GEOMETRY OF LIMITS OF ZEROS OF POLYNOMIAL SEQUENCES OF TYPE $(1, 1)$

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Abstract. In this paper, we study the root distribution of some univariate polynomials satisfying a recurrence of order two with linear polynomial coefficients. We show that the set of non-isolated limits of zeros of the polynomials is either an arc, or a circle, or a "lollipop", or an interval. As an application, we discover a sufficient and necessary condition for the universal real-rootedness of the polynomials, subject to certain sign condition on the coefficients of the recurrence. Moreover, we obtain the sharp bound for all the zeros when they are real.

1. INTRODUCTION

Root distribution of polynomials in a sequence discover intensive information about the interrelations of the polynomials in the sequence, especially when the sequence satisfies a recurrence. Stanley [\[14\]](#page-12-0) provides some figures for the root distribution of some polynomials in a sequence arising from combinatorics.

In the study of the root distribution of sequential polynomials, both the realrootedness and the limiting distribution of zeros of the polynomials receive much attention. Some evidence for the significance of real-rootedness of polynomials can be found in Stanley $[15, §4]$ $[15, §4]$. Bleher and Mallison $[6]$ consider the zeros of Taylor polynomials, and the asymptotics of the zeros for linear combinations of exponentials. Some study on certain "zero attractor" of particular sequences of polynomials can be found in [\[7,](#page-12-2) [10\]](#page-12-3). The exploration of zero attractors of Appell polynomials has been regarded as "gems in experimental mathematics" in [\[8\]](#page-12-4). Limiting distribution of zeros has been used to study the four-color theorem via the chromatic polynomials initiated by Birkhoff $[5]$, which amounts to the nonexistence of a chromatic polynomial with a zero at the point 4. Beraha and Kahane $\boxed{2}$ examine the limits of zeros for the sequence of chromatic polynomials of a special family of 3-regular graphs, described as to consist of an inner and outer square

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separated by n 4-rings. It turns out that the number 4 is a limit of zeros of polynomials in this family.

Motived by the LCGD conjecture from topological graph theory, Gross, Mansour, Tucker and the first author $[11, 12]$ $[11, 12]$ $[11, 12]$ study the root distribution of polynomials satisfying the recurrence

(1.1)
$$
W_n(z) = A(z)W_{n-1}(z) + B(z)W_{n-2}(z),
$$

where the functions $A(z)$ and $B(z)$ are polynomials such that one of them is linear and that the other is constant. They established the real-rootedness subject to some sign conditions of the coefficients of $A(z)$ and $B(z)$. Since the real-rootedness implies the log-concavity, they confirm the LCGD conjecture for many graph families whose genus polynomials satisfy Recurrence [\(1.1\)](#page-1-0) with the sign conditions. Orthogonal polynomials and quasi-orthogonal polynomials have closed relations with Recurrence [\(1.1\)](#page-1-0); see Andrews, Richard and Ranjan [\[1\]](#page-12-9) and Brezinski, Driver and Redivo-Zaglia [\[9\]](#page-12-10). Jin and Wang [\[13\]](#page-12-11) characterized the common zeros of polynomials $W_n(z)$ for general $A(z)$ and $B(z)$.

Following Gross et al. [\[11\]](#page-12-7), a sequence $\{W_n(z)\}_n$ of polynomials satisfying Recurrence [\(1.1\)](#page-1-0) is said to be of type $(\deg A(z), \deg B(z))$. It is normalized if $W_0(z) = 1$ and $W_1(z) = z$. When $A(z) = az + b$ and $B(z) = cz + d$ are linear, Recurrence [\(1.1\)](#page-1-0) reduces to

(1.2)
$$
W_n(z) = (az + b)W_{n-1}(z) + (cz + d)W_{n-2}(z).
$$

Concentrating on the root distribution, and considering the polynomials defined by $(-1)^n W_n(-z)$, one may suppose without loss of generality that $c \geq 0$. We use a quadruple $(\text{sgn}(a), \text{sgn}(b), \text{sgn}(c), \text{sgn}(d))$, each coordinate of which is either + or − or 0, to denote the combination of signs of the numbers a, b, c, d .

