# A FORMULA FOR THE ASSOCIATED BUCHSBAUM-RIM MULTIPLICITIES OF A DIRECT SUM OF CYCLIC MODULES II

FUTOSHI HAYASAKA

Abstract. The associated Buchsbaum-Rim multiplicities of a module are a descending sequence of non-negative integers. These invariants of a module are a generalization of the classical Hilbert-Samuel multiplicity of an ideal. In this article, we compute the associated Buchsbaum-Rim multiplicity of a direct sum of cyclic modules and give a formula for the second to last positive Buchsbaum-Rim multiplicity in terms of the ordinary Buchsbaum-Rim and Hilbert-Samuel multiplicities. This is a natural generalization of a formula given by Kirby and Rees.

#### 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  of dimension  $d > 0$ . The associated Buchsbaum-Rim multiplicities of an  $R$ -module  $C$  of finite length, which is denoted by  $\{e^{j}(C)\}_{0\leq j\leq d+r-1}$ , are a sequence of integers. These are invariants of C introduced by Kleiman-Thorup [\[10\]](#page-15-0) and Kirby-Rees [\[8\]](#page-15-1) independently. For an R-module C of finite length with a minimal free presentation  $R^n \stackrel{\varphi}{\to} R^r \to C \to 0$ , the multiplicities are defined by the so-called Buchsbaum-Rim function of two variables

$$
\Lambda(p,q) := \ell_R(S_{p+q}/M^p S_q),
$$

where  $S_p$  (resp.  $M^p$ ) is a homogeneous component of degree p of  $S = \text{Sym}_R(F)$  (resp.  $R[M] = \text{Im} \text{Sym}_R(\varphi)$ . The function  $\Lambda(p,q)$  is eventually a polynomial of total degree  $d + r - 1$ , and then the associated Buchsbaum-Rim multiplicities are defined as for  $j =$  $0, 1, \ldots, d + r - 1,$ 

 $e^j(C) :=$  (The coefficient of  $p^{d+r-1-j}q^j$  in the polynomial)  $\times (d+r-1-j)!j!$ .

These are a descending sequence of non-negative integers with  $e^{r-1}(C)$  is positive, and  $e^{j}(C) = 0$  for  $j \geq r$ . This was proved by Kleiman-Thorup [\[10\]](#page-15-0) and Kirby-Rees [\[8\]](#page-15-1) independently. Moreover, they proved that the first multiplicity  $e^0(C)$  coincides with the ordinary Buchsbaum-Rim multiplicity  $e(C)$  of C introduced in [\[2\]](#page-15-2), which is the normalized leading coefficient of the polynomial function  $\lambda(p) = \Lambda(p, 0) = \ell_R(S_p/M^p)$  of degree  $d + r - 1$  for  $p \gg 0$ . Namely,

$$
e(C) = e^{0}(C) \ge e^{1}(C) \ge \cdots \ge e^{r-1}(C) > e^{r}(C) = \cdots = e^{d+r-1}(C) = 0.
$$

Note that the ordinary Buchsbaum-Rim multiplicity  $e(R/I)$  of a cyclic module defined by an m-primary ideal I coincides with the classical Hilbert-Samuel multiplicity  $e(I)$  of I. Thus, the ordinary Buchsbaum-Rim multiplicity  $e^0(C) = e(C)$  and the associated one  $e^{j}(C)$  are a generalization of the classical Hilbert-Samuel multiplicity. However, as

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compared to the classical Hilbert-Samuel multiplicity, the Buchsbaum-Rim multiplicities are not well-understood.

There are some cases where the computation of the ordinary Buchsbaum-Rim multiplicity is possible (see  $[1, 3, 6, 7, 8]$  $[1, 3, 6, 7, 8]$  $[1, 3, 6, 7, 8]$  $[1, 3, 6, 7, 8]$  $[1, 3, 6, 7, 8]$  for instance). In particular, in the case where C is a direct sum of cyclic modules, there is an interesting relation between the ordinary Buchsbaum-Rim multiplicity and the mixed multiplicities of ideals. Let  $I_1, \ldots, I_r$  be m-primary ideals in R. Then Kirby and Rees proved that

$$
e(R/I_1 \oplus \cdots \oplus R/I_r) = \sum_{\substack{i_1,\ldots,i_r \geq 0 \\ i_1+\cdots+i_r=d}} e_{i_1\cdots i_r}(I_1,\ldots,I_r),
$$

where  $e_{i_1\cdots i_r}(I_1,\ldots,I_r)$  is the mixed multiplicity of  $I_1,\ldots,I_r$  of type  $(i_1,\ldots,i_r)$  (see [\[7,](#page-15-6) [8\]](#page-15-1) and also [\[1\]](#page-15-3)). Then we are interested in the other associated Buchsbaum-Rim multiplicities in this case.

The starting point of this research is the following interesting formula which was also discovered by Kirby-Rees [\[7,](#page-15-6) [8\]](#page-15-1). Suppose that  $I_1 \subset I_2 \subset \cdots \subset I_r$ . Then they proved that for any  $j = 1, \ldots, r - 1$ , the j<sup>th</sup> Buchsbaum-Rim multiplicity can be expressed as the ordinary Buchsbaum-Rim multiplicity of a direct sum of  $(r - j)$  cyclic modules defined by the last  $(r - j)$  ideals:

$$
e^j(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_{j+1}\oplus\cdots\oplus R/I_r).
$$

In particular, the last positive one  $e^{r-1}$  can be expressed as the classical Hilbert-Samuel multiplicity  $e(I_r)$  of the largest ideal:

$$
e^{r-1}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_r).
$$

Then it is natural to ask the formula for  $e^{j}(R/I_1 \oplus \cdots \oplus R/I_r)$  without the assumption  $I_1 \subset I_2 \subset \cdots \subset I_r$ . However, as compared to the special case considered in [\[7,](#page-15-6) [8\]](#page-15-1), it seems that the problem is more complicated, and we need a different approach to obtain the formula in general. Recently, we tried to compute the function  $\Lambda(p,q)$  directly using some ideas and obtained the formula for  $e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r)$  without the assumption  $I_1$  ⊂ · · · ⊂  $I_r$ . Indeed, we proved in our previous work [\[5,](#page-15-7) Theorem 1.3] that for any mprimary ideals  $I_1, \ldots, I_r$ , the last positive Buchsbaum-Rim multiplicity can be expressed as the classical Hilbert-Samuel multiplicity  $e(I_1 + \cdots + I_r)$  of the sum of all ideals:

$$
e^{r-1}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/[I_1+\cdots+I_r]).
$$

The present purpose is to improve the method of computation given in [\[5\]](#page-15-7) towards a formula for not only the last positive Buchsbaum-Rim multiplicity  $e^{r-1}(R/I_1 \oplus \cdots \oplus R/I_r)$ but also the next one  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$  in terms of the ordinary Buchsbaum-Rim and Hilbert-Samuel multiplicities. Here is the main result.

