^{k+1} \zeta^{\text{dr}}(s+k+1) \in \widehat{\mathcal{P}}^{\text{dr}}$$

to

$$\left(p^s \sum_{n=1}^{pr} \frac{1}{n^s} \right)_p \in \mathcal{R}.$$

Remark 7.2. Conjecture 2 for \mathcal{M} is equivalent to the statement $\text{per}_{\widehat{\mathcal{R}}}$ induces an isomorphism of filtered algebras $\widehat{\mathcal{P}}^{\text{dr}} \cong \mathcal{P}_{\widehat{\mathcal{R}}}(\mathcal{M})$. In particular, Conjecture 2 would imply $\mathcal{P}_{\widehat{\mathcal{R}}}(\mathcal{M})$ is complete with respect to Fil^\bullet .

8. MIXED TATE MOTIVES OVER \mathbb{Z}

The theory of mixed Tate motives and their periods is well-developed. In this section, we compute the sequential periods of $\mathcal{MT}(\mathbb{Z})$. We also compute the \mathcal{A} -valued and $\widehat{\mathcal{R}}$ -valued periods of $\mathcal{MT}(\mathbb{Z})$.

Mixed Tate motives are described in [Del89]. In [DG05] a construction is given for the category of mixed Tate motives unramified over the ring of S -integers in a number field. Theory of motivic iterated integrals appears in [Gon05]. A computation of the periods of $\mathcal{MT}(\mathbb{Z})$ is given in [Bro12]. The crystalline realization (for each prime p) on $\mathcal{MT}(\mathbb{Z})$ is constructed in [Yam10], and these assemble to give an Ogus realization.

Recall that a *composition* is a finite ordered list $\underline{s} = (s_1, \dots, s_k)$ of positive integers. The *weight* of \underline{s} is $|\underline{s}| := s_1 + \dots + s_k$. The various types of periods of $\mathcal{MT}(\mathbb{Z})$ are indexed by compositions.

8.1. Motivic and complex periods. For $\underline{s} = (s_1, \dots, s_k)$ a composition satisfying $s_1 \geq 2$, the *multiple zeta value* (or *MZV*) is

$$\zeta(\underline{s}) := \sum_{n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}} \in \mathbb{R}.$$

Using the so-called shuffle regularization, it is possible to extend the definition to the case $s_1 = 1$. It is known the the MZVs are periods of mixed Tate motives over \mathbb{Z} .

The ring \mathcal{P}^m of motivic periods of $\mathcal{MT}(\mathbb{Z})$ contains elements $\zeta^m(\underline{s})$, the *motivic MZVs*, as \underline{s} ranges over the compositions. The period map takes $\zeta^m(\underline{s})$ to $\zeta(\underline{s})$. The elements $\zeta^m(\underline{s})$ satisfy many algebraic relations, and a relation satisfied by $\zeta(\underline{s})$ is called *motivic* if the corresponding relation holds for $\zeta^m(\underline{s})$. There is also an invertible element $\mathbb{L} \in \mathcal{P}^m$, the *motivic Lefschetz period*, which maps to $2\pi i$ under the period map. Euler's calculation that $\zeta(2) = \pi^2/6$ lifts to the motivic setting, and we have $\zeta^m(2) = -(\mathbb{L}^m)^2/24$.

Proposition 8.1 (Brown [Bro12]). *The ring of motivic periods of $\mathcal{MT}(\mathbb{Z})$ is spanned as a $\mathbb{Q}[\mathbb{L}^m, (\mathbb{L}^m)^{-1}]$ -module by the motivic MZVs.*

Corollary 8.2. *The ring of complex periods of $\mathcal{MT}(\mathbb{Z})$ is spanned as a $\mathbb{Q}[2\pi i, (2\pi i)^{-1}]$ -module by the MZVs.*

There is a \mathbb{Z} -grading on \mathcal{P}^m called the grading by weight,³ where \mathbb{L} has weight 1 and $\zeta^m(\underline{s})$ has weight \underline{s} . One often works with the subalgebra $\mathcal{H} \subset \mathcal{P}^m$ spanned by the motivic MZVs. The ring \mathcal{H} is called the *ring of motivic MZVs*, and it has the advantage of being \mathbb{N} -graded with finite-dimensional graded pieces.

8.2. De Rham and p -adic periods. For \underline{s} a composition, there are p -adic analogues $\zeta_p(\underline{s}) \in \mathbb{Q}_p$ of the MZVs,⁴ arising from the action of Frobenius on the crystalline fundamental group of the thrice punctured line (see [Del02], or §5.28 of [DG05]). The p -adic MZVs are p -adic periods of $\mathcal{MT}(\mathbb{Z})$. Results of Jarossay [Jar15a] give explicit computations of $\zeta_p(\underline{s})$.

