Central Values of $GL(2) \times GL(3)$ Rankin-Selberg L-functions with Applications [1]

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Abstract Let f be a normalized holomorphic cusp form for $SL_2(\mathbb{Z})$ of weight k with $k \equiv 0 \bmod 4$. By the Kuznetsov trace formula for $GL_3(\mathbb{R})$, we obtain the first moment of central values of $L(s, f \otimes \phi)$, where ϕ varies over Hecke-Maass cusp forms for $SL_3(\mathbb{Z})$. As an application, we obtain a non-vanishing result for $L(1/2, f \otimes \phi)$ and show that such f is determined by $\{L(1/2, f \otimes \phi)\}$ as ϕ varies.

Keywords: central values, the Rankin-Selberg L-function, the Kunznetsov trace formula
MSC 11F11, 11F67

1. Introduction

Special values of L-functions are expected to carry important information on relevant arithmetic and geometric objects. In 1997, Luo and Ramakrishnan [LR1997] asked the question that to what extent modular forms are actually characterized by their special L-values. In the same paper, they considered the moment of $\chi_d(p)L(1/2, f \otimes \chi_d)$ as d varies, and showed that a cuspidal normalized holomorphic Hecke newform f is uniquely determined by the family $\{L(1/2, f \otimes \chi_d)\}$ for all quadratic characters χ_d. Since then, this problem has been studied by many authors ([Lu1999], [CD2005], [Li2007], [Li2009], [GHS2009], [Mu2010], [Liu2010], [Pi2010], [Liu2011], [Zh2011], [Ma2014], [Pi2014], [Su2014],[MS2015]).

Let f be a normalized holomorphic Hecke-cusp form for $SL_2(\mathbb{Z})$ of fixed weight k with $k \equiv 0 \bmod 4$. Let $\{\phi\}$ be a Hecke basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$. In this paper, we consider central values of Rankin-Selberg L-functions $L(s, f \otimes \phi)$. By calculating the twisted moment of $A_\phi(p,p)L(1/2, f \otimes \phi)$ where $A_\phi(p,p)$ is the Hecke eigenvalue of ϕ at (p,p), we show that f is uniquely determined by the family $\{L(1/2, f \otimes \phi)\}$ as ϕ varies over a Hecke basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$.

To state our result, we give the following notations.

- For ϕ a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$, let $\boldsymbol{\mu}_\phi = (\mu_1, \mu_2, \mu_3)$ be the Langlands parameter of ϕ. We know that $\boldsymbol{\mu}_\phi$ is a point in the region
$$\Lambda'_{1/2} := \left\{ (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad \begin{matrix} |\mathrm{Re}(\mu_j)| \le \dfrac{1}{2}, \quad \mu_1 + \mu_2 + \mu_3 = 0, \\ \{\mu_1, \mu_2, \mu_3\} = \{-\overline{\mu}_1, -\overline{\mu}_2, -\overline{\mu}_3\} \end{matrix} \right\}$$
in the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Let
$$\nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = -\nu_1 - \nu_2$$
which are known as the spectral coordinates.
- Fix a point $\boldsymbol{\mu}^0 \in \Lambda'_{1/2}$ such that
$$|\mu_j^0| \asymp \|\boldsymbol{\mu}^0\| := T, \quad 1 \le j \le 3.$$

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As in [BB2015] (or see [HLZ2017]), we choose the test function $h(\boldsymbol{\mu})$ to localize at a ball of radius $M = T^\theta$ with $0 < \theta < 1$ about $w(\boldsymbol{\mu}^0)$, where w are elements in the Weyl group W. For a precise definition of $h(\boldsymbol{\mu})$, we refer to section 2.3.

- Let $d\boldsymbol{\mu} = d\mu_1 d\mu_2$ and $d_{\mathrm{spec}}\boldsymbol{\mu} = \mathrm{spec}(\boldsymbol{\mu}) d\boldsymbol{\mu}$ with

$$\mathrm{spec}(\boldsymbol{\mu}) = \prod_{j=1}^{3} \left(3\nu_j \tan\left(\frac{3\pi}{2}\nu_j\right)\right).$$

Our main result is in the following.

Theorem 1.1. *Let f be a normalized holomorphic Hecke cusp form for $SL_2(\mathbb{Z})$ of weight k with $k \equiv 0 \bmod 4$. Let $\{\phi\}$ be a Hecke basis of the space of cusp forms for $SL_3(\mathbb{Z})$ and $A_\phi(p,p)$ be the Hecke eigenvalue of ϕ at (p,p). One has*

$$\sum_{\phi} \frac{h(\boldsymbol{\mu}_\phi)}{\mathcal{N}_\phi} A_\phi(p,p) L(1/2, f \otimes \phi) = \frac{\lambda_f(p)}{p^{3/2}} M_k(h) + O_{k,\epsilon}(p^{\frac{7}{32}+\epsilon} T^{\frac{5}{2}+\epsilon} M^2) \tag{1.1}$$

for $T \gg_{k,\epsilon} p^{3+\frac{7}{16}+\epsilon}$. Here $\mathcal{N}_\phi$ is the normalized factor defined in (2.5) and

$$M_k(h) = \frac{1}{192\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) \left(1 + \prod_{j=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_j)}{\Gamma(\frac{k}{2} - \mu_j)}\right) d_{\mathrm{spec}}\boldsymbol{\mu}.$$

Note that $M_k(h) \asymp_k T^3 M^2$. On taking $p = 1$, the above theorem implies the existence of non-vanishing of $L(1/2, f \otimes \phi)$ as ϕ varies. Moreover, by the strong multiplicity one theorem (see [PS1979]), we have the following corollary.

Corollary 1. *Let f and f' be two normalized holomorphic cusp forms for $SL_2(\mathbb{Z})$ of fixed weight k with $k \equiv 0 \bmod 4$. If $L(1/2, f \otimes \phi) = L(1/2, f' \otimes \phi)$ for all Hecke-Maass cusp forms ϕ for $SL_3(\mathbb{Z})$, then $f = f'$.*

We remark that central values of $L(s, f \otimes \phi)$ vanish for $k \equiv 2 \bmod 4$. In this case we can consider $\frac{d}{ds} L(1/2, f \otimes \phi)$ instead of $L(1/2, f \otimes g)$ as in [Zh2011]. But we do not address this here.

This paper is arranged as follows. In section 2, we review the Kuznetsov trace formula in the version of [Bu2014], choose the test function and give the approximate functional equation of the Rankin-Selberg L-function. Theorem 1.1 will be proved in section 3, where we apply the approximate functional equation and the Kuznetsov trace formula, and give estimations on each terms. The main term in (1.1) comes from the geometric term associated to the trivial Weyl's element, and the error term comes from the maximal Eisenstein series in the continuous spectrum.

2. Preliminaries

In this section, we review the definition of automorphic forms on $SL(3,\mathbb{Z})$ in [Bl2013] (or see [Go2006]), the Kuznetsov's trace formula in [Bu2014] (or see [BB2015]) and the approximate functional equation of Rankin-Selberg L-functions.

Let

$$\begin{aligned} \mathfrak{h}^3 &= \left\{ z = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}, \quad x_1, x_2, x_3 \in \mathbb{R}, y_1, y_2 \in \mathbb{R}^+ \right\} \\ &\simeq GL_3(\mathbb{R})/O_3(\mathbb{R})Z(\mathbb{R}) \end{aligned}$$

be the generalized Poincare upper half plane. Given a spectral parameter $(\nu_1,\nu_2)\in\mathbb{C}^2$, the function I_{ν_1,ν_2} on $\mathfrak{h}^3$ is defined by

$$I_{\nu_1,\nu_2}(z)=y_1^{1+2\nu_1+\nu_2}y_2^{1+\nu_1+2\nu_2}$$

and the Jacquet-Whittaker function is defined by

$$\mathcal{W}^{\pm}_{\nu_1,\nu_2}(z):=\int_{\mathbb{R}^3} I_{\nu_1,\nu_2}\left(\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}\begin{pmatrix}1 & u_2 & u_3\\ & 1 & u_1\\ & & 1\end{pmatrix}z\right)e(-(u_1\pm u_2))du_1du_2du_3$$

where $e(x)=\exp(2\pi ix)$.