<span id="page-1-0"></span>**Theorem 1.1.** Let  $I_1, \ldots, I_r$  be arbitrary m-primary ideals in R. Then we have a formula

$$
e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r) = E_{r-1}(I_1, \ldots, I_r) - (d+1)(r-1)e(R/[I_1 + \cdots + I_r]),
$$

where  $E_{r-1}(I_1,\ldots,I_r)$  is a sum of the ordinary Buchsbaum-Rim multiplicities of two cyclic modules defined by the ideals  $I_1 + \cdots + \widehat{I}_j + \cdots + I_r$  and  $I_1 + \cdots + I_r$ :

$$
E_{r-1}(I_1,\ldots,I_r) := \sum_{j=1}^r e(R/[I_1 + \cdots + \widehat{I}_j + \cdots + I_r] \oplus R/[I_1 + \cdots + I_r]).
$$

Let me illustrate the formula when  $r = 3$ . Let  $C = R/I_1 \oplus R/I_2 \oplus R/I_3$ . It is known that  $e^0(C)$  coincides with the ordinary Buchsbaum-Rim multiplicity by [\[8,](#page-15-1) [10\]](#page-15-0), and  $e^2(C)$ can be expressed as the ordinary Hilbert-Samuel multiplicity of the sum of all ideals by [\[5\]](#page-15-7). Theorem [1.1](#page-1-0) tells us that there is a similar expression for the remaining multiplicity  $e^{1}(C)$ . Namely, if we put  $I_{123} := I_1 + I_2 + I_3$  and  $I_{ij} := I_i + I_j$  for  $1 \leq i < j \leq 3$ , then we can write all the multiplicities in terms of ordinary Buchsbaum-Rim multiplicities and hence mixed multiplicities.

$$
e^{0}(C) = e(R/I_{1} \oplus R/I_{2} \oplus R/I_{3})
$$
  
\n
$$
e^{1}(C) = e(R/I_{23} \oplus R/I_{123}) + e(R/I_{13} \oplus R/I_{123}) + e(R/I_{12} \oplus R/I_{123}) - 2(d+1)e(R/I_{123})
$$
  
\n
$$
e^{2}(C) = e(R/I_{123}).
$$

Our formula can be viewed as a natural generalization of the above mentioned Kirby-Rees formula for  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$  in a special case where  $I_1 \subset I_2 \subset \cdots \subset I_r$ . Indeed, as an immediate consequence of Theorem [1.1,](#page-1-0) we get the following.

Corollary 4.2. Let  $I_1, \ldots, I_r$  be m-primary ideals in R and assume that  $I_1, \ldots, I_{r-1} \subset I_r$ , that is, the ideal  $I_r$  is the largest ideal. Then we have a formula

$$
e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/[I_1+\cdots+I_{r-1}]\oplus R/I_r).
$$

In particular, if  $I_1 \subset I_2 \subset \cdots \subset I_r$ , then

$$
e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_{r-1}\oplus R/I_r).
$$

The contents of the article are organized as follows. In the next section 2, we will recall some necessary notation and results from our previous work [\[5\]](#page-15-7). In section 3, we will compute the Buchsbaum-Rim function of two variables by improving the method in [\[5\]](#page-15-7). In the last section 4, we will give a proof of Theorem [1.1](#page-1-0) and its consequence Corollary [4.2.](#page-13-0) We will also discuss the remaining multiplicities  $e^{j}(C)$  for  $j = 1, ..., r - 3$ .

Throughout this article, we will work in the same manner in our previous work [\[5\]](#page-15-7). Let  $(R, \mathfrak{m})$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  of dimension  $d > 0$ . Let  $r > 0$  be a fixed positive integer and let  $[r] = \{1, \ldots, r\}$ . For a finite set  $A$ ,  $^{\sharp}A$ denotes the number of elements of A. Vectors are always written in bold-faced letters, e.g.,  $\mathbf{i} = (i_1, \ldots, i_r)$ . We work in the usual multi-index notation. Let  $I_1, \ldots, I_r$  be ideals in R and let  $t_1, \ldots, t_r$  be indeterminates. Then for a vector  $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}_{\geq 0}^r$ , we denote  $I^i = I_1^{i_1} \cdots I_r^{i_r}, t^i = t_1^{i_1} \cdots t_r^{i_r}$  and  $|i| = i_1 + \cdots + i_r$ . For vectors  $a, b \in \mathbb{Z}^r$ ,  $\boldsymbol{a} \geq \boldsymbol{b} \stackrel{\text{def}}{\Leftrightarrow} a_i \geq b_i$  for all  $i = 1, \ldots, r$ . Let  $\boldsymbol{0} = (0, \ldots, 0)$  be the zero vector in  $\mathbb{Z}_{\geq 0}^r$ . By convention, empty sum is defined to be zero.

#### 2. Preliminaries

In this section, we give a few elementary facts to compute the associated Buchsbaum-Rim multiplicities. See also [\[5,](#page-15-7) section 2] for the related facts and the details.

In what follows, let  $I_1, \ldots, I_r$  be m-primary ideals in R and let  $C = R/I_1 \oplus \cdots \oplus R/I_r$ . Let  $S = R[t_1, \ldots, t_r]$  be a polynomial ring over R and let  $R[M] = R[I_1t_1, \ldots, I_rt_r]$  be the multi-Rees algebra of  $I_1, \ldots, I_r$ . Let  $S_p$  (resp.  $M^p$ ) be a homogeneous component of degree p of S (resp.  $R[M]$ ). Then it is easy to see that the function  $\Lambda(p,q)$  can be expressed as

$$
\Lambda(p,q) = \sum_{\boldsymbol{n} \in H_{p,q}} \ell_R(R/J_{p,q}(\boldsymbol{n}))
$$

where  $H_{p,q} := \{ \boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r \mid |\boldsymbol{n}| = p + q \}$  and  $J_{p,q}(\boldsymbol{n}) := \sum_{\boldsymbol{n} \in \mathbb{Z}_{\geq 0}^r} \prod_{i=1}^r \binom{p_i}{\boldsymbol{n}}$  $|i|=p$  $0\leq i \leq n$  $I^i$  for  $n \in H_{p,q}$ . For a subset

 $\Delta \subset H_{p,q}$ , we set

$$
\Lambda_{\Delta}(p,q):=\sum_{\boldsymbol{n}\in\Delta}\ell_R(R/J_{p,q}(\boldsymbol{n})).
$$

As in [\[5\]](#page-15-7), we consider the following special subsets of  $H_{p,q}$ , which will play a basic role in our computation of  $\Lambda(p,q)$ . For  $p,q>0$  and  $k=1,\ldots,r$ , let

$$
\Delta_{p,q}^{(k)} := \{ \boldsymbol{n} \in H_{p,q} \mid n_1,\ldots,n_k > p, n_{k+1} + \cdots + n_r \leq p \}.
$$

Then the function  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$  can be described explicitly as follows.

