# A FORMULA FOR THE ASSOCIATED BUCHSBAUM-RIM MULTIPLICITIES OF A DIRECT SUM OF CYCLIC MODULES II

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ABSTRACT. The associated Buchsbaum-Rim multiplicities of a module are a descending sequence of non-negative integers. These invariants of a module are a generalization of the classical Hilbert-Samuel multiplicity of an ideal. In this article, we compute the associated Buchsbaum-Rim multiplicity of a direct sum of cyclic modules and give a formula for the second to last positive Buchsbaum-Rim multiplicity in terms of the ordinary Buchsbaum-Rim and Hilbert-Samuel multiplicities. This is a natural generalization of a formula given by Kirby and Rees.

#### 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  of dimension d > 0. The associated Buchsbaum-Rim multiplicities of an *R*-module *C* of finite length, which is denoted by  $\{e^j(C)\}_{0 \le j \le d+r-1}$ , are a sequence of integers. These are invariants of *C* introduced by Kleiman-Thorup [10] and Kirby-Rees [8] independently. For an *R*-module *C* of finite length with a minimal free presentation  $R^n \xrightarrow{\varphi} R^r \to C \to 0$ , the multiplicities are defined by the so-called Buchsbaum-Rim function of two variables

$$\Lambda(p,q) := \ell_R(S_{p+q}/M^p S_q),$$

where  $S_p$  (resp.  $M^p$ ) is a homogeneous component of degree p of  $S = \text{Sym}_R(F)$  (resp.  $R[M] = \text{Im Sym}_R(\varphi)$ ). The function  $\Lambda(p,q)$  is eventually a polynomial of total degree d + r - 1, and then the associated Buchsbaum-Rim multiplicities are defined as for  $j = 0, 1, \ldots, d + r - 1$ ,

$$e^{j}(C) := (\text{The coefficient of } p^{d+r-1-j}q^{j} \text{ in the polynomial}) \times (d+r-1-j)!j!$$

These are a descending sequence of non-negative integers with  $e^{r-1}(C)$  is positive, and  $e^j(C) = 0$  for  $j \ge r$ . This was proved by Kleiman-Thorup [10] and Kirby-Rees [8] independently. Moreover, they proved that the first multiplicity  $e^0(C)$  coincides with the ordinary Buchsbaum-Rim multiplicity e(C) of C introduced in [2], which is the normalized leading coefficient of the polynomial function  $\lambda(p) = \Lambda(p, 0) = \ell_R(S_p/M^p)$  of degree d + r - 1 for  $p \gg 0$ . Namely,

$$e(C) = e^{0}(C) \ge e^{1}(C) \ge \dots \ge e^{r-1}(C) > e^{r}(C) = \dots = e^{d+r-1}(C) = 0.$$

Note that the ordinary Buchsbaum-Rim multiplicity e(R/I) of a cyclic module defined by an **m**-primary ideal I coincides with the classical Hilbert-Samuel multiplicity e(I) of I. Thus, the ordinary Buchsbaum-Rim multiplicity  $e^0(C) = e(C)$  and the associated one  $e^j(C)$  are a generalization of the classical Hilbert-Samuel multiplicity. However, as

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compared to the classical Hilbert-Samuel multiplicity, the Buchsbaum-Rim multiplicities are not well-understood.

There are some cases where the computation of the ordinary Buchsbaum-Rim multiplicity is possible (see [1, 3, 6, 7, 8] for instance). In particular, in the case where C is a direct sum of cyclic modules, there is an interesting relation between the ordinary Buchsbaum-Rim multiplicity and the mixed multiplicities of ideals. Let  $I_1, \ldots, I_r$  be m-primary ideals in R. Then Kirby and Rees proved that

$$e(R/I_1 \oplus \cdots \oplus R/I_r) = \sum_{\substack{i_1, \dots, i_r \ge 0\\i_1 + \dots + i_r = d}} e_{i_1 \cdots i_r}(I_1, \dots, I_r),$$

where  $e_{i_1\cdots i_r}(I_1,\ldots,I_r)$  is the mixed multiplicity of  $I_1,\ldots,I_r$  of type  $(i_1,\ldots,i_r)$  (see [7, 8] and also [1]). Then we are interested in the other associated Buchsbaum-Rim multiplicities in this case.

The starting point of this research is the following interesting formula which was also discovered by Kirby-Rees [7, 8]. Suppose that  $I_1 \subset I_2 \subset \cdots \subset I_r$ . Then they proved that for any  $j = 1, \ldots, r - 1$ , the *j*th Buchsbaum-Rim multiplicity can be expressed as the ordinary Buchsbaum-Rim multiplicity of a direct sum of (r-i) cyclic modules defined by the last (r-j) ideals:

$$e^{j}(R/I_{1}\oplus\cdots\oplus R/I_{r})=e(R/I_{j+1}\oplus\cdots\oplus R/I_{r}).$$

In particular, the last positive one  $e^{r-1}$  can be expressed as the classical Hilbert-Samuel multiplicity  $e(I_r)$  of the largest ideal:

$$e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r) = e(R/I_r).$$

Then it is natural to ask the formula for  $e^j(R/I_1 \oplus \cdots \oplus R/I_r)$  without the assumption  $I_1 \subset I_2 \subset \cdots \subset I_r$ . However, as compared to the special case considered in [7, 8], it seems that the problem is more complicated, and we need a different approach to obtain the formula in general. Recently, we tried to compute the function  $\Lambda(p,q)$  directly using some ideas and obtained the formula for  $e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r)$  without the assumption  $I_1 \subset \cdots \subset I_r$ . Indeed, we proved in our previous work [5, Theorem 1.3] that for any  $\mathfrak{m}$ primary ideals  $I_1, \ldots, I_r$ , the last positive Buchsbaum-Rim multiplicity can be expressed as the classical Hilbert-Samuel multiplicity  $e(I_1 + \cdots + I_r)$  of the sum of all ideals:

$$e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r) = e(R/[I_1 + \cdots + I_r]).$$

The present purpose is to improve the method of computation given in [5] towards a formula for not only the last positive Buchsbaum-Rim multiplicity  $e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r)$ but also the next one  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$  in terms of the ordinary Buchsbaum-Rim and Hilbert-Samuel multiplicities. Here is the main result.

