ABELIAN IDEALS OF A BOREL SUBALGEBRA AND ROOT SYSTEMS, II

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ABSTRACT. Let g be a simple Lie algebra with a Borel subalgebra b and 21b the set of abelian ideals of b. Let Δ^+ be the corresponding set of positive roots. We continue our study of combinatorial properties of the partition of Ab parameterised by the long positive roots. In particular, the union of an arbitrary set of maximal abelian ideals is described, if $\mathfrak{g} \neq \mathfrak{sl}_n$. We also characterise the greatest lower bound of two positive roots, when it exists, and point out interesting subposets of Δ^+ that are modular lattices.

INTRODUCTION

Let g be a simple Lie algebra over $\mathbb C$, with a triangular decomposition $\mathfrak g = \mathfrak u \oplus \mathfrak t \oplus \mathfrak u^-$. Here t is a Cartan and $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is a fixed Borel subalgebra. The theory of abelian ideals of b is based on their relationship, due to D. Peterson, with the *minuscule elements* of the affine Weyl group W (see Kostant's account in [\[6\]](#page-11-0); another approach is presented in [\[2\]](#page-11-1)). In this note, we elaborate on some topics related to the combinatorial theory of abelian ideals, which can be regarded as a sequel to [\[11\]](#page-11-2). We mostly work in the combinatorial setting, i.e., the abelian ideals of b, which are sums of root spaces of u, are identified with the corresponding sets of positive roots.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ in the vector space $V = \mathfrak{t}_{\mathbb{R}}^*$, Δ^+ the set of positive roots in Δ corresponding to u, Π the set of simple roots in Δ^+ , and θ the highest root in Δ^+ . Then W is the Weyl group and (,) is a W-invariant scalar product on V. We equip Δ^+ with the usual partial ordering ' \succ '. An *upper ideal* (or just an *ideal*) of (Δ^+, \succ) is a subset $I \subset \Delta^+$ such that if $\gamma \in I, \nu \in \Delta^+$, and $\nu + \gamma \in \Delta^+$, then $\nu + \gamma \in I$. An upper ideal I is *abelian*, if $\gamma' + \gamma'' \notin \Delta^+$ for all $\gamma', \gamma'' \in I$. The set of minimal elements of *I* is denoted by $\min(I)$. It also makes sense to consider the maximal elements of the complement of I , denoted max $(\Delta^+ \setminus I)$.

Write $\mathfrak A$ b (resp. $\mathfrak A$ o) for the set of all abelian (resp. all upper) ideals of Δ^+ and think of them as posets with respect to inclusion. The upper ideal *generated by* γ is $I\langle \succcurlyeq \gamma \rangle =$ $\{\nu \in \Delta^+ \mid \nu \succcurlyeq \gamma\}$. Then $\min(I \langle \succcurlyeq \gamma \rangle) = \{\gamma\}$. A root $\gamma \in \Delta^+$ is said to be *commutative*, if $I\langle \succcurlyeq\gamma\rangle\in\mathfrak{Ab}.$ Write $\Delta_{\sf com}^+$ for the set of all commutative roots. This notion was introduced

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in [\[9\]](#page-11-3), and the subset $\Delta_{\sf com}^+$ for each Δ is explicitly described in [\[9,](#page-11-3) Theorem 4.4]. Note that $\Delta_{\sf com}^+\in {\mathfrak A}{\mathfrak d}.$

Let \mathfrak{Ab}^o denote the set of nonempty abelian ideals and Δ_l^+ $_l^+$ the set of long positive roots. In [\[8,](#page-11-4) Sect. 2], we defined a mapping $\tau : \mathfrak{Ab}^o \to \Delta_l^+$ $_l^+$, which is onto. Letting $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$, we get a partition of \mathfrak{Ab}^o parameterised by Δ_l^+ $_l^+$. Each \mathfrak{Ab}_{μ} is a subposet of \mathfrak{Ab} and, moreover, \mathfrak{Ab}_{μ} has a unique minimal and unique maximal element (ideal) [\[8,](#page-11-4) Sect. 3]. These extreme abelian ideals in \mathfrak{Ab}_{μ} are denoted by $I(\mu)_{\min}$ and $I(\mu)_{\max}$. Then $\{I(\alpha)_{\max} \mid \alpha \in \Pi_l\}$ are exactly the maximal abelian ideals of b.

In this article, we first establish a property of (Δ^+, \geq) , which seems to be new. It was proved in [\[11,](#page-11-2) Appendix] that, for any $\eta_1, \eta_2 \in \Delta^+$, there exists the *least upper bound*, denoted $\eta_1 \vee \eta_2$. Moreover, an explicit formula for $\eta_1 \vee \eta_2$ is also given. Here we prove that the *greatest lower bound*, $\eta_1 \wedge \eta_2$, exists if and only if supp(γ_1) \cap supp(γ_2) $\neq \emptyset$. Furthermore, if $\eta_i = \sum_{\alpha \in \Pi} c_{i\alpha} \alpha$, then $\eta_1 \wedge \eta_2 = \sum_{\alpha \in \Pi} \min \{c_{1\alpha}, c_{2\alpha}\} \alpha$. This also implies that $I \langle \succcurlyeq \eta \rangle$ is a modular lattice for any $\eta \in \Delta^+$, see Theorem [2.4.](#page-3-0) Another example a modular lattice inside Δ^+ is the subposet $\Delta_{\alpha}(i) = \{ \gamma \in \Delta^+ \mid \text{ht}_{\alpha}(\gamma) = i \}$, where $\alpha \in \Pi$ and $\text{ht}_{\alpha}(\gamma)$ is the coefficient of α in the expression of γ via Π.

Using properties of '∨' and '∧' and \mathbb{Z} -gradings of \mathfrak{g} , we prove uniformly that if Δ is not of type \mathbf{A}_n , then $\Delta_{\text{nc}}^+ := \Delta^+ \setminus \Delta_{\text{com}}^+$ has the unique maximal element, which is $\lfloor \theta/2 \rfloor :=$ $\sum_{\alpha\in\Pi}$ [ht $_{\alpha}(\theta)/2$] α , see Section [3.](#page-4-0) In particular, $\lfloor \theta/2 \rfloor$ is a root. (Note that if Δ is of type **A**_n, then $\lfloor \theta/2 \rfloor = 0$ and $\Delta_{\text{nc}}^+ = \emptyset$.) We also describe the maximal abelian ideals $I(\alpha)_{\text{max}}$ if $\mathsf{ht}_{\alpha}(\theta)$ is odd.

In Section [4,](#page-7-0) we study the sets of maximal and minimal elements related to abelian ideals of the form $I(\alpha)_{\min}$ and $I(\alpha)_{\max}$, with $\alpha \in \Pi_l := \Pi \cap \Delta_l^+$ $\frac{+}{l}$.