The ring \mathcal{P}^{dr} of de Rham periods of $\mathcal{MT}(\mathbb{Z})$ contains elements $\zeta^{\text{dr}}(\underline{s})$, the *de Rham MZVs*, as \underline{s} ranges over the compositions. Objects of $\mathcal{MT}(\mathbb{Z})$ are unramified at every prime, so for each p there is a p -adic period map $\text{per}_p : \mathcal{P}^{\text{dr}} \rightarrow \mathbb{Q}_p$, which takes $\zeta^{\text{dr}}(\underline{s})$ to $\zeta_p(\underline{s})$. There is an invertible element \mathbb{L}^{dr} , the de Rham Lefschetz period, which the period map takes to p .

Proposition 8.3 (Brown [Bro14]). *The ring of de Rham periods of $\mathcal{MT}(\mathbb{Z})$ is spanned as a $\mathbb{Q}[\mathbb{L}^{\text{dr}}, (\mathbb{L}^{\text{dr}})^{-1}]$ -module by the de Rham MZVs.*

Corollary 8.4. *The ring of p -adic periods of $\mathcal{MT}(\mathbb{Z})$ is spanned over \mathbb{Q} by the p -adic MZVs.*

The de Rham MZVs satisfy the same \mathbb{Q} -linear relations as the motivic MZVs, along with the additional relation $\zeta^{\text{dr}}(2) = 0$, thus the p -adic MZVs also satisfy these relations.

³The weight here is one half of the motivic weight

⁴The p -adic MZVs we use here were constructed by Deligne, but there is a different version due to Furusho [Fur04].

Remark 8.5. The p -adic period map is not injective because it kills $\mathbb{L}^{\text{dr}} - p$. It is conjectured that for each p , the kernel of per_p is generated as an ideal by $\mathbb{L}^{\text{dr}} - p$.

8.3. Sequential periods. The sequential period map $\text{per}_{\mathcal{R}}$ takes $\zeta^{\text{dr}}(\underline{s})$ to the *sequential MZV*

$$\zeta_{\mathbf{p}}(\underline{s}) := (\zeta_p(\underline{s}))_{\mathbf{p}} \in \mathcal{R},$$

and takes \mathbb{L}^{dr} to \mathbf{p} .

Proposition 8.6. *The ring of sequential periods of $\mathcal{MT}(\mathbb{Z})$ is spanned as a $\mathbb{Q}[\mathbf{p}, \mathbf{p}^{-1}]$ -module by the sequential MZVs.*

Proof. Follows from Proposition 8.3 by applying $\text{per}_{\mathcal{R}}$. \square

Like \mathcal{P}^m , the ring \mathcal{P}^{dr} is graded by weight, where \mathbb{L}^{dr} has weight 1 and $\zeta^{\text{dr}}(\underline{s})$ has weight $|\underline{s}|$. The Hodge filtration is induced by the grading: $\text{Fil}^n \mathcal{P}^{\text{dr}}$ is spanned by elements of weight n and higher. It is known [Cha17] that the p -adic multiple zeta values satisfy

$$\zeta_p(\underline{s}) \in p^{|\underline{s}|} \mathbb{Z}_p \tag{8.1}$$

for all $p > |\underline{s}|$, which means that the Ogus realization on $\mathcal{MT}(\mathbb{Z})$ is divisible, i.e. $\text{per}_{\mathcal{R}}(\text{Fil}^n \mathcal{P}^{\text{dr}}) \subset \text{Fil}^n \mathcal{R}$ for all n . So \mathcal{A} -valued periods (Sec. 6) and $\widehat{\mathcal{R}}$ -valued periods (Sec. 7) are defined for $\mathcal{MT}(\mathbb{Z})$.

The statement that $\text{per}_{\widehat{\mathcal{R}}}$ is injective for $\mathcal{MT}(\mathbb{Z})$, which is equivalent to Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$, was stated by the author in [Ros18a] (Conjecture 1.3). A related conjecture can be found in [Jar16] (Conjecture 7.7).

8.4. Multiple harmonic sums. Multiple harmonic sums are truncated versions of the MZVs. As we will see, they are related to the \mathcal{A} -valued periods and $\widehat{\mathcal{R}}$ -valued periods of $\mathcal{MT}(\mathbb{Z})$.

