ABELIAN IDEALS OF A BOREL SUBALGEBRA AND ROOT SYSTEMS, II

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ABSTRACT. Let $\mathfrak g$ be a simple Lie algebra with a Borel subalgebra $\mathfrak b$ and $\mathfrak A\mathfrak b$ the set of abelian ideals of $\mathfrak b$. Let Δ^+ be the corresponding set of positive roots. We continue our study of combinatorial properties of the partition of $\mathfrak A\mathfrak b$ parameterised by the long positive roots. In particular, the union of an arbitrary set of maximal abelian ideals is described, if $\mathfrak g \neq \mathfrak s\mathfrak l_n$. We also characterise the greatest lower bound of two positive roots, when it exists, and point out interesting subposets of Δ^+ that are modular lattices.

INTRODUCTION

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} , with a triangular decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{t} \oplus \mathfrak{u}^-$. Here \mathfrak{t} is a Cartan and $\mathfrak{b} = \mathfrak{u} \oplus \mathfrak{t}$ is a fixed Borel subalgebra. The theory of abelian ideals of \mathfrak{b} is based on their relationship, due to D. Peterson, with the *minuscule elements* of the affine Weyl group \widehat{W} (see Kostant's account in [6]; another approach is presented in [2]). In this note, we elaborate on some topics related to the combinatorial theory of abelian ideals, which can be regarded as a sequel to [11]. We mostly work in the combinatorial setting, i.e., the abelian ideals of \mathfrak{b} , which are sums of root spaces of \mathfrak{u} , are identified with the corresponding sets of positive roots.

Let Δ be the root system of $(\mathfrak{g},\mathfrak{t})$ in the vector space $V=\mathfrak{t}_{\mathbb{R}}^*$, Δ^+ the set of positive roots in Δ corresponding to \mathfrak{u} , Π the set of simple roots in Δ^+ , and θ the highest root in Δ^+ . Then W is the Weyl group and $(\ ,\)$ is a W-invariant scalar product on V. We equip Δ^+ with the usual partial ordering ' \succcurlyeq '. An *upper ideal* (or just an *ideal*) of (Δ^+, \succcurlyeq) is a subset $I \subset \Delta^+$ such that if $\gamma \in I, \nu \in \Delta^+$, and $\nu + \gamma \in \Delta^+$, then $\nu + \gamma \in I$. An upper ideal I is *abelian*, if $\gamma' + \gamma'' \not\in \Delta^+$ for all $\gamma', \gamma'' \in I$. The set of minimal elements of I is denoted by $\min(I)$. It also makes sense to consider the maximal elements of the complement of I, denoted $\max(\Delta^+ \setminus I)$.

Write \mathfrak{Ab} (resp. \mathfrak{Ad}) for the set of all abelian (resp. all upper) ideals of Δ^+ and think of them as posets with respect to inclusion. The upper ideal *generated by* γ is $I\langle \succcurlyeq \gamma \rangle = \{\nu \in \Delta^+ \mid \nu \succcurlyeq \gamma\}$. Then $\min(I\langle \succcurlyeq \gamma \rangle) = \{\gamma\}$. A root $\gamma \in \Delta^+$ is said to be *commutative*, if $I\langle \succcurlyeq \gamma \rangle \in \mathfrak{Ab}$. Write Δ^+_{com} for the set of all commutative roots. This notion was introduced

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in [9], and the subset Δ_{com}^+ for each Δ is explicitly described in [9, Theorem 4.4]. Note that $\Delta_{\text{com}}^+ \in \mathfrak{A}\mathfrak{d}$.

Let \mathfrak{Ab}^o denote the set of nonempty abelian ideals and Δ_l^+ the set of long positive roots. In [8, Sect. 2], we defined a mapping $\tau:\mathfrak{Ab}^o\to\Delta_l^+$, which is onto. Letting $\mathfrak{Ab}_\mu=\tau^{-1}(\mu)$, we get a partition of \mathfrak{Ab}^o parameterised by Δ_l^+ . Each \mathfrak{Ab}_μ is a subposet of \mathfrak{Ab} and, moreover, \mathfrak{Ab}_μ has a unique minimal and unique maximal element (ideal) [8, Sect. 3]. These extreme abelian ideals in \mathfrak{Ab}_μ are denoted by $I(\mu)_{\min}$ and $I(\mu)_{\max}$. Then $\{I(\alpha)_{\max} \mid \alpha \in \Pi_l\}$ are exactly the maximal abelian ideals of \mathfrak{b} .

In this article, we first establish a property of (Δ^+, \succcurlyeq) , which seems to be new. It was proved in [11, Appendix] that, for any $\eta_1, \eta_2 \in \Delta^+$, there exists the *least upper bound*, denoted $\eta_1 \vee \eta_2$. Moreover, an explicit formula for $\eta_1 \vee \eta_2$ is also given. Here we prove that the *greatest lower bound*, $\eta_1 \wedge \eta_2$, exists if and only if $\operatorname{supp}(\gamma_1) \cap \operatorname{supp}(\gamma_2) \neq \varnothing$. Furthermore, if $\eta_i = \sum_{\alpha \in \Pi} c_{i\alpha}\alpha$, then $\eta_1 \wedge \eta_2 = \sum_{\alpha \in \Pi} \min\{c_{1\alpha}, c_{2\alpha}\}\alpha$. This also implies that $I\langle \succcurlyeq \eta \rangle$ is a modular lattice for any $\eta \in \Delta^+$, see Theorem 2.4. Another example a modular lattice inside Δ^+ is the subposet $\Delta_\alpha(i) = \{\gamma \in \Delta^+ \mid \operatorname{ht}_\alpha(\gamma) = i\}$, where $\alpha \in \Pi$ and $\operatorname{ht}_\alpha(\gamma)$ is the coefficient of α in the expression of γ via Π .

Using properties of 'V' and 'A' and \mathbb{Z} -gradings of \mathfrak{g} , we prove uniformly that if Δ is not of type \mathbf{A}_n , then $\Delta_{\rm nc}^+ := \Delta^+ \setminus \Delta_{\rm com}^+$ has the unique maximal element, which is $\lfloor \theta/2 \rfloor := \sum_{\alpha \in \Pi} \lfloor \operatorname{ht}_{\alpha}(\theta)/2 \rfloor \alpha$, see Section 3. In particular, $\lfloor \theta/2 \rfloor$ is a root. (Note that if Δ is of type \mathbf{A}_n , then $\lfloor \theta/2 \rfloor = 0$ and $\Delta_{\rm nc}^+ = \varnothing$.) We also describe the maximal abelian ideals $I(\alpha)_{\rm max}$ if $\operatorname{ht}_{\alpha}(\theta)$ is odd.