<span id="page-3-0"></span>**Proposition 2.1.** ([\[5,](#page-15-7) Proposition 2.3]) Let  $p, q > 0$  with  $q \geq (p+1)r$  and let  $k = 1, ..., r$ . Then

$$
\Lambda_{\Delta_{p,q}^{(k)}}(p,q)=\sum_{\substack{n_{k+1},\ldots,n_r\ge 0\\ n_{k+1}+\cdots+n_r\le p}} \binom{q-(k-1)p-1-(n_{k+1}+\cdots+n_r)}{k-1} \ell_R(R/\mathfrak{a}),
$$

where **a** is an ideal depending on  $n_{k+1}, \ldots, n_r$ :

$$
\mathfrak{a} := (I_1 + \cdots + I_k)^{p - (n_{k+1} + \cdots + n_r)} \prod_{j=k+1}^r (I_1 + \cdots + I_k + I_j)^{n_j}.
$$

Here we make a slightly different description of the above mentioned basic functions  $\Lambda_{\Delta^{(k)}_{p,q}}(p,q)$ . To state it, we first recall some elementary facts about the ordinary Buchsbaum-Rim functions and multiplicities of a direct sum of cyclic modules. The ordinary Buchsbaum-Rim function  $\lambda(p)$  of  $C = R/I_1 \oplus \cdots \oplus R/I_r$  (we will often denote it  $\lambda_C(p)$  to emphasize the defining module  $C$ ) can be expressed as follows:

$$
\lambda(p) = \ell_R(S_p/M^p)
$$
  
= 
$$
\sum_{\substack{i \geq 0 \\ |i|=p}} \ell_R(R/I^i)
$$
  
= 
$$
\sum_{\substack{i \geq 0 \\ |i|=p}} \ell_R(R/I_1^{i_1} \cdots I_r^{i_r}).
$$

In particular, if we consider the case where  $I_1 = \cdots = I_r =: I$ , then

$$
\lambda(p) = \binom{p+r-1}{r-1} \ell_R(R/I^p).
$$

The function  $\ell_R(R/I^p)$  is just the Hilbert-Samuel function of I so that it is a polynomial for all large enough  $p$ , and one can write

$$
\ell_R(R/I^p) = \frac{e(R/I)}{d!}p^d + \text{(lower terms)},
$$

where  $e(R/I)$  is the usual Hilbert-Samuel multiplicity of I. Therefore, the ordinary Buchsbaum-Rim function can be expressed as

$$
\lambda(p) = \frac{e(R/I)}{d!(r-1)!}p^{d+r-1} + \text{(lower terms)}.
$$

This implies the following elementary formula for the ordinary Buchsbaum-Rim multiplicity:

<span id="page-4-1"></span>(1) 
$$
e(C) = e(\underbrace{R/I \oplus \cdots \oplus R/I}_{r}) = {d+r-1 \choose r-1} e(R/I).
$$

Now, let me give another description of  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$ .

<span id="page-4-0"></span>**Proposition 2.2.** Let  $p, q > 0$  with  $q \geq (p+1)r$  and let  $k = 1, \ldots, r$ . Then

$$
\begin{split} \Lambda_{\Delta_{p,q}^{(k)}}(p,q) &= \binom{q-(k-1)p-1}{k-1} \lambda_{L_k}(p) \\ &- \sum_{\substack{n_{k+1},\dots,n_r\geq 0 \\ n_{k+1}+\dots+n_r\leq p}} \sum_{i=0}^{n_{k+1}+\dots+n_r-1} \binom{q-(k-1)p-2-i}{k-2} \ell_R(R/\mathfrak{a}), \end{split}
$$

where  $L_k := R/[I_1 + \cdots + I_k] \oplus \bigoplus^r$  $j = k+1$  $R/[I_1 + \cdots + I_k + I_j]$  is a direct sum of  $(r - k + 1)$ cyclic modules and  $\mathfrak{a} := (I_1 + \cdots + I_k)^{p-(n_{k+1}+\cdots+n_r)}$  $j=k+1$  $(I_1 + \cdots + I_k + I_j)^{n_j}$  is an ideal depending on  $n_{k+1}, \ldots, n_r$ .

*Proof.* The case where  $k = 1$  follows from Proposition [2.1.](#page-3-0) Indeed,

$$
\Lambda_{\Delta_{p,q}^{(1)}}(p,q) = \sum_{\substack{n_2,\ldots,n_r \geq 0 \\ n_2+\cdots+n_r \leq p}} \ell_R\left(R/I_1^{p-(n_2+\cdots+n_r)}\prod_{j=2}^r (I_1+I_j)^{n_j}\right)
$$
  

$$
= \sum_{\substack{i \geq 0 \\ |i|=p}} \ell_R\left(R/I_1^{i_1}(I_1+I_2)^{i_2}\cdots (I_1+I_r)^{i_r}\right)
$$
  

$$
= \lambda_{L_1}(p).
$$

Suppose that  $k \geq 2$ . By using an elementary combinatorial formula  $\binom{m-\ell}{n} = \binom{m}{n}$  $\sum_{i=0}^{\ell-1} \binom{m-\ell+i}{n-1}$ , one can see that

$$
\begin{aligned}\n\left( \begin{matrix} q - (k-1)p - 1 - (n_{k+1} + \dots + n_r) \\ k-1 \end{matrix} \right) \\
= \left( \begin{matrix} q - (k-1)p - 1 \\ k-1 \end{matrix} \right) - \sum_{j=0}^{n_{k+1} + \dots + n_r - 1} \left( \begin{matrix} q - (k-1)p - 1 - (n_{k+1} + \dots + n_r) + j \\ k-2 \end{matrix} \right) \\
= \left( \begin{matrix} q - (k-1)p - 1 \\ k-1 \end{matrix} \right) - \sum_{j=0}^{n_{k+1} + \dots + n_r - 1} \left( \begin{matrix} q - (k-1)p - 2 + j - (n_{k+1} + \dots + n_r - 1) \\ k-2 \end{matrix} \right) \\
= \left( \begin{matrix} q - (k-1)p - 1 \\ k-1 \end{matrix} \right) - \sum_{i=0}^{n_{k+1} + \dots + n_r - 1} \left( \begin{matrix} q - (k-1)p - 2 - i \\ k-2 \end{matrix} \right).\n\end{aligned}
$$