**Theorem 1.1.** Let  $I_1, \ldots, I_r$  be arbitrary  $\mathfrak{m}$ -primary ideals in R. Then we have a formula

$$e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r) = E_{r-1}(I_1, \dots, I_r) - (d+1)(r-1)e(R/[I_1 + \dots + I_r]),$$

where  $E_{r-1}(I_1, \ldots, I_r)$  is a sum of the ordinary Buchsbaum-Rim multiplicities of two cyclic modules defined by the ideals  $I_1 + \cdots + \widehat{I_j} + \cdots + I_r$  and  $I_1 + \cdots + I_r$ :

$$E_{r-1}(I_1,\ldots,I_r) := \sum_{j=1}^r e(R/[I_1+\cdots+\widehat{I_j}+\cdots+I_r] \oplus R/[I_1+\cdots+I_r]).$$

Let me illustrate the formula when r = 3. Let  $C = R/I_1 \oplus R/I_2 \oplus R/I_3$ . It is known that  $e^0(C)$  coincides with the ordinary Buchsbaum-Rim multiplicity by [8, 10], and  $e^2(C)$ can be expressed as the ordinary Hilbert-Samuel multiplicity of the sum of all ideals by [5]. Theorem 1.1 tells us that there is a similar expression for the remaining multiplicity  $e^1(C)$ . Namely, if we put  $I_{123} := I_1 + I_2 + I_3$  and  $I_{ij} := I_i + I_j$  for  $1 \le i < j \le 3$ , then we can write all the multiplicities in terms of ordinary Buchsbaum-Rim multiplicities and hence mixed multiplicities.

$$e^{0}(C) = e(R/I_{1} \oplus R/I_{2} \oplus R/I_{3})$$
  

$$e^{1}(C) = e(R/I_{23} \oplus R/I_{123}) + e(R/I_{13} \oplus R/I_{123}) + e(R/I_{12} \oplus R/I_{123}) - 2(d+1)e(R/I_{123})$$
  

$$e^{2}(C) = e(R/I_{123}).$$

Our formula can be viewed as a natural generalization of the above mentioned Kirby-Rees formula for  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$  in a special case where  $I_1 \subset I_2 \subset \cdots \subset I_r$ . Indeed, as an immediate consequence of Theorem 1.1, we get the following.

**Corollary 4.2.** Let  $I_1, \ldots, I_r$  be  $\mathfrak{m}$ -primary ideals in R and assume that  $I_1, \ldots, I_{r-1} \subset I_r$ , that is, the ideal  $I_r$  is the largest ideal. Then we have a formula

$$e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r) = e(R/[I_1 + \cdots + I_{r-1}] \oplus R/I_r).$$

In particular, if  $I_1 \subset I_2 \subset \cdots \subset I_r$ , then

$$e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_{r-1}\oplus R/I_r).$$

The contents of the article are organized as follows. In the next section 2, we will recall some necessary notation and results from our previous work [5]. In section 3, we will compute the Buchsbaum-Rim function of two variables by improving the method in [5]. In the last section 4, we will give a proof of Theorem 1.1 and its consequence Corollary 4.2. We will also discuss the remaining multiplicities  $e^j(C)$  for  $j = 1, \ldots, r-3$ .

Throughout this article, we will work in the same manner in our previous work [5]. Let  $(R, \mathfrak{m})$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  of dimension d > 0. Let r > 0 be a fixed positive integer and let  $[r] = \{1, \ldots, r\}$ . For a finite set A,  $^{\sharp}A$  denotes the number of elements of A. Vectors are always written in bold-faced letters, e.g.,  $\mathbf{i} = (i_1, \ldots, i_r)$ . We work in the usual multi-index notation. Let  $I_1, \ldots, I_r$  be ideals in R and let  $t_1, \ldots, t_r$  be indeterminates. Then for a vector  $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}_{\geq 0}^r$ , we denote  $\mathbf{I}^{\mathbf{i}} = I_1^{i_1} \cdots I_r^{i_r}, \mathbf{t}^{\mathbf{i}} = t_1^{i_1} \cdots t_r^{i_r}$  and  $|\mathbf{i}| = i_1 + \cdots + i_r$ . For vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^r$ ,  $\mathbf{a} \geq \mathbf{b} \stackrel{\text{def}}{\Leftrightarrow} a_i \geq b_i$  for all  $i = 1, \ldots, r$ . Let  $\mathbf{0} = (0, \ldots, 0)$  be the zero vector in  $\mathbb{Z}_{\geq 0}^r$ . By convention, empty sum is defined to be zero.

#### 2. Preliminaries

In this section, we give a few elementary facts to compute the associated Buchsbaum-Rim multiplicities. See also [5, section 2] for the related facts and the details.