In Section 4, we study the sets of maximal and minimal elements related to abelian ideals of the form $I(\alpha)_{\min}$ and $I(\alpha)_{\max}$, with $\alpha \in \Pi_l := \Pi \cap \Delta_l^+$.

Theorem 0.1. If $S \subset \Pi_l$ is arbitrary and Δ is not of type \mathbf{A}_n , then there is the bijection

$$\eta \in \min \bigl(\bigcap_{\alpha \in S} I(\alpha)_{\min}\bigr) \overset{1:1}{\longmapsto} \eta' = \theta - \eta \in \max \bigl(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}\bigr).$$

Our proof is conceptual and relies on the fact θ is a multiple of a fundamental weight if Δ is not of type \mathbf{A}_n . For \mathbf{A}_n , the same bijection holds if S is a **connected** subset on the Dynkin diagram. The case in which #S=1 was considered earlier in [11, Theorem 4.7]. This has some interesting consequences if $S=\Pi_l$ and hence $\bigcup_{\alpha\in\Pi_l}I(\alpha)_{\max}=\Delta_{\mathsf{com}}^+$, see Proposition 4.6.

In Section 5, we describe the interval $[\lfloor \theta/2 \rfloor, \theta - \lfloor \theta/2 \rfloor]$ inside the poset Δ^+ .

1. Preliminaries

We have $\Pi = \{\alpha_1, \dots, \alpha_n\}$, the vector space $V = \bigoplus_{i=1}^n \mathbb{R}\alpha_i$, the Weyl group W generated by simple reflections s_{α} ($\alpha \in \Pi$), and a W-invariant inner product (,) on V. Set $\rho =$

 $\frac{1}{2}\sum_{\nu\in\Delta^+}\nu$. The partial ordering ' \preccurlyeq ' in Δ^+ is defined by the rule that $\mu\preccurlyeq\nu$ if $\nu-\mu$ is a non-negative integral linear combination of simple roots. Write $\mu\prec\nu$, if $\mu\preccurlyeq\nu$ and $\mu\neq\nu$. If $\mu=\sum_{i=1}^nc_i\alpha_i\in\Delta$, then $\mathsf{ht}_{\alpha_i}(\mu):=c_i$, $\mathsf{ht}(\mu):=\sum_{i=1}^nc_i$ and $\mathsf{supp}(\mu)=\{\alpha_i\in\Pi\mid c_i\neq 0\}$.

The Heisenberg ideal $\mathcal{H} := \{ \gamma \in \Delta^+ \mid (\gamma, \theta) \neq 0 \} = \{ \gamma \in \Delta^+ \mid (\gamma, \theta) > 0 \} \in \mathfrak{Ad}$ plays a prominent role in the theory of abelian ideals and posets $\mathfrak{Ab}_{\mu} = \tau^{-1}(\mu)$.

Let us collect some known results that are frequently used below.

- If $I \in \mathfrak{Ad}$ is not abelian, then there exist $\eta, \eta' \in I$ such that $\eta + \eta' = \theta$, see [8, p. 1897]. Therefore, $I \notin \mathfrak{Ab}$ if and only if $I \cap \mathfrak{H} \notin \mathfrak{Ab}$.
- $I = I(\mu)_{\min}$ for some $\mu \in \Delta_I^+$ if and only if $I \subset \mathcal{H}$ [8, Theorem 4.3];
- $\#I(\mu)_{\min} = (\rho, \theta^{\vee} \mu^{\vee}) + 1$ [8, Theorem 4.2(4)];
- For $I \in \mathfrak{Ab}^o$, we have $I \in \mathfrak{Ab}_{\mu}$ if and only if $I \cap \mathcal{H} = I(\mu)_{\min}$ [11, Prop. 3.2];
- The set of (globally) maximal abelian ideals is $\{I(\alpha)_{max} \mid \alpha \in \Pi_l\}$ [8, Corollary 3.8].
- For any $\mu \in \Delta_l^+$, there is a unique element of minimal length in W that takes θ to μ [8, Theorem 4.1]. Writing w_μ for this element, one has $\ell(w_\mu) = (\rho, \theta^\vee \mu^\vee)$ [8, Theorem 4.1].
- Let $\mathcal{N}(w)$ be the inversion set of $w \in W$. By [12, Lemma 1.1],

$$I(\mu)_{\min} = \{\theta\} \cup \{\theta - \gamma \mid \gamma \in \mathcal{N}(w_{\mu})\}.$$

For each $\eta \in \mathcal{H} \setminus \{\theta\}$ there is a unique $\eta' \in \mathcal{H} \setminus \{\theta\}$ such that $\eta + \eta'$ is a root, and this root is θ . It is well known that $\#\mathcal{H} = 2(\rho, \theta^{\vee}) - 1 = 2h^* - 3$, where h^* is the *dual Coxeter number* of Δ . Since $\#I(\alpha)_{\min} = (\rho, \theta^{\vee}) = h^* - 1$ for $\alpha \in \Pi_l$, the ideal $I(\alpha)_{\min}$ contains θ and exactly a half of elements of $\mathcal{H} \setminus \{\theta\}$, cf. also [11, Lemma 3.3].

Although the affine Weyl group and minuscule elements are not explicitly used in this paper, their use is hidden in properties of the posets \mathfrak{Ab}_{μ} , $\mu \in \Delta_l^+$, and ideals $I(\mu)_{\text{min}}$, $I(\mu)_{\text{max}}$. Important properties of the maximal abelian ideals are also obtained in [3, 16].

We refer to [1], [4, $\S 3.1$] for standard results on root systems and Weyl groups and to [15, Chapter 3] for posets.

2. The greatest lower bound in Δ^+

It is proved in [8, Appendix] that the poset (Δ^+, \succcurlyeq) is a join-semilattice. i.e., for any pair $\eta, \eta' \in \Delta^+$, there is the least upper bound (= join), denoted $\eta \vee \eta'$. Furthermore, there is a simple explicit formula for ' \vee ', see [8, Theorem A.1]. However, Δ^+ is not a meet-semilattice. We prove below that under a natural constraint the greatest lower bound (= meet) exists and can explicitly be described. Afterwards, we provide some applications of this property in the theory of abelian ideals.

Definition 1. Let $\eta, \eta' \in \Delta^+$. The root ν is the greatest lower bound (or *meet*) of η and η' if

• $\eta \succcurlyeq \nu, \eta' \succcurlyeq \nu$;

• if $\eta \succcurlyeq \kappa$ and $\eta' \succcurlyeq \kappa$, then $\nu \succcurlyeq \kappa$.

The meet of η and η' , if it exists, is denoted by $\eta \wedge \eta'$.