Theorem 0.1. If $S \subset \Pi_l$ is arbitrary and Δ is not of type \mathbf{A}_n , then there is the bijection

$$
\eta \in \min\left(\bigcap_{\alpha \in S} I(\alpha)_{\min}\right) \xrightarrow{1:1} \eta' = \theta - \eta \in \max\left(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}\right).
$$

Our proof is conceptual and relies on the fact θ is a multiple of a fundamental weight if Δ is not of type A_n . For A_n , the same bijection holds if S is a **connected** subset on the Dynkin diagram. The case in which $\#S = 1$ was considered earlier in [\[11,](#page-11-2) Theorem 4.7]. This has some interesting consequences if $S = \Pi_l$ and hence $\bigcup_{\alpha \in \Pi_l} I(\alpha)_{\max} = \Delta_{\textsf{com}}^+$, see Proposition [4.6.](#page-9-0)

In Section [5,](#page-10-0) we describe the interval $[|\theta/2|, \theta - |\theta/2|]$ inside the poset Δ^+ .

1. PRELIMINARIES

We have $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, the vector space $V = \bigoplus_{i=1}^n \mathbb{R} \alpha_i$, the Weyl group W generated by simple reflections s_{α} ($\alpha \in \Pi$), and a W-invariant inner product (,) on V. Set $\rho =$

1 $\frac{1}{2}\sum_{\nu\in\Delta^+}\nu$. The partial ordering ' \preccurlyeq' in Δ^+ is defined by the rule that $\mu \preccurlyeq \nu$ if $\nu-\mu$ is a non-negative integral linear combination of simple roots. Write $\mu \prec \nu$, if $\mu \preccurlyeq \nu$ and $\mu \neq \nu$. If $\mu = \sum_{i=1}^n c_i \alpha_i \in \Delta$, then $\mathrm{ht}_{\alpha_i}(\mu) := c_i$, $\mathrm{ht}(\mu) := \sum_{i=1}^n c_i$ and $\mathrm{supp}(\mu) = \{ \alpha_i \in \Pi \mid c_i \neq 0 \}.$

The Heisenberg ideal $\mathcal{H} := \{ \gamma \in \Delta^+ \mid (\gamma, \theta) \neq 0 \} = \{ \gamma \in \Delta^+ \mid (\gamma, \theta) > 0 \} \in \mathfrak{A}$ plays a prominent role in the theory of abelian ideals and posets $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$.

Let us collect some known results that are frequently used below.

- If $I \in \mathfrak{A}$ is not abelian, then there exist $\eta, \eta' \in I$ such that $\eta + \eta' = \theta$, see [\[8,](#page-11-4) p. 1897]. Therefore, $I \notin \mathfrak{Ab}$ if and only if $I \cap \mathfrak{H} \notin \mathfrak{Ab}$.
- $I = I(\mu)$ _{min} for some $\mu \in \Delta_l^+$ $\frac{1}{l}$ if and only if $I \subset \mathcal{H}$ [\[8,](#page-11-4) Theorem 4.3];
- $\#I(\mu)_{\min} = (\rho, \theta^{\vee} \mu^{\vee}) + 1$ [\[8,](#page-11-4) Theorem 4.2(4)];
- For $I \in \mathfrak{Ab}^o$, we have $I \in \mathfrak{Ab}_\mu$ if and only if $I \cap \mathfrak{H} = I(\mu)_{\min}$ [\[11,](#page-11-2) Prop. 3.2];
- The set of (globally) maximal abelian ideals is $\{I(\alpha)_{\text{max}} \mid \alpha \in \Pi_l\}$ [\[8,](#page-11-4) Corollary 3.8].
- For any $\mu \in \Delta_l^+$ $_l^+$, there is a unique element of minimal length in W that takes θ to μ [\[8,](#page-11-4) Theorem 4.1]. Writing w_{μ} for this element, one has $\ell(w_{\mu}) = (\rho, \theta^{\vee} - \mu^{\vee})$ [8, Theorem 4.1].
- Let $\mathcal{N}(w)$ be the inversion set of $w \in W$. By [\[12,](#page-11-5) Lemma 1.1],

$$
I(\mu)_{\min} = \{\theta\} \cup \{\theta - \gamma \mid \gamma \in \mathcal{N}(w_{\mu})\}.
$$

For each $\eta \in \mathcal{H} \setminus \{\theta\}$ there is a unique $\eta' \in \mathcal{H} \setminus \{\theta\}$ such that $\eta + \eta'$ is a root, and this root is θ . It is well known that $\#\mathcal{H} = 2(\rho, \theta^{\vee}) - 1 = 2h^* - 3$, where h^* is the *dual Coxeter number* of Δ . Since $\#I(\alpha)_{\min}=(\rho,\theta^{\vee})=h^*-1$ for $\alpha\in\Pi_l$, the ideal $I(\alpha)_{\min}$ contains θ and exactly a half of elements of $\mathcal{H} \setminus \{\theta\}$, cf. also [\[11,](#page-11-2) Lemma 3.3].

Although the affine Weyl group and minuscule elements are not explicitly used in this paper, their use is hidden in properties of the posets \mathfrak{Ab}_{μ} , $\mu \in \Delta_l^+$ $_l^+$, and ideals $I(\mu)_{\sf min}$, $I(\mu)_{\text{max}}$. Important properties of the maximal abelian ideals are also obtained in [\[3,](#page-11-6) [16\]](#page-11-7).

We refer to [\[1\]](#page-11-8), [\[4,](#page-11-9) § 3.1] for standard results on root systems and Weyl groups and to [\[15,](#page-11-10) Chapter 3] for posets.

2. THE GREATEST LOWER BOUND IN Δ^+

It is proved in [\[8,](#page-11-4) Appendix] that the poset (Δ^+, \geq) is a join-semilattice. i.e., for any pair $\eta, \eta' \in \Delta^+$, there is the least upper bound (= *join*), denoted $\eta \vee \eta'$. Furthermore, there is a simple explicit formula for ' \vee' , see [\[8,](#page-11-4) Theorem A.1]. However, Δ^+ is not a meetsemilattice. We prove below that under a natural constraint the greatest lower bound (= *meet*) exists and can explicitly be described. Afterwards, we provide some applications of this property in the theory of abelian ideals.

Definition 1. Let $\eta, \eta' \in \Delta^+$. The root ν is the greatest lower bound (or *meet*) of η and η' if

 \bullet $\eta \succcurlyeq \nu$, $\eta' \succcurlyeq \nu$;

• if $\eta \succcurlyeq \kappa$ and $\eta' \succcurlyeq \kappa$, then $\nu \succcurlyeq \kappa$.

The meet of η and η' , if it exists, is denoted by $\eta \wedge \eta'$.