In what follows, let  $I_1, \ldots, I_r$  be m-primary ideals in R and let  $C = R/I_1 \oplus \cdots \oplus R/I_r$ . Let  $S = R[t_1, \ldots, t_r]$  be a polynomial ring over R and let  $R[M] = R[I_1t_1, \ldots, I_rt_r]$  be the multi-Rees algebra of  $I_1, \ldots, I_r$ . Let  $S_p$  (resp.  $M^p$ ) be a homogeneous component of degree p of S (resp. R[M]). Then it is easy to see that the function  $\Lambda(p,q)$  can be expressed as

$$\Lambda(p,q) = \sum_{\boldsymbol{n} \in H_{p,q}} \ell_R(R/J_{p,q}(\boldsymbol{n}))$$

where  $H_{p,q} := \{ \boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r \mid |\boldsymbol{n}| = p + q \}$  and  $J_{p,q}(\boldsymbol{n}) := \sum_{\substack{|\boldsymbol{i}| = p \\ \boldsymbol{0} \leq \boldsymbol{i} \leq \boldsymbol{n}}} \boldsymbol{I}^{\boldsymbol{i}}$  for  $\boldsymbol{n} \in H_{p,q}$ . For a subset

 $\Delta \subset H_{p,q}$ , we set

$$\Lambda_{\Delta}(p,q) := \sum_{\boldsymbol{n} \in \Delta} \ell_R(R/J_{p,q}(\boldsymbol{n}))$$

As in [5], we consider the following special subsets of  $H_{p,q}$ , which will play a basic role in our computation of  $\Lambda(p,q)$ . For p,q > 0 and  $k = 1, \ldots, r$ , let

$$\Delta_{p,q}^{(k)} := \{ \boldsymbol{n} \in H_{p,q} \mid n_1, \dots, n_k > p, n_{k+1} + \dots + n_r \le p \}.$$

Then the function  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$  can be described explicitly as follows.

**Proposition 2.1.** ([5, Proposition 2.3]) Let p, q > 0 with  $q \ge (p+1)r$  and let  $k = 1, \ldots, r$ . Then

$$\Lambda_{\Delta_{p,q}^{(k)}}(p,q) = \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \binom{q-(k-1)p-1-(n_{k+1}+\dots+n_r)}{k-1} \ell_R(R/\mathfrak{a}),$$

where  $\mathfrak{a}$  is an ideal depending on  $n_{k+1}, \ldots, n_r$ :

$$\mathfrak{a} := (I_1 + \dots + I_k)^{p - (n_{k+1} + \dots + n_r)} \prod_{j=k+1}^r (I_1 + \dots + I_k + I_j)^{n_j}$$

Here we make a slightly different description of the above mentioned basic functions  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$ . To state it, we first recall some elementary facts about the ordinary Buchsbaum-Rim functions and multiplicities of a direct sum of cyclic modules. The ordinary Buchsbaum-Rim function  $\lambda(p)$  of  $C = R/I_1 \oplus \cdots \oplus R/I_r$  (we will often denote it  $\lambda_C(p)$  to emphasize the defining module C) can be expressed as follows:

$$\begin{split} \lambda(p) &= \ell_R(S_p/M^p) \\ &= \sum_{\substack{\boldsymbol{i} \geq \mathbf{0} \\ |\boldsymbol{i}| = p}} \ell_R(R/\boldsymbol{I}^{\boldsymbol{i}}) \\ &= \sum_{\substack{\boldsymbol{i} \geq \mathbf{0} \\ |\boldsymbol{i}| = p}} \ell_R(R/I_1^{i_1}\cdots I_r^{i_r}). \end{split}$$

In particular, if we consider the case where  $I_1 = \cdots = I_r =: I$ , then

$$\lambda(p) = \binom{p+r-1}{r-1} \ell_R(R/I^p).$$

The function  $\ell_R(R/I^p)$  is just the Hilbert-Samuel function of I so that it is a polynomial for all large enough p, and one can write

$$\ell_R(R/I^p) = \frac{e(R/I)}{d!} p^d + (\text{lower terms}),$$

where e(R/I) is the usual Hilbert-Samuel multiplicity of I. Therefore, the ordinary Buchsbaum-Rim function can be expressed as

$$\lambda(p) = \frac{e(R/I)}{d!(r-1)!} p^{d+r-1} + (\text{lower terms}).$$

This implies the following elementary formula for the ordinary Buchsbaum-Rim multiplicity:

(1) 
$$e(C) = e(\underbrace{R/I \oplus \cdots \oplus R/I}_{r}) = \binom{d+r-1}{r-1}e(R/I).$$

Now, let me give another description of  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$ .

**Proposition 2.2.** Let p, q > 0 with  $q \ge (p+1)r$  and let  $k = 1, \ldots, r$ . Then

$$\begin{split} \Lambda_{\Delta_{p,q}^{(k)}}(p,q) &= \binom{q - (k-1)p - 1}{k-1} \lambda_{L_k}(p) \\ &- \sum_{\substack{n_{k+1}, \dots, n_r \ge 0\\n_{k+1} + \dots + n_r \le p}} \sum_{i=0}^{n_{k+1} + \dots + n_r - 1} \binom{q - (k-1)p - 2 - i}{k-2} \ell_R(R/\mathfrak{a}), \end{split}$$

where  $L_k := R/[I_1 + \dots + I_k] \oplus \bigoplus_{j=k+1}^r R/[I_1 + \dots + I_k + I_j]$  is a direct sum of (r - k + 1)cyclic modules and  $\mathfrak{a} := (I_1 + \dots + I_k)^{p-(n_{k+1} + \dots + n_r)} \prod_{j=k+1}^r (I_1 + \dots + I_k + I_j)^{n_j}$  is an ideal depending on  $n_{k+1}, \dots, n_r$ .