Obviously, if $\alpha, \alpha' \in \Pi$, then their meet does not exist. But as we see below, the only reason for such a failure is that their supports are disjoint.

Lemma 2.1 (see [14, Lemma 3.1]). Suppose that $\gamma \in \Delta^+$ and $\alpha, \beta \in \Pi$. If $\gamma - \alpha, \gamma - \beta \in \Delta^+$, then either $\gamma - \alpha - \beta \in \Delta^+$ or $\gamma = \alpha + \beta$ and hence α, β are adjacent in the Dynkin diagram.

Lemma 2.2 (see [13, Lemma 3.2]). *Suppose that* $\gamma \in \Delta^+$ *and* $\alpha, \beta \in \Pi$. *If* $\gamma + \alpha, \gamma + \beta \in \Delta^+$, *then* $\gamma + \alpha + \beta \in \Delta^+$.

Let us provide a reformulation of these lemmata in terms of ' \vee ' and ' \wedge '. To this end, we note that in the previous lemma, $(\gamma + \alpha) \wedge (\gamma + \beta) = \gamma$.

Proposition 2.3. Let $\eta_1, \eta_2 \in \Delta^+$.

- (i) If $\eta_1 \vee \eta_2$ covers both η_1 and η_2 , then either $\eta_1 \vee \eta_2 = \alpha + \beta = \eta_1 + \eta_2$ for some adjacent $\alpha, \beta \in \Pi$, or η_1 and η_2 both cover $\eta_1 \wedge \eta_2$;
- (ii) If $\eta_1 \wedge \eta_2$ exists and η_1 and η_2 both cover $\eta_1 \wedge \eta_2$, then $\eta_1 \vee \eta_2$ covers both η_1 and η_2 .

For any two roots $\eta = \sum_{\alpha \in \Pi} c_{\alpha} \alpha$ and $\eta' = \sum_{\alpha \in \Pi} c'_{\alpha} \alpha$, one defines two elements of the root lattice, $\min(\eta, \eta') = \sum_{\alpha \in \Pi} \min\{c_{\alpha}, c'_{\alpha}\}\alpha$ and $\max(\eta, \eta') = \sum_{\alpha \in \Pi} \max\{c_{\alpha}, c'_{\alpha}\}\alpha$. Recall that the poset (Δ^+, \succcurlyeq) is graded and the rank function is the usual *height* of a root, i.e., $\operatorname{ht}(\eta) = \sum_{\alpha \in \Pi} c_{\alpha}$. We also set $\operatorname{ht}_{\alpha}(\eta) := c_{\alpha}$.

Theorem 2.4.

- 1) For any $\gamma \in \Delta^+$, the upper ideal $I(\succcurlyeq \gamma)$ is a modular lattice;
- 2) the meet $\gamma_1 \wedge \gamma_2$ exists if and only if $supp(\gamma_1) \cap supp(\gamma_2) \neq \emptyset$. In this case, one has $\gamma_1 \wedge \gamma_2 = min(\gamma_1, \gamma_2)$.

Proof. 1) By [8, Theorem A.1(i)], the join always exists in Δ^+ and formulae for 'V' show that $\gamma_1 \vee \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$ whenever $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$. Therefore, $I \langle \succcurlyeq \gamma \rangle$ is a join-semilattice with a unique minimal element. Hence the meet also exists for any $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$, see [15, Prop. 3.3.1]. That is, $I \langle \succcurlyeq \gamma \rangle$ is a lattice. Note that, for $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$, the first possibility in Proposition 2.3(i) does not realise. Therefore, using Proposition 2.3 with $I \langle \succcurlyeq \gamma \rangle$ in place of Δ^+ and [15, Prop. 3.3.2], we conclude that $I \langle \succcurlyeq \gamma \rangle$ is a modular lattice.

Yet, this does not provide a formula for the meet and leaves a theoretical possibility that $\gamma_1 \wedge \gamma_2$ depends on γ .

2) If $\operatorname{supp}(\gamma_1) \cap \operatorname{supp}(\gamma_2) = \emptyset$, then there are no roots ν such that $\gamma_1 \succcurlyeq \nu$ and $\gamma_2 \succcurlyeq \nu$. Conversely, if $\operatorname{supp}(\gamma_1) \cap \operatorname{supp}(\gamma_2) \neq \emptyset$, then $\gamma_1, \gamma_2 \in I \langle \succcurlyeq \gamma \rangle$ for some γ . Using again [15, Prop. 3.3.2], the modularity of the lattice $I \langle \succcurlyeq \gamma \rangle$ implies that $\operatorname{ht}(\gamma_1 \vee \gamma_2) + \operatorname{ht}(\gamma_1 \wedge \gamma_2) = \emptyset$

 $\operatorname{ht}(\gamma_1) + \operatorname{ht}(\gamma_2)$, where $\gamma_1 \wedge \gamma_2$ is taken inside $I \langle \not \succ \gamma \rangle$. It is clear that $\gamma_1 \wedge \gamma_2 \preccurlyeq \min(\gamma_1, \gamma_2)$. Moreover, in this situation, the formulae of [8, Theorem A.1(i)] imply that $\gamma_1 \vee \gamma_2 = \max(\gamma_1, \gamma_2)$. Therefore, $\operatorname{ht}(\min(\gamma_1, \gamma_2)) = \operatorname{ht}(\gamma_1 \wedge \gamma_2)$ and thereby $\min(\gamma_1, \gamma_2) = \gamma_1 \wedge \gamma_2$.

Remark 2.5. A special class of modular lattices inside Δ^+ occurs in connection with \mathbb{Z} -gradings of \mathfrak{g} . For $\alpha \in \Pi$, set $\Delta_{\alpha}(i) = \{ \gamma \in \Delta \mid \operatorname{ht}_{\alpha}(\gamma) = i \}$. It is known that $\Delta_{\alpha}(i)$ has a unique minimal and a unique maximal element, see Section 3. It is also clear that $\gamma_1 \wedge \gamma_2$ and $\gamma_1 \vee \gamma_2 \in \Delta_{\alpha}(i)$ for all $\gamma_1, \gamma_2 \in \Delta_{\alpha}(i)$. Hence $\Delta_{\alpha}(i)$ is a **modular** lattice. (It was already noticed in [8, Appendix] that $\Delta_{\alpha}(i)$ is a lattice.)