By Proposition [2.1,](#page-3-0) we can write the function  $\Lambda_{\Delta_{p,q}^{(k)}}(p,q)$  as follows:

$$
\begin{split}\n&\Delta_{\Delta_{p,q}^{(k)}}(p,q) \\
&= \sum_{\substack{n_{k+1},\ldots,n_r\geq 0\\n_{k+1}+\ldots+n_r\leq p}} \left( q - (k-1)p - 1 - (n_{k+1} + \ldots + n_r) \right) \ell_R(R/\mathfrak{a}) \\
&= \sum_{\substack{n_{k+1},\ldots,n_r\geq 0\\n_{k+1}+\ldots+n_r\leq p}} \left[ \left( q - (k-1)p - 1 \right) - \sum_{i=0}^{n_{k+1}+\ldots+n_r-1} \left( q - (k-1)p - 2 - i \right) \right] \ell_R(R/\mathfrak{a}) \\
&= \left( q - (k-1)p - 1 \right) \sum_{\substack{n_{k+1},\ldots,n_r\geq 0\\n_{k+1}+\ldots+n_r\leq p}} \ell_R(R/\mathfrak{a}) \\
&\quad - \sum_{\substack{n_{k+1},\ldots,n_r\geq 0\\n_{k+1}+\ldots+n_r\leq p}} \sum_{i=0}^{n_{k+1}+\ldots+n_r-1} \left( q - (k-1)p - 2 - i \right) \ell_R(R/\mathfrak{a}) \\
&\quad - \sum_{\substack{n_{k+1},\ldots,n_r\geq 0\\n_{k+1}+\ldots+n_r\leq p}} \sum_{i=0}^{n_{k+1}+\ldots+n_r-1} \left( q - (k-1)p - 2 - i \right) \ell_R(R/\mathfrak{a}),\n\end{split}
$$

where  $L_k := R/[I_1 + \cdots + I_k] \oplus \left(\bigoplus_{r=1}^r A_r\right)$  $_{j=k+1}$  $R/[I_1 + \cdots + I_k + I_j]$  is a direct sum of  $(r - k + 1)$ cyclic modules and  $\mathfrak{a} := (I_1 + \cdots + I_k)^{p-(n_{k+1}+\cdots+n_r)}$  $_{j=k+1}$  $(I_1 + \cdots + I_k + I_j)^{n_j}$  is an ideal depending on  $n_{k+1}, \ldots, n_r$ .

## 3. A computation of the Buchsbaum-Rim functions

In this section, we compute the function  $\Lambda(p,q)$  by improving the method in [\[5\]](#page-15-7) towards a formula for  $e^{r-2}(R/I_1 \oplus \cdots \oplus R/I_r)$ . The notation we will use here is under the same manner in [\[5\]](#page-15-7). See also [\[5,](#page-15-7) Section 3] for more detailed observations.

In order to compute the multiplicity defined by the asymptotic function  $\Lambda(p,q)$ , we may assume that  $q \ge (p+1)r \gg 0$ . In what follows, let p, q be fixed integers satisfying  $q \geq (p+1)r \gg 0$ . We put  $H := H_{p,q}$  for short. Then the set H can be divided by r-regions

$$
H = \coprod_{k=1}^{r} H^{(k)},
$$

where  $H^{(k)} := \{ n \in H \mid \frac{\sharp}{i} \mid n_i > p \} = k \}.$  Moreover, we divide each  $H^{(k)}$  into  $\binom{r}{k}$  $\binom{r}{k}$ -regions

$$
H^{(k)}=\coprod_{\stackrel{A\subset [r]}{\sharp}_A= r-k}D_A^{(k)},
$$

where  $D_A^{(k)}$  $A^{(k)}_{A} := \{ n \in H^{(k)} \mid n_i > p \text{ for } i \notin A, n_i \leq p \text{ for } i \in A \}$  and  $D_{\emptyset}^{(r)} = H^{(r)}$ . Then

$$
H = \coprod_{k=1}^r \coprod_{\substack{A \subset [r] \\ \sharp A = r - k}} D_A^{(k)}.
$$

Let me illustrate this decomposition when  $r = 3$ . Figure [1](#page-6-0) below is the picture where  $H^{(3)} = D_{\emptyset}^{(3)}$  $\psi_{\emptyset}^{(3)}$  is the region of the pattern of dots,  $H^{(2)} = D_{\{1\}}^{(2)}$  $\{ \begin{smallmatrix} (2) \ 1 \end{smallmatrix} \coprod D_{\{2\}}^{(2)}$  $\{2\}\amalg D_{\{3\}}^{(2)}$  $\binom{2}{3}$  is the region of no pattern, and  $H^{(1)} = D_{f_1}^{(1)}$  $\{_{1,2\}}^{\left( 1\right) }\coprod D_{\left\{ 1,\right\} }^{\left( 1\right) }$  $\{_{1,3\}}\coprod D_{\{2,} }^{(1)}$  $\binom{1}{2,3}$  is the region of lines.



<span id="page-6-0"></span>FIGURE 1. A decomposition of H when  $r = 3$ 

Therefore, the computation of  $\Lambda(p,q)$  can be reduced to the one of each  $\Lambda_{D_A^{(k)}}(p,q)$ :

$$
\Lambda(p,q) = \sum_{k=1}^{r} \Lambda_{H^{(k)}}(p,q) \n= \sum_{k=1}^{r} \sum_{\substack{A \subset [r] \\ \sharp A = r - k}} \Lambda_{D_A^{(k)}}(p,q).
$$

When  $k = r$ ,  $D_{\emptyset}^{(r)} = H^{(r)} = \Delta_{p,q}^{(r)}$  so that we get the explicit description of  $\Lambda_{H^{(r)}}(p,q)$  by Proposition [2.2.](#page-4-0) Similarly, when  $k = r - 1$ ,  $D_{\{r\}}^{(r-1)} = \Delta_{p,q}^{(r-1)}$  so that we get the explicit description of  $\Lambda_{D_{\{r\}}^{(r-1)}}(p,q)$  by Proposition [2.2](#page-4-0) and hence the one of  $\Lambda_{H^{(r-1)}}(p,q)$ .