*Proof.* The case where k = 1 follows from Proposition 2.1. Indeed,

$$\Lambda_{\Delta_{p,q}^{(1)}}(p,q) = \sum_{\substack{n_2,\dots,n_r \ge 0\\n_2+\dots+n_r \le p}} \ell_R \Big( R/I_1^{p-(n_2+\dots+n_r)} \prod_{j=2}^r (I_1+I_j)^{n_j} \Big)$$
  
$$= \sum_{\substack{i\ge 0\\|i|=p}} \ell_R \Big( R/I_1^{i_1} (I_1+I_2)^{i_2} \cdots (I_1+I_r)^{i_r} \Big)$$
  
$$= \lambda_{L_1}(p).$$

Suppose that  $k \ge 2$ . By using an elementary combinatorial formula  $\binom{m-\ell}{n} = \binom{m}{n} - \sum_{i=0}^{\ell-1} \binom{m-\ell+i}{n-1}$ , one can see that

$$\begin{pmatrix} q - (k-1)p - 1 - (n_{k+1} + \dots + n_r) \\ k - 1 \end{pmatrix}$$

$$= \begin{pmatrix} q - (k-1)p - 1 \\ k - 1 \end{pmatrix} - \sum_{j=0}^{n_{k+1} + \dots + n_r - 1} \begin{pmatrix} q - (k-1)p - 1 - (n_{k+1} + \dots + n_r) + j \\ k - 2 \end{pmatrix}$$

$$= \begin{pmatrix} q - (k-1)p - 1 \\ k - 1 \end{pmatrix} - \sum_{j=0}^{n_{k+1} + \dots + n_r - 1} \begin{pmatrix} q - (k-1)p - 2 + j - (n_{k+1} + \dots + n_r - 1) \\ k - 2 \end{pmatrix}$$

$$= \begin{pmatrix} q - (k-1)p - 1 \\ k - 1 \end{pmatrix} - \sum_{i=0}^{n_{k+1} + \dots + n_r - 1} \begin{pmatrix} q - (k-1)p - 2 - i \\ k - 2 \end{pmatrix}.$$

By Proposition 2.1, we can write the function  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$  as follows:

$$\begin{split} & = \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \binom{q-(k-1)p-1-(n_{k+1}+\dots+n_r)}{k-1} \ell_R(R/\mathfrak{a}) \\ & = \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \left[ \binom{q-(k-1)p-1}{k-1} - \sum_{i=0}^{n_{k+1}+\dots+n_r-1} \binom{q-(k-1)p-2-i}{k-2} \ell_R(R/\mathfrak{a}) \\ & = \binom{q-(k-1)p-1}{k-1} \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \ell_R(R/\mathfrak{a}) \\ & - \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \sum_{i=0}^{n_{k+1}+\dots+n_r-1} \binom{q-(k-1)p-2-i}{k-2} \ell_R(R/\mathfrak{a}) \\ & = \binom{q-(k-1)p-1}{k-1} \lambda_{L_k}(p) \\ & - \sum_{\substack{n_{k+1},\dots,n_r \ge 0\\n_{k+1}+\dots+n_r \le p}} \sum_{i=0}^{n_{k+1}+\dots+n_r-1} \binom{q-(k-1)p-2-i}{k-2} \ell_R(R/\mathfrak{a}), \end{split}$$

where  $L_k := R/[I_1 + \dots + I_k] \oplus \bigoplus_{j=k+1}^r R/[I_1 + \dots + I_k + I_j]$  is a direct sum of (r - k + 1)cyclic modules and  $\mathfrak{a} := (I_1 + \dots + I_k)^{p-(n_{k+1} + \dots + n_r)} \prod_{j=k+1}^r (I_1 + \dots + I_k + I_j)^{n_j}$  is an ideal depending on  $n_{k+1}, \dots, n_r$ .

### 3. A COMPUTATION OF THE BUCHSBAUM-RIM FUNCTIONS

In this section, we compute the function  $\Lambda(p,q)$  by improving the method in [5] towards a formula for  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$ . The notation we will use here is under the same manner in [5]. See also [5, Section 3] for more detailed observations.

In order to compute the multiplicity defined by the asymptotic function  $\Lambda(p,q)$ , we may assume that  $q \ge (p+1)r \gg 0$ . In what follows, let p,q be fixed integers satisfying  $q \ge (p+1)r \gg 0$ . We put  $H := H_{p,q}$  for short. Then the set H can be divided by r-regions

$$H = \coprod_{k=1}^r H^{(k)},$$

where  $H^{(k)} := \{ n \in H \mid \sharp \{ i \mid n_i > p \} = k \}$ . Moreover, we divide each  $H^{(k)}$  into  $\binom{r}{k}$ -regions

$$H^{(k)} = \coprod_{\substack{A \subset [r] \\ \sharp A = r-k}} D_A^{(k)},$$

where  $D_A^{(k)} := \{ \boldsymbol{n} \in H^{(k)} \mid n_i > p \text{ for } i \notin A, n_i \leq p \text{ for } i \in A \}$  and  $D_{\emptyset}^{(r)} = H^{(r)}$ . Then

$$H = \prod_{k=1}^{r} \prod_{\substack{A \subset [r] \\ \sharp_A = r-k}} D_A^{(k)}.$$

Let me illustrate this decomposition when r = 3. Figure 1 below is the picture where  $H^{(3)} = D_{\emptyset}^{(3)}$  is the region of the pattern of dots,  $H^{(2)} = D_{\{1\}}^{(2)} \coprod D_{\{2\}}^{(2)} \coprod D_{\{3\}}^{(2)}$  is the region of no pattern, and  $H^{(1)} = D_{\{1,2\}}^{(1)} \coprod D_{\{1,3\}}^{(1)} \coprod D_{\{2,3\}}^{(1)}$  is the region of lines.