Remark 2.6. In what follows, we have to distinguish the \mathbf{A}_n -case from the other types. One the reasons is that θ is not a multiple of a fundamental weight only for \mathbf{A}_n . In all other types, there is a unique $\alpha_{\theta} \in \Pi$ such that $(\theta, \alpha_{\theta}) \neq 0$. For the \mathbb{Z} -grading associated with α_{θ} , one then has $\Delta_{\alpha_{\theta}}(1) = \mathcal{H} \setminus \{\theta\}$ and $\Delta_{\alpha_{\theta}}(2) = \{\theta\}$. That is, $\mathcal{H} \setminus \{\theta\}$ (or just \mathcal{H}) has a unique minimal element, which is α_{θ} , if and only if Δ is not of type \mathbf{A}_n . This provides the following consequence of Theorem 2.4:

If Δ is not of type \mathbf{A}_n , then for all $\eta_1, \eta_2 \in \mathcal{H} \setminus \{\theta\}$, the meet $\eta_1 \wedge \eta_2$ exists and lies in $\mathcal{H} \setminus \{\theta\}$. This is going to be used several times in Section 4.

3. \mathbb{Z} -gradings and non-commutative roots

If $\gamma \in \Delta_{\text{com}}^+$, then γ belongs to a maximal abelian ideal. Since $I(\alpha)_{\text{max}}$, $\alpha \in \Pi_l$, are all the maximal abelian ideals in Δ^+ , we have

$$\Delta_{\mathsf{com}}^+ = \bigcup_{\alpha \in \Pi_l} I(\alpha)_{\mathsf{max}}.$$

Set $\Delta_{nc}^+ = \Delta^+ \setminus \Delta_{com}^+$. In this section, we obtain an *a priori* description of Δ_{nc}^+ . Let us introduce special elements of the root lattice

$$(3\cdot 1) \qquad \qquad \lfloor \theta/2 \rfloor = \sum_{\alpha \in \Pi} \lfloor \mathsf{ht}_{\alpha}(\theta)/2 \rfloor \alpha \ \ \mathsf{and} \ \ \lceil \theta/2 \rceil = \sum_{\alpha \in \Pi} \lceil \mathsf{ht}_{\alpha}(\theta)/2 \rceil \alpha.$$

Hence $\lfloor \theta/2 \rfloor + \lceil \theta/2 \rceil = \theta$. Note that $\lfloor \theta/2 \rfloor = 0$ if and only if $\theta = \sum_{\alpha \in \Pi} \alpha$, i.e., Δ is of type \mathbf{A}_n .

Lemma 3.1. *Suppose that* Δ *is not of type* \mathbf{A}_n *, so that* $\lfloor \theta/2 \rfloor \neq 0$ *.*

- (1) If $\gamma \in \Delta_{\mathsf{nc}}^+$, then $\mathsf{ht}_{\alpha} \gamma \leqslant \lfloor \mathsf{ht}_{\alpha}(\theta)/2 \rfloor$ for all $\alpha \in \Pi$, i.e., $\gamma \preccurlyeq \lfloor \theta/2 \rfloor$.
- (2) If $\gamma_1, \gamma_2 \leq \lfloor \theta/2 \rfloor$, then $\gamma_1 \vee \gamma_2 \leq \lfloor \theta/2 \rfloor$.

Proof. (1) Obvious.

(2) By [8, Theorem A.1], if $supp(\gamma_1) \cup supp(\gamma_2)$ is connected, then $\gamma_1 \vee \gamma_2 = \max(\gamma_1, \gamma_2)$ and the assertion is clear. Otherwise, $\gamma_1 \vee \gamma_2 = \gamma_1 + (connecting\ root) + \gamma_2$. Recall that if the union of supports is not connected, then there is a (unique) chain of simple roots

that connects them. If this chain consists of $\alpha_{i_1}, \ldots, \alpha_{i_s}$, then the "connecting root" is $\alpha_{i_1} + \cdots + \alpha_{i_s}$. Here we only need the condition that $\operatorname{ht}_{\alpha}(\theta) \geqslant 2$ for any α in the connecting chain. Indeed, the roots in this chain are not extreme in the Dynkin diagram, and outside type \mathbf{A}_n the coefficients of non-extreme simple roots are always $\geqslant 2$.

Remark. For \mathbf{A}_n , $\lfloor \theta/2 \rfloor = 0$ and hence $\Delta_{\mathsf{nc}}^+ = \varnothing$.

Set $\mathcal{A} = \{ \gamma \in \Delta^+ \mid \gamma \leq \lfloor \theta/2 \rfloor \}$. Then $\mathcal{A} \neq \emptyset$ if and only if Δ is not of type \mathbf{A}_n . It follows from Lemma 3.1 that

- $\Delta_{nc}^+ \subset \mathcal{A}$;
- A has a unique maximal element.

Our goal is to prove that $\Delta_{nc}^+ = \mathcal{A}$ and $\max(\mathcal{A}) = \{\lfloor \theta/2 \rfloor\}$. The latter essentially boils down to the assertion that $\lfloor \theta/2 \rfloor$ is a root.

For an arbitrary $\alpha \in \Pi$, consider the \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\alpha}(i)$ corresponding to α . That is, the set of roots of $\mathfrak{g}_{\alpha}(i)$ is $\Delta_{\alpha}(i)$, see Remark 2.5. In particular, $\alpha \in \Delta_{\alpha}(1)$ and $\Pi \setminus \{\alpha\} \subset \Delta_{\alpha}(0)$. Here $\mathfrak{l} := \mathfrak{g}_{\alpha}(0)$ is reductive and contains the Cartan subalgebra \mathfrak{t} . By an old result of Kostant (see [7] and Joseph's exposition in [5, 2.1]), each $\mathfrak{g}_{\alpha}(i)$, $i \neq 0$, is a simple \mathfrak{l} -module. Therefore, $\Delta_{\alpha}(i)$ has a unique minimal and a unique maximal element. The following is a particular case of Theorem 2.3 in [7].

Proposition 3.2. If
$$i + j \leq \mathsf{ht}_{\alpha}(\theta)$$
, then $0 \neq [\mathfrak{g}_{\alpha}(i), \mathfrak{g}_{\alpha}(j)] = \mathfrak{g}_{\alpha}(i + j)$.

Once one has proved that $[\mathfrak{g}_{\alpha}(i),\mathfrak{g}_{\alpha}(j)] \neq 0$, the equality $[\mathfrak{g}_{\alpha}(i),\mathfrak{g}_{\alpha}(j)] = \mathfrak{g}_{\alpha}(i+j)$ stems from the fact that $\mathfrak{g}_{\alpha}(i+j)$ is a simple I-module. We derive from this result two corollaries.

Corollary 3.3. For any $\mu \in \Delta_{\alpha}(i)$, there is $\nu \in \Delta_{\alpha}(j)$ such that $\mu + \nu \in \Delta_{\alpha}(i+j)$.