Obviously, if $\alpha, \alpha' \in \Pi$, then their meet does not exist. But as we see below, the only reason for such a failure is that their supports are disjoint.

Lemma 2.1 (see [\[14,](#page-11-11) Lemma 3.1]). *Suppose that* $\gamma \in \Delta^+$ *and* $\alpha, \beta \in \Pi$ *.* If $\gamma - \alpha, \gamma - \beta \in \Delta^+$ *, then either* $\gamma - \alpha - \beta \in \Delta^+$ *or* $\gamma = \alpha + \beta$ *and hence* α, β *are adjacent in the Dynkin diagram.*

Lemma 2.2 (see [\[13,](#page-11-12) Lemma 3.2]). *Suppose that* $\gamma \in \Delta^+$ *and* $\alpha, \beta \in \Pi$ *. If* $\gamma + \alpha, \gamma + \beta \in \Delta^+$ *, then* $\gamma + \alpha + \beta \in \Delta^+$ *.*

Let us provide a reformulation of these lemmata in terms of ' \vee ' and ' \wedge '. To this end, we note that in the previous lemma, $(\gamma + \alpha) \wedge (\gamma + \beta) = \gamma$.

Proposition 2.3. *Let* $\eta_1, \eta_2 \in \Delta^+$ *.*

- (i) If $\eta_1 \vee \eta_2$ *covers both* η_1 *and* η_2 *, then either* $\eta_1 \vee \eta_2 = \alpha + \beta = \eta_1 + \eta_2$ for some adjacent $\alpha, \beta \in \Pi$, or η_1 and η_2 both cover $\eta_1 \wedge \eta_2$;
- (ii) *If* $\eta_1 \wedge \eta_2$ *exists and* η_1 *and* η_2 *both cover* $\eta_1 \wedge \eta_2$ *, then* $\eta_1 \vee \eta_2$ *covers both* η_1 *and* η_2 *.*

For any two roots $\eta=\sum_{\alpha\in\Pi}c_\alpha\alpha$ and $\eta'=\sum_{\alpha\in\Pi}c'_\alpha\alpha$, one defines two elements of the root lattice, $\min(\eta, \eta') = \sum_{\alpha \in \Pi} \min\{c_\alpha, c'_\alpha\} \alpha$ and $\max(\eta, \eta') = \sum_{\alpha \in \Pi} \max\{c_\alpha, c'_\alpha\} \alpha$. Recall that the poset (Δ^+, \succ) is graded and the rank function is the usual *height* of a root, i.e., $\mathsf{ht}(\eta) = \sum_{\alpha \in \Pi} c_{\alpha}$. We also set $\mathsf{ht}_{\alpha}(\eta) := c_{\alpha}$.

Theorem 2.4.

- 1) *For any* $\gamma \in \Delta^+$, the upper ideal $I\langle \succcurlyeq \gamma \rangle$ is a modular lattice;
- 2) *the meet* $\gamma_1 \wedge \gamma_2$ *exists if and only if* $\text{supp}(\gamma_1) \cap \text{supp}(\gamma_2) \neq \emptyset$ *. In this case, one has* $\gamma_1 \wedge \gamma_2 = \min(\gamma_1, \gamma_2)$.

Proof. 1) By [\[8,](#page-11-4) Theorem A.1(i)], the join always exists in Δ^+ and formulae for ' \vee ' show that $\gamma_1 \vee \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$ whenever $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$. Therefore, $I \langle \succcurlyeq \gamma \rangle$ is a join-semilattice with a unique minimal element. Hence the meet also exists for any $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$, see [\[15,](#page-11-10) Prop. 3.3.1]. That is, $I\langle \succcurlyeq \gamma \rangle$ is a lattice. Note that, for $\gamma_1, \gamma_2 \in I\langle \succcurlyeq \gamma \rangle$, the first possibility in Proposition [2.3\(](#page-3-1)i) does not realise. Therefore, using Proposition [2.3](#page-3-1) with $I\langle \succcurlyeq \gamma \rangle$ in place of Δ^+ and [\[15,](#page-11-10) Prop. 3.3.2], we conclude that $I\langle \succcurlyeq \gamma \rangle$ is a modular lattice.

Yet, this does not provide a formula for the meet and leaves a theoretical possibility that $\gamma_1 \wedge \gamma_2$ depends on γ .

2) If supp(γ_1) \cap supp(γ_2) = \emptyset , then there are no roots ν such that $\gamma_1 \succcurlyeq \nu$ and $\gamma_2 \succcurlyeq \nu$. Conversely, if $\text{supp}(\gamma_1) \cap \text{supp}(\gamma_2) \neq \emptyset$, then $\gamma_1, \gamma_2 \in I \Leftrightarrow \gamma$ for some γ . Using again [\[15,](#page-11-10) Prop. 3.3.2], the modularity of the lattice $I\langle \succcurlyeq \gamma \rangle$ implies that ht $(\gamma_1 \vee \gamma_2) + \text{ht}(\gamma_1 \wedge \gamma_2) =$

 $\mathsf{ht}(\gamma_1)+\mathsf{ht}(\gamma_2)$, where $\gamma_1\wedge\gamma_2$ is taken inside $I\langle\succcurlyeq\gamma\rangle$. It is clear that $\gamma_1\wedge\gamma_2\preccurlyeq \min(\gamma_1,\gamma_2)$. More-over, in this situation, the formulae of [\[8,](#page-11-4) Theorem A.1(i)] imply that $\gamma_1 \vee \gamma_2 = \max(\gamma_1, \gamma_2)$. Therefore, $\text{ht}(\min(\gamma_1, \gamma_2)) = \text{ht}(\gamma_1 \land \gamma_2)$ and thereby $\min(\gamma_1, \gamma_2) = \gamma_1 \land \gamma_2$.

Remark 2.5. A special class of modular lattices inside Δ^+ occurs in connection with \mathbb{Z} gradings of g. For $\alpha \in \Pi$, set $\Delta_{\alpha}(i) = \{\gamma \in \Delta \mid \text{ht}_{\alpha}(\gamma) = i\}$. It is known that $\Delta_{\alpha}(i)$ has a unique minimal and a unique maximal element, see Section [3.](#page-4-0) It is also clear that $\gamma_1 \wedge \gamma_2$ and $\gamma_1 \vee \gamma_2 \in \Delta_{\alpha}(i)$ for all $\gamma_1, \gamma_2 \in \Delta_{\alpha}(i)$. Hence $\Delta_{\alpha}(i)$ is a **modular** lattice. (It was already noticed in [\[8,](#page-11-4) Appendix] that $\Delta_{\alpha}(i)$ is a lattice.)