<span id="page-7-0"></span>**Proposition 3.1.** We have the following description of  $\Lambda_{H(k)}(p,q)$  when  $k = r, r - 1$ . (1) The case where  $k = r$ :

$$
\Lambda_{H^{(r)}}(p,q) = \binom{q-(r-1)p-1}{r-1} \lambda_L(p),
$$

where  $L := R/[I_1 + \cdots + I_r]$  is a cyclic module. (2) The case where  $k = r - 1$ :

$$
\Lambda_{H^{(r-1)}}(p,q) = {q-(r-2)p-1 \choose r-2} \sum_{j=1}^{r} \lambda_{L_j}(p)
$$
  
- 
$$
\sum_{j=1}^{r} \sum_{n=0}^{p} \sum_{i=0}^{n-1} {q-(r-2)p-2-i \choose r-3} \ell_R(R/\mathfrak{a}_j(n))
$$

where  $L_j := R/[I_1 + \cdots + I_j + \cdots + I_r] \oplus R/[I_1 + \cdots + I_r]$  is a direct sum of two cyclic modules and  $\mathfrak{a}_j(n) := (I_1 + \cdots + \widehat{I}_j + \cdots + I_r)^{p-n} (I_1 + \cdots + I_r)^n$  is an ideal depending on j and n.

*Proof.* These follow directly from Proposition [2.2.](#page-4-0)  $\Box$ 

We now turn to investigate the remaining functions  $\Lambda_{H^{(k)}}(p,q)$  when  $k = 1, 2, \ldots, r-2$ . These cases seem to be more complicated than the case of  $k = r, r - 1$ . Suppose that  $k = 1, 2, \ldots, r - 2$  and let A be a subset of  $[r]$  with  $^{\sharp}A = r - k$ . Then we divide the set  $D_A^{(k)}$  $\Lambda^{(k)}$  into 2-parts as follows:

$$
D_A^{(k)} = E_{A-}^{(k)} \coprod E_{A+}^{(k)},
$$

where

$$
E_{A-}^{(k)} := \{ \mathbf{n} \in D_A^{(k)} \mid \sum_{i \in A} n_i \le p \},
$$
  

$$
E_{A+}^{(k)} := \{ \mathbf{n} \in D_A^{(k)} \mid \sum_{i \in A} n_i > p \}.
$$

Let

$$
H_-^{(k)}:=\coprod_{\substack{A\subset [r]\\ \sharp A=r-k\\ 8}}E_{A-}^{(k)},
$$

$$
H_{+}^{(k)} := \coprod_{\substack{A \subset [r] \\ \sharp A = r - k}} E_{A+}^{(k)}.
$$

Then

$$
\Lambda_{H^{(k)}}(p,q)=\Lambda_{H^{(k)}_-}(p,q)+\Lambda_{H^{(k)}_+}(p,q).
$$

Let me illustrate this decomposition when  $r = 3$ . Figure [2](#page-8-0) below is the picture where  $H_{-}^{(1)}\;=\;E_{\{1,\,\cdot\,}}^{(1)}$  $\mathop{\prod}\limits_{\{1,2\} - \,}^{(1)} \, \prod E_{\{1,2\}}^{(1)}$  $E_{\{1,3\}-}^{(1)} \coprod E_{\{2,3\}-}^{(1)}$  is the region of the pattern of lines, and  $H_{+}^{(1)} =$  $E_{\ell1}^{(1)}$  $\mathcal{F}^{(1)}_{\{1,2\}+}\coprod E^{(1)}_{\{1,2\}}$  $\mathcal{F}^{(1)}_{\{1,3\}+}\coprod E^{(1)}_{\{2,1\}}$  ${2,3}_{2,3}^{(1)}$  is the region of the pattern of dots.



<span id="page-8-0"></span>FIGURE 2. A decomposition of  $H^{(1)}$  when  $r = 3$ 

Here we note that  $E_{\{k+1,\ldots,r\}-}^{(k)} = \Delta_{p,q}^{(k)}$  for any  $k = 1, 2, \ldots, r-2$ . Thus, the function  $\Lambda_{H_{-}^{(k)}}(p,q)$  can be expressed explicitly as follows, similar to the one of  $\Lambda_{H^{(r)}}(p,q)$  and  $\Lambda_{H^{(r-1)}}(p,q).$ 

<span id="page-8-1"></span>**Proposition 3.2.** For any  $k = 1, 2, ..., r - 2$ , we have the following description.

$$
\Lambda_{H_{-}^{(k)}}(p,q) = {q-(k-1)p-1 \choose k-1} \sum_{\substack{A \subset [r] \\ \sharp A = r-k}} \lambda_{L_A(p)} \\
\quad - \sum_{\substack{A \subset [r] \\ \sharp A = r-k}} \sum_{\substack{n_j \ge 0 (j \in A) \\ (\sum_{j \in A} n_j) \le p}} \sum_{i=0}^{(\sum_{j \in A} n_j)-1} {q-(k-1)p-2-i \choose k-2} \ell_R(R/\mathfrak{a}),
$$

where  $L_A := \left( R / \left| \right| \sum_{n=1}^{\infty} \right)$  $s\in[r]\backslash A$  $I_s$ ]  $\oplus \bigoplus$ j∈A  $\sqrt{2}$  $R/\lceil \sum$  $s\in[r]\backslash A$  $\left\{I_s + I_j\right\}$  is a direct sum of  $(r - k + 1)$ cyclic modules and <sup>a</sup> := <sup>X</sup>  $s\in[r]\backslash A$  $I_s\big)^{p-(\sum_{j\in A}n_j)}$   $\prod$ j∈A  $(\nabla)$  $s\in[r]\backslash A$  $\left(I_s+I_j\right)^{n_j}$  is an ideal depending on A and  $n_j$   $(j \in A)$ .

*Proof.* This follows directly from Proposition [2.2.](#page-4-0)  $\Box$ 

On the other hand, the function  $\Lambda_{H_{+}^{(k)}}(p,q)$  seems to be more complicated than the one  $\Lambda_{H_{-}^{(k)}}(p,q)$ . We do not get the explicit description, but we have the following inequality.