Proof. Let $e_{\mu} \in \mathfrak{g}_{\alpha}(i)$ be a root vector for μ . Assume that the property in question does not hold. Then $[e_{\mu}, \mathfrak{g}_{\alpha}(j)] = 0$. Hence $[L \cdot e_{\mu}, \mathfrak{g}_{\alpha}(j)] = 0$, where $L \subset G$ is the connected reductive group with Lie algebra \mathfrak{l} . Since the linear span of an L-orbit in a simple L-module is the whole space, this implies that $[\mathfrak{g}_{\alpha}(i), \mathfrak{g}_{\alpha}(j)] = 0$, which contradicts the proposition. \square

Set $d_{\alpha} = \lfloor \operatorname{ht}_{\alpha}(\theta)/2 \rfloor$, and let $\mu_{d_{\alpha}}$ be the lowest weight in $\Delta_{\alpha}(d_{\alpha})$.

Corollary 3.4. $\mu_{d_{\alpha}} \in \Delta_{\mathrm{nc}}^+$.

Proof. By Corollary 3.3, there is $\lambda \in \Delta_{\alpha}(d_{\alpha})$ such that $\mu_{d_{\alpha}} + \lambda$ is a root in $\Delta_{\alpha}(2d_{\alpha})$. Since $\mu_{d_{\alpha}} \leq \gamma$, the upper ideal in Δ^+ generated by $\mu_{d_{\alpha}}$ is not abelian.

This allows us to obtain the promised characterisation of Δ_{nc}^+ .

Theorem 3.5. If $\lfloor \theta/2 \rfloor \neq 0$, i.e., Δ is not of type \mathbf{A}_n , then $\lfloor \theta/2 \rfloor$ is the unique maximal element of Δ_{nc}^+ . Furthermore, $\lfloor \theta/2 \rfloor \in \mathcal{H}$.

Proof. It was noticed above that $\Delta_{\mathsf{nc}}^+ \subset \mathcal{A}$, \mathcal{A} has a unique maximal element, say $\hat{\nu}$, and $\hat{\nu} \preccurlyeq \lfloor \theta/2 \rfloor$. By Corollary 3.4, for any $\alpha \in \Pi$, there is $\mu_{\alpha} \in \Delta_{\mathsf{nc}}^+$ such that $\mathsf{ht}_{\alpha}(\mu_{\alpha}) = d_{\alpha}$. Therefore $\bigvee_{\alpha \in \Pi} \mu_{\alpha} \succcurlyeq \lfloor \theta/2 \rfloor$. On the other hand, $\mu_{\alpha} \preccurlyeq \hat{\nu}$ for each α and hence $\bigvee_{\alpha \in \Pi} \mu_{\alpha} \preccurlyeq \hat{\nu} \preccurlyeq \lfloor \theta/2 \rfloor$. Thus, $\lfloor \theta/2 \rfloor = \hat{\nu}$ is a root. If α_{θ} is the unique simple root such that $(\theta, \alpha_{\theta}) \neq 0$, then $\mathsf{ht}_{\alpha_{\theta}}(\theta) = 2$. Therefore $\lfloor \theta/2 \rfloor \in \mathcal{H}$ whenever Δ is not \mathbf{A}_n .

The fact that $\lfloor \theta/2 \rfloor$ is the unique maximal non-commutative root has been observed in [9, Sect. 4] via a case-by-case analysis.

Example 3.6. If
$$\Delta$$
 is of type \mathbf{E}_8 , then $\theta = \frac{2345642}{3}$ and $\lfloor \theta/2 \rfloor = \frac{1122321}{1}$.

Remark 3.7. In the proof of Corollary 3.4 and then Theorem 3.5, we only need the property, which follows from Proposition 3.2, that $[\mathfrak{g}_{\alpha}(d_{\alpha}),\mathfrak{g}_{\alpha}(d_{\alpha})] = \mathfrak{g}_{\alpha}(2d_{\alpha})$.

For $\alpha \in \Pi$ with $\operatorname{ht}_{\alpha}(\theta) = 2$ or 3, this means that $[\mathfrak{g}_{\alpha}(1), \mathfrak{g}_{\alpha}(1)] = \mathfrak{g}_{\alpha}(2)$, which is obvious. This covers all classical simple Lie algebras, \mathbf{E}_{6} , and \mathbf{G}_{2} . For \mathbf{E}_{7} , \mathbf{E}_{8} , and \mathbf{F}_{4} , there are $\alpha \in \Pi$ such that $\operatorname{ht}_{\alpha}(\theta) \in \{4, 5, 6\}$. Then the required relation is $[\mathfrak{g}_{\alpha}(2), \mathfrak{g}_{\alpha}(2)] = \mathfrak{g}_{\alpha}(4)$ or $[\mathfrak{g}_{\alpha}(3), \mathfrak{g}_{\alpha}(3)] = \mathfrak{g}_{\alpha}(6)$. This can easily be verified case-by-case. However, our intention is to provide a case-free treatment of this property.

Another consequence of Kostant's theory [7] is that one obtains an explicit presentation of some maximal abelian ideals.

Proposition 3.8. Suppose that $\operatorname{ht}_{\alpha}(\theta) = 2d_{\alpha} + 1$ is odd. Then $\mathfrak{a} := \bigoplus_{j \geqslant d_{\alpha} + 1} \mathfrak{g}_{\alpha}(j)$ (i.e., $\Delta_{\mathfrak{a}} := \bigcup_{j \geqslant d_{\alpha} + 1} \Delta_{\alpha}(j)$ in the combinatorial set up) is a maximal abelian ideal of \mathfrak{b} .

Proof. Obviously, \mathfrak{a} is abelian. Let $\lambda \in \Delta_{\alpha}(d_{\alpha})$ be the highest weight. It follows from the simplicity of all \mathfrak{l} -modules $\mathfrak{g}_{\alpha}(i)$ that λ is the only maximal element of $\Delta^+ \setminus \Delta_{\mathfrak{a}}$. Therefore, it suffices to prove that the upper ideal $\Delta_{\mathfrak{a}} \cup \{\lambda\}$ is not abelian. Indeed, there is $\nu \in \Delta_{\alpha}(d_{\alpha}+1)$ such that $\nu + \lambda$ is a root (apply Corollary 3.3 with $i = d_{\alpha}$ and $j = d_{\alpha} + 1$.)

This prompts the following question. Suppose that $\operatorname{ht}_{\alpha}(\theta) = 2d_{\alpha} + 1$. Then $\mathfrak{a} = I(\beta)_{\max}$ for some $\beta \in \Pi_l$. What is the relationship between α and β ? We say below that $\alpha \in \Pi$ is odd, if $\operatorname{ht}_{\alpha}(\theta)$ is odd.