Remark 2.6. In what follows, we have to distinguish the A_n -case from the other types. One the reasons is that θ is not a multiple of a fundamental weight only for A_n . In all other types, there is a unique $\alpha_{\theta} \in \Pi$ such that $(\theta, \alpha_{\theta}) \neq 0$. For the Z-grading associated with α_{θ} , one then has $\Delta_{\alpha_{\theta}}(1) = \mathfrak{H} \setminus \{\theta\}$ and $\Delta_{\alpha_{\theta}}(2) = \{\theta\}$. That is, $\mathfrak{H} \setminus \{\theta\}$ (or just \mathfrak{H}) has a unique minimal element, which is α_{θ} , if and only if Δ is not of type \mathbf{A}_n . This provides the following consequence of Theorem [2.4:](#page-3-0)

If Δ *is not of type* \mathbf{A}_n *, then for all* $\eta_1, \eta_2 \in \mathcal{H} \setminus \{\theta\}$ *, the meet* $\eta_1 \wedge \eta_2$ *exists and lies in* $\mathcal{H} \setminus \{\theta\}$ *.* This is going to be used several times in Section [4.](#page-7-0)

3. Z-GRADINGS AND NON-COMMUTATIVE ROOTS

If $\gamma \in \Delta_{\sf com}^+$, then γ belongs to a maximal abelian ideal. Since $I(\alpha)_{\sf max}$, $\alpha \in \Pi_l$, are all the maximal abelian ideals in Δ^+ , we have

$$
\Delta_{\text{com}}^+ = \bigcup_{\alpha \in \Pi_l} I(\alpha)_{\text{max}}.
$$

Set $\Delta_{\text{nc}}^+ = \Delta^+ \setminus \Delta_{\text{com}}^+$. In this section, we obtain an a priori description of Δ_{nc}^+ . Let us introduce special elements of the root lattice

(3.1)
$$
\lfloor \theta/2 \rfloor = \sum_{\alpha \in \Pi} \lfloor \mathsf{ht}_{\alpha}(\theta)/2 \rfloor \alpha \text{ and } \lceil \theta/2 \rceil = \sum_{\alpha \in \Pi} \lceil \mathsf{ht}_{\alpha}(\theta)/2 \rceil \alpha.
$$

Hence $\lfloor \theta/2 \rfloor + \lceil \theta/2 \rceil = \theta$. Note that $\lfloor \theta/2 \rfloor = 0$ if and only if $\theta = \sum_{\alpha \in \Pi} \alpha$, i.e., Δ is of type \mathbf{A}_n .

Lemma 3.1. *Suppose that* Δ *is not of type* \mathbf{A}_n *, so that* $\lfloor \theta/2 \rfloor \neq 0$ *.*

- (1) *If* $\gamma \in \Delta_{\text{nc}}^+$, then $\text{ht}_{\alpha} \gamma \leq \lfloor \text{ht}_{\alpha}(\theta)/2 \rfloor$ *for all* $\alpha \in \Pi$, *i.e.*, $\gamma \leq \lfloor \theta/2 \rfloor$ *.*
- (2) *If* $\gamma_1, \gamma_2 \preccurlyeq \lfloor \theta/2 \rfloor$ *, then* $\gamma_1 \vee \gamma_2 \preccurlyeq \lfloor \theta/2 \rfloor$ *.*

Proof. (1) Obvious.

(2) By [\[8,](#page-11-4) Theorem A.1], if $\text{supp}(\gamma_1) \cup \text{supp}(\gamma_2)$ is connected, then $\gamma_1 \vee \gamma_2 = \max(\gamma_1, \gamma_2)$ and the assertion is clear. Otherwise, $\gamma_1 \vee \gamma_2 = \gamma_1 + ($ connecting root $)+ \gamma_2$. Recall that if the union of supports is not connected, then there is a (unique) chain of simple roots

that connects them. If this chain consists of $\alpha_{i_1}, \ldots, \alpha_{i_s}$, then the "connecting root" is $\alpha_{i_1}+\cdots+\alpha_{i_s}.$ Here we only need the condition that ht $_\alpha(\theta)\geqslant 2$ for any α in the connecting chain. Indeed, the roots in this chain are not extreme in the Dynkin diagram, and outside type \mathbf{A}_n the coefficients of non-extreme simple roots are always ≥ 2 .

Remark. For \mathbf{A}_n , $\lfloor \theta/2 \rfloor = 0$ and hence $\Delta_{nc}^+ = \varnothing$.

Set $A = \{ \gamma \in \Delta^+ \mid \gamma \preccurlyeq \lfloor \theta/2 \rfloor \}.$ Then $A \neq \emptyset$ if and only if Δ is not of type \mathbf{A}_n . It follows from Lemma [3.1](#page-4-1) that

- $\Delta_{\mathsf{nc}}^+ \subset \mathcal{A}$;
- A has a unique maximal element.

Our goal is to prove that $\Delta_{\text{nc}}^+ = A$ and $\max(A) = \{ \lfloor \theta/2 \rfloor \}$. The latter essentially boils down to the assertion that $|\theta/2|$ is a root.

For an arbitrary $\alpha \in \Pi$, consider the Z-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\alpha}(i)$ corresponding to α . That is, the set of roots of $\mathfrak{g}_{\alpha}(i)$ is $\Delta_{\alpha}(i)$, see Remark [2.5.](#page-4-2) In particular, $\alpha \in \Delta_{\alpha}(1)$ and $\Pi \setminus \{\alpha\} \subset \Delta_{\alpha}(0)$. Here $\mathfrak{l} := \mathfrak{g}_{\alpha}(0)$ is reductive and contains the Cartan subalgebra t. By an old result of Kostant (see [\[7\]](#page-11-13) and Joseph's exposition in [\[5,](#page-11-14) 2.1]), each $\mathfrak{g}_{\alpha}(i)$, $i \neq 0$, is a simple *l*-module. Therefore, $\Delta_{\alpha}(i)$ has a unique minimal and a unique maximal element. The following is a particular case of Theorem 2.3 in [\[7\]](#page-11-13).

Proposition 3.2. *If* $i + j \leq h$ t_α(θ)*, then* $0 \neq [g_{\alpha}(i), g_{\alpha}(j)] = g_{\alpha}(i + j)$ *.*

Once one has proved that $[\mathfrak{g}_{\alpha}(i), \mathfrak{g}_{\alpha}(j)] \neq 0$, the equality $[\mathfrak{g}_{\alpha}(i), \mathfrak{g}_{\alpha}(j)] = \mathfrak{g}_{\alpha}(i + j)$ stems from the fact that $\mathfrak{g}_{\alpha}(i+j)$ is a simple *l*-module. We derive from this result two corollaries.