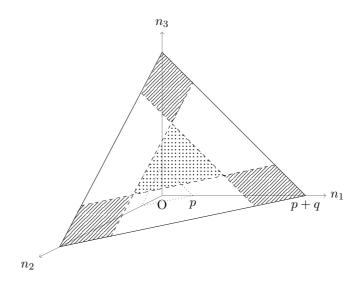


FIGURE 1. A decomposition of H when r = 3

Therefore, the computation of  $\Lambda(p,q)$  can be reduced to the one of each  $\Lambda_{D_A^{(k)}}(p,q)$ :

$$\begin{split} \Lambda(p,q) &= \sum_{k=1}^r \Lambda_{H^{(k)}}(p,q) \\ &= \sum_{k=1}^r \sum_{\substack{A \subset [r] \\ \sharp_A = r-k}} \Lambda_{D^{(k)}_A}(p,q) \end{split}$$

When k = r,  $D_{\emptyset}^{(r)} = H^{(r)} = \Delta_{p,q}^{(r)}$  so that we get the explicit description of  $\Lambda_{H^{(r)}}(p,q)$  by Proposition 2.2. Similarly, when k = r - 1,  $D_{\{r\}}^{(r-1)} = \Delta_{p,q}^{(r-1)}$  so that we get the explicit description of  $\Lambda_{D_{\{r\}}^{(r-1)}}(p,q)$  by Proposition 2.2 and hence the one of  $\Lambda_{H^{(r-1)}}(p,q)$ .

**Proposition 3.1.** We have the following description of  $\Lambda_{H^{(k)}}(p,q)$  when k = r, r - 1. (1) The case where k = r:

$$\Lambda_{H^{(r)}}(p,q) = \binom{q - (r-1)p - 1}{r-1} \lambda_L(p),$$

where  $L := R/[I_1 + \dots + I_r]$  is a cyclic module. (2) The case where k = r - 1:

$$\begin{split} \Lambda_{H^{(r-1)}}(p,q) &= \binom{q - (r-2)p - 1}{r-2} \sum_{j=1}^{r} \lambda_{L_j}(p) \\ &- \sum_{j=1}^{r} \sum_{n=0}^{p} \sum_{i=0}^{n-1} \binom{q - (r-2)p - 2 - i}{r-3} \ell_R(R/\mathfrak{a}_j(n)) \end{split}$$

where  $L_j := R/[I_1 + \dots + \widehat{I_j} + \dots + I_r] \oplus R/[I_1 + \dots + I_r]$  is a direct sum of two cyclic modules and  $\mathfrak{a}_j(n) := (I_1 + \dots + \widehat{I_j} + \dots + I_r)^{p-n}(I_1 + \dots + I_r)^n$  is an ideal depending on j and n.

*Proof.* These follow directly from Proposition 2.2.

We now turn to investigate the remaining functions  $\Lambda_{H^{(k)}}(p,q)$  when k = 1, 2, ..., r-2. These cases seem to be more complicated than the case of k = r, r-1. Suppose that k = 1, 2, ..., r-2 and let A be a subset of [r] with  ${}^{\sharp}A = r-k$ . Then we divide the set  $D_A^{(k)}$  into 2-parts as follows:

$$D_A^{(k)} = E_{A-}^{(k)} \coprod E_{A+}^{(k)},$$

where

$$E_{A-}^{(k)} := \{ \boldsymbol{n} \in D_A^{(k)} \mid \sum_{i \in A} n_i \le p \},\$$
$$E_{A+}^{(k)} := \{ \boldsymbol{n} \in D_A^{(k)} \mid \sum_{i \in A} n_i > p \}.$$

Let

$$H_{-}^{(k)} := \prod_{\substack{A \subset [r] \\ \#_{A=r-k} \\ 8}} E_{A-}^{(k)},$$

$$H_{+}^{(k)} := \prod_{\substack{A \subset [r] \\ \sharp_{A=r-k}}} E_{A+}^{(k)}.$$

Then

$$\Lambda_{H^{(k)}}(p,q) = \Lambda_{H^{(k)}}(p,q) + \Lambda_{H^{(k)}}(p,q).$$

Let me illustrate this decomposition when r = 3. Figure 2 below is the picture where  $H_{-}^{(1)} = E_{\{1,2\}-}^{(1)} \coprod E_{\{1,3\}-}^{(1)} \coprod E_{\{2,3\}-}^{(1)}$  is the region of the pattern of lines, and  $H_{+}^{(1)} = E_{\{1,2\}+}^{(1)} \coprod E_{\{1,3\}+}^{(1)} \coprod E_{\{2,3\}+}^{(1)}$  is the region of the pattern of dots.

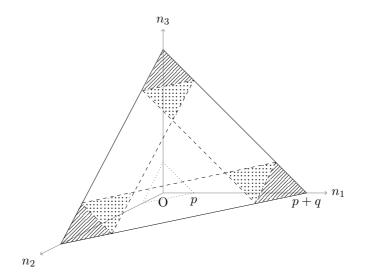


FIGURE 2. A decomposition of  $H^{(1)}$  when r = 3

Here we note that  $E_{\{k+1,\ldots,r\}}^{(k)} = \Delta_{p,q}^{(k)}$  for any  $k = 1, 2, \ldots, r-2$ . Thus, the function  $\Lambda_{H_{-}^{(k)}}(p,q)$  can be expressed explicitly as follows, similar to the one of  $\Lambda_{H^{(r)}}(p,q)$  and  $\Lambda_{H^{(r-1)}}(p,q)$ .

**Proposition 3.2.** For any k = 1, 2, ..., r - 2, we have the following description.

$$\begin{split} \Lambda_{H_{-}^{(k)}}(p,q) &= \binom{q - (k-1)p - 1}{k-1} \sum_{\substack{A \subset [r] \\ \sharp A = r-k}} \lambda_{L_A}(p) \\ &- \sum_{\substack{A \subset [r] \\ \sharp A = r-k}} \sum_{\substack{n_j \ge 0(j \in A) \\ (\sum_{j \in A} n_j) \le p}} \sum_{i=0}^{(\sum_{j \in A} n_j) - 1} \binom{q - (k-1)p - 2 - i}{k-2} \ell_R(R/\mathfrak{a}), \end{split}$$

where  $L_A := \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s \Big] \right) \oplus \bigoplus_{j \in A} \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s + I_j \Big] \right)$  is a direct sum of (r - k + 1)cyclic modules and  $\mathfrak{a} := \left( \sum_{s \in [r] \setminus A} I_s \right)^{p - (\sum_{j \in A} n_j)} \prod_{j \in A} \left( \sum_{s \in [r] \setminus A} I_s + I_j \right)^{n_j}$  is an ideal depending on A and  $n_j$   $(j \in A)$ .