Let $\nu_3=-\nu_1-\nu_2$ and

$$\mu_1=\nu_1+2\nu_2,\quad \mu_2=\nu_1-\nu_2,\quad \mu_3=-2\nu_1-\nu_2. \tag{2.1}$$

We will simultaneously use $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)$ and (ν_1,ν_2,ν_3) coordinates,

$$\nu_1=\frac{1}{3}(\mu_1-\mu_2),\quad \nu_2=\frac{1}{3}(\mu_2-\mu_3),\quad \nu_3=-\nu_1-\nu_2.$$

2.1. **Automorphic forms for $SL(3,\mathbb{Z})$.**

2.1.1. *Hecke-Maass cusp forms.* A Hecke-Maass cusp form for $\Gamma=SL_3(\mathbb{Z})$ of type $(\frac{1}{3}+\nu_1,\frac{1}{3}+\nu_2)$ is a function $\phi:\Gamma\backslash\mathfrak{h}^3\to\mathbb{C}$ which has the Fourier expansion

$$\phi(z)=\sum_{m_1=1}^{\infty}\sum_{m_2\neq 0}\frac{A_\phi(m_1,m_2)}{m_1|m_2|}\sum_{\gamma\in U_2\backslash SL_2(\mathbb{Z})}\mathcal{W}^{\mathrm{sgn}(m_2)}_{\nu_1,\nu_2}\left(\begin{pmatrix}m_1|m_2| & & \\ & m_1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\gamma & \\ & 1\end{pmatrix}z\right)c_{\nu_1,\nu_2}.$$

Here $A_\phi(m_1,m_2)$ are eigenvalues of ϕ at (m_1,m_2) satisfying

$$A_\phi(m_1,m_2)\ll m_1m_2,$$

$\mathcal{W}^{\pm}_{\nu_1,\nu_2}(z)$ is the Jacquet-Whittaker function and c_{ν_1,ν_2} is a constant depending only on ν_1 and ν_2 (see formula (2.13) in [Bl2013]).

Let $\boldsymbol{\mu}_\phi=(\mu_1,\mu_2,\mu_3)$ be the Langlands parameter of ϕ where μ_j are given by (2.1). The L-function associated to ϕ is defined by

$$L(s,\phi):=\sum_{m\geq 1}\frac{A_\phi(1,m)}{m^s}$$

for $\mathrm{Re}(s)>2$. It has analytic continuation for $s\in\mathbb{C}$ and satisfies the functional equation

$$\Lambda(s,\phi)=\prod_{j=1}^{3}\Gamma_{\mathbb{R}}(s+\mu_j)L(s,\phi)=\Lambda(1-s,\phi^\vee).$$

Here $\Gamma_{\mathbb{R}}(s)=\pi^{-\frac{s}{2}}\Gamma(s/2)$ and $\phi^\vee$ is the dual Hecke-Maass cusp form of ϕ with

$$A_{\phi^\vee}(m_1,m_2)=A_\phi(m_2,m_1),\quad \boldsymbol{\mu}_{\phi^\vee}=(-\mu_1,-\mu_2,-\mu_3).$$

2.1.2. *The minimal Eisenstein series.* Let $P_{1,1,1}$ be the standard minimal parabolic subgroup of GL_3 and U_3 be the unipotent radical of $P_{1,1,1}$. Given a spectral parameter $(\nu_1,\nu_2)\in\mathbb{C}^2$, let $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)$ be the Langlands parameter given by (2.1). The minimal Eisenstein series

$$E^{\min}_{\nu_1,\nu_2}(z):=\sum_{\gamma\in U_3(\mathbb{Z})\backslash\Gamma} I_{\nu_1,\nu_2}(\gamma z)$$

is defined for $\mathrm{Re}(\nu_1)$ and $\mathrm{Re}(\nu_2)$ sufficient large and has meromorphic continuation to all $(\nu_1,\nu_2)\in\mathbb{C}^2$. The Hecke eigenvalues $A^{\min}_{\nu_1,\nu_2}(m,n)$ of $E^{\min}_{\nu_1,\nu_2}(z)$ at (m,n) are defined by

$$A^{\min}_{\nu_1,\nu_2}(1,n)=\sum_{d_1d_2d_3=n} d_1^{-\mu_1}d_2^{-\mu_2}d_3^{-\mu_3}$$

and by Hecke relations

$$\begin{aligned}A^{\min}_{\nu_1,\nu_2}(m,1) &= \overline{A^{\min}_{\nu_1,\nu_2}(1,m)},\\ A^{\min}_{\nu_1,\nu_2}(m_1,m_2) &= \sum_{d|(m_1,m_2)}\mu(d)A^{\min}_{\nu_1,\nu_2}\left(\frac{m_1}{d},1\right)A^{\min}_{\nu_1,\nu_2}\left(1,\frac{m_2}{d}\right).\end{aligned}$$

The L-function associated to $E^{\min}_{\nu_1,\nu_2}(z)$ is

$$L(s,E^{\min}_{\nu_1,\nu_2}):=\sum_{m\geq 1}\frac{A^{\min}_{\nu_1,\nu_2}(1,m)}{m^s}=\zeta(s+\mu_1)\zeta(s+\mu_2)\zeta(s+\mu_3)$$

where μ_i are given by (2.1).

2.1.3. *The Maximal Eisenstein series.* Let $g: SL_2(\mathbb{Z})\backslash\mathfrak{h}^2\to\mathbb{C}$ be a Hecke-Maass cusp form with the spectral parameter $it_g\in i\mathbb{R}$ and Hecke eigenvalues $\lambda_g(m)$. We assume that g is normalized by $\|g\|=1$. Let

$$P_{2,1}=\begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix}$$

be the standard maximal parabolic subgroup of GL_3. For $u\in\mathbb{C}$, the maximal Eisenstein series

$$E^{\max}_{u,g}(z):=\sum_{\gamma\in P_{2,1}(\mathbb{Z})\backslash\Gamma}\det(\gamma z)^{\frac12+u}g(\mathfrak{m}_{P_{2,1}}(\gamma z))$$

is defined for $\mathrm{Re}(u)$ sufficient large. Here $\mathfrak{m}_{P_{2,1}}$ is the restriction to the upper left corner,

$$\mathfrak{m}_{P_{2,1}}:\mathfrak{h}^3\to\mathfrak{h}^2,\qquad \begin{pmatrix} y_1y_2 & y_1x_2 & x_3\\ & y_1 & x_1\\ & & 1\end{pmatrix}\mapsto\begin{pmatrix} y_2 & x_2\\ & 1\end{pmatrix}.$$

The Hecke eigenvalue $A^{\max}_{u,g}(m,n)$ of $E^{\max}_{u,g}$ at (m,n) is defined by

$$A^{\max}_{u,g}(1,n)=\sum_{d_1d_2=|n|}\lambda_g(d)d_1^{-u}d_2^{2u} \tag{2.2}$$

and by the Hecke relations

$$\begin{aligned}A^{\max}_{u,g}(m,1) &= \overline{A^{\max}_{u,g}(1,m)},\\ A^{\max}_{u,g}(m_1,m_2) &= \sum_{d|(m_1,m_2)}\mu(d)A^{\max}_{u,g}\left(\frac{m_1}{d},1\right)A^{\max}_{u,g}\left(1,\frac{m_2}{d}\right).\end{aligned} \tag{2.3}$$

The L-function associated to $E_{u,g}^{\max}(z)$ is

$$L(s, E_{u,g}^{\max}) = \sum_{m\geq 1} \frac{A_{u,g}^{\max}(1,m)}{m^s} = \zeta(s-2u)L(s+u,g)$$

and the complete L-function is

$$\Lambda(s, E_{u,g}^{\max}) = \prod_{i=1}^{3} \Gamma_{\mathbb{R}}(s+\mu_i')L(s, E_{u,g}^{\max}) = \Lambda(1-s, E_{-u,g}^{\max}),$$

where

$$\mu_1' = u + it_g, \quad \mu_2' = u - it_g, \quad \mu_3' = -2u. \tag{2.4}$$

2.2. **The Kuznetsov trace formula.** We recall the Kuznetsov trace formula in the version of [Bu2014]. Let $d\boldsymbol{\mu} = d\mu_1 d\mu_2$ be the standard measure on the Lie algebra

$$\Lambda_\infty := \{\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad \mu_1 + \mu_2 + \mu_3 = 0\}.$$

We set $d_{\mathrm{spec}}(\boldsymbol{\mu}) = \mathrm{spec}(\boldsymbol{\mu})d\boldsymbol{\mu}$ with

$$\mathrm{spec}(\boldsymbol{\mu}) := \prod_{j=1}^{3} \left(3\nu_j \tan\left(\frac{3\pi}{2}\nu_j\right)\right).$$

2.2.1. *Normalized factors.* The normalized factors are defined as follows.