<span id="page-9-0"></span>**Proposition 3.3.** For any  $k = 1, 2, ..., r - 2$ , there exists a polynomial  $g_k^{\circ}(X) \in \mathbb{Q}[X]$  of degree  $d + r - k$  such that

$$
\Lambda_{H_+^{(k)}}(p,q)\leq \binom{q-(k-1)p-1}{k-1}g_k^\circ(p).
$$

Proof. This follows from [\[5,](#page-15-7) Lemma 3.5].

−

Here we consider the following functions  $g_k(p)$  and  $h_k(p,q)$  appeared in Propositions [3.1](#page-7-0) and [3.2,](#page-8-1) which will be used in the next section. For any  $k = 1, \ldots, r - 1$ , we define

<span id="page-9-2"></span>(2) 
$$
g_k(p) := \sum_{\substack{A \subset [r] \\ \sharp A = r - k}} \lambda_{L_A}(p)
$$

$$
(3) \qquad h_{k}(p,q):=\sum_{A\subset [r]\atop \#A=r-k}\sum_{\substack{n_{j}\geq 0 (j\in A)\\ \vdots \\ (\sum_{j\in A}n_{j})\leq p}}\sum_{i=0}^{(\sum_{j\in A}n_{j})-1}\binom{q-(k-1)p-2-i}{k-2}\ell_{R}(R/\mathfrak{a})
$$

where  $L_A := \left( R / \left| \right. \sum_{n=1}^{\infty} \right)$  $s\in[r]\backslash A$  $I_s$ ]  $\oplus \bigoplus$ j∈A  $\sqrt{ }$  $R/\lceil \sum$  $s\in[r]\backslash A$  $I_s + I_j$  is a direct sum of  $(r - k + 1)$ cyclic modules and <sup>a</sup> := <sup>X</sup>  $s\in[r]\backslash A$  $I_s\Big)^{p-(\sum_{j\in A}n_j)}$   $\prod$ j∈A  $(\nabla)$  $s\in[r]\backslash A$  $I_s + I_j \bigg)^{n_j}$ . When  $k = r$ , we set  $g_r(p) = \lambda_{R/[I_1 + \dots + I_r]}(p)$  and  $h_r(p,q) = 0$ . Note that for  $p, q \gg 0$ ,  $g_k(p)$  is a polynomial function of degree  $d + r - k$ , and  $h_k(p,q)$  is a non-negative integer valued function.

Then, the above two Propositions [3.2](#page-8-1) and [3.3](#page-9-0) imply the following.

<span id="page-9-1"></span>Corollary 3.4. For any  $k = 1, 2, ..., r - 2$ , there exists a polynomial  $f_k(X) \in \mathbb{Q}[X]$  of degree  $d + r - k$  such that

$$
\Lambda_{H^{(k)}}(p,q)\leq \binom{q-(k-1)p-1}{k-1}f_k(p).
$$



Proof. By Propositions [3.2](#page-8-1) and [3.3,](#page-9-0)

$$
\Lambda_{H^{(k)}}(p,q) = \Lambda_{H_{-}^{(k)}}(p,q) + \Lambda_{H_{+}^{(k)}}(p,q)
$$
\n
$$
\leq {q - (k-1)p - 1 \choose k-1} g_k(p) - h_k(p,q) + {q - (k-1)p - 1 \choose k-1} g_k^{\circ}(p)
$$
\n
$$
= {q - (k-1)p - 1 \choose k-1} (g_k(p) + g_k^{\circ}(p)) - h_k(p,q)
$$
\n
$$
\leq {q - (k-1)p - 1 \choose k-1} (g_k(p) + g_k^{\circ}(p)).
$$

Thus,  $f_k(X) := g_k(X) + g_k^{\circ}(X)$  is our desired polynomial.

# 4. Proof of Theorem [1.1](#page-1-0)

We give a proof of Theorem [1.1.](#page-1-0) In this section, we work in the same situation and under the same notation as in the previous sections. For  $k = 1, 2, \ldots, r$ , we consider the following function:

$$
F_k(p,q) := \Lambda(p,q) - \sum_{i=1}^k {q - (r-i)p - 1 \choose r-i} g_{r-i+1}(p),
$$

which is a polynomial function for  $p, q \gg 0$  with the total degree is at most  $d + r - 1$ . We begin with the following.

<span id="page-10-2"></span>Proposition 4.1. Suppose that p is a large enough fixed integer. Then

$$
\lim_{q \to \infty} \frac{1}{q^{r-2}} F_2(p, q) = 0.
$$

*Proof.* Fix  $p \gg 0$ . By Proposition [3.1](#page-7-0) and Corollary [3.4,](#page-9-1) we have the following equalities and inequality.

$$
F_2(p,q) + h_{r-1}(p,q) = \Lambda(p,q) - \Lambda_{H^{(r)}}(p,q) - \Lambda_{H^{(r-1)}}(p,q)
$$
  
= 
$$
\sum_{k=1}^{r-2} \Lambda_{H^{(k)}}(p,q)
$$
  

$$
\leq \sum_{k=1}^{r-2} {q - (k-1)p - 1 \choose k-1} f_k(p).
$$

Hence, we have that

$$
-h_{r-1}(p,q) \le F_2(p,q) \le \sum_{k=1}^{r-2} {q-(k-1)p-1 \choose k-1} f_k(p).
$$

Therefore, it is enough to show that

<span id="page-10-0"></span>(4) 
$$
\lim_{q \to \infty} \frac{1}{q^{r-2}} \sum_{k=1}^{r-2} {q - (k-1)p - 1 \choose k-1} f_k(p) = 0, \text{ and}
$$

<span id="page-10-1"></span>(5) 
$$
\lim_{q \to \infty} \frac{1}{q^{r-2}} h_{r-1}(p, q) = 0.
$$

The first assertion [\(4\)](#page-10-0) is clear because the degree of a polynomial function

$$
\sum_{k=1}^{r-2} {q - (k-1)p - 1 \choose k-1} f_k(p)
$$

with respect to q is at most  $(r-2)-1=r-3$ . We show the second assertion [\(5\)](#page-10-1). Then one can see that

$$
h_{r-1}(p,q) = \sum_{j=1}^{r} \sum_{n=0}^{p} \sum_{i=0}^{n-1} {q - (r-2)p - 2 - i \choose r-3} \ell_R(R/\mathfrak{a}_j(n))
$$
  
\n
$$
\leq \sum_{j=1}^{r} \sum_{n=0}^{p} n {q - (r-2)p - 2 \choose r-3} \ell_R(R/\mathfrak{a}_j(n))
$$
  