Gross et al. [\[11,](#page-12-7)[12\]](#page-12-8), establish the real-rootedness for Cases $(+,*,0,-)$, $(0,+,+,+)$ and $(0, +, +, -)$, where the symbol $*$ indicates that the number b might be of any sign. In Case $(-, -, +, -)$, Wang and Zhang [\[17\]](#page-13-1) establish the real-rootedness of all polynomials $W_n(z)$ for when $\Delta_g > 0$, where $\Delta_g = (b+c)^2 + 4d(1-a)$. In Case $(+, +, +, +)$, they $\boxed{18}$ show that every polynomial $W_n(z)$ is real-rooted if and only if $ad < bc$.

According to Beraha, Kahane, and Weiss' result $[3, 4]$ $[3, 4]$ $[3, 4]$ on limits of zeros of polynomials satisfying Recurrence [\(1.1\)](#page-1-0), polynomials satisfying Recurrence [\(1.2\)](#page-1-1) have at most two isolated limits of zeros. In this paper, we show that the set of non-isolated limits of zeros of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) is either an arc, or a circle, or a "lollipop", or an interval. As an application, we can show that in Case $(+, -, +, -)$, every polynomial is real-rooted if and only if $ad \leq bc$. Moreover, when the isolated limits are real, the zeros approach to them in an oscillating manner in Cases $(0, +, +, +)$ and $(+, +, +, +)$, that is, from both the

left and right sides of the isolated limits, while the convergence way is from only one side in Case $(+, -, +, -)$; see Theorem [3.4.](#page-10-0)

We should mention that the generating function of the normalized polynomials satisfying Recurrence [\(1.1\)](#page-1-0) is

$$
\sum_{n\geq 0} W_n(z)t^n = \frac{1 + (z - A(z))t}{1 - A(z)t - B(z)t^2}.
$$

In comparison, the root distribution of the polynomials generated by the function

$$
\sum_{n\geq 0} W_n(z)t^n = \frac{1}{1 - A(z)t - B(z)t^2}
$$

has been investigated in $[16]$, in which Tran found an algebraic curve containing the zeros of all polynomials $W_n(z)$ with large subscript n.

This paper is organised as follows. After reviewing necessary notion and and notation, we interpret Beraha et al.'s characterization for polynomials satisfying Recurrence (1.2) in Theorem [2.3.](#page-3-0) In §[3,](#page-7-0) we provide a sufficient and necessary condition of real-rootedness in Case $(+, -, +, -)$, and the root distribution when they are real-rooted as an application of Theorem [2.3.](#page-3-0)

2. Geometry of the limits of zeros

Throughout this paper, we let $a, b, c, d \in \mathbb{R}$, $ac \neq 0$, and let $\{W_n(z)\}_{n\geq 0}$ be a sequence of polynomials satisfying Recurrence (1.2) . Then the polynomial $W_n(z)$ has leading term $a^{n-1}z^n$. For any complex number $z = re^{i\theta}$ with $\theta \in (-\pi, \pi]$, we the square root notation \sqrt{z} to denote the number $\sqrt{r}e^{i\theta/2}$, which lies in the right half-plane $\theta \in (-\pi/2, \pi/2]$. The general formula in Lemma [2.1](#page-2-0) is the base of our study, which can be found in [\[11,](#page-12-7) [12\]](#page-12-8).