- For ϕ a Hecke-Maass cusp form with $\boldsymbol{\mu}_\phi = (\mu_1, \mu_2, \mu_3)$, we denote by

$$\mathcal{N}_\phi := \|\phi\|^2 \prod_{j=1}^{3} \cos\left(\frac{3}{2}\pi\nu_j\right). \tag{2.5}$$

  Note that for $\boldsymbol{\mu}_\phi = (\mu_1, \mu_2, \mu_3)$ with $\mu_i \asymp T$, one has

$$\mathcal{N}_\phi \asymp \mathrm{Res}_{s=1} L(s, \phi \otimes \phi^\vee) \ll T^\epsilon.$$

- For $E_{\nu_1,\nu_2}^{\min}(z)$ the minimal Eisenstein series with the Langlands parameter $\boldsymbol{\mu}(E_{\nu_1,\nu_2}^{\min}) = (\mu_1, \mu_2, \mu_3)$, the normalized factor is defined by

$$\mathcal{N}_{\nu_1,\nu_2}^{\min} := \frac{1}{16} \prod_{j=1}^{3} |\zeta(1+3\nu_j)|^2.$$

- For $E_{u,g}^{\max}(z)$ the maximal Eisenstein series, we define

$$\mathcal{N}_{u,g}^{\max} := 8L(1, \mathrm{Ad}^2 g)|L(1+3u, g)|^2.$$

2.2.2. *Kloosteman Sums.* Two type of Kloosterman sums are defined as follows. Assume $D_1 \mid D_2$, we have the incomplete Kloosterman sum

$$\tilde{S}(n_1, n_2, m_1, D_1, D_2) := \sum_{\substack{C_1 (\bmod D_1), C_2 (\bmod D_2) \\ (C_1, D_1) = (C_2, D_2/D_1) = 1}} e\left(n_2 \frac{\overline{C_1} C_2}{D_1} + m_1 \frac{\overline{C_2}}{D_2/D_1} + n_1 \frac{C_1}{D_1}\right).$$

The complete Kloosterman sum is defined by

$$\begin{aligned} & S(n_1, n_2, m_1, m_2, D_1, D_2) \\ := & \sum_{\substack{B_1, C_1 \bmod D_1 \\ B_2, C_2 \bmod D_2 \\ D_1C_2 + B_1B_2 + D_2C_1 = 0 \bmod D_1D_2 \\ (B_j, C_j, D_j) = 1}} e\left(\frac{n_1B_1 + m_1(Y_1D_2 - Z_1B_2)}{D_1} + \frac{m_2B_2 + n_2(Y_2D_1 - Z_2B_1)}{D_2}\right) \end{aligned}$$

where $Y_jB_j + Z_jC_j \equiv 1 \bmod D_j$ for $j = 1, 2$.

By the standard (Weil-type) bounds we have (see formulas 3.1 and 3.2 in [BB2015])

$$\tilde{S}(n_1, n_2, m_1, D_1, D_2) \ll \left((m_1, D_2/D_1)D_1^2, (n_1, n_2, D_1)D_2)\right)(D_1D_2)^{\epsilon}$$

and

$$S(n_1, n_2, m_1, m_2, D_1, D_2) \ll (D_1D_2)^{1/2+\epsilon}\left\{(D_1, D_2)(m_1n_1, [D_1, D_2])(m_2n_2, [D_1, D_2])\right\}^{1/2}.$$

2.2.3. *Integral kernels.* Following Theorems 2 and 3 in [Bu2014], the integral kernels are given as follows. For $s \in \mathbb{C}$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, we let

$$\tilde{G}^{\pm}(s, \boldsymbol{\mu}) := \frac{\pi^{-3s}}{12288\pi^{7/2}}\left(\prod_{j=1}^{3}\frac{\Gamma(\frac{s-\mu_j}{2})}{\Gamma(\frac{1-s+\mu_j}{2})} \pm i\prod_{j=1}^{3}\frac{\Gamma\left(\frac{1+s-\mu_j}{2}\right)}{\Gamma\left(\frac{2-s+\mu_j}{2}\right)}\right).$$

The integral kernel associated to w_4 is defined by

$$K_{w_4}(y; \boldsymbol{\mu}) = \int_{-i\infty}^{i\infty}|y|^{-s}\tilde{G}^{\epsilon}(s, \boldsymbol{\mu})\frac{ds}{2\pi i}$$

for $y \in \mathbb{R} - \{0\}$ with $\epsilon = \mathrm{sgn}(y)$.

For $(s_1, s_2) \in \mathbb{C}^2$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, we let

$$G(s_1, s_2, \boldsymbol{\mu}) := \frac{1}{\Gamma(s_1 + s_2)}\prod_{j=1}^{3}\Gamma(s_1 - \mu_j)\Gamma(s_2 + \mu_j).$$

We also define the following trigonometric functions

$$\begin{aligned} S^{++}(s_1, s_2; \boldsymbol{\mu}) &= \frac{1}{24\pi^2}\prod_{j=1}^{3}\cos\left(\frac{3}{2}\pi\nu_j\right), \\ S^{+-}(s_1, s_2; \boldsymbol{\mu}) &= -\frac{1}{32\pi^2}\frac{\cos(\frac{3}{2}\pi\nu_2)\sin(\pi(s_1 - \mu_1))\sin(\pi(s_2 + \mu_2))\sin(\pi(s_2 + \mu_3))}{\sin(\frac{3}{2}\pi\nu_1)\sin(\frac{3}{2}\pi\nu_3)\sin(\pi(s_1 + s_2))}, \\ S^{-+}(s_1, s_2; \boldsymbol{\mu}) &= -\frac{1}{32\pi^2}\frac{\cos(\frac{3}{2}\pi\nu_1)\sin(\pi(s_1 - \mu_1))\sin(\pi(s_1 - \mu_2))\sin(\pi(s_2 + \mu_3))}{\sin(\frac{3}{2}\pi\nu_2)\sin(\frac{3}{2}\pi\nu_3)\sin(\pi(s_1 + s_2))}, \\ S^{--}(s_1, s_2; \boldsymbol{\mu}) &= \frac{1}{32\pi^2}\frac{\cos(\frac{3}{2}\pi\nu_3)\sin(\pi(s_1 - \mu_2))\sin(\pi(s_2 + \mu_2))}{\sin(\frac{3}{2}\pi\nu_2)\sin(\frac{3}{2}\pi\nu_1)}. \end{aligned}$$

The integral kernel associated to the longest Weyl's element w_l is defined by

$$K_{w_l}^{\epsilon_1, \epsilon_2}(y_1, y_2; \boldsymbol{\mu}) = \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}|4\pi^2y_1|^{-s_1}|4\pi^2y_2|^{-s_2}G(s_1, s_2; \boldsymbol{\mu})S^{\epsilon_1, \epsilon_2}(s_1, s_2; \boldsymbol{\mu})\frac{ds_1ds_2}{(2\pi i)^2}$$

for $(y_1, y_2) \in (\mathbb{R} - \{0\})^2$ with $\epsilon_i = \mathrm{sgn}(y_i)$.

2.2.4. *The Kuznetsov's trace formula.* Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and let $h(\boldsymbol{\mu})$ be a function that is holomorphic on

$$\Lambda_{1/2+\delta} = \left\{ \boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad \mu_1 + \mu_2 + \mu_3 = 0, \mathrm{Re}(\mu_j) \leq \frac{1}{2} + \delta \right\}$$

for some $\delta > 0$, symmetric under the Weyl group W, rapidly decaying as $|\mathrm{Im}\mu_j| \to \infty$ and satisfies

$$h(3\nu_1 + 1, 3\nu_2 + 1, 3\nu_3 + 1) = 0.$$

Then one has

$$\mathcal{C} + \mathcal{E}_{min} + \mathcal{E}_{max} = \Delta + \Sigma_4 + \Sigma_5 + \Sigma_l,$$

where

$$\begin{aligned}
\mathcal{C} &= \sum_{\phi} \frac{h(\boldsymbol{\mu}_\phi)}{\mathcal{N}_\phi} A_\phi(n_1, n_2) \overline{A_\phi(m_1, m_2)}, \\
\mathcal{E}_{max} &= \frac{1}{2\pi i} \sum_{g} \int_{\mathrm{Re}(u)=0} \frac{h(u + it_g, u - it_g, -2u)}{\mathcal{N}_{u,g}^{\max}} A_{u,g}^{\max}(n_1, n_2) \overline{A_{u,g}^{\max}(m_1, m_2)} du, \\
\mathcal{E}_{min} &= \frac{1}{24(2\pi i)^2} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} \frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\nu_1,\nu_2}^{\min}} A_{\boldsymbol{\mu}}^{\min}(n_1, n_2) \overline{A_{\boldsymbol{\mu}}^{\min}(m_1, m_2)} d\boldsymbol{\mu},
\end{aligned}$$

and

$$\begin{aligned}
\Delta &= \delta_{m_1,n_1} \delta_{m_2,n_2} \frac{1}{192\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) d_{\mathrm{spec}} \boldsymbol{\mu}, \\
\Sigma_4 &= \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_2 | D_1 \\ m_2 D_1 = n_1 D_2^2}} \frac{\tilde{S}(-\epsilon n_2, m_2, m_1, D_2, D_1)}{D_1 D_2} \Phi_{w_4}\left(\frac{\epsilon m_1 m_2 n_2}{D_1 D_2}; h \right), \\
\Sigma_5 &= \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_1 | D_2 \\ m_1 D_2 = n_2 D_1^2}} \frac{\tilde{S}(-\epsilon n_1, m_1, m_2, D_1, D_2)}{D_1 D_2} \Phi_{w_5}\left(\frac{\epsilon n_1 m_1 m_2}{D_1 D_2}; h \right), \\
\Sigma_l &= \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \sum_{D_1, D_2} \frac{S(\epsilon_2 n_2, \epsilon_1 n_1; m_1, m_2; D_1, D_2)}{D_1 D_2} \Phi_{w_l}\left(-\frac{\epsilon_2 m_1 n_2 D_2}{D_1^2}, -\frac{\epsilon_1 m_2 n_1 D_1}{D_2^2}; h \right).
\end{aligned}$$