Example 3.9. 1) If $ht_{\alpha}(\theta) = 1$, i.e., $d_{\alpha} = 0$, then \mathfrak{a} is the (abelian) nilradical of the corresponding maximal parabolic subalgebra. Then $\beta = \alpha$. This covers all simple roots and all maximal abelian ideals in type \mathbf{A}_n .

- 2) For Δ of type \mathbf{D}_n or \mathbf{E}_n , there are exactly three odd simple roots α .
- For \mathbf{D}_n , these are the endpoints of the Dynkin diagram and $d_{\alpha}=0$. That is, again $\alpha=\beta$ in these cases.
 - For \mathbf{E}_n , there are also odd simple roots with $d_{\alpha} \geqslant 1$ and then $\beta \neq \alpha$.

Nevertheless, the related maximal abelian ideals always correspond to the extreme nodes

of the Dynkin diagram! Moreover, one always has $ht_{\beta}(\theta) = d_{\alpha} + 1$. (Similar things happen for \mathbf{F}_4 and \mathbf{G}_2 .) It might be interesting to find a reason behind it.

Below is the table of all exceptional cases with $d_{\alpha} \geqslant 1$. The numbering of simple roots follows [4, Tables]. In particular, the numbering for \mathbf{E}_8 is $^{1234567}_{8}$ and the extreme nodes correspond to $\alpha_1, \alpha_7, \alpha_8$.

	,			,				\mathbf{G}_2
$egin{array}{c} lpha \ d_lpha \ eta \ \mathrm{ht}_eta(heta) \end{array}$	α_3	α_3	α_5	α_2	α_4	α_8	α_3	α_1
d_{α}	1	1	1	1	2	1	1	1
β	α_6	α_7	α_6	α_1	α_8	α_7	α_4	α_2
$ht_eta(heta)$	2	2	2	2	3	2	2	2

4. BIJECTIONS RELATED TO THE MAXIMAL ABELIAN IDEALS

In this section, we consider abelian ideals of the form $I(\alpha)_{\min}$ and $I(\alpha)_{\max}$ for $\alpha \in \Pi_l$, and their derivatives (intersections and unions).

The following is Theorem 4.7 in [11].

Theorem 4.1. For any $\alpha \in \Pi_l$, there is a one-to-one correspondence between $\min(I(\alpha)_{\min})$ and $\max(\Delta^+ \setminus I(\alpha)_{\max})$. Namely, if $\eta \in \max(\Delta^+ \setminus I(\alpha)_{\max})$, then $\eta' := \theta - \eta \in \min(I(\alpha)_{\min})$, and vice versa.

It formally follows from this theorem that $\min(I(\alpha)_{\min})$ and $\max(\Delta^+ \setminus I(\alpha)_{\max})$ both belong to \mathcal{H} . This is clear for the former, since $I(\alpha)_{\min} \subset \mathcal{H}$. And the key point in the proof of Theorem 4.1 was to demonstrate a priori that $\max(\Delta^+ \setminus I(\alpha)_{\max}) \subset \mathcal{H}$.

Below, we provide a generalisation of Theorem 4.1, which is even more general than [11, Theorem 4.9], i.e., we will **not** assume that $S \subset \Pi_l$ be connected. Another improvement is that we give a conceptual proof of that generalisation, while Theorem 4.9 in [11] was proved case-by-case and no details has been given there.

The following is a key step for our generalisation of Theorem 4.1.

Theorem 4.2. Suppose that $S \subset \Pi_l$ and $\gamma \in \max(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max})$. If Δ is not of type \mathbf{A}_n , then $\gamma \in \mathcal{H}$.

Proof. Here we have to distinguish two possibilities: either $\gamma \in \Delta_{nc}^+$ or $\gamma \in \Delta_{com}^+$.

(1) Suppose that $\gamma \in \Delta_{\rm nc}^+$ and assume that $\gamma \not\in \mathcal{H}$. Then there are $\eta, \eta' \succ \gamma$ such that $\eta + \eta' = \theta$, see [8, p. 1897]. Here both η and η' belong to $\mathcal{H} \cap \left(\bigcup_{\alpha \in S} I(\alpha)_{\rm max}\right) = \bigcup_{\alpha \in S} I(\alpha)_{\rm min}$. Since Δ is not of type \mathbf{A}_n , \mathcal{H} has a unique minimal element (= the unique simple root that is not orthogonal to θ). Therefore, $\mu := \eta \wedge \eta'$ exists and belongs to \mathcal{H} . (The existence of $\eta \wedge \eta'$ also follows from Theorem 2.4(1).) Since $\eta, \eta' \succcurlyeq \mu$, we have $\mu \in \Delta_{\rm nc}^+$. This implies

that $\mu \notin \bigcup_{\alpha \in S} I(\alpha)_{\min}$ and hence $\mu \notin \bigcup_{\alpha \in S} I(\alpha)_{\max}$. By the definition of meet, $\gamma \preccurlyeq \mu$. Furthermore, $\gamma \notin \mathcal{H}$ and $\mu \in \mathcal{H}$. Hence $\gamma \prec \mu$ and γ is not maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$. A contradiction!

(2) Suppose that $\gamma \in \Delta_{\mathsf{com}}^+$. Consider the abelian ideal $J = I \langle \succcurlyeq \gamma \rangle$. By the assumption, $J \setminus \{\gamma\} \subset \bigcup_{\alpha \in S} I(\alpha)_{\max}$. On the other hand, since $J \not\subset I(\alpha)_{\max}$ for each $\alpha \in S$, we conclude that

$$J \cap \mathcal{H} \not\subset I(\alpha)_{\max} \cap \mathcal{H} = I(\alpha)_{\min}$$

see [11, Prop. 3.2]. For each $\alpha \in S$, we pick $\eta_{\alpha} \in (J \cap \mathcal{H}) \setminus I(\alpha)_{\min}$. Then $\eta_{\alpha} \succcurlyeq \gamma$. Since Δ is not of type \mathbf{A}_n , the meet $\eta := \bigwedge_{\alpha \in S} \eta_{\alpha}$ exists and belong to \mathcal{H} (Remark 2.6) and also $\eta \succcurlyeq \gamma$. Note also that $\eta \not\in I(\alpha)_{\min}$ for each $\alpha \in S$. (Otherwise, if $\eta \in I(\alpha_0)_{\min}$, then $\eta_{\alpha_0} \in I(\alpha_0)_{\min}$ as well.) Therefore, $\eta \not\in \bigcup_{\alpha \in S} I(\alpha)_{\min}$ and hence $\eta \not\in \bigcup_{\alpha \in S} I(\alpha)_{\max}$ (because $\eta \in \mathcal{H}$). As γ is assumed to be maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$, we must have $\gamma = \eta \in \mathcal{H}$.

Remark. For A_n , this theorem remains true if we add the hypothesis that $S \subset \Pi_l$ is a *connected* subset in the Dynkin diagram, see also Example 4.4.