\n
$$
\leq \sum_{j=1}^{r} \sum_{n=0}^{p} p {q - (r-2)p - 2 \choose r-3} \ell_R(R/\mathfrak{a}_j(n))
$$
  
\n
$$
= p {q - (r-2)p - 2 \choose r-3} \sum_{j=1}^{r} \sum_{n=0}^{p} \ell_R(R/\mathfrak{a}_j(n)),
$$

where  $\mathfrak{a}_j(n) := (I_1 + \cdots + \widehat{I}_j + \cdots + I_r)^{p-n} (I_1 + \cdots + I_r)^n$ . Note that

$$
\sum_{j=1}^{r} \sum_{n=0}^{p} \ell_R(R/\mathfrak{a}_j(n)) = \sum_{j=1}^{r} \lambda_{L_j}(p)
$$

is a sum of the ordinary Buchsbaum-Rim functions of two cyclic modules, where

$$
L_j = R/[I_1 + \cdots + \widehat{I}_j + \cdots + I_r] \oplus R/[I_1 + \cdots + I_r].
$$

Hence, noting that  $h_{r-1}(p,q) \geq 0$ , we have that

$$
0 \le h_{r-1}(p,q) \le \binom{q - (r-2)p - 2}{r-3} u(p)
$$

for some polynomial function  $u(p)$  of degree  $(d+1)+1=d+2$ . Therefore,

$$
\lim_{q \to \infty} \frac{1}{q^{r-2}} {q - (r-2)p - 2 \choose r-3} u(p) = 0
$$

so that  $\lim_{q\to\infty} \frac{1}{q^r}$  $\frac{1}{q^{r-2}}h_{r-1}(p,q) = 0.$ 

We are now ready to prove Theorem [1.1.](#page-1-0)

*Proof of Theorem [1.1.](#page-1-0)* The degree of  $\Lambda(p,q)$  with respect to q is at most  $r-1$  so that one can write

$$
\Lambda(p,q) = \sum_{i=0}^{r-1} a_i q^i
$$

where each  $a_i$  is a polynomial function of p with degree at most  $d + r - 1 - i$ . Similarly, we can write

$$
\binom{q - (r - 1)p - 1}{r - 1} g_r(p) = \sum_{j=0}^{r-1} b_j q^j
$$

$$
\binom{q - (r - 2)p - 1}{r - 2} g_{r-1}(p) = \sum_{k=0}^{r-2} c_k q^k
$$

where each  $b_j$  (resp.  $c_k$ ) is a polynomial function of p with degree at most  $d + r - 1 - j$ (resp.  $d + r - 1 - k$ ). Then

$$
F_2(p,q) = (a_{r-1} - b_{r-1})q^{r-1} + (a_{r-2} - b_{r-2} - c_{r-2})q^{r-2} + \text{(lower terms in } q\text{)}.
$$

By Proposition [4.1,](#page-10-2) we have the equalities as polynomials of  $p$ ,

<span id="page-12-0"></span>(6) 
$$
a_{r-1} = b_{r-1}
$$
, and

<span id="page-12-1"></span>(7) 
$$
a_{r-2} = b_{r-2} + c_{r-2}.
$$

Note that the first equality [\(6\)](#page-12-0) implies a formula  $e^{r-1}(C) = e(R/[I_1 + \cdots + I_r])$  which is our previous result in [\[5\]](#page-15-7). We then look at the second equality [\(7\)](#page-12-1). Since the total degree  $\Lambda(p,q)$  is  $d+r-1$ , and the coefficient of  $p^{d+1}q^{r-2}$  is non-zero, which is  $\frac{e^{r-2}(C)}{(d+1)!(r-2)!}$ , the polynomial  $a_{r-2}$  is of the form:

$$
a_{r-2} = \frac{e^{r-2}(C)}{(d+1)!(r-2)!}p^{d+1} + (\text{lower terms in } p).
$$

Since  $g_r(p) = \lambda_{R/[I_1 + \cdots + I_r]}(p)$  is the Hilbert-Samuel function of  $I_1 + \cdots + I_r$ ,

$$
\begin{aligned}\n&\left(\begin{array}{c} q - (r-1)p - 1 \\ r - 1 \end{array}\right) g_r(p) \\
&= \left(\begin{array}{c} q - (r-1)p - 1 \\ r - 1 \end{array}\right) \left(\frac{e(R/[I_1 + \dots + I_r])}{d!} p^d + (\text{lower terms in } p)\right) \\
&= \frac{(q - (r-1)p)^{r-1}}{(r-1)!} \cdot \frac{e(R/[I_1 + \dots + I_r])}{d!} p^d + (\text{lower terms}) \\
&= \frac{e(R/[I_1 + \dots + I_r])}{d! (r-1)!} p^d q^{r-1} - \frac{(r-1)e(R/[I_1 + \dots + I_r])}{d!(r-2)!} p^{d+1} q^{r-2} + (\text{lower terms in } q)\n\end{aligned}
$$

so that

$$
b_{r-2} = -\frac{(r-1)e(R/[I_1 + \dots + I_r])}{d!(r-2)!}p^{d+1}.
$$

Similarly, since  $g_{r-1}(p) = \sum_{j=1}^r \lambda_{L_j}(p)$ , and its normalized leading coefficient is

$$
E_{r-1} := E_{r-1}(I_1, \dots, I_r) := \sum_{j=1}^r e(L_j),
$$

where

$$
L_j = R/[I_1 + \cdots + \widehat{I}_j + \cdots + I_r] \oplus R/[I_1 + \cdots + I_r],
$$
  
<sub>13</sub>

we have that

$$
\begin{aligned}\n\binom{q-(r-2)p-1}{r-2}g_{r-1}(p) &= \binom{q-(r-2)p-1}{r-2}\left(\frac{E_{r-1}}{(d+1)!}p^{d+1} + (\text{lower terms in } p)\right) \\
&= \frac{(q-(r-2)p)^{r-2}}{(r-2)!} \cdot \frac{E_{r-1}}{(d+1)!}p^{d+1} + (\text{lower terms}) \\
&= \frac{E_{r-1}}{(d+1)!(r-2)!}p^{d+1}q^{r-2} + (\text{lower terms in } q).\n\end{aligned}
$$

Therefore, we get that

$$
c_{r-2} = \frac{E_{r-1}}{(d+1)!(r-2)!}p^{d+1}.
$$

By comparing the coefficient of  $p^{d+1}$  in the equation [\(7\)](#page-12-1), we have the equality

$$
\frac{e^{r-2}(C)}{(d+1)!(r-2)!} = -\frac{(r-1)e(R/[I_1+\cdots+I_r])}{d!(r-2)!} + \frac{E_{r-1}(I_1,\ldots,I_r)}{(d+1)!(r-2)!}.
$$

By multiplying  $(d+1)!(r-2)!$  to the above equation, we get the desired formula.  $\square$ 

As stated in the proof, the proof of Theorem [1.1](#page-1-0) contains our previous result in [\[5\]](#page-15-7). Moreover, the obtained formula for  $e^{r-2}(C)$  can be viewed as a natural generalization of the Kirby-Rees formula given in [\[8\]](#page-15-1).