Here

$$\begin{aligned}
\Phi_{w_4}(y; h) &= \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) K_{w_4}(y; \boldsymbol{\mu}) d_{\mathrm{spec}} \boldsymbol{\mu}, \\
\Phi_{w_5}(y; h) &= \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) K_{w_4}(-y; -\boldsymbol{\mu}) d_{\mathrm{spec}} \boldsymbol{\mu}, \\
\Phi_{w_l}(y_1, y_2; h) &= \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) K_{w_l}^{\mathrm{sgn}(y_1), \mathrm{sgn}(y_2)}(y_1, y_2; \boldsymbol{\mu}) d_{\mathrm{spec}} \boldsymbol{\mu}.
\end{aligned}$$

2.3. **The choice of the test function.** By unitarity and the Jacquet-Shalika's bounds, the Langlands parameter $\boldsymbol{\mu}_\phi$ of a Hecke-Maass cusp form ϕ for $SL_3(\mathbb{Z})$ is contained in

$$\Lambda'_{1/2} := \left\{ (\mu_1, \mu_2, \mu_3) \in \mathbb{C}^3, \quad \begin{array}{l} |\mathrm{Re}(\mu_j)| \le \frac{1}{2}, \quad \mu_1 + \mu_2 + \mu_3 = 0, \\ \{\mu_1, \mu_2, \mu_3\} = \{\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3\} \end{array} \right\}.$$

Let $\boldsymbol{\mu}^0 = (\mu_1^0, \mu_2^0, \mu_3^0)$ be in generic position in $\Lambda'_{1/2}$, i.e.

$$|\mu_j^0| \asymp \|\boldsymbol{\mu}^0\| := T, \quad 1 \le j \le 3.$$

Following [BB2015](or see [HLZ2017]), we choose a test function $h(\boldsymbol{\mu})$ to localizes at a ball of radius $M = T^\theta$ with $0 < \theta < 1$ about $w(\boldsymbol{\mu}^0)$ for each $w \in W$. It is defined by

$$h(\boldsymbol{\mu}) := P(\boldsymbol{\mu})^2 \left(\sum_{w \in W} \psi\left(\frac{w(\boldsymbol{\mu}) - \boldsymbol{\mu}^0}{M} \right) \right)^2,$$

where $\psi(\boldsymbol{\mu}) = \exp\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right)$ and

$$P(\boldsymbol{\mu}) = \prod_{0 \le n \le A_0} \prod_{j=1}^{3} \frac{\left(\nu_j - \frac{1}{3}(1+2n)\right)\left(\nu_j + \frac{1}{3}(1+2n)\right)}{|\nu_j^0|^2}$$

for some fixed large $A_0 > 0$. Here

$$\begin{aligned} W \quad = \quad & \left\{ I, w_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}, w_3 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, w_4 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \right. \\ & \left. w_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, w_l = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \right\} \end{aligned}$$

is the Weyl group for $GL_3(\mathbb{R})$.

We need the following two lemmas in [BB2015], which are used in truncating summations in geometric terms after the application of the Kuznetsov's trace formula.

Lemma 2.1. *Let $0 < |y| \le T^{3-\epsilon}$. Then for any constant $A \ge 0$ one has*

$$\Phi_{w_4}(y; h) \ll_{\epsilon, B} T^{-A}.$$

If $|y| > T^{3-\epsilon}$ then

$$|y|^j \Phi_{w_4}^{(j)}(y; h) \ll_{j,\epsilon} T^3 M^2 (T + |y|^{1/3})^j$$

for any $j \in \mathbb{N}_0$.

Lemma 2.2. *Let $\mathcal{Y} := \min\{|y_1|^{1/3}|y_2|^{1/6}, |y_1|^{1/6}|y_2|^{1/3}\}$. If $\mathcal{Y} \le T^{1-\epsilon}$, then*

$$\Phi_{w_l}(y_1, y_2; h) \ll_{B,\epsilon} T^{-A}$$

for any fixed constant $A \ge 0$. If $\mathcal{Y} \gg T^{1-\epsilon}$, then

$$\begin{aligned} & |y_1|^{j_1} |y_2|^{j_2} \frac{\partial^{j_1}}{\partial y_1^{j_1}} \frac{\partial^{j_2}}{\partial y_1^{j_2}} \Phi_{w_l}(y_1, y_2) \\ \ll_{j_1, j_2, \epsilon} \quad & T^3 M^2 (T + |y_1|^{1/2} + |y_1|^{1/3}|y_2|^{1/6})^{j_1} (T + |y_2|^{1/2} + |y_2|^{1/3}|y_1|^{1/6})^{j_2} \end{aligned}$$

for all $j_1, j_2 \in \mathbb{N}_0$.

2.4. **Rankin-Selberg L-functions.** We recall holomorphic Hecke cusp forms in [Iw1997]. Let f be a normalized holomorphic Hecke cusp form of weight k for $SL_2(\mathbb{Z})$ such that f has the Fourier expansion

$$f(z) = \sum_{m \geq 1} \lambda_f(m) m^{\frac{k-1}{2}} e(mz),$$

where $\lambda_f(m)$ are Hecke eigenvalues of the Hecke operators $T(m)$. The L-function associated to f is

$$L(s, f) = \sum_{m \geq 1} \frac{\lambda_f(m)}{m^s}$$

which is absolutely convergent for $\mathrm{Re}(s) > 1$ by the Ramanujan-Deligne's bound $\lambda_f(m) \ll m^{\epsilon}$. It has analytic continuation for all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, f) := \Gamma_{\mathbb{R}}\left(s + \frac{k-1}{2}\right) \Gamma_{\mathbb{R}}\left(s + \frac{k+1}{2}\right) L(s, f) = i^k \Lambda(1-s, f).$$

Let f be as above and ϕ be a Hecke-Maass cusp form for $SL_3(\mathbb{Z})$ with Langlands parameter $\boldsymbol{\mu}_\phi = (\mu_1, \mu_2, \mu_3)$. The Rankin-Selberg L-function $L(s, f \otimes \phi)$ is defined by (see Section 12.2 in [Go2006])

$$L(s, f \otimes \phi) := \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) \overline{A_\phi(m_1, m_2)}}{(m_1^2 m_2)^s}$$

for $\mathrm{Re}(s)$ sufficient large. It has analytic continuation for all $s \in \mathbb{C}$ and satisfies the functional equation

$$\begin{aligned}
\Lambda(s, f \otimes \phi) &= \prod_{i=1}^{3} \Gamma_{\mathbb{R}}\left(s + \frac{k-1}{2} - \mu_i\right) \Gamma_{\mathbb{R}}\left(s + \frac{k+1}{2} - \mu_i\right) L(s, f \otimes \phi) \\
&= (i^k)^3 \Lambda(1-s, f \otimes \phi^{\vee}),
\end{aligned}$$

where $\phi^{\vee}$ is the dual Maass cusp form associated to ϕ.

Let $E^{\min}_{\nu_1,\nu_2}(z)$ be the minimal Eisenstein series with the Langlands parameter $\boldsymbol{\mu}(E^{\min}_{\nu_1,\nu_2})$. By Euler products of $L(s, f)$ and $L(s, E^{\min}_{\nu_1,\nu_2})$, we have

$$\begin{aligned}
L(s, f \otimes E^{\min}_{\nu_1,\nu_2}) : &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) \overline{A^{\min}_{\nu_1,\nu_2}(m_1, m_2)}}{(m_1^2 m_2)^s} \\
&= L(s - \mu_1, f) L(s - \mu_2, f) L(s - \mu_3, f).
\end{aligned}$$

It satisfies the functional equation

$$\begin{aligned}
\Lambda(s, f \otimes E^{\min}_{\nu_1,\nu_2}) : &= \prod_{j=1}^{3} \Gamma_{\mathbb{R}}\left(s + \frac{k-1}{2} - \mu_j\right) \Gamma_{\mathbb{R}}\left(s + \frac{k+1}{2} - \mu_j\right) L(s, f \times E^{\min}_{\nu_1,\nu_2}) \\
&= \Lambda(1-s, f \otimes E^{\min}_{-\nu_1,-\nu_2}).
\end{aligned}$$

For $E_{u,g}^{\max}(z)$ the maximal Eisenstein series with $\boldsymbol{\mu}(E_{u,g}^{\max}) = (\mu_1', \mu_2', \mu_3')$ where μ_j' are given by (2.4), we have

$$\begin{aligned} L(s, f \otimes E_{u,g}^{\max}) : &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)\overline{A_{\nu,u}^{\max}(m_1, m_2)}}{(m_1^2 m_2)^s} \\ &= L(s + 2u, f) L(s - u, f \otimes g), \end{aligned}$$

where $L(s, f \otimes g)$ is the Rankin-Selberg function associated to f and g. The complete L-function is

$$\begin{aligned} \Lambda(s, f \otimes E_{u,g}^{\max}) &= \prod_{j=1}^{3} \Gamma_{\mathbb{R}} \left(s + \frac{k-1}{2} - \mu_j' \right) \left(s + \frac{k+1}{2} - \mu_j' \right) L(s, f \otimes E_{u,g}^{\max}) \\ &= i^k \Lambda(1 - s, f \otimes E_{-u,g}^{\max}). \end{aligned}$$

2.5. **The approximate functional equation.** For the Rankin-Selberg L-function defined in the previous section, we have the following approximate functional equation (see Theorem 5.3 in [IK2004]).