Definition 8.7. Let $\underline{s} = (s_1, \dots, s_k)$ be a composition and N a positive integer. The quantity

$$H_N(s_1, \dots, s_k) := \sum_{N \geq n_1 > \dots > n_k \geq 1} \frac{1}{n_1^{s_1} \dots n_k^{s_k}} \in \mathbb{Q}$$

is called a *multiple harmonic sum*.

Multiple harmonic sums are known to have interesting arithmetic properties, particularly in the case $N = p - 1$ with p prime (see [Hof04], [Zha08]). The following formula of Jarossay [Jar15b] expresses multiple harmonic sums $H_{p-1}(\underline{s})$ in terms of p -adic MZVs.

Theorem 8.8. *Let $\underline{s} = (s_1, \dots, s_k)$ be a composition. There is a p -adically convergent series identity*

$$p^{|\underline{s}|} H_{p-1}(\underline{s}) = \sum_{i=0}^k \sum_{\ell_1, \dots, \ell_i \geq 0} (-1)^{s_1 + \dots + s_i} \prod_{j=1}^i \binom{s_j + \ell_j - 1}{\ell_j} \zeta_p(s_i + \ell_i, \dots, s_1 + \ell_1) \zeta_p(s_{i+1}, \dots, s_k). \quad (8.2)$$

It follows from (8.1) that the convergence of the infinite series on the right hand side of (8.2) is uniform in p , in the sense that it induces a convergent series identity in $\widehat{\mathcal{R}}$.

8.5. \mathcal{A} -valued periods. To study congruences between multiple harmonic sums H_{p-1} modulo p , Kaneko and Zagier made the following definition.

Definition 8.9 (Kaneko-Zagier, unpublished). For \underline{s} a composition, the *finite multiple zeta value* is defined to be

$$\zeta_{\mathcal{A}}(\underline{s}) := (H_{p-1}(\underline{s}) \pmod{p})_p \in \mathcal{A}.$$

An equality between finite multiple zeta values corresponds to a congruence modulo p that holds for all but finitely many p .

Theorem 8.10. *The ring of \mathcal{A} -valued periods of $\mathcal{MT}(\mathbb{Z})$ is spanned as a vector space over \mathbb{Q} by the finite MZVs.*

Proof. The degree 0 graded piece of \mathcal{P}^{dr} is the \mathbb{Q} -span of the elements

$$(\mathbb{L}^{\text{dr}})^{-|\underline{s}|} \zeta^{\text{dr}}(\underline{s}), \quad (8.3)$$

as \underline{s} ranges through the compositions. The formula (8.2) implies that the image of (8.3) under the map $\text{per}_{\mathcal{R}}$ is in the \mathbb{Q} -span of the finite MZVs. A result of Yasuda [Yas16] implies that conversely, we can solve for the finite MZVs as a \mathbb{Q} -linear combination of the images of (8.3) under $\text{per}_{\mathcal{R}}$. \square

Kaneko-Zagier conjectured that relations between the finite multiple zeta values are governed by relations among the real multiple zeta values. Specifically, the conjecture says that the \mathbb{Q} -linear relations satisfied by the finite multiple zeta values are precisely the same as the \mathbb{Q} -linear relations satisfied by the *symmetrized multiple zeta values*

$$\zeta^s(s_1, \dots, s_k) := \sum_{i=0}^k (-1)^{s_1 + \dots + s_k} \zeta(s_i, \dots, s_1) \zeta(s_{i+1}, \dots, s_k)$$

modulo $\zeta(2)$. We show that Kaneko-Zagier's conjecture is essentially equivalent to Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$.

Theorem 8.11. *The finite multiple zeta values satisfy every relation satisfied by the motivic version of the symmetrized multiple zeta values modulo $\zeta^{\mathfrak{m}}(2)$. Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$ is equivalent to the assertion that the finite MZVs satisfy precisely the same relations as the motivic symmetrized MZVs modulo $\zeta^{\mathfrak{m}}(2)$.*

Proof. Yasuda's result [Yas16] implies that $\mathcal{P}_0^{\text{dr}}$ is the \mathbb{Q} -span of the the elements

$$\bar{\zeta}^{\text{dr},s}(s_1, \dots, s_k) := (\mathbb{L}^{\text{dr}})^{-|\underline{s}|} \sum_{i=0}^k (-1)^{s_1+\dots+s_k} \zeta^{\text{dr}}(s_i, \dots, s_1) \zeta^{\text{dr}}(s_{i+1}, \dots, s_k). \quad (8.4)$$