Corollary 3.3. *For any* $\mu \in \Delta_{\alpha}(i)$ *, there is* $\nu \in \Delta_{\alpha}(j)$ *such that* $\mu + \nu \in \Delta_{\alpha}(i + j)$ *.*

Proof. Let $e_{\mu} \in \mathfrak{g}_{\alpha}(i)$ be a root vector for μ . Assume that the property in question does not hold. Then $[e_\mu, \mathfrak{g}_\alpha(j)] = 0$. Hence $[L \cdot e_\mu, \mathfrak{g}_\alpha(j)] = 0$, where $L \subset G$ is the connected reductive group with Lie algebra l. Since the linear span of an L-orbit in a simple L-module is the whole space, this implies that $[g_{\alpha}(i), g_{\alpha}(j)] = 0$, which contradicts the proposition. \square

Set $d_{\alpha} = \lfloor \frac{ht_{\alpha}(\theta)}{2} \rfloor$, and let $\mu_{d_{\alpha}}$ be the lowest weight in $\Delta_{\alpha}(d_{\alpha})$.

Corollary 3.4. $\mu_{d_{\alpha}} \in \Delta_{\text{nc}}^{+}$.

Proof. By Corollary [3.3,](#page-5-0) there is $\lambda \in \Delta_{\alpha}(d_{\alpha})$ such that $\mu_{d_{\alpha}} + \lambda$ is a root in $\Delta_{\alpha}(2d_{\alpha})$. Since $\mu_{d_{\alpha}} \preccurlyeq \gamma$, the upper ideal in Δ^+ generated by $\mu_{d_{\alpha}}$ is not abelian.

This allows us to obtain the promised characterisation of $\Delta_{\sf nc}^+$.

Theorem 3.5. *If* $\lfloor \theta/2 \rfloor \neq 0$, *i.e.*, Δ *is not of type* \mathbf{A}_n , *then* $\lfloor \theta/2 \rfloor$ *is the unique maximal element of* Δ_{nc}^{+} *. Furthermore,* $\lfloor \theta/2 \rfloor \in \mathcal{H}$ *.*

Proof. It was noticed above that $\Delta_{\sf nc}^+ \subset A$, A has a unique maximal element, say $\hat{\nu}$, and $\hat{\nu} \preccurlyeq$ $\lfloor \theta/2 \rfloor$. By Corollary [3.4,](#page-5-1) for any $\alpha \in \Pi$, there is $\mu_\alpha \in \Delta^+_{nc}$ such that $\text{ht}_\alpha(\mu_\alpha) = d_\alpha$. Therefore $\bigvee \mu_\alpha \succcurlyeq \lfloor \theta/2 \rfloor$. On the other hand, $\mu_\alpha \preccurlyeq \hat{\nu}$ for each α and hence $\bigvee \mu_\alpha \preccurlyeq \hat{\nu} \preccurlyeq \lfloor \theta/2 \rfloor$. Thus, $\alpha \in \Pi$ and $\alpha \in \Pi$ $\lfloor \theta/2 \rfloor = \hat{\nu}$ is a root. If α_{θ} is the unique simple root such that $(\theta, \alpha_{\theta}) \neq 0$, then $\text{ht}_{\alpha_{\theta}}(\theta) = 2$. Therefore $|\theta/2| \in \mathcal{H}$ whenever Δ is not \mathbf{A}_n .

The fact that $|\theta/2|$ is the unique maximal non-commutative root has been observed in [\[9,](#page-11-3) Sect. 4] via a case-by-case analysis.

Example 3.6. If
$$
\triangle
$$
 is of type **E**₈, then $\theta = \frac{2345642}{3}$ and $\lfloor \theta/2 \rfloor = \frac{1122321}{1}$.

Remark 3.7*.* In the proof of Corollary [3.4](#page-5-1) and then Theorem [3.5,](#page-5-2) we only need the property, which follows from Proposition [3.2,](#page-5-3) that $[\mathfrak{g}_{\alpha}(d_{\alpha}), \mathfrak{g}_{\alpha}(d_{\alpha})] = \mathfrak{g}_{\alpha}(2d_{\alpha}).$

For $\alpha \in \Pi$ with $\text{ht}_{\alpha}(\theta) = 2$ or 3, this means that $[\mathfrak{g}_{\alpha}(1), \mathfrak{g}_{\alpha}(1)] = \mathfrak{g}_{\alpha}(2)$, which is obvious. This covers all classical simple Lie algebras, E_6 , and G_2 . For E_7 , E_8 , and F_4 , there are $\alpha \in \Pi$ such that $\text{ht}_{\alpha}(\theta) \in \{4, 5, 6\}$. Then the required relation is $[\mathfrak{g}_{\alpha}(2), \mathfrak{g}_{\alpha}(2)] = \mathfrak{g}_{\alpha}(4)$ or $[\mathfrak{g}_{\alpha}(3), \mathfrak{g}_{\alpha}(3)] = \mathfrak{g}_{\alpha}(6)$. This can easily be verified case-by-case. However, our intention is to provide a case-free treatment of this property.

Another consequence of Kostant's theory [\[7\]](#page-11-13) is that one obtains an explicit presentation of some maximal abelian ideals.

Proposition 3.8. *Suppose that* $ht_\alpha(\theta) = 2d_\alpha + 1$ *is odd. Then* $\mathfrak{a} := \bigoplus_{j \geqslant d_\alpha + 1} \mathfrak{g}_\alpha(j)$ *(i.e.,* $\Delta_{\mathfrak{a}} :=$ $\bigcup_{j \geqslant d_\alpha+1} \Delta_\alpha(j)$ *in the combinatorial set up) is a maximal abelian ideal of* b.

Proof. Obviously, a is abelian. Let $\lambda \in \Delta_{\alpha}(d_{\alpha})$ be the highest weight. It follows from the simplicity of all l-modules $\mathfrak{g}_{\alpha}(i)$ that λ is the only maximal element of $\Delta^+\setminus\Delta_{\mathfrak{a}}$. Therefore, it suffices to prove that the upper ideal $\Delta_{\mathfrak{a}} \cup \{\lambda\}$ is not abelian. Indeed, there is $\nu \in \Delta_{\alpha}(d_{\alpha}+1)$ such that $\nu + \lambda$ is a root (apply Corollary [3.3](#page-5-0) with $i = d_{\alpha}$ and $j = d_{\alpha} + 1$.)

This prompts the following question. Suppose that $\text{ht}_{\alpha}(\theta) = 2d_{\alpha} + 1$. Then $\mathfrak{a} = I(\beta)_{\text{max}}$ for some $\beta\in\Pi_l.$ What is the relationship between α and β ? We say below that $\alpha\in\Pi$ is *odd*, if ht_{α}(θ) is odd.

Example 3.9. 1) If $ht_{\alpha}(\theta) = 1$, i.e., $d_{\alpha} = 0$, then a is the (abelian) nilradical of the corresponding maximal parabolic subalgebra. Then $\beta = \alpha$. This covers all simple roots and all maximal abelian ideals in type A_n .