Lemma 2.3. *Let $G(s) = e^{s^2}$. We have*

$$\begin{aligned} L\left(\frac{1}{2}, f \otimes \phi\right) &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2) A_\phi(m_2, m_1)}{(m_1^2 m_2)^{1/2}} V_k(m_1^2 m_2, \boldsymbol{\mu}_\phi) \\ &\quad + i^k \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_1) A_\phi(m_1, m_2)}{(m_1^2 m_2)^{1/2}} \tilde{V}(m_1^2 m_2; k, \boldsymbol{\mu}_\phi), \end{aligned}$$

where

$$V_k(y, \boldsymbol{\mu}) = \frac{1}{2\pi i} \int_{(3)} y^{-s} \prod_{i=1}^{3} \frac{\Gamma_{\mathbb{R}}\left(s + \frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_{\mathbb{R}}\left(s + \frac{1}{2} + \frac{k+1}{2} - \mu_i\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_{\mathbb{R}}\left(\frac{1}{2} + \frac{k+1}{2} - \mu_i\right)} G(s) \frac{ds}{s} \tag{2.6}$$

and

$$\tilde{V}_k(y, \boldsymbol{\mu}) = \frac{1}{2\pi i} \int_{(3)} y^{-s} \prod_{i=1}^{3} \frac{\Gamma_{\mathbb{R}}\left(s + \frac{1}{2} + \frac{k-1}{2} + \mu_i\right) \Gamma_{\mathbb{R}}\left(s + \frac{1}{2} + \frac{k+1}{2} + \mu_i\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2} + \frac{k-1}{2} - \mu_i\right) \Gamma_{\mathbb{R}}\left(\frac{1}{2} + \frac{k+1}{2} - \mu_i\right)} G(s) \frac{ds}{s}.$$

The functions $V_k(y, \boldsymbol{\mu})$ and $\tilde{V}_k(y, \boldsymbol{\mu})$ have the following properties, which can be proved by the method in Proposition 5.4 in [IK2004].

Lemma 2.4. *Assume that $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ with $\mu_i \asymp T$. One has*

$$y^a \frac{\partial^a}{\partial y^a} V_k(y, \boldsymbol{\mu}) \ll_k \left(\frac{y}{T^3}\right)^{-A}, \quad y^a \frac{\partial^a}{\partial y^a} \tilde{V}_k(y, \boldsymbol{\mu}) \ll_k \left(\frac{y}{T^3}\right)^{-A}$$

for any large number $A > 0$ and any $a \in \mathbb{N}_0$. Moreover, for $y \gg T^3$,

$$\begin{aligned} V_k(y, \boldsymbol{\mu}) &= 1 + O_{B,k}\left(\frac{T^3}{y}\right)^{-B} \\ \tilde{V}_k(y, \boldsymbol{\mu}) &= \prod_{i=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_i)}{\Gamma(\frac{k}{2} - \mu_i)} + O_{B,k}\left(\frac{T^3}{y}\right)^{-B} \end{aligned}$$

for any $0 < B < \frac{k-1}{2}$.

3. Proof of Theorem 1.1

Let $k \equiv 0 \bmod 4$. For $h(\boldsymbol{\mu})$ defined in section 2.3, we consider

$$\mathcal{A} = \sum_{\phi} \frac{h(\boldsymbol{\mu}_\phi)}{\mathcal{N}_\phi} A_\phi(p,p) L(1/2, f \otimes \phi),$$

where ϕ runs over a Hecke-Maass basis of the space of Maass cusp forms for $SL_3(\mathbb{Z})$. By the approximate functional equation in Lemma 2.3, one has

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$$

where

$$\begin{aligned}
\mathcal{A}_1 &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\phi} \frac{h(\boldsymbol{\mu}_\phi) V_k(m_1^2 m_2, \boldsymbol{\mu_\phi})}{\mathcal{N}_j} A_\phi(m_2, m_1) A_\phi(p,p), \\
\mathcal{A}_2 &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\phi} \frac{h(\boldsymbol{\mu}_\phi) \tilde{V}_k(m_1^2 m_2, \boldsymbol{\mu_\phi})}{\mathcal{N}_j} A_\phi(m_1, m_2) A_\phi(p,p).
\end{aligned}$$

Thus Theorem 1.1 follows from

$$\mathcal{A}_1 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) d_{\mathrm{spec}}(\boldsymbol{\mu}) + O_{k,\epsilon}(p^{\frac{7}{32}+\epsilon} T^{\frac{5}{2}+\epsilon} M^2), \tag{3.1}$$

$$\mathcal{A}_2 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) \prod_{j=1}^{3} \frac{\Gamma(\frac{k}{2} + \mu_j)}{\Gamma(\frac{k}{2} - \mu_j)} d_{\mathrm{spec}}(\boldsymbol{\mu}) + O_{k,\epsilon}(p^{\frac{7}{32}+\epsilon} T^{\frac{5}{2}+\epsilon} M^2). \tag{3.2}$$

Since the proof of (3.2) is the same as that of (3.1). We only prove (3.1).

For $\mathcal{A}_1$, by letting

$$H_y(\boldsymbol{\mu}) := h(\boldsymbol{\mu}) V_k(y; \boldsymbol{\mu})$$

and applying the Kunzetsov's trace formula in section 2.2, one has

$$\mathcal{A}_1 = \mathcal{D}_1 + \mathcal{R}_{1,w_4} + \mathcal{R}_{1,w_l} - \mathcal{E}_{1,\max} - \mathcal{E}_{1,\min},$$

where

$$\begin{aligned}
\mathcal{D}_1 &= \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) V_k(p^3, \boldsymbol{\mu}) d_{\mathrm{spec}} \boldsymbol{\mu}, \\
\mathcal{R}_{1,w_4} &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_2 | D_1 \\ p D_1 = m_2 D_2^2}} \frac{\tilde{S}(-\epsilon m_1, p, p; D_2, D_1)}{D_1 D_2} \Phi_{w_4}\left(\frac{\epsilon m_1 p^2}{D_1 D_2}; H_{m_1^2 m_2} \right), \\
\mathcal{R}_{1,w_5} &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_1 | D_2 \\ p D_2 = m_1 D_1^2}} \frac{\tilde{S}(\epsilon m_2, p, p; D_1, D_2)}{D_1 D_2} \Phi_{w_5}\left(\frac{\epsilon m_2 p^2}{D_1 D_2}; H_{m_1^2 m_2} \right), \\
\mathcal{R}_{1,w_l} &= \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \sum_{D_1, D_2} \frac{S(\epsilon_2 m_1, \epsilon_1 m_2, p, p; D_1, D_2)}{D_1 D_2} \\
&\quad \times \Phi_{w_l}\left(-\frac{\epsilon_2 p m_1 D_2}{D_1^2}, -\frac{\epsilon_1 p m_2 D_1}{D_2^2}; H_{m_1^2 m_2} \right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_{1,\max} &= \sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}}\sum_g\frac{1}{2\pi i}\\
&\qquad\int_{\mathrm{Re}(u)=0}\frac{H_{m_1^2m_2}(u+it_g,u-it_g,-2u)}{\mathcal{N}_{u,g}^{\max}}A_{u,g}^{\max}(m_2,m_1)A_{u,g}^{\max}(p,p)du,\\
\mathcal{E}_{1,\min} &= \sum_{m_1\geq 1}\sum_{m_2\geq 1}\frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}}\frac{1}{24(2\pi i)^2}\\
&\qquad\iint_{\mathrm{Re}(\boldsymbol{\mu})=0}\frac{H_{m_1^2m_2}(\boldsymbol{\mu})}{\mathcal{N}_{\nu_1,\nu_2}^{\min}}A_{\nu_1,\nu_2}^{\min}(m_2,m_1)A_{\nu_1,\nu_2}^{\min}(p,p)d\boldsymbol{\mu}
\end{aligned}$$

The main term in (3.1) comes from the estimation on $\mathcal{D}_1$ in (3.5), and the error term comes from the contribution of $\mathcal{E}_{1,\max}$ in (3.4). For $\mathcal{E}_{1,\min}$ and $\mathcal{R}_{1,w_4}$, $\mathcal{R}_{1,w_5}$, $\mathcal{R}_{1,w_6}$, we will show that their contribution is negligible under the condition in Theorem 1.1.