The elements (8.4) satisfy precisely the same relations as the motivic symmetrized multiple zeta values modulo $\zeta^{\mathfrak{m}}(2)$. Jarossay's formula (8.2) implies that $\text{per}_{\mathcal{A}}$ takes $\bar{\zeta}^{\text{dr},s}(\underline{s})$ to $\zeta_{\mathcal{A}}(\underline{s})$. This proves that the finite multiple zeta values satisfy every relations satisfied by the motivic symmetrized MZVs modulo $\zeta^{\mathfrak{m}}(2)$. The converse is the statement that $\text{per}_{\mathcal{A}} : \mathcal{P}_0^{\text{dr}} \rightarrow \mathcal{A}$ is injective, which by Theorem 6.4 is equivalent to Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$. \square

We can now prove the modulo p independence statement for Bernoulli numbers, which was stated in Sec. 5.2.

Proof of Theorem 5.3. For $n \geq 3$ odd, the elements

$$n(\mathbb{L}^{\text{dr}})^{-n} \zeta^{\text{dr}}(n) \in \mathcal{P}_0^{\text{dr}} \quad (8.5)$$

are algebraically independent. If we assume Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$, then by Theorem 6.4, $\text{per}_{\mathcal{R}}$ maps (8.5) to algebraically independent elements of \mathcal{R} . The Theorem now follow from the well-known formula

$$np^{-n} \zeta_p(n) \equiv B_{p-n} \pmod{p}$$

for p sufficiently large. \square

8.6. $\widehat{\mathcal{R}}$ -valued periods. Jarossay's formula (8.2) expresses $H_{p-1}(\underline{s})$ as a uniformly convergent series in terms of p -adic multiple zeta values, which implies that

$$H_{\mathfrak{p}-1}(\underline{s}) := (H_{p-1}(\underline{s})) \in \mathcal{R}$$

is in the image of the completed sequential period map (it is the image of the element of $\widehat{\mathcal{P}}^{\text{dr}}$ obtained by replacing each ζ_p on the right hand side of (8.2) with ζ^{dr}). Many other combinatorially defined quantities depending on p can be expressed in terms of the H_{p-1} , and can be shown to be $\widehat{\mathcal{R}}$ -valued periods of $\mathcal{MT}(\mathbb{Z})$. An explicit description of the ring of $\widehat{\mathcal{R}}$ -valued periods of $\mathcal{MT}(\mathbb{Z})$ was computed in [Ros18a].

Definition 8.12 ([Ros18a], Theorem 3.3). The *MHS algebra* is the subalgebra of $\widehat{\mathcal{R}}$ consisting of those $\alpha \in \widehat{\mathcal{R}}$ for which there exist sequences $a_n \in \mathbb{Q}$, $b_n \in \mathbb{Z}$ with $b_n \rightarrow \infty$, and compositions \underline{s}_n , such that

$$\alpha = \sum_{n=0}^{\infty} a_n \mathbf{p}^{b_n} H_{\mathbf{p}-1}(\underline{s}_n). \quad (8.6)$$

Observe that the condition $b_n \rightarrow \infty$ guarantees that (8.6) converges.

Theorem 8.13 ([Ros18a], Theorem 3.3). *The ring of $\widehat{\mathcal{R}}$ -valued periods of the category $\mathcal{MT}(\mathbb{Z})$ is the MHS algebra.*

As a consequence, elements of the MHS algebra can be lifted to $\widehat{\mathcal{P}}^{\text{dr}}$, and if we assume the truth of Conjecture 2 for $\mathcal{MT}(\mathbb{Z})$, we get a Galois theory for the MHS algebra. Some aspects of the Galois action are computed in [Ros18a].

Many combinatorially-defined sequences can be shown to be in the MHS algebra (several examples are given in [Ros18a], §7). For example, if $f(x), g(x) \in \mathbb{Z}[x]$ have positive leading coefficient, then the sequence of binomial coefficients

$$\binom{f(\mathbf{p})}{g(\mathbf{p})} := \binom{\binom{f(p)}{g(p)}}{p} \in \mathcal{R}$$

is in the MHS algebra ([Ros16], Theorem 7.11). In [Ros18a], §4, there is an algorithm (along with a link to an implementation) that takes as input a positive integer n and two elements $(a_p), (b_p)$ of the MHS algebra, and gives as output either a proof that $a_p \equiv b_p \pmod{p^n}$ for p large, or a proof that Conjecture 2 implies there are infinitely many p for which $a_p \not\equiv b_p \pmod{p^n}$.

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