2) For Δ of type \mathbf{D}_n or \mathbf{E}_n , there are exactly three odd simple roots α .

– For \mathbf{D}_n , these are the endpoints of the Dynkin diagram and $d_\alpha = 0$. That is, again $\alpha = \beta$ in these cases.

– For **E**_n, there are also odd simple roots with $d_{\alpha} \geq 1$ and then $\beta \neq \alpha$.

Nevertheless, the related maximal abelian ideals always correspond to the extreme nodes

of the Dynkin diagram! Moreover, one always has $\text{ht}_{\beta}(\theta) = d_{\alpha} + 1$. (Similar things happen for \mathbf{F}_4 and \mathbf{G}_2 .) It might be interesting to find a reason behind it.

Below is the table of all exceptional cases with $d_{\alpha} \geq 1$. The numbering of simple roots follows [\[4,](#page-11-9) Tables]. In particular, the numbering for \mathbf{E}_8 is $\frac{1234567}{8}$ and the extreme nodes correspond to $\alpha_1, \alpha_7, \alpha_8$.

				$\begin{vmatrix} E_6 \end{vmatrix} E_7$ $\begin{vmatrix} E_8 \end{vmatrix}$ $\begin{vmatrix} F_4 \end{vmatrix} G_2$
$\begin{array}{ c c c c c c } \hline \hline \alpha & \alpha_3 & \alpha_3 & \alpha_5 & \alpha_2 & \alpha_4 & \alpha_8 & \alpha_3 & \alpha_1 \ \hline d_\alpha & 1 & 1 & 1 & 2 & 1 & 1 & 1 \ \beta & \alpha_6 & \alpha_7 & \alpha_6 & \alpha_1 & \alpha_8 & \alpha_7 & \alpha_4 & \alpha_2 \ \hline \text{ht}_\beta(\theta) & 2 & 2 & 2 & 2 & 3 & 2 & 2 & 2 \ \hline \end{array}$				

4. BIJECTIONS RELATED TO THE MAXIMAL ABELIAN IDEALS

In this section, we consider abelian ideals of the form $I(\alpha)_{\sf min}$ and $I(\alpha)_{\sf max}$ for $\alpha\in\Pi_l$, and their derivatives (intersections and unions).

The following is Theorem 4.7 in [\[11\]](#page-11-2).

Theorem 4.1. For any $\alpha \in \Pi_l$, there is a one-to-one correspondence between $\min(I(\alpha)_{\min})$ and $\max(\Delta^+ \setminus I(\alpha)_{\max})$. Namely, if $\eta \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, then $\eta' := \theta - \eta \in \min(I(\alpha)_{\min})$, and *vice versa.*

It formally follows from this theorem that $\min(I(\alpha)_{\min})$ and $\max(\Delta^+ \setminus I(\alpha)_{\max})$ both belong to H. This is clear for the former, since $I(\alpha)_{\min} \subset H$. And the key point in the proof of Theorem [4.1](#page-7-1) was to demonstrate a priori that $\max(\Delta^+ \setminus I(\alpha)_{\max}) \subset \mathcal{H}$.

Below, we provide a generalisation of Theorem [4.1,](#page-7-1) which is even more general than [\[11,](#page-11-2) Theorem 4.9], i.e., we will **not** assume that $S \subset \Pi_l$ be connected. Another improvement is that we give a conceptual proof of that generalisation, while Theorem 4.9 in [\[11\]](#page-11-2) was proved case-by-case and no details has been given there.

The following is a key step for our generalisation of Theorem [4.1.](#page-7-1)

Theorem 4.2. *Suppose that* $S \subset \Pi_l$ *and* $\gamma \in \max(\Delta^+) \setminus \Box$ $\alpha{\in}S$ $I(\alpha)_{\text{max}}$). If Δ *is not of type* \mathbf{A}_n , *then* $\gamma \in \mathcal{H}$ *.*

Proof. Here we have to distinguish two possibilities: either $\gamma \in \Delta_{\text{nc}}^+$ or $\gamma \in \Delta_{\text{com}}^+$.

(1) Suppose that $\gamma \in \Delta_{\text{nc}}^+$ and assume that $\gamma \notin \mathcal{H}$. Then there are $\eta, \eta' \succ \gamma$ such that $\eta+\eta'=\theta$, see [\[8,](#page-11-4) p. 1897]. Here both η and η' belong to $\mathcal{H}\cap(\bigcup_{\alpha\in S}I(\alpha)_{\max})=\bigcup_{\alpha\in S}I(\alpha)_{\min}$. Since Δ is not of type \mathbf{A}_n , \mathcal{H} has a unique minimal element (= the unique simple root that is not orthogonal to θ). Therefore, $\mu := \eta \wedge \eta'$ exists and belongs to \mathcal{H} . (The existence of $\eta \wedge \eta'$ also follows from Theorem [2.4\(](#page-3-0)1).) Since $\eta, \eta' \succcurlyeq \mu$, we have $\mu \in \Delta_{\text{nc}}^+$. This implies that $\mu \notin \bigcup_{\alpha \in S} I(\alpha)_{\min}$ and hence $\mu \notin \bigcup_{\alpha \in S} I(\alpha)_{\max}$. By the definition of meet, $\gamma \preccurlyeq \mu$. Furthermore, $\gamma \notin \mathcal{H}$ and $\mu \in \mathcal{H}$. Hence $\gamma \prec \mu$ and γ is not maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$. A contradiction!

(2) Suppose that $\gamma \in \Delta_{\text{com}}^+$. Consider the abelian ideal $J = I \langle \succcurlyeq \gamma \rangle$. By the assumption, $J\setminus\{\gamma\}\subset\bigcup_{\alpha\in S}I(\alpha)_{\max}$. On the other hand, since $J\not\subset I(\alpha)_{\max}$ for each $\alpha\in S$, we conclude that

$$
J \cap \mathcal{H} \not\subset I(\alpha)_{\max} \cap \mathcal{H} = I(\alpha)_{\min},
$$

see [\[11,](#page-11-2) Prop. 3.2]. For each $\alpha \in S$, we pick $\eta_{\alpha} \in (J \cap \mathcal{H}) \setminus I(\alpha)_{\min}$. Then $\eta_{\alpha} \succcurlyeq \gamma$. Since Δ is not of type \mathbf{A}_n , the meet $\eta := \bigwedge_{\alpha \in S} \eta_\alpha$ exists and belong to $\mathcal H$ (Remark [2.6\)](#page-4-3) and also $\eta \succcurlyeq \gamma$. Note also that $\eta \notin I(\alpha)_{\text{min}}$ for each $\alpha \in S$. (Otherwise, if $\eta \in I(\alpha_0)_{\text{min}}$, then $\eta_{\alpha_0} \in I(\alpha_0)_{\text{min}}$ as well.) Therefore, $\eta \notin \bigcup_{\alpha \in S} I(\alpha)_{\min}$ and hence $\eta \notin \bigcup_{\alpha \in S} I(\alpha)_{\max}$ (because $\eta \in \mathcal{H}$). As γ is assumed to be maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$, we must have $\gamma = \eta \in \mathcal{H}$.