<span id="page-13-0"></span>Corollary 4.2. Let  $I_1, \ldots, I_r$  be m-primary ideals in R and assume that  $I_1, \ldots, I_{r-1} \subset I_r$ , that is, the ideal  $I_r$  is the largest ideal. Then we have a formula

$$
e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/[I_1+\cdots+I_{r-1}]\oplus R/I_r).
$$

In particular, if  $I_1 \subset I_2 \subset \cdots \subset I_r$ , then

$$
e^{r-2}(R/I_1\oplus\cdots\oplus R/I_r)=e(R/I_{r-1}\oplus R/I_r).
$$

*Proof.* Suppose that  $I_1, \ldots, I_{r-1} \subset I_r$ . Then by Theorem [1.1,](#page-1-0)

$$
e^{r-2}(C) = \sum_{j=1}^{r} e(R/[I_1 + \dots + \widehat{I_j} + \dots + I_r] \oplus R/[I_1 + \dots + I_r])
$$
  
\n
$$
- (d+1)(r-1)e(R/[I_1 + \dots + I_r])
$$
  
\n
$$
= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r) + (r-1)e(R/I_r \oplus R/I_r)
$$
  
\n
$$
- (d+1)(r-1)e(R/I_r)
$$
  
\n
$$
= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r) + (r-1)(d+1)e(R/I_r)
$$
  
\n
$$
- (d+1)(r-1)e(R/I_r)
$$
  
\n
$$
= e(R/[I_1 + \dots + I_{r-1}] \oplus R/I_r).
$$

Here the third equality follows from the elementary formula  $(1)$ .

Before closing this article, we would like to give a few observations on the remaining multiplicities. We first recall the polynomial function  $F_k(p,q)$  defined at the beginning of this section:

$$
F_k(p,q) := \Lambda(p,q) - \sum_{i=1}^k {q - (r-i)p - 1 \choose r-i} g_{r-i+1}(p).
$$

The key of our proof of Theorem [1.1](#page-1-0) is the fact that  $\deg_q F_2(p,q) \leq r-3$  (Proposition [4.1\)](#page-10-2). It would be interesting to know whether this kind of property holds true or not for which  $k$ .

<span id="page-14-0"></span>Question 4.3. Let p be a fixed large enough integer. Then for which  $k = 1, 2, ..., r - 1$ , does the following hold true?

<span id="page-14-1"></span>
$$
\lim_{q \to \infty} \frac{1}{q^{r-k}} F_k(p,q) = 0.
$$

In other word, is the degree of  $F_k(p,q)$  with respect to q at most  $r - k - 1$ ?

This holds true when  $k = 2$  (and also  $k = 1$ ) by Proposition [4.1.](#page-10-2) We are interested in the remaining cases. Suppose that  $k \geq 3$ . The affirmative answer to Question [4.3](#page-14-0) will tell us that for any  $1 \leq j \leq k$ , the  $(r - j)$ th associated Buchsbaum-Rim multiplicity  $e^{r-j}(C)$ is determined by the polynomial

(8) 
$$
\sum_{i=1}^{k} {q - (r - i)p - 1 \choose r - i} g_{r-i+1}(p).
$$

Then we will be able to describe the multiplicity  $e^{r-j}(C)$  as a sum of the ordinary Buchsbaum-Rim multiplicities of a direct sum of at most  $(r-j)$  cyclic modules in the same manner. Here we would like to record the expected formula. Note that the polynomial  $g_{r-i+1}(p)$  defined in [\(2\)](#page-9-2) is of the form

$$
g_{r-i+1}(p) = \frac{1}{(d+i-1)!} \sum_{\substack{A \subset [r] \\ \sharp A=i-1}} e(L_A) \cdot p^{d+i-1} + \text{(lower terms)}
$$

where 
$$
L_A := \left( R / \Big[ \sum_{s \in [r] \setminus A} I_s \Big] \right) \oplus \bigoplus_{j \in A} \left( R / \Big[ \sum_{s \in [r] \setminus A} I_s + I_j \Big] \right)
$$
. We put  

$$
E_{r-i+1} := E_{r-i+1}(I_1, \dots, I_r) := \sum_{\substack{A \subset [r] \\ \sharp A = i-1}} e(L_A).
$$

Then for any  $1 \leq j \leq k$ , the coefficient of  $p^{d+j-1}q^{r-j}$  in the polynomial [\(8\)](#page-14-1) is

$$
\sum_{i=1}^{j} \frac{E_{r-i+1}}{(d+i-1)!(r-i)!} \binom{r-i}{r-j} \left( -(r-i) \right)^{j-i}.
$$

If Question [4.3](#page-14-0) is affirmative, then the above coefficient coincides with

$$
\frac{e^{r-j}(C)}{(d+j-1)!(r-j)!}
$$

so that we can get the formula for  $e^{r-j}(C)$ . Therefore, we can ask the following.

<span id="page-14-2"></span>Question 4.4. Under the same notation as above, does the formula

$$
e^{r-j}(R/I_1 \oplus \cdots \oplus R/I_r) = \sum_{i=1}^j {d+j-1 \choose j-i} (-(r-i))^{j-i} E_{r-i+1}(I_1, \ldots, I_r)
$$

hold true?

This is affirmative when  $j = 1$  ([\[4,](#page-15-8) Theorem 1.3]) and  $j = 2$  (Theorem [1.1\)](#page-1-0). Note that the affirmative answer to Question [4.3](#page-14-0) for some k implies the affirmative one to Question [4.4](#page-14-2) for any  $1 \leq j \leq k$ .

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Department of Environmental and Mathematical Sciences, Okayama University, 3-1-1 Tsushimanaka, Kita-ku, Okayama, 700-8530, JAPAN

E-mail address: hayasaka@okayama-u.ac.jp