Lemma 2.1. Let $A, B \in \mathbb{C}$. Suppose that $W_0 = 1$ and $W_n = AW_{n-1} + BW_{n-2}$ for $n > 2$. Then

$$
W_n = \begin{cases} \alpha_+ \lambda_+^n + \alpha_- \lambda_-^n, & \text{if } \Delta \neq 0, \\ \frac{A + nh}{2} \cdot \left(\frac{A}{2}\right)^{n-1}, & \text{if } \Delta = 0, \end{cases}
$$

for $n \geq 0$, where $h = 2W_1 - A$ and

$$
\lambda_{\pm} = \frac{A \pm \sqrt{\Delta}}{2}
$$
, $\alpha_{\pm} = \frac{\sqrt{\Delta} \pm h}{2\sqrt{\Delta}}$, $with \Delta = A^2 + 4B$.

Accordingly, we employ the notations

$$
\Delta(z) = A(z)^{2} + 4B(z) = a^{2}z^{2} + (2ab + 4c)z + (b^{2} + 4d),
$$

$$
h(z) = 2W_1(z) - A(z) = (2 - a)z - b,
$$

\n
$$
\lambda_{\pm}(z) = \frac{A(z) \pm \sqrt{\Delta(z)}}{2},
$$

\n
$$
\alpha_{\pm}(z) = \frac{\sqrt{\Delta(z)} \pm h(z)}{2\sqrt{\Delta(z)}},
$$

\n
$$
g(z) = -\alpha_{+}(z)\alpha_{-}(z)\Delta(z) = \frac{h^2(z) - \Delta(z)}{4} = (1 - a)z^2 - (b + c)z - d.
$$

Denote by $x_A = -b/a$ and $x_B = -d/c$ the zeros of $A(z)$ and $B(z)$ respectively. The function $\Delta(z)$ has two zeros

$$
x_{\Delta}^{\pm} = x_A + \frac{-2c \pm 2\sqrt{\Delta_{\Delta}}}{a^2},
$$

where $\Delta_{\Delta} = c^2 - a^2 B(x_A)$ is the discriminant of $\Delta(z)$. A number $z^* \in \mathbb{C}$ is a limit of zeros of the sequence $\{W_n(z)\}_n$ of polynomials if there is a zero z_n of $W_n(z)$ for each *n* such that $\lim_{n\to\infty} z_n = z^*$.

Lemma 2.2 (Beraha et al. $[3]$). Under the non-degeneracy conditions

(N-i) the sequence ${W_n(z)}_n$ does not satisfy a recurrence of order less than two, (N-ii) $\lambda_+(z) \neq \omega \lambda_-(z)$ for some $z \in \mathbb{C}$ and some constant ω such that $|\omega| = 1$,

a number z is a limit of zeros if and only if it satisfies one of the following conditions:

(C-i)
$$
\alpha_-(z) = 0
$$
 and $\lambda_+(z) < \lambda_-(z)$;
\n(C-ii) $\alpha_+(z) = 0$ and $\lambda_+(z) > \lambda_-(z)$;
\n(C-iii) $\lambda_+(z) = \lambda_-(z)$.

A limit z of zeros is said to be *non-isolated* if it satisfies Condition $(C$ -iii), and to be *isolated* if it satisfies Condition $(C-i)$ or Condition $(C-i)$. We denote the set of non-isolated limits of zeros of the polynomials $W_n(z)$ by \clubsuit , and denote the set of isolated limits of zeros by \spadesuit . The clover symbol \clubsuit is adopted for the leaflets of a clover are not alone, while the spade symbol ♠ appearing as a single leaflet represents isolation in comparison.

Theorem 2.3. Let $a, b, c, d \in \mathbb{R}$ and $ac \neq 0$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. Then the sets of isolated and non-isolated limits of zeros of ${W_n(z)}_n$ are respectively

$$
\spadesuit = \{ z \in \mathbb{C} \colon g(z) = 0, \, \Re(A(z)\overline{h(z)}) < 0 \} \qquad \text{and} \qquad
$$