Remark. For \mathbf{A}_n , this theorem remains true if we add the hypothesis that $S \subset \Pi_l$ is a connected subset in the Dynkin diagram, see also Example [4.4.](#page-8-0)

Theorem 4.3. If $S \subset \Pi_l$ is arbitrary and Δ is not of type \mathbf{A}_n , then there is the bijection

$$
\eta \in \min\left(\bigcap_{\alpha \in S} I(\alpha)_{\min}\right) \stackrel{1:1}{\longmapsto} \eta' = \theta - \eta \in \max\left(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}\right).
$$

Proof. (1) Suppose that $\eta \in \min(\bigcap_{\alpha \in S} I(\alpha)_{\min})$. As Δ is not of type \mathbf{A}_n , there is a unique $\alpha_{\theta} \in \Pi$ such that $(\theta, \alpha_{\theta}) \neq 0$. Then $\theta - \alpha_{\theta} \in \mathcal{H}$ is the only root covered by θ . Therefore, $\theta - \alpha_{\theta} \in I(\alpha)_{\min}$ for all $\alpha \in \Pi_l$. Hence $\eta \neq \theta$ and hence $\eta' = \theta - \eta$ is a root (in \mathcal{H}). Since $\eta \in I(\alpha)$ _{min}, we have $\eta' \notin I(\alpha)$ _{min}, see [\[11,](#page-11-2) Lemma 3.3]. And this holds for each $\alpha \in S$. Hence $\eta' \not\in \bigcup_{\alpha \in S} I(\alpha)_{\text{min}}$ and thereby $\eta' \not\in \bigcup_{\alpha \in S} I(\alpha)_{\text{max}}$.

Assume that η' is not maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$, i.e., $\eta' + \beta \not\in \bigcup_{\alpha \in S} I(\alpha)_{\max}$ for some $β ∈ Π$. Again, $η' \prec θ - αθ$, hence $η' + β ∈ θ \setminus {θ}$. Then $θ - (η' + β) = η - β ∈ θ$ and arguing "backwards" we obtain that $\eta-\beta\in\bigcap_{\alpha\in S}I(\alpha)_{\min}$, which contradicts the fact that η is minimal.

(2) By Theorem [4.2,](#page-7-2) if $\eta' \in \max(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max})$, then $\eta' \in \mathcal{H}$. Under these circumstances, the previous part of the proof can be reversed. \Box

Example 4.4. Suppose that Δ is of type A_n , with the usual numbering of simple roots. Then $I(\alpha_i)_{\text{max}} = I \langle \succcurlyeq \alpha_i \rangle$ for all i and $\mathcal{H} = I(\alpha_1)_{\text{max}} \cup I(\alpha_n)_{\text{max}}$, where

$$
I(\alpha_1)_{\min} = I(\alpha_1)_{\max} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_1 - \varepsilon_n, \varepsilon_1 - \varepsilon_{n+1} = \theta\},
$$

$$
I(\alpha_n)_{\min} = I(\alpha_n)_{\max} = \{\varepsilon_n - \varepsilon_{n+1}, \dots, \varepsilon_2 - \varepsilon_{n+1}, \varepsilon_1 - \varepsilon_{n+1}\}.
$$

If $S = \{\alpha_1, \alpha_n\}$, then S is not connected for $n \geqslant 3$, $I(\alpha_1)_{\min} \cap I(\alpha_n)_{\min} = \{\theta\}$, and $\max(\Delta^+)$ $(I(\alpha_1)_{\max} \cup I(\alpha_n)_{\max}) = {\varepsilon_2 - \varepsilon_n}.$ That is, Theorems [4.2](#page-7-2) and [4.3](#page-8-1) do not apply here. However, both remain true if S is assumed to be connected and $S \neq \Pi$. For instance, suppose that $S = {\alpha_i, \alpha_{i+1}, \ldots, \alpha_j}$ with $1 < i < j < n$. Then $\min(\bigcap_{\alpha \in S} I(\alpha)_{\min}) =$ $\{\varepsilon_1-\varepsilon_{j+1},\varepsilon_i-\varepsilon_{n+1}\}\$ and $\max(\Delta^+\setminus\bigcup_{\alpha\in S}I(\alpha)_{\max})=\{\varepsilon_1-\varepsilon_i,\varepsilon_{j+1}-\varepsilon_{n+1}\}.$ If $S = \Pi$, then $\bigcap_{\alpha \in \Pi} I(\alpha)_{\min} = \{\theta\}$ and $\Delta^+ = \bigcup_{\alpha \in \Pi} I(\alpha)_{\max}$.

As a by-product of Theorem [4.3,](#page-8-1) we derive a property of maximal abelian ideals outside type **A**. Given $S \subset \Pi_l$, let $\langle S \rangle$ be the smallest connected subset of Π_l containing S.

Theorem 4.5. Let $S \subset \Pi_l$. Then

- (i) $\bigcap_{\alpha \in S} I(\alpha)_{\min} = \bigcap_{\alpha \in \langle S \rangle} I(\alpha)_{\min}$;
- (ii) if $\Delta \neq \mathbf{A}_n$, then $\bigcup_{\alpha \in S} I(\alpha)_{\max} = \bigcup_{\alpha \in \langle S \rangle} I(\alpha)_{\max}$.

Proof. (i) By [\[11,](#page-11-2) Theorem 2.1], \bigcap $\alpha{\in}S$ $I(\alpha)_{\min} = I(\gamma)_{\min}$, where $\gamma = \sqrt{\frac{\beta}{\alpha}}$ $_{\alpha \in S}$ α . It remains to notice that $\sqrt{}$ $_{\alpha \in S}$ $\alpha = \sum_{\alpha \in \langle S \rangle} \alpha = \bigvee$ $\alpha{\in}\langle S\rangle$ α.