3.1. **Estimation on the continuous spectrum.** We consider $\mathcal{E}_{1,\min}$ firstly. Note that $H_y(\boldsymbol{\mu}) = h(\boldsymbol{\mu})V_k(y,\boldsymbol{\mu})$. By the integral expression of $V_k(y,\boldsymbol{\mu})$ in (2.6) and the fact that

$$\sum_{m_1,m_2\geq 1}\frac{\lambda_f(m_2)A_{\nu_1,\nu_2}^{\min}(m_2,m_1)}{(m_1^2m_2)^{s+\frac{1}{2}}} = L\left(\frac{1}{2}+s-\mu_1,f\right)L\left(\frac{1}{2}+s-\mu_2,f\right)L\left(\frac{1}{2}+s-\mu_3,f\right)$$

for $\mathrm{Re}(s)=3$, one has

$$\mathcal{E}_{1,\min} = \frac{1}{24(2\pi i)^2}\iint_{\mathrm{Re}(\boldsymbol{\mu})=0}A_{\nu_1,\nu_2}^{\min}(p,p)\frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\nu_1,\nu_2}^{\min}}\mathcal{I}_k^{\min}(\boldsymbol{\mu})d\boldsymbol{\mu},$$

where

$$\mathcal{I}_k^{\min}(\boldsymbol{\mu}) = \frac{1}{2\pi i}\int_{(3)}G(s)\prod_{i=1}^{3}\frac{\Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k-1}{2}-\mu_i\right)\Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k-1}{2}-\mu_i\right)\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)}L\left(\frac{1}{2}+s-\mu_i,f\right)\frac{ds}{s}.$$

For $\mathcal{I}_k^{\min}(\boldsymbol{\mu})$, moving the line of integration to $\mathrm{Re}(s)=\epsilon$ and applying the subconvexity bound (see [Go1982])

$$L(1/2+it,f)\ll_k(1+|t|)^{1/3+\epsilon},$$

one has

$$\mathcal{I}_k^{\min}(\boldsymbol{\mu})\ll_{\epsilon,k}\prod_{j=1}^{3}(1+|\mathrm{Im}(\mu_j)|)^{\frac{1}{3}+\epsilon}.$$

It gives that

$$\mathcal{E}_{1,\min}\ll_{k,\epsilon}\iint_{\mathrm{Re}(\boldsymbol{\mu})=0}A_{\nu_1,\nu_2}^{\min}(p,p)\frac{h(\boldsymbol{\mu})}{\mathcal{N}_{\nu_1,\nu_2}^{\min}}\prod_j(1+|\mathrm{Im}(\mu_j)|)^{\frac{1}{3}+\epsilon}d\boldsymbol{\mu}.$$

Note that $A_{\nu_1,\nu_2}^{\min}(p,p)=O(1)$ and

$$\mathcal{N}_{\nu_1,\nu_2}^{\min} = \frac{1}{16}\prod_{j=1}^{3}|\zeta(1+3\nu_{\pi,j})|^2\gg\prod_{j=1}^{3}\left(\frac{1}{\log(1+3\mathrm{Im}\nu_{\pi,j})}\right)^2.$$

One has

$$\mathcal{E}_{1,\min} \ll_{k,\epsilon} T^{1+\epsilon} M^2. \tag{3.3}$$

Next we consider $\mathcal{E}_{1,\max}$. By similar argument as above one has

$$\mathcal{E}_{1,\max} = \sum_g \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=0} A^{\max}_{u,g}(p,p) \frac{h(u+it_g, u-it_g, -2u)}{\mathcal{N}^{\max}_{u,g}} \mathcal{I}^{\max}_k(u+it_g, u-it_g, -2u) du,$$

where

$$\begin{aligned} \mathcal{I}^{\max}_k(\boldsymbol{\mu}) \quad = \quad & \frac{1}{2\pi i} \int_{(3)} \prod_{i=1}^{3} \frac{\Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k-1}{2}-\mu_i\right) \Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k-1}{2}-\mu_i\right) \Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)} \\ & L\left(\frac{1}{2}+s+2u, f\right) L\left(\frac{1}{2}+s-u, f\otimes g\right) G(s) \frac{ds}{s}. \end{aligned}$$

For $\mathcal{I}^{\max}_k(\boldsymbol{\mu})$, by moving the line of integration to $\mathrm{Re}(s) = \frac{1}{2}+\epsilon$ and applying the fact that

$$L(1+\epsilon+2u, f) \ll 1, \quad L\left(1+\epsilon-u, f\otimes g\right) \ll 1,$$

which follow from the Ramanujar-Deligue's bound and the property of Rankin-Selberg L-functions (see [RS1996]), one has

$$\begin{aligned} \mathcal{I}^{\max}_k(\boldsymbol{\mu}) \quad \ll \quad & \int_{(\frac{1}{2}+\epsilon)} \prod_{i=1}^{3} \frac{\Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k-1}{2}-\mu_i\right) \Gamma_{\mathbb{R}}\left(s+\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)}{\Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k-1}{2}-\mu_i\right) \Gamma_{\mathbb{R}}\left(\frac{1}{2}+\frac{k+1}{2}-\mu_i\right)} G(s) \frac{ds}{s} \\ \ll_{k,\epsilon} \quad & \prod_{j=1}^{3} \left(1+|\mathrm{Im}\mu_j|\right)^{\frac{1}{2}+\epsilon}. \end{aligned}$$

Moreover, by the definition of $A^{\max}_{u,g}(m,n)$ in (2.2) and (2.3), and the bound $\lambda_g(p) \ll p^{\frac{7}{64}+\epsilon}$ in [KS2003], one has $A^{\max}_{u,g}(p,p) \ll p^{\frac{7}{32}+\epsilon}$. These together with

$$\mathcal{N}^{\max}_{u,g} = 8L(1, \mathrm{Ad}^2 g)|L(1+3u, g)|^2 \gg \left(\frac{1}{1+\log|u|}\right)$$

give that

$$\begin{aligned} \mathcal{E}_{1,\max} \quad \ll_{k,\epsilon} \quad & p^{\frac{7}{32}+\epsilon} \sum_g \int_{\mathrm{Re}(u)=0} \frac{h(u+it_g, u-it_g, -2u)}{\mathcal{N}_{u,g}} \left(1+|\mathrm{Im}u+t_g|\right)^{\frac{1}{2}+\epsilon} \\ & \left(1+|\mathrm{Im}u-t_g|\right)^{\frac{1}{2}+\epsilon} \left(1+|2\mathrm{Im}u|\right)^{\frac{1}{2}+\epsilon} d\mu \\ \ll_{k,\epsilon} \quad & p^{\frac{7}{32}+\epsilon} T^{\frac{3}{2}+\epsilon} M \sum_{\substack{g \\ T-M \le it_g \le T+M}} 1 \\ \ll_{k,\epsilon} \quad & p^{\frac{7}{32}+\epsilon} T^{\frac{5}{2}+\epsilon} M^2, \end{aligned} \tag{3.4}$$

where we have used the Weyl's law for Hecke-Mass cusp forms for $SL_2(\mathbb{Z})$ (see [Iw2002]).

3.2. **Estimation on the diagonal term $\mathcal{D}_1$.** For the diagonal term $\mathcal{D}_1$, by Lemma 2.4, we have

$$\mathcal{D}_1 = \frac{\lambda_f(p)}{p^{3/2}} \frac{1}{192\pi^5} \left(1 + O_B\left(\frac{p}{T}\right)^{3B}\right) \iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) d_{\mathrm{spec}}(\boldsymbol{\mu}) \tag{3.5}$$

for $0 < B < \frac{k-1}{2}$. The choice of $h(\boldsymbol{\mu})$ in section 2.3 gives

$$\iint_{\mathrm{Re}(\boldsymbol{\mu})=0} h(\boldsymbol{\mu}) d_{\mathrm{spec}}(\boldsymbol{\mu}) \asymp T^3 M^2.$$

Recall that $k \geq 12$. By (3.3) and (3.4), D_1 gives the main term in (3.1) if

$$T \gg_{k,\epsilon} p^{3+\frac{7}{16}+\epsilon}. \tag{3.6}$$

3.3. **Estimation on other geometric terms.** In this subsection, we show that the contribution from other geometric terms are negligible. For $\mathcal{R}_{1,w_4}$ and $\mathcal{R}_{1,w_l}$, it follows immediately from the application of the truncation Lemmas 2.1 and 2.2, respectively. To show that $\mathcal{R}_{1,w_5}$ is negligible, one needs to open the incomplete Kloosterman sum, rearrange the summation and apply the Voronoi formula for GL_2.