*Proof.* This follows directly from Proposition 2.2.

On the other hand, the function  $\Lambda_{H^{(k)}_+}(p,q)$  seems to be more complicated than the one  $\Lambda_{H^{(k)}_+}(p,q)$ . We do not get the explicit description, but we have the following inequality.

**Proposition 3.3.** For any k = 1, 2, ..., r - 2, there exists a polynomial  $g_k^{\circ}(X) \in \mathbb{Q}[X]$  of degree d + r - k such that

$$\Lambda_{H^{(k)}_+}(p,q) \leq \binom{q-(k-1)p-1}{k-1}g^\circ_k(p).$$

*Proof.* This follows from [5, Lemma 3.5].

Here we consider the following functions  $g_k(p)$  and  $h_k(p,q)$  appeared in Propositions 3.1 and 3.2, which will be used in the next section. For any k = 1, ..., r - 1, we define

(2) 
$$g_k(p) := \sum_{\substack{A \subset [r] \\ \sharp A = r-k}} \lambda_{L_A}(p)$$

(3) 
$$h_k(p,q) := \sum_{\substack{A \subset [r] \\ \#A = r-k}} \sum_{\substack{n_j \ge 0(j \in A) \\ (\sum_{j \in A} n_j) \le p}} \sum_{i=0}^{(\sum_{j \in A} n_j)-1} \binom{q - (k-1)p - 2 - i}{k-2} \ell_R(R/\mathfrak{a})$$

where  $L_A := \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s \Big] \right) \oplus \bigoplus_{j \in A} \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s + I_j \Big] \right)$  is a direct sum of (r - k + 1)cyclic modules and  $\mathfrak{a} := \left( \sum_{s \in [r] \setminus A} I_s \right)^{p - (\sum_{j \in A} n_j)} \prod_{j \in A} \left( \sum_{s \in [r] \setminus A} I_s + I_j \right)^{n_j}$ . When k = r, we set  $g_r(p) = \lambda_{R/[I_1 + \dots + I_r]}(p)$  and  $h_r(p,q) = 0$ . Note that for  $p,q \gg 0$ ,  $g_k(p)$  is a polynomial function of degree d + r - k, and  $h_k(p,q)$  is a non-negative integer valued function.

Then, the above two Propositions 3.2 and 3.3 imply the following.

**Corollary 3.4.** For any k = 1, 2, ..., r - 2, there exists a polynomial  $f_k(X) \in \mathbb{Q}[X]$  of degree d + r - k such that

$$\Lambda_{H^{(k)}}(p,q) \le \binom{q - (k-1)p - 1}{k-1} f_k(p).$$

*Proof.* By Propositions 3.2 and 3.3,

$$\begin{split} \Lambda_{H^{(k)}}(p,q) &= \Lambda_{H^{(k)}_{-}}(p,q) + \Lambda_{H^{(k)}_{+}}(p,q) \\ &\leq \binom{q - (k-1)p - 1}{k-1} g_{k}(p) - h_{k}(p,q) + \binom{q - (k-1)p - 1}{k-1} g_{k}^{\circ}(p) \\ &= \binom{q - (k-1)p - 1}{k-1} (g_{k}(p) + g_{k}^{\circ}(p)) - h_{k}(p,q) \\ &\leq \binom{q - (k-1)p - 1}{k-1} (g_{k}(p) + g_{k}^{\circ}(p)). \end{split}$$

Thus,  $f_k(X) := g_k(X) + g_k^{\circ}(X)$  is our desired polynomial.

## 

## 4. Proof of Theorem 1.1

We give a proof of Theorem 1.1. In this section, we work in the same situation and under the same notation as in the previous sections. For k = 1, 2, ..., r, we consider the following function:

$$F_k(p,q) := \Lambda(p,q) - \sum_{i=1}^k \binom{q - (r-i)p - 1}{r-i} g_{r-i+1}(p),$$

which is a polynomial function for  $p, q \gg 0$  with the total degree is at most d + r - 1. We begin with the following.

**Proposition 4.1.** Suppose that *p* is a large enough fixed integer. Then

$$\lim_{q \to \infty} \frac{1}{q^{r-2}} F_2(p,q) = 0.$$

*Proof.* Fix  $p \gg 0$ . By Proposition 3.1 and Corollary 3.4, we have the following equalities and inequality.

$$F_{2}(p,q) + h_{r-1}(p,q) = \Lambda(p,q) - \Lambda_{H^{(r)}}(p,q) - \Lambda_{H^{(r-1)}}(p,q)$$
  
$$= \sum_{k=1}^{r-2} \Lambda_{H^{(k)}}(p,q)$$
  
$$\leq \sum_{k=1}^{r-2} \binom{q - (k-1)p - 1}{k-1} f_{k}(p).$$

Hence, we have that

$$-h_{r-1}(p,q) \le F_2(p,q) \le \sum_{k=1}^{r-2} \binom{q-(k-1)p-1}{k-1} f_k(p).$$

Therefore, it is enough to show that

(4) 
$$\lim_{q \to \infty} \frac{1}{q^{r-2}} \sum_{k=1}^{r-2} \binom{q - (k-1)p - 1}{k-1} f_k(p) = 0, \text{ and}$$