$$
\clubsuit = \begin{cases} \widehat{x_{\Delta}} x_{A} x_{\Delta}^{+}, & \text{if } \Delta_{\Delta} < 0; \\ C_{0}, & \text{if } \Delta_{\Delta} = 0; \\ J_{\Delta} \cup C_{0}, & \text{if } \Delta_{\Delta} > 0 \text{ and } B(x_{A}) > 0; \\ J_{\Delta}, & \text{if } \Delta_{\Delta} > 0 \text{ and } B(x_{A}) \leq 0; \end{cases}
$$

where \overline{z} denotes the complex conjugate of z, $x_{\Delta}^- x_{A} x_{\Delta}^+$ stands for the circular arc connecting the points x_{Δ}^- and x_{Δ}^+ , through the point x_A ,

$$
C_0 = \{ z \in \mathbb{C} \colon |z - x_B| = |x_A - x_B| \}
$$

is the circle with center x_B and radius $|x_A - x_B|$, and

$$
J_{\Delta} = \{x \in \mathbb{R} \colon x_{\Delta}^- \le x \le x_{\Delta}^+\}
$$

is an interval.

Proof. Condition $(N-i)$ is satisfied since otherwise one would have $W_n(z) = z^n$ for each n, contradicting the fact $W_2(z) = az^2 + (b+c)z + d$. Condition [\(N-ii\)](#page-3-5) holds true since $|\lambda_-(x)| \neq |\lambda_+(x)|$ for sufficiently large real number x.

Suppose that $z \in \spadesuit$. From definition, we have $\alpha_-(z)\alpha_+(z) = 0$, which implies

$$
0 = g(z) = \frac{h^2(z) - \Delta(z)}{4}.
$$

Thus $\sqrt{\Delta(z)} \in {\pm h(z)}$. If $\sqrt{\Delta(z)} = h(z)$, then $\alpha(z) = 0$ from definition. By Lemma [2.2,](#page-3-6) we have $\lambda_+(z) < \lambda_-(z)$, i.e., $\Re(A(z)\overline{h(z)}) < 0$. Along the same line we can handle the other case $\sqrt{\Delta(z)} = -h(z)$.

It is clear that $\{x_A, x_\Delta^-, x_\Delta^+\} \subseteq \clubsuit$. Let $z = x + yi \in \clubsuit$ such that $A(z)\Delta(z) \neq 0$, where $x, y \in \mathbb{R}$. If $y = 0$, then $z, A(z), \Delta(z) \in \mathbb{R}$. In this case, we can infer that

$$
\lambda_-(z) = \lambda_+(z) \iff \Delta(z) < 0 \iff \Delta_{\Delta} > 0 \text{ and } x \in (x_{\Delta}^-, x_{\Delta}^+).
$$

Otherwise $y \neq 0$. We can infer that

$$
\lambda_{-}(z) = \lambda_{+}(z) \iff \text{the vectors } A(z) \text{ and } \sqrt{\Delta(z)} \text{ are orthogonal}
$$

\n
$$
\iff \text{the vectors } A^{2}(z) \text{ and } \Delta(z) \text{ have opposite directions}
$$

\n
$$
\iff A^{2}(z) \text{ and } B(z) \text{ have opposite directions, } |A^{2}(z)| < |4B(z)|
$$

\n
$$
\iff \begin{cases} \Re A^{2}(z) \cdot \Im B(z) = \Re B(z) \cdot \Im A^{2}(z) \\ \Im A^{2}(z) \cdot \Im B(z) < 0 \\ |\Im A^{2}(z)| < 4|\Im B(z)| \end{cases}
$$

$$
\iff \begin{cases} (x - x_B)^2 + y^2 = (x_A - x_B)^2 \\ (x - x_A)(x - x_A + 2c/a^2) < 0 \end{cases}
$$
\n
$$
\iff z \in C_0 \cap S_0 \setminus \{x_A, x_\Delta^-, x_\Delta^+\},
$$

where $S_0 = \{z \in \mathbb{C} : |\Re z - x_A| \leq |2c/a^2|, c \cdot (\Re z - x_A) \leq 0\}$ is the vertical strip with boundaries $\Re z = x_A$ and $\Re z = x_A - 2c/a^2$. It is clear that the boundary $\Re z = x_A$ intersects the circle C_0 at the point x_A . To figure out the intersection of the other boundary with C_0 , we proceed according to the sign of Δ_{Δ} .