3.3.1. *The term $\mathcal{R}_{1,w_4}$.* Consider $\mathcal{R}_{1,w_4}$ firstly. By the property of $V_k(y;\boldsymbol{\mu})$ in lemma 2.4, the terms in summations over m_1 and m_2 are negligible for those $m_1^2 m_2 > T^{3+\epsilon}$. By Lemma 2.1, the contribution of terms in summations over D_1 and D_2 is negligible if

$$\frac{p^2 m_1}{D_1 D_2} = \frac{p^{3/2} m_1 \sqrt{m_2}}{D_1^{3/2}} \leq T^{3-\epsilon}.$$

Thus one needs only to consider

$$\sum_{\substack{m_1, m_2 \geq 1 \\ m_1^2 m_2 \leq T^{3+\epsilon}}} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon \in \{\pm 1\}} \sum_{\substack{D_2 | D_1 \\ pD_1 = m_2 D_2^2 \\ 1 \leq D_1 \leq \frac{p(m_1^2 m_2)^{1/3}}{T^{2-\epsilon}}}} \frac{\tilde{S}(-\epsilon m_1, p, p; D_2, D_1)}{D_1 D_2} \Phi_{w_4}\left(\frac{\epsilon p^2 m_1}{D_1 D_2}; H_{m_1^2 m_2}\right).$$

Note that $m_1^2 m_2 \leq T^{3+\epsilon}$ and $1 \leq D_1 \leq \frac{p(m_1^2 m_2)^{1/3}}{T^{2-\epsilon}}$ give $p \geq T^{1-\epsilon}$, which contradicts with (3.6). Thus these terms vanish and $\mathcal{R}_{1,w_4}$ is negligible.

3.3.2. *The term $\mathcal{R}_{1,w_l}$.* For $\mathcal{R}_{1,w_l}$, by the property of $V_k(y,\boldsymbol{\mu})$ in lemma 2.4, the terms in summations over m_1 and m_2 are negligible for those $m_1^2 m_1 \leq T^{3+\epsilon}$. Let

$$\mathcal{Y} := p^{1/2} \min\left\{\frac{m_1^{1/3} m_2^{1/6}}{D_1^{1/2}}, \frac{m_2^{1/3} m_1^{1/6}}{D_2^{1/2}}\right\}.$$

By lemma 2.2, the contribution is negligible for those terms in summations over D_1 and D_2 satisfying $\mathcal{Y} \leq T^{1-\epsilon}$. Thus we need only to estimate

$$\sum_{\substack{m_1, m_2 \geq 1 \\ m_1^2 m_2 \leq T^{3+\epsilon}}} \frac{\lambda_f(m_2)}{(m_1^2 m_2)^{1/2}} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \sum_{\substack{D_1, D_2 \\ \mathcal{Y} > T^{1-\epsilon}}} \frac{S(\epsilon_2 m_1, \epsilon_1 m_2, p, p; D_1, D_2)}{D_1 D_2}$$

$$\Phi_{w_l}\left(-\frac{\epsilon_2 p m_1 D_2}{D_1^2}, -\frac{\epsilon_1 p m_2 D_1}{D_2^2}; H_{m_1^2 m_2}\right).$$

Note that $m_1^2m_2 \le T^{3+\epsilon}$ and $\mathcal{Y} > T^{1-\epsilon}$ give $p \ge T^{1-\epsilon}$, which contradicts with (3.6). Thus these terms vanish and $\mathcal{R}_{1,w_l}$ is negligible.

3.3.3. *The term $\mathcal{R}_{w_5}$.* Consider $\mathcal{R}_{w_5}$. By the similar argument in previous sections, one needs only to consider the contribution of

$$\begin{aligned}\mathcal{R}^* := \quad & \sum_{\substack{m_1,m_2\ge 1\\ T^{\frac{8}{3}}\le m_1^2m_2\le T^{3+\epsilon}}} \frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}} \sum_{\epsilon\in\{\pm1\}} \\ & \sum_{\substack{D_1|D_2\\ pD_2=m_1D_1^2\\ 1\le D_2\le \frac{p(m_2^2m_1)^{1/3}}{T^{2-\epsilon}}}} \frac{\tilde{S}(-\epsilon m_1,p,p;D_2,D_1)}{D_1D_2}\Phi_{w_4}\left(\frac{\epsilon p^2 m_1}{D_1D_2}; H_{m_1^2m_2}\right),\end{aligned}$$

since other terms either vanish or are negligible.

We show that $\mathcal{R}^*$ is also negligible. Recall the smooth partition of unity

$$1 = \sum_{\alpha\ge 0}\omega\left(\frac{m_1^2m_2}{N_\alpha}\right),$$

where ω is a function which is smooth and compactly supported on $[\frac{1}{2},\frac{5}{2}]$ and $N_\alpha = 2^\alpha$. One has

$$\begin{aligned}\mathcal{R}^* \ll \quad & \sum_{\substack{\alpha\ge 0\\ T^{\frac{8}{3}}\ll N_\alpha\ll T^{3+\epsilon}}} \sum_{m_1,m_2\ge 1}\omega\left(\frac{m_1^2m_2}{N_\alpha}\right)\frac{\lambda_f(m_2)}{(m_1^2m_2)^{1/2}}\sum_{\epsilon\in\{\pm1\}} \\ & \sum_{\substack{D_1|D_2\\ pD_2=m_1D_1^2}} \frac{\tilde{S}(\epsilon m_2,p,p;D_1,D_2)}{D_1D_2}\Phi_{w_5}\left(\frac{\epsilon m_2p^2}{D_1D_2}; H_{m_1^2m_2}\right).\end{aligned}$$

Let $D_2 = D_1\delta$. We open the incomplete Kloosterman sum, rearrange the summation and then obtain

$$\begin{aligned}\mathcal{R}^* \ll \quad & \sum_{T^{8/3}\ll N_\alpha\ll T^{3+\epsilon}}\sum_{m_1\ge 1}\frac{1}{m_1}\sum_{\epsilon\in\{\pm1\}}\sum_{\substack{\delta,D_1\ge 1\\ p\delta=m_1D_1}}\frac{1}{D_1^2\delta}\sum_{\substack{C_1(\bmod D_1),\,C_2(\bmod D_1\delta)\\ (C_1,D_1)=(C_2,\delta)=1}} e\left(\frac{p\overline{C}_1C_2}{D_1}+p\frac{\overline{C}_2}{\delta}\right) \\ & \sum_{m_2\ge 1}\omega\left(\frac{m_1^2m_2}{N_\alpha}\right)\frac{\lambda_f(m_2)}{\sqrt{m_2}}\Phi_{w_5}\left(\frac{\epsilon p^2m_2}{D_1^2\delta}, H_{m_1^2m_2}\right)e\left(\epsilon m_2\frac{C_1}{D_1}\right). \qquad (3.7)\end{aligned}$$

Thus one can apply the following $GL(2)$ Voronoi formula (see formula (4.71) in [IK2004]).

Lemma 3.1. *Let $c \ge 1$ and $(a,c) = 1$. Let F be a smooth, compactly supported function on $\mathbb{R}^+$. One has*

$$\sum_{m\ge 1}\lambda_f(m)e\left(\frac{am}{c}\right)F(m) = \frac{1}{c}\sum_{n\ge 1}\lambda_f(n)e\left(-\frac{\overline{a}n}{c}\right)G(n),$$

where $G(y) = 2\pi i^k\int_0^\infty F(x)J_{k-1}\left(\frac{4\pi\sqrt{xy}}{c}\right)dx$. Here $J_k(y)$ is the J-Bessel function.

For the summation over m_2 in (3.7), we apply the Voronoi formula in the above lemma and obtain

$$\begin{aligned}
&\sum_{m_2\geq 1}\omega\left(\frac{m_1^2m_2}{N_\alpha}\right)\frac{\lambda_f(m_2)}{\sqrt{m_2}}\Phi_{w_5}\left(\frac{\epsilon p^2m_2}{D_1^2\delta},H_{m_1^2m_2}\right)e\left(\epsilon m_2\frac{C_1}{D_1}\right)\\
=\;& \frac{1}{D_1}\sum_{m_2\geq 1}\lambda_f(m_2)e\left(-\frac{\epsilon\overline{C}_1m_2}{D_1}\right)G(m_2),
\end{aligned}$$

where

$$G(m_2)=2\pi i^k\int_0^\infty\omega\left(\frac{m_1^2x}{N_\alpha}\right)\frac{1}{x^{1/2}}\Phi_{w_5}\left(\frac{\epsilon p^2x}{D_1^2\delta},H_{m_1^2m_2}\right)J_{k-1}\left(\frac{4\pi\sqrt{xm_2}}{D_1}\right)dx.$$

Lemma 3.2. *We have*

$$G(m_2)\ll_{j,k,\epsilon}\frac{\sqrt{N_\alpha}}{m_1}\left(\frac{p^{1+\epsilon}}{N_\alpha^{\frac16-\epsilon}m_2^{\frac12}}\right)^j$$

for any $j\in\mathbb{N}_0$.