(ii) This follows from (i) and Theorem [4.3.](#page-8-1) Namely, if Δ is not of type \mathbf{A}_n , then

$$
\max(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}) = \max(\Delta^+ \setminus \bigcup_{\alpha \in \langle S \rangle} I(\alpha)_{\max}).
$$

Hence both unions also coincide. \Box

The equality \bigcap $\alpha \in S$ $I(\alpha)_{\min} = I(\bigvee$ $\alpha{\in}S$ $\alpha)_{\sf min}$ has interesting consequences. By [\[8,](#page-11-4) Prop. 4.6], the minimal elements of the abelian ideal $I(\gamma)_{\text{min}}$ have the following description:

Let $w_{\gamma} \in W$ *be a unique element of minimal length such that* $w_{\gamma}(\theta) = \gamma$ *. If* $\beta \in \Pi$ *and* $(\beta, \gamma^{\vee}) = -1$, then $w_{\gamma}^{-1}(\beta + \gamma) = w_{\gamma}^{-1}(\beta) + \theta \in \min(I(\gamma)_{\min})$. Conversely, any element of $\min(I(\gamma)_{\min})$ *is obtained in this way.*

For any γ of the form $\sqrt{\alpha}$, the required simple roots β are easily determined, which yields the maximal elements of $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}.$ We consider below the particular case in which $S=\Pi_l$.

Proposition 4.6. Set $|\Pi_l| = \sum_{\alpha \in \Pi_l} \alpha$. If $|\Pi_l| \neq \theta$, i.e., Δ is not of type \mathbf{A}_n , then there is a unique $\hat{\boldsymbol{\alpha}} \in \Pi$ such that $|\Pi_l| + \hat{\boldsymbol{\alpha}}$ is a root. More precisely,

– *if* $\Delta \in \{D - E\}$, then $\hat{\alpha}$ *is the branching point in the Dynkin diagram;*

– *if* ∆ ∈ {B-C-F-G}*, then* αˆ *is the unique short root that is adjacent to a long root in the Dynkin diagram.*

In all these cases, $w_{\text{III},i}^{-1}$ $\frac{-1}{|\Pi_l|}(\boldsymbol{\hat{\alpha}}) = -\lfloor \theta/2 \rfloor.$

Proof. If $S = \Pi_l$, then $\bigcup_{\alpha \in \Pi_l} I(\alpha)_{\max} = \Delta_{\text{com}}^+$. Hence $\max(\Delta^+ \setminus \bigcup_{\alpha \in \Pi_l} I(\alpha)_{\max}) = {\lfloor \theta/2 \rfloor }$, see Theorem [3.5.](#page-5-2) Therefore, by Theorem [4.3,](#page-8-1) the unique minimal element of $I(|\Pi_l|)_{\sf min} =$ $\bigcap_{\alpha\in\Pi_l}I(\alpha)_{\min}$ is $\theta-\lfloor\theta/2\rfloor=:\lceil\theta/2\rceil.$ This means that there is a unique simple root $\hat{\alpha}$ such that $(|\Pi_l|^\vee,\hat\alpha)=-1$, i.e., $|\Pi_l|+\hat\alpha$ is a root. Since $w_{|\Pi_l}^{-1}$ $\frac{-1}{|\Pi_l|}(|\Pi_l|+\bm{\hat{\alpha}})=\theta+w_{|\Pi_l}^{-1}$ $\frac{-1}{|\Pi_l|}(\boldsymbol{\hat{\alpha}})=\theta-\lfloor \theta/2 \rfloor$, the last assertion follows.

Clearly, $\hat{\boldsymbol{\alpha}}$ specified in the statement satisfies the condition that $(|\Pi_l|, \hat{\boldsymbol{\alpha}}) < 0.$

The A_n -case can partially be included in the DE-picture, if we formally assume that $\hat{\alpha} = 0$ (because there is no branching point).

5. ON THE INTERVAL $[|\theta/2|, [\theta/2]|$

In this section, we first assume that Δ is not of type \mathbf{A}_n . Since $\lfloor \theta/2 \rfloor \in \mathcal{H}$, we have $\lceil \theta/2 \rceil = \theta - |\theta/2| \in \mathcal{H}$ and also $\lceil \theta/2 \rceil \leq \lceil \theta/2 \rceil$. We consider the interval between $\lceil \theta/2 \rceil$ and $\lceil \theta/2 \rceil$ in Δ^+ . Let *h* be the Coxeter number of Δ .

Proposition 5.1. *Set* $\mathfrak{J} = \{ \gamma \in \Delta^+ \mid |\theta/2| \leq \gamma \leq |\theta/2| \}$ *.* \rightarrow *if* ∆ ∈ {**D-E**}, then $\mathfrak{J} \simeq \mathbb{B}^3$ and $\mathsf{ht}([\theta/2]) = (h/2) + 1;$

– *if* $\Delta \in \{B-C-F-G\}$ *, then* \Im *is a segment and* $\text{ht}([\theta/2]) = h/2$ *.*

Proof. This can be verified case-by-case, but we also provide some ^a priori hints. It follows from the definition of $\lfloor \theta/2 \rfloor$, see Eq. (3-[1\)](#page-4-4), that

$$
\lceil \theta/2 \rceil - \lfloor \theta/2 \rfloor = \theta - 2 \lfloor \theta/2 \rfloor = 2 \lceil \theta/2 \rceil - \theta = \sum_{\alpha: \text{ ht}_{\alpha}(\theta) \text{ odd}} \alpha,
$$

the sum of all odd simple roots. Let $\mathcal{O} \subset \Pi$ denote the set of odd roots. Then ht($\lceil \theta/2 \rceil$) – $\text{ht}(|\theta/2|) = \#\mathcal{O}.$

• In the simply-laced case, $(\theta - 2\lfloor \theta/2 \rfloor, \lfloor \theta/2 \rfloor^{\vee}) = 1 - 4 = -3$. Therefore, there are at least three $\alpha \in \mathcal{O}$ such that $(\alpha, \lfloor \theta/2 \rfloor^{\vee}) = -1$, i.e., $\lfloor \theta/2 \rfloor + \alpha \in \Delta^+$. On the other hand, for any $\gamma \in \Delta^+$, there are at most three $\alpha \in \Pi$ such that $\gamma + \alpha \in \Delta^+$ [\[10,](#page-11-15) Theorem 3.1(i)]. Thus, there are exactly three odd roots α_i such that $|\theta/2| + \alpha_i \in \Delta^+$. Actually, there are only three odd roots in the ${D-E}$ -case. Hence every odd root can be added to $\lfloor \theta/2 \rfloor$. Likewise, $(2\lceil \theta/2 \rceil - \theta, \lceil \theta/2 \rceil) = 3$ and the same three roots can be subtracted from $\lceil \theta/2 \rceil$. This yields all six roots strictly between $\lfloor \theta/2 \rfloor$ and $\lfloor \theta/2 \rfloor$. If $\mathcal{O} = \{\beta_1, \beta_2, \beta_3\}$, then $\mathfrak J$ is as follows:

• In the non-simply laced cases, there is always a unique odd root and hence $\mathfrak{J} =$ $\{[\theta/2], [\theta/2]\}.$

Remark 5.2. If Δ is of type \mathbf{A}_n , then $|\theta/2| = 0$ and $\theta/2| = \theta$. Then $\mathfrak{J} = \Delta^+ \cup \{0\}$. However, this poset is not a modular lattice.

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