Suppose that Δ_{Δ} < 0. Then $J_{\Delta} = \emptyset$ from definition, and

$$
\Re(x_{\Delta}^{\pm}) = x_A - \frac{2c}{a^2}
$$
 and $\Im(x_{\Delta}^{\pm}) = \pm \frac{2\sqrt{-\Delta_{\Delta}}}{a^2}$.

It follows that

$$
\left(x_{\Delta}^{\pm} - x_B\right)^2 = \left(x_A - \frac{2c}{a^2} - x_B\right)^2 + \left(\frac{2\sqrt{-\Delta_{\Delta}}}{a^2}\right)^2 = (x_A - x_B)^2.
$$

Thus the points x^{\pm}_{Δ} lie on the intersection of the boundary $\Re z = x_A - 2c/a^2$ and the circle C_0 . Since the intersection contains at most two points, the points x^{\pm}_{Δ} consitute the intersection. Hence the set $\clubsuit = C_0 \cap S_0$ is the circular arc $x_{\Delta}^- x_A x_{\Delta}^+$.

When $\Delta_{\Delta} = 0$, the points $x_{\Delta}^{\pm} = x_A - 2c/a^2$ coincide with each other. As a consequence, we have $C_0 \cap S_0 = C_0$ and $\clubsuit = J_\Delta \cup C_0 = C_0$.

Below we can suppose that $\Delta_{\Delta} > 0$. Note that

(2.1)
$$
B(x_A) = c(x_A - x_B).
$$

When $B(x_A) \leq 0$, we claim that $C_0 \cap S_0 = \{x_A\}$. Let $z \in C_0 \cap S_0$. If $c > 0$, then $x_A \leq x_B$ by Eq. [\(2.1\)](#page-5-0). Since $z \in C_0$, we have $\Re z \geq x_A$. Since $z \in S_0$, we have $c(\Re z - x_A) \leq 0$. Therefore, we infer that $\Re z = x_A$, and $z = x_A$ consequently. Otherwise $c < 0$. Then $x_A \ge x_B$ by Eq. [\(2.1\)](#page-5-0). In this case, $z \in C_0$ implies $\Re z \le x_A$, and $z \in S_0$ implies $\Re z \geq x_A$. Hence $z = x_A$ for the same reason. This proves the claim. Since $\Delta(x_A) = 4B(x_A) \leq 0$, we have $x_A \in J_\Delta$. Hence $\clubsuit = J_\Delta$.

When $B(x_A) > 0$, we claim that $C_0 \subset S_0$. Let $z \in C_0$. One may show $c(\Re z-x_A) \leq 0$ in the same fashion as when $B(x_A) < 0$. By geometric interpretation and the condition $\Delta_{\Delta} > 0$, we deduce that

$$
|\Re z - x_A| \leq
$$
 (the diameter of C_0) = $2|x_A - x_B|$ < $|2c/a^2|$.

This proves the claim and hence $\clubsuit = J_\Delta \cup C_0$.

We remark that $z \in \spadesuit$ if and only if $\overline{z} \in \spadesuit$. Since $\Delta_{\Delta} \leq 0$ implies $B(x_A) > 0$, the case " $\Delta > 0$ and $B(x_A) \leq 0$ " in Theorem [2.3](#page-3-0) can be reduced to " $B(x_A) \leq 0$ ". Corollary 2.4. Let $a, b, c, d \in \mathbb{R}$ and $ac \neq 0$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. If every polynomial $W_n(z)$ for large n is real-rooted, then $B(x_A) \leq 0$, and $\Delta \geq 0$ as a consequence.