Proof. For $G(m_2)$, we change the variable $t=\frac{m_1^2}{N_\alpha}x$ to obtain

$$G(m_2)=2\pi i^k\frac{\sqrt{N_\alpha}}{m_1}\int_0^\infty\omega(t)\Phi_{w_5}\left(\frac{\epsilon p^2N_\alpha}{D_1^2\delta m_1^2}t,H_{m_1^2m_2}\right)J_{k-1}\left(\frac{4\pi\sqrt{N_\alpha m_2t}}{m_1D_1}\right)\frac{dt}{\sqrt{t}}.$$

Let $R=\frac{4\pi\sqrt{N_\alpha m_2}}{m_1D_1}$. By applying the recurrence formula of the J-Bessel function

$$\frac{d}{dy}\left((R\sqrt{y})^{s+1}J_{s+1}(R\sqrt{y})\right)=\frac{R^2}{2}(R\sqrt{y})^sJ_s(R\sqrt{y}),$$

one has

$$\begin{aligned}
G(m_2)\;&=\;2\pi i^k\frac{\sqrt{N_\alpha}}{m_1}\frac{1}{R^{k-1}}\frac{-2}{R^2}\int_0^\infty\left(t^{-\frac{k}{2}}\omega(t)\Phi_{w_5}\left(\frac{\epsilon p^2N_\alpha}{D_1^2\delta m_1^2}t,H_{m_1^2m_2}\right)\right)'\left(R\sqrt{t}\right)^kJ_k(R\sqrt{t})dt\\
&=\;2\pi i^k\frac{\sqrt{N_\alpha}}{m_1}\frac{1}{R^{k-1}}\left(-\frac{2}{R^2}\right)^j\\
&\quad\int_0^\infty\left(t^{-\frac{k}{2}}\omega(t)\Phi_{w_5}\left(\frac{\epsilon p^2N_\alpha}{D_1^2\delta m_1^2}t,H_{m_1^2m_2}\right)\right)^{(j)}\left(R\sqrt{t}\right)^{k+j-1}J_{k+j-1}(R\sqrt{t})dt
\end{aligned}$$

for any $j\in\mathbb{N}_0$. Note that Φ_{w_5} also satisfies Lemma 2.1 and one has

$$\left(\Phi_{w_5}\left(\frac{\epsilon p^2N_\alpha}{D_1^2\delta m_1^2}t,H_{m_1^2m_2}\right)\right)^{(j)}\ll T^3M^2\left(\frac{p^2N_\alpha}{D_1^2\delta m_1^2}t\right)^{j\left(\frac13+\epsilon\right)}.$$

It gives that

$$\begin{aligned}
G(m_2)\;&\ll_{k,j,\epsilon}\;\frac{\sqrt{N_\alpha}}{m_1}\left(\left(\frac{p^2N}{D_1^2\delta m_1^2}\right)^{\left(\frac13+\epsilon\right)}\Big/R\right)^j\\
&\ll_{k,j,\epsilon}\;\frac{\sqrt{N_\alpha}}{m_1}\left(\frac{p^{\frac23+\epsilon}}{N_\alpha^{\frac16-\epsilon}m_2^{\frac12}}\left(\frac{m_1D_1}{\delta}\right)^{\frac13}\right)^j
\end{aligned}$$

since $R = \frac{4\pi\sqrt{N_\alpha m_2}}{m_1 D_1}$. The lemma follows immediately from the fact that $m_1 D_1 = p\delta$. $\square$

By lemma 3.2, the contribution is negligible for those terms in $\mathcal{R}^*$ satisfying

$$\frac{p^{1+\epsilon}}{N_\alpha^{\frac{1}{6}-\epsilon} m_2^{\frac{1}{2}}} \ll_{k,\epsilon} T^{-\epsilon}.$$

Note that $N_\alpha \gg T^{\frac{8}{3}}$. Thus one needs only to consider terms in $\mathcal{R}^*$ satisfying the condition

$$p^{1+\epsilon} \gg_{k,\epsilon} N_\alpha^{\frac{1}{6}-\epsilon} T^{-\epsilon} \gg T^{\frac{4}{9}-\epsilon},$$

which contradicts with (3.6). Thus the contribution of $\mathcal{R}^*$ is negligible.

References

[Bl2013] V. Blomer, *Applications of the Kuznetsov formula on $GL(3)$*, Inventiones Mathematicae 194(3), 673-729, 2013.

[BB2015] V. Blomer and J. Buttcane, *On the subconvexity problem for L-functions on $GL(3)$*, arXiv:1504.02667, 2015.

[Bu2014] J. Buttcane, *The Spectral Kuznetsov Formula on $SL(3)$*, Transactions of the American Mathematical Society, 2014.

[CD2005] G. Chinta and A. Diaconu, *Determination of a GL_3 cuspform by twists of central L-values*, Int. Math. Res. Notices 2005, 2941-2967, 2005.

[GHS2009] S. Ganguly, J. Hoffstein and J. Sengupta, *Determining modular forms on $SL_2(\mathbb{Z})$ by central values of convolution L-functions*, Math. Ann. 345, 843-857, 2009.

[Go2006] D. Goldfeld, *Automorphic forms and L-functions for the group $GL(n,\mathbb{R})$*, Cambridge University Press, 2006.

[Go1982] A. Good, *The square mean of Dirichlet series associated with cusp forms*, Mathematika 29 (2), 278-295, 1982.

[HLZ2017] B. Huang, S. Liu and Z. Xu, *Mollification and non-vanishing of automorphic L-functions on GL(3)*, arXiv:1704.00314, 2017.

[Iw1997] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, volume 17, American Mathematical Society, Providence, RI, 1997.

[Iw2002] H. Iwaniec, *Spectral methods of automorphic forms*, Second edition. Graduate Studies in Mathematics, volume 53. American Mathematical Society, Providence, RI; Revista Mathemtica Iberoamericana, Madrid, 2002.

[IK2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications 53, American Mathematical Society, Providence, RI, 2004.

[KS2003] H. Kim and P. Sarnak, *Appendix: Refined estimates towards the Ramanujan and Selberg conjectures*, J. Amer. Math. Soc 16.1, 175-181, 2003.

[Li2007] J. Li, *Determination of a GL_2 automorphic cuspidal representation by twists of critical L-values*, J. Number Theory 123 (2), 255-289, 2007.

[Li2009] X. Li, *The Central value of the RankinCSelberg L-Functions*, Geometric & Functional Analysis 18.5, 1660-1695, 2009.

[Liu2010] S. C. Liu, *Determination of $GL(3)$ cusp forms by central values of $GL(3) \times GL(2)$ L-functions*, Int. Math. Res. Notices 2010 (21), 4025-4041, 2010.

[Liu2011] S. C. Liu, *Determination of $GL(3)$ cusp forms by central values of $GL(3) \times GL(2)$ L-functions, level aspect*, J. Number Theory 133 (8), 1397-1408, 2011.

[Lu1999] W. Luo, *Special L-values of Rankin-Selberg convolutons*, Math. Ann. 314 (3), 591-600, 1999.

[LR1997] W. Luo and D. Ramakrishnan, *Determination of modular forms by twists of critical L-values*, Invent. Math. 130 (2), 371-398, 1997.

[Ma2014] R. Matsuda, *Determination of $GL(3)$ Hecke-Maass forms from twisted central values*, Mathematics 148.B11, 272C287, 2014.

[Mu2010] R. Munshi, *On effective determination of modular forms by twists of critical L-values*, Math. Ann. 347 (4), 963-978, 2010.

[MS2015] R. Munshi and J. Sengupta, *On effective determination of Maass forms from central values of Rankin-Selberg L-function*, Forum Mathematicum 27(1), 467-484, 2015.

[Pi2010] Q. Pi, *Determining cusp forms by central values of Rankin-Selberg L-functions*, J. Number Theory 130(10), 2283-2292, 2010.

[Pi2014] Q. Pi *On effective determination of cusp forms by L-values, level aspect*, Journal of Number Theory 142(6), 305-321, 2014.

[PS1979] I. Piatetski-Shapiro, *Multiplicity one theorems*, Automorphic Forms, Representations and L-Functions, Proceedings of the Symposium on Pure Mathematics, XXXIII, American Mathematical Society, 1979.

[RS1996] Z. Rudnick and P. Sarnak, *Zeros of principal L -functions and random matrix theory*, Duke Mathematical Journal 81(2), 269-322, 1996.

[Su2014] Q. Sun, *On effective determination of symmetric-square lifts, level aspect*, International Journal of Number Theory 12(7), 976-990, 2014.

[Zh2011] Y. Zhang, *Determining modular forms of general level by central values of convolution L-functions*, ACTA ARITHMETICA 150(1), 93-103, 2011.

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