(5) 
$$\lim_{q \to \infty} \frac{1}{q^{r-2}} h_{r-1}(p,q) = 0.$$

The first assertion (4) is clear because the degree of a polynomial function

$$\sum_{k=1}^{r-2} \binom{q - (k-1)p - 1}{k-1} f_k(p)$$

with respect to q is at most (r-2) - 1 = r - 3. We show the second assertion (5). Then one can see that

$$h_{r-1}(p,q) = \sum_{j=1}^{r} \sum_{n=0}^{p} \sum_{i=0}^{n-1} \binom{q - (r-2)p - 2 - i}{r-3} \ell_R(R/\mathfrak{a}_j(n))$$

$$\leq \sum_{j=1}^{r} \sum_{n=0}^{p} n \binom{q - (r-2)p - 2}{r-3} \ell_R(R/\mathfrak{a}_j(n))$$

$$\leq \sum_{j=1}^{r} \sum_{n=0}^{p} p \binom{q - (r-2)p - 2}{r-3} \ell_R(R/\mathfrak{a}_j(n))$$

$$= p \binom{q - (r-2)p - 2}{r-3} \sum_{j=1}^{r} \sum_{n=0}^{p} \ell_R(R/\mathfrak{a}_j(n)),$$

where  $\mathfrak{a}_j(n) := (I_1 + \dots + \widehat{I_j} + \dots + I_r)^{p-n} (I_1 + \dots + I_r)^n$ . Note that

$$\sum_{j=1}^r \sum_{n=0}^p \ell_R(R/\mathfrak{a}_j(n)) = \sum_{j=1}^r \lambda_{L_j}(p)$$

is a sum of the ordinary Buchsbaum-Rim functions of two cyclic modules, where

$$L_j = R/[I_1 + \dots + \widehat{I}_j + \dots + I_r] \oplus R/[I_1 + \dots + I_r].$$

Hence, noting that  $h_{r-1}(p,q) \ge 0$ , we have that

$$0 \le h_{r-1}(p,q) \le \binom{q - (r-2)p - 2}{r-3}u(p)$$

for some polynomial function u(p) of degree (d+1) + 1 = d + 2. Therefore,

$$\lim_{q \to \infty} \frac{1}{q^{r-2}} \binom{q - (r-2)p - 2}{r-3} u(p) = 0$$

so that  $\lim_{q \to \infty} \frac{1}{q^{r-2}} h_{r-1}(p,q) = 0.$ 

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The degree of  $\Lambda(p,q)$  with respect to q is at most r-1 so that one can write

$$\Lambda(p,q) = \sum_{i=0}^{r-1} a_i q^i$$

where each  $a_i$  is a polynomial function of p with degree at most d + r - 1 - i. Similarly, we can write

$$\binom{q - (r-1)p - 1}{r-1}g_r(p) = \sum_{j=0}^{r-1} b_j q^j$$
$$\binom{q - (r-2)p - 1}{r-2}g_{r-1}(p) = \sum_{k=0}^{r-2} c_k q^k$$

where each  $b_j$  (resp.  $c_k$ ) is a polynomial function of p with degree at most d + r - 1 - j (resp. d + r - 1 - k). Then

$$F_2(p,q) = (a_{r-1} - b_{r-1})q^{r-1} + (a_{r-2} - b_{r-2} - c_{r-2})q^{r-2} + (\text{lower terms in } q).$$

By Proposition 4.1, we have the equalities as polynomials of p,

(6) 
$$a_{r-1} = b_{r-1}$$
, and

(7) 
$$a_{r-2} = b_{r-2} + c_{r-2}$$

Note that the first equality (6) implies a formula  $e^{r-1}(C) = e(R/[I_1 + \dots + I_r])$  which is our previous result in [5]. We then look at the second equality (7). Since the total degree  $\Lambda(p,q)$  is d + r - 1, and the coefficient of  $p^{d+1}q^{r-2}$  is non-zero, which is  $\frac{e^{r-2}(C)}{(d+1)!(r-2)!}$ , the polynomial  $a_{r-2}$  is of the form:

$$a_{r-2} = \frac{e^{r-2}(C)}{(d+1)!(r-2)!}p^{d+1} + (\text{lower terms in } p).$$

Since  $g_r(p) = \lambda_{R/[I_1 + \dots + I_r]}(p)$  is the Hilbert-Samuel function of  $I_1 + \dots + I_r$ ,

$$\begin{pmatrix} q - (r-1)p - 1 \\ r - 1 \end{pmatrix} g_r(p)$$

$$= \begin{pmatrix} q - (r-1)p - 1 \\ r - 1 \end{pmatrix} \left( \frac{e(R/[I_1 + \dots + I_r])}{d!} p^d + (\text{lower terms in } p) \right)$$

$$= \frac{(q - (r-1)p)^{r-1}}{(r-1)!} \cdot \frac{e(R/[I_1 + \dots + I_r])}{d!} p^d + (\text{lower terms})$$

$$= \frac{e(R/[I_1 + \dots + I_r])}{d!(r-1)!} p^d q^{r-1} - \frac{(r-1)e(R/[I_1 + \dots + I_r])}{d!(r-2)!} p^{d+1} q^{r-2} + (\text{lower terms in } q)$$

so that

$$b_{r-2} = -\frac{(r-1)e(R/[I_1 + \dots + I_r])}{d!(r-2)!}p^{d+1}.$$

Similarly, since  $g_{r-1}(p) = \sum_{j=1}^{r} \lambda_{L_j}(p)$ , and its normalized leading coefficient is

$$E_{r-1} := E_{r-1}(I_1, \dots, I_r) := \sum_{j=1}^r e(L_j),$$

where

$$L_j = R/[I_1 + \dots + \widehat{I_j} + \dots + I_r] \oplus R/[I_1 + \dots + I_r],$$

we have that

$$\binom{q - (r - 2)p - 1}{r - 2} g_{r-1}(p) = \binom{q - (r - 2)p - 1}{r - 2} \binom{E_{r-1}}{(d + 1)!} p^{d+1} + (\text{lower terms in } p)$$

$$= \frac{(q - (r - 2)p)^{r-2}}{(r - 2)!} \cdot \frac{E_{r-1}}{(d + 1)!} p^{d+1} + (\text{lower terms})$$

$$= \frac{E_{r-1}}{(d + 1)!(r - 2)!} p^{d+1} q^{r-2} + (\text{lower terms in } q).$$