Proof. Since every polynomial $W_n(z)$ for large n is real-rooted, we have $\bigoplus \emptyset \subset \mathbb{R}$. By Theorem [2.3,](#page-3-0) we find either $\clubsuit = J_{\Delta}$, or $\clubsuit = C_0$ and C_0 degenerates to a single point. In the former case, we find $B(x_A) \leq 0$. In the latter case, we have $\Delta_{\Delta} = 0$ and $x_A = x_B$, which is impossible since otherwise

$$
0 = \Delta_{\Delta} = c^2 - a^2 B(x_A) = c^2,
$$

a contradiction. This completes the proof.

When $\clubsuit = J_{\Delta} \cup C_0$, it turns out that the set \clubsuit looks like a lollipop; see Fig. [2.1.](#page-6-0)

FIGURE 2.1. The zero distribution of $W_{30}(z)$ for the parameters $(a, b, c, d) = (1, -2, 2, -1)$ and $(a, b, c, d) = (1, 2, -2, -1)$, for each of which we have $x_A = -2$, $x_B = -1/2$, and $B(x_A) = 3$.

Theorem 2.5. Suppose $\Delta_{\Delta} > 0$ and $B(x_A) > 0$. Then $J_{\Delta} \cap C_0 = \{2x_B - x_A\}$, and the part of J_{Δ} outside the circle C_0 is longer than the part of J_{Δ} inside C_0 .

Proof. By Theorem [2.3,](#page-3-0) we have $\clubsuit = J_{\Delta} \cup C_0$. First of all, denote $x_0 = 2x_B - x_A$ to be one of the two real points on C_0 , other than x_A . Since

$$
\Delta(x_0) = -\frac{4B(x_A)\Delta_{\Delta}}{c^2} < 0,
$$

we have $x_0 \in J_\Delta$. Second, the centre of the circle C_0 is not on the interval J_Δ since $\Delta(x_B) = A^2(x_B) > 0$. It follows that $J_\Delta \cap C_0 = \{x_0\}$. Thirdly, note that

(2.2)
$$
x_0 - \frac{x_{\Delta}^- + x_{\Delta}^+}{2} = \frac{1}{c} \cdot \frac{2\Delta_{\Delta}}{a^2}.
$$

If $c > 0$, then $x_B < x_A$ by Eq. [\(2.1\)](#page-5-0). It follows that $x_0 < x_B$. Thus the interval J_{Δ} intersects the circle C_0 from the left of C_0 . By Eq. [\(2.2\)](#page-6-1), we have $x_0 > (x_{\Delta}^- + x_{\Delta}^+)/2$. Thus the part of J_{Δ} outside the circle C_0 is longer than the part of J_{Δ} inside. The other case $c < 0$ can be handled in the same way.

3. THE INTERLACING ZEROS FOR CASE $(+, -, +, -)$

Here is the main result of this section.

Theorem 3.1. Let $a, c > 0$ and $b, d < 0$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. Then $W_n(z)$ is realrooted if and only if $x_A \leq x_B$.

The necessity part of Theorem [3.1](#page-7-1) can be seen directly from Corollary [2.4.](#page-5-1) The sufficiency part will be handled for the case $x_A < x_B$ in Theorem [3.4,](#page-10-0) and for the case $x_A = x_B$ in Theorem [3.6.](#page-11-0) Throughout this section, we suppose that $x_A \le x_B$, which implies that $\Delta_{\Delta} > 0$ and $x_{\Delta}^{\pm} \in \mathbb{R}$. The zeros of the function $g(z)$ are

$$
x_g^{\pm} = \begin{cases} \frac{b+c}{2(1-a)} \pm \frac{\sqrt{\Delta_g}}{2|1-a|}, & \text{if } a \neq 1, \\ -\frac{d}{b+c}, & \text{if } a = 1 \text{ and } b+c \neq 0, \end{cases}
$$