Theorem 4.3. If $S \subset \Pi_l$ is arbitrary and Δ is not of type \mathbf{A}_n , then there is the bijection

$$\eta \in \min \left(\bigcap_{\alpha \in S} I(\alpha)_{\min}\right) \xrightarrow{1:1} \eta' = \theta - \eta \in \max \left(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}\right).$$

Proof. (1) Suppose that $\eta \in \min(\bigcap_{\alpha \in S} I(\alpha)_{\min})$. As Δ is not of type \mathbf{A}_n , there is a unique $\alpha_{\theta} \in \Pi$ such that $(\theta, \alpha_{\theta}) \neq 0$. Then $\theta - \alpha_{\theta} \in \mathcal{H}$ is the only root covered by θ . Therefore, $\theta - \alpha_{\theta} \in I(\alpha)_{\min}$ for all $\alpha \in \Pi_l$. Hence $\eta \neq \theta$ and hence $\eta' = \theta - \eta$ is a root (in \mathcal{H}). Since $\eta \in I(\alpha)_{\min}$, we have $\eta' \notin I(\alpha)_{\min}$, see [11, Lemma 3.3]. And this holds for each $\alpha \in S$. Hence $\eta' \notin \bigcup_{\alpha \in S} I(\alpha)_{\min}$ and thereby $\eta' \notin \bigcup_{\alpha \in S} I(\alpha)_{\max}$.

Assume that η' is not maximal in $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$, i.e., $\eta' + \beta \not\in \bigcup_{\alpha \in S} I(\alpha)_{\max}$ for some $\beta \in \Pi$. Again, $\eta' \prec \theta - \alpha_{\theta}$, hence $\eta' + \beta \in \mathcal{H} \setminus \{\theta\}$. Then $\theta - (\eta' + \beta) = \eta - \beta \in \mathcal{H}$ and arguing "backwards" we obtain that $\eta - \beta \in \bigcap_{\alpha \in S} I(\alpha)_{\min}$, which contradicts the fact that η is minimal.

(2) By Theorem 4.2, if $\eta' \in \max(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max})$, then $\eta' \in \mathcal{H}$. Under these circumstances, the previous part of the proof can be reversed.

Example 4.4. Suppose that Δ is of type \mathbf{A}_n , with the usual numbering of simple roots. Then $I(\alpha_i)_{\mathsf{max}} = I\langle \succcurlyeq \alpha_i \rangle$ for all i and $\mathfrak{H} = I(\alpha_1)_{\mathsf{max}} \cup I(\alpha_n)_{\mathsf{max}}$, where

$$I(\alpha_1)_{\min} = I(\alpha_1)_{\max} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_1 - \varepsilon_n, \varepsilon_1 - \varepsilon_{n+1} = \theta\},$$

$$I(\alpha_n)_{\min} = I(\alpha_n)_{\max} = \{\varepsilon_n - \varepsilon_{n+1}, \dots, \varepsilon_2 - \varepsilon_{n+1}, \varepsilon_1 - \varepsilon_{n+1}\}.$$

If $S = \{\alpha_1, \alpha_n\}$, then S is not connected for $n \geqslant 3$, $I(\alpha_1)_{\min} \cap I(\alpha_n)_{\min} = \{\theta\}$, and $\max(\Delta^+ \setminus (I(\alpha_1)_{\max} \cup I(\alpha_n)_{\max})) = \{\varepsilon_2 - \varepsilon_n\}$. That is, Theorems 4.2 and 4.3 do not apply here. However, both remain true if S is assumed to be connected and $S \neq \Pi$. For instance,

suppose that $S = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_j\}$ with 1 < i < j < n. Then $\min(\bigcap_{\alpha \in S} I(\alpha)_{\min}) = \{\varepsilon_1 - \varepsilon_{j+1}, \varepsilon_i - \varepsilon_{n+1}\}$ and $\max(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}) = \{\varepsilon_1 - \varepsilon_i, \varepsilon_{j+1} - \varepsilon_{n+1}\}$. If $S = \Pi$, then $\bigcap_{\alpha \in \Pi} I(\alpha)_{\min} = \{\theta\}$ and $\Delta^+ = \bigcup_{\alpha \in \Pi} I(\alpha)_{\max}$.

As a by-product of Theorem 4.3, we derive a property of maximal abelian ideals outside type **A**. Given $S \subset \Pi_l$, let $\langle S \rangle$ be the smallest connected subset of Π_l containing S.

Theorem 4.5. Let $S \subset \Pi_l$. Then

- (i) $\bigcap_{\alpha \in S} I(\alpha)_{\min} = \bigcap_{\alpha \in \langle S \rangle} I(\alpha)_{\min}$;
- (ii) if $\Delta \neq \mathbf{A}_n$, then $\bigcup_{\alpha \in S} I(\alpha)_{\max} = \bigcup_{\alpha \in \langle S \rangle} I(\alpha)_{\max}$.

Proof. (i) By [11, Theorem 2.1], $\bigcap_{\alpha \in S} I(\alpha)_{\min} = I(\gamma)_{\min}$, where $\gamma = \bigvee_{\alpha \in S} \alpha$. It remains to notice that $\bigvee_{\alpha \in S} \alpha = \sum_{\alpha \in \langle S \rangle} \alpha = \bigvee_{\alpha \in \langle S \rangle} \alpha$.

(ii) This follows from (i) and Theorem 4.3. Namely, if Δ is not of type \mathbf{A}_n , then

$$\max \left(\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max} \right) = \max \left(\Delta^+ \setminus \bigcup_{\alpha \in \langle S \rangle} I(\alpha)_{\max} \right).$$

Hence both unions also coincide.

The equality $\bigcap_{\alpha \in S} I(\alpha)_{\min} = I(\bigvee_{\alpha \in S} \alpha)_{\min}$ has interesting consequences. By [8, Prop. 4.6], the minimal elements of the abelian ideal $I(\gamma)_{\min}$ have the following description:

Let $w_{\gamma} \in W$ be a unique element of minimal length such that $w_{\gamma}(\theta) = \gamma$. If $\beta \in \Pi$ and $(\beta, \gamma^{\vee}) = -1$, then $w_{\gamma}^{-1}(\beta + \gamma) = w_{\gamma}^{-1}(\beta) + \theta \in \min(I(\gamma)_{\min})$. Conversely, any element of $\min(I(\gamma)_{\min})$ is obtained in this way.

For any γ of the form $\bigvee_{\alpha \in S} \alpha$, the required simple roots β are easily determined, which yields the maximal elements of $\Delta^+ \setminus \bigcup_{\alpha \in S} I(\alpha)_{\max}$. We consider below the particular case in which $S = \Pi_l$.