Therefore, we get that

$$c_{r-2} = \frac{E_{r-1}}{(d+1)!(r-2)!}p^{d+1}.$$

By comparing the coefficient of  $p^{d+1}$  in the equation (7), we have the equality

$$\frac{e^{r-2}(C)}{(d+1)!(r-2)!} = -\frac{(r-1)e(R/[I_1+\dots+I_r])}{d!(r-2)!} + \frac{E_{r-1}(I_1,\dots,I_r)}{(d+1)!(r-2)!}$$

By multiplying (d+1)!(r-2)! to the above equation, we get the desired formula.

As stated in the proof, the proof of Theorem 1.1 contains our previous result in [5]. Moreover, the obtained formula for  $e^{r-2}(C)$  can be viewed as a natural generalization of the Kirby-Rees formula given in [8].

**Corollary 4.2.** Let  $I_1, \ldots, I_r$  be m-primary ideals in R and assume that  $I_1, \ldots, I_{r-1} \subset I_r$ , that is, the ideal  $I_r$  is the largest ideal. Then we have a formula

$$e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/[I_1+\cdots+I_{r-1}]\oplus R/I_r)$$

In particular, if  $I_1 \subset I_2 \subset \cdots \subset I_r$ , then

$$e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_{r-1}\oplus R/I_r).$$

*Proof.* Suppose that  $I_1, \ldots, I_{r-1} \subset I_r$ . Then by Theorem 1.1,

$$\begin{split} e^{r-2}(C) &= \sum_{j=1}^{'} e(R/[I_1 + \dots + \widehat{I_j} + \dots + I_r] \oplus R/[I_1 + \dots + I_r]) \\ &\quad -(d+1)(r-1)e(R/[I_1 + \dots + I_r]) \\ &= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r) + (r-1)e(R/I_r \oplus R/I_r) \\ &\quad -(d+1)(r-1)e(R/I_r) \\ &= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r) + (r-1)(d+1)e(R/I_r) \\ &\quad -(d+1)(r-1)e(R/I_r) \\ &= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r). \end{split}$$

Here the third equality follows from the elementary formula (1).

Before closing this article, we would like to give a few observations on the remaining multiplicities. We first recall the polynomial function  $F_k(p,q)$  defined at the beginning of this section:

$$F_k(p,q) := \Lambda(p,q) - \sum_{i=1}^k \binom{q - (r-i)p - 1}{r-i} g_{r-i+1}(p).$$

The key of our proof of Theorem 1.1 is the fact that  $\deg_q F_2(p,q) \leq r-3$  (Proposition 4.1). It would be interesting to know whether this kind of property holds true or not for which k.

**Question 4.3.** Let p be a fixed large enough integer. Then for which k = 1, 2, ..., r - 1, does the following hold true?

$$\lim_{q \to \infty} \frac{1}{q^{r-k}} F_k(p,q) = 0.$$

In other word, is the degree of  $F_k(p,q)$  with respect to q at most r-k-1?

This holds true when k = 2 (and also k = 1) by Proposition 4.1. We are interested in the remaining cases. Suppose that  $k \ge 3$ . The affirmative answer to Question 4.3 will tell us that for any  $1 \le j \le k$ , the (r - j)th associated Buchsbaum-Rim multiplicity  $e^{r-j}(C)$ is determined by the polynomial

(8) 
$$\sum_{i=1}^{k} \binom{q - (r - i)p - 1}{r - i} g_{r-i+1}(p).$$

Then we will be able to describe the multiplicity  $e^{r-j}(C)$  as a sum of the ordinary Buchsbaum-Rim multiplicities of a direct sum of at most (r-j) cyclic modules in the same manner. Here we would like to record the expected formula. Note that the polynomial  $g_{r-i+1}(p)$  defined in (2) is of the form

$$g_{r-i+1}(p) = \frac{1}{(d+i-1)!} \sum_{\substack{A \subset [r] \\ \#A=i-1}} e(L_A) \cdot p^{d+i-1} + (\text{lower terms})$$
  
where  $L_A := \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s \Big] \Big) \oplus \bigoplus_{j \in A} \left( R \Big/ \Big[ \sum_{s \in [r] \setminus A} I_s + I_j \Big] \right)$ . We put  
 $E_{r-i+1} := E_{r-i+1}(I_1, \dots, I_r) := \sum_{\substack{A \subset [r] \\ \#A=i-1}} e(L_A).$ 

Then for any  $1 \leq j \leq k$ , the coefficient of  $p^{d+j-1}q^{r-j}$  in the polynomial (8) is

$$\sum_{i=1}^{j} \frac{E_{r-i+1}}{(d+i-1)!(r-i)!} \binom{r-i}{r-j} \left(-(r-i)\right)^{j-i}.$$

If Question 4.3 is affirmative, then the above coefficient coincides with

$$\frac{e^{r-j}(C)}{(d+j-1)!(r-j)!}$$

so that we can get the formula for  $e^{r-j}(C)$ . Therefore, we can ask the following.

Question 4.4. Under the same notation as above, does the formula

$$e^{r-j}(R/I_1 \oplus \dots \oplus R/I_r) = \sum_{i=1}^{j} {d+j-1 \choose j-i} (-(r-i))^{j-i} E_{r-i+1}(I_1, \dots, I_r)$$

hold true?

This is affirmative when j = 1 ([4, Theorem 1.3]) and j = 2 (Theorem 1.1). Note that the affirmative answer to Question 4.3 for some k implies the affirmative one to Question 4.4 for any  $1 \le j \le k$ .

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