where $\Delta_g = (b+c)^2 + 4d(1-a)$. We define two numbers u and v by

(3.1)
$$
(u, v) = \begin{cases} (x_{\Delta}^-, x_{\Delta}^+), & \text{if } a < 2 \text{ and } F \le 0; \\ (x_g^-, x_g^+), & \text{if } a > 2 \text{ and } F < 0; \\ (x_g^+, x_{\Delta}^+), & \text{if } a < 1 \text{ and } F > 0; \\ (x_g^-, x_{\Delta}^+), & \text{otherwise}; \end{cases}
$$

where $F = \Delta_g - \Delta_{\Delta} = d(a-2)^2 + bc(2-a) + b^2$. Note that $(u, v) = (x_{\Delta}^-, x_{\Delta}^+)$ if $a = 1$ and $b + c = 0$. Furthermore, we have $u, v \in \mathbb{R}$ since $\Delta_q > \Delta_{\Delta} > 0$ whenever $a \geq 2$ or $F > 0$. As will be seen in Theorems [3.4](#page-10-0) and [3.6,](#page-11-0) we have $u < v$ and the interval (u, v) is the best bound for the zeros of $W_n(z)$.

3.1. Case $x_A < x_B$. We determine the signs of $W_n(u)$ and $W_n(v)$ in Lemma [3.2.](#page-7-2)

Lemma 3.2. Let $a, c > 0$ and $b, d < 0$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. Suppose that $x_A < x_B$. Then we have

- (3.2) $u \leq x_{\Delta}^- < x_A < x_{\Delta}^+ \leq v < x_B$,
- (3.3) $u < 0 < v$,

(3.4)
$$
W_n(u)(-1)^n > 0,
$$

$$
(3.5) \t Wn(v) > 0, \t and
$$

$$
(3.6) \qquad \{u, v\} \subseteq \spadesuit \cup \clubsuit.
$$

Proof. The premise $x_A < x_B$ implies $\Delta(x_A) = 4B(x_A) < 0$. It follows that

$$
x_A \in (x_\Delta^-, x_\Delta^+),
$$
 $x_\Delta^+ > 0,$ $A(x_\Delta^+) > 0 > A(x_\Delta^-),$ and

(3.7)
$$
h(x_{\Delta}^{+}) = (2 - a)x_{\Delta}^{+} - b \ge -b > 0 \quad \text{if } a \le 2.
$$

Since $\Delta(x_B) = A^2(x_B) > 0$ and $x_\Delta^- < x_A < x_B$, we have $x_\Delta^+ < x_B$.

To confirm Relation (3.6) , by Theorem [2.3,](#page-3-0) it suffices to show that

(3.8)
$$
A(x)h(x) < 0, \quad \text{for any } x \in \{u, v\} \setminus \{x_\Delta^-, x_\Delta^+\}.
$$

Let x_h be the unique zero of the function $h(z)$ when $a \neq 2$. Then $x_h = b/(2 - a)$. We proceed according to the definition of the numbers u and v .

Case 3.2.1. $a < 2, F \leq 0$ and $[u, v] = J_{\Delta}$. It is routine to compute that

(3.9)
$$
h(x_{\Delta}^{-})h(x_{\Delta}^{+}) = \frac{4F}{a^{2}}.
$$

Together with Ineq. [\(3.7\)](#page-8-1), we have $h(x_{\Delta}^-) \leq 0$ and thus

$$
x_{\Delta}^- \le x_h = \frac{b}{2-a} < 0,
$$

verifying Ineq. [\(3.3\)](#page-7-3). By Lemma [2.1,](#page-2-0) we have

(3.10)
$$
W_n(x_\Delta^{\pm}) = \frac{A(x_\Delta^{\pm}) + nh(x_\Delta^{\pm})}{2} \cdot \left(\frac{A(x_\Delta^{\pm})}{2}\right)^{n-1},
$$

which implies Ineqs. (3.4) and (3.5) .