Proposition 4.6. Set $|\Pi_l| = \sum_{\alpha \in \Pi_l} \alpha$. If $|\Pi_l| \neq \theta$, i.e., Δ is not of type \mathbf{A}_n , then there is a unique $\hat{\boldsymbol{\alpha}} \in \Pi$ such that $|\Pi_l| + \hat{\boldsymbol{\alpha}}$ is a root. More precisely,

- if $\Delta \in \{D-E\}$, then $\hat{\alpha}$ is the branching point in the Dynkin diagram;
- if $\Delta \in \{B-C-F-G\}$, then $\hat{\alpha}$ is the unique short root that is adjacent to a long root in the Dynkin diagram.

In all these cases, $w_{|\Pi_l|}^{-1}(\hat{\boldsymbol{\alpha}}) = -\lfloor \theta/2 \rfloor$.

Proof. If $S = \Pi_l$, then $\bigcup_{\alpha \in \Pi_l} I(\alpha)_{\max} = \Delta_{\text{com}}^+$. Hence $\max(\Delta^+ \setminus \bigcup_{\alpha \in \Pi_l} I(\alpha)_{\max}) = \{\lfloor \theta/2 \rfloor\}$, see Theorem 3.5. Therefore, by Theorem 4.3, the unique minimal element of $I(|\Pi_l|)_{\min} = \bigcap_{\alpha \in \Pi_l} I(\alpha)_{\min}$ is $\theta - \lfloor \theta/2 \rfloor =: \lceil \theta/2 \rceil$. This means that there is a unique simple root $\hat{\boldsymbol{\alpha}}$ such that $(|\Pi_l|^\vee, \hat{\boldsymbol{\alpha}}) = -1$, i.e., $|\Pi_l| + \hat{\boldsymbol{\alpha}}$ is a root. Since $w_{|\Pi_l|}^{-1}(|\Pi_l| + \hat{\boldsymbol{\alpha}}) = \theta + w_{|\Pi_l|}^{-1}(\hat{\boldsymbol{\alpha}}) = \theta - \lfloor \theta/2 \rfloor$, the last assertion follows.

Clearly, $\hat{\boldsymbol{\alpha}}$ specified in the statement satisfies the condition that $(|\Pi_l|, \hat{\boldsymbol{\alpha}}) < 0$.

The \mathbf{A}_n -case can partially be included in the **DE**-picture, if we formally assume that $\hat{\boldsymbol{\alpha}} = 0$ (because there is no branching point).

5. On the interval
$$[\lfloor \theta/2 \rfloor, \lceil \theta/2 \rceil]$$

In this section, we first assume that Δ is not of type \mathbf{A}_n . Since $\lfloor \theta/2 \rfloor \in \mathcal{H}$, we have $\lceil \theta/2 \rceil = \theta - \lfloor \theta/2 \rfloor \in \mathcal{H}$ and also $\lfloor \theta/2 \rfloor \preccurlyeq \lceil \theta/2 \rceil$. We consider the interval between $\lfloor \theta/2 \rfloor$ and $\lceil \theta/2 \rceil$ in Δ^+ . Let h be the Coxeter number of Δ .

Proposition 5.1. *Set* $\mathfrak{J} = \{ \gamma \in \Delta^+ \mid |\theta/2| \leq \gamma \leq \lceil \theta/2 \rceil \}.$

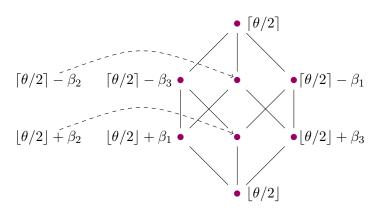
- if $\Delta \in \{ \mathbf{D-E} \}$, then $\mathfrak{J} \simeq \mathbb{B}^3$ and $\operatorname{ht}(\lceil \theta/2 \rceil) = (h/2) + 1$;
- if $\Delta \in \{B\text{-}C\text{-}F\text{-}G\}$, then \mathfrak{J} is a segment and $\mathsf{ht}(\lceil \theta/2 \rceil) = h/2$.

Proof. This can be verified case-by-case, but we also provide some *a priori* hints. It follows from the definition of $\lfloor \theta/2 \rfloor$, see Eq. (3·1), that

$$\lceil \theta/2 \rceil - \lfloor \theta/2 \rfloor = \theta - 2\lfloor \theta/2 \rfloor = 2\lceil \theta/2 \rceil - \theta = \sum_{\alpha: \, \mathsf{ht}_{\alpha}(\theta) \, \mathsf{odd}} \alpha,$$

the sum of all odd simple roots. Let $\mathcal{O} \subset \Pi$ denote the set of odd roots. Then $\mathsf{ht}(\lceil \theta/2 \rceil) - \mathsf{ht}(\lceil \theta/2 \rceil) = \#\mathcal{O}$.

• In the simply-laced case, $(\theta - 2\lfloor \theta/2 \rfloor, \lfloor \theta/2 \rfloor^{\vee}) = 1 - 4 = -3$. Therefore, there are at least three $\alpha \in \mathcal{O}$ such that $(\alpha, \lfloor \theta/2 \rfloor^{\vee}) = -1$, i.e., $\lfloor \theta/2 \rfloor + \alpha \in \Delta^+$. On the other hand, for any $\gamma \in \Delta^+$, there are at most three $\alpha \in \Pi$ such that $\gamma + \alpha \in \Delta^+$ [10, Theorem 3.1(i)]. Thus, there are exactly three odd roots α_i such that $\lfloor \theta/2 \rfloor + \alpha_i \in \Delta^+$. Actually, there are only three odd roots in the $\{\mathbf{D}\mathbf{-E}\}$ -case. Hence every odd root can be added to $\lfloor \theta/2 \rfloor$. Likewise, $(2\lceil \theta/2 \rceil - \theta, \lceil \theta/2 \rceil^{\vee}) = 3$ and the same three roots can be subtracted from $\lceil \theta/2 \rceil$. This yields all six roots strictly between $\lfloor \theta/2 \rfloor$ and $\lceil \theta/2 \rceil$. If $\mathcal{O} = \{\beta_1, \beta_2, \beta_3\}$, then \mathfrak{J} is as follows:



• In the non-simply laced cases, there is always a unique odd root and hence $\mathfrak{J} = \{\lfloor \theta/2 \rfloor, \lceil \theta/2 \rceil\}$.

Remark 5.2. If Δ is of type \mathbf{A}_n , then $\lfloor \theta/2 \rfloor = 0$ and $\lceil \theta/2 \rceil = \theta$. Then $\mathfrak{J} = \Delta^+ \cup \{0\}$. However, this poset is not a modular lattice.

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