Case 3.2.2. $a > 2$, $F < 0$ and $[u, v] = [x_g^-, x_g^+]$. Observe that

(3.11)
$$
g(x_{\Delta}^{\pm}) = \frac{h^2(x_{\Delta}^{\pm})}{4} \ge 0.
$$

Since the polynomial $q(z)$ is quadratic with leading coefficient negative, we can derive all inequalities in [\(3.2\)](#page-7-4) except $v < x_B$. Since $F < 0$, we have $d(a-2)-bc < 0$ and thus

$$
g(x_B) = \frac{-d}{c^2}((a-1)d - bc) < \frac{-d}{c^2}((a-2)d - bc) < 0.
$$

Since $x_g^- < x_A < x_B$, we infer that $x_g^+ < x_B$.

On the other hand, by Vièta's theorem, we have

(3.12)
$$
x_g^- x_g^+ = \frac{d}{a-1},
$$

whose negativity verifies Ineq. (3.3) . By Lemma [2.1,](#page-2-0) we have

(3.13)
$$
W_n(x_g^{\pm}) = (x_g^{\pm})^n,
$$

which implies Ineqs. (3.4) and (3.5) . It is routine to compute that

(3.14)
$$
h(x_g^-)h(x_g^+) = \frac{F}{a-1} \quad \text{if } a \neq 1.
$$

Thus $h(v) < 0 < h(u)$. By [\(3.2\)](#page-7-4), we have $A(u) < 0 < A(v)$. This proves Ineq. [\(3.8\)](#page-8-4).

Case 3.2.3. $a < 1, F > 0$ and $[u, v] = [x_g^+, x_\Delta^+]$. In view of Eqs. [\(3.10\)](#page-8-5) and [\(3.13\)](#page-9-0) and Ineq. (3.7) , to confirm Ineqs. (3.2) to (3.5) and (3.8) , we shall show that

$$
x_g^+ \le x_\Delta^-,
$$
 $x_g^+ < 0$, and $h(x_g^+) > 0$.

In fact, we note that the polynomial $g(z)$ is quadratic with leading coefficient positive. On the one hand, Eq. (3.14) gives $x_h \in (x_g^-, x_g^+)$. This confirms $h(x_g^+) > 0$ immediately. By Eq. [\(3.9\)](#page-8-6), we can deduce that $x_h < x_{\Delta}$, since otherwise one would have the absurd inequality

$$
0 < x^+_\Delta < x_h = \frac{b}{2 - a} < 0.
$$

Thus Ineq. [\(3.11\)](#page-8-7) implies $(x_g^-, x_g^+) \cap J_\Delta = \emptyset$. Moreover, the whole interval (x_g^-, x_g^+) lies to the left of J_{Δ} . This proves $x_g^+ \leq x_{\Delta}^-$. On the other hand, by Ineq. [\(3.12\)](#page-8-8) we have $x_g^- x_g^+ > 0$. Since $x_g^- < x_h < 0$, we find $x_g^+ < 0$.

Case 3.2.4. For all remaining cases we have $[u, v] = [x_g^-, x_\Delta^+]$. This time, to confirm Ineqs. (3.2) to (3.5) and (3.8) , we shall show that

$$
x_g^-\leq x_\Delta^-,\qquad x_g^-\,<0,\qquad h(x_\Delta^+)\geq 0,\quad \text{ and }\quad h(x_g^-)>0.
$$

In fact, when $a = 1$, in view of Case [3.2.1,](#page-8-9) we now have $F > 0$ and thus $b + c < 0$. Note that $g(z) = -(b+c)z - d$. It follows from Ineq. [\(3.11\)](#page-8-7) that $x_g^- \le x_\Delta^-$. Since $g(0) = -d > 0$, we obtain $x_g^- < 0$. By Ineq. [\(3.7\)](#page-8-1), we have $h(x_\Delta^+) \geq 0$. It is routine to compute that

$$
h(x_g^-) = x_g^- - b = -\frac{d}{b+c} - b = -\frac{F}{b+c} > 0.
$$

Now, in view of Cases [3.2.1](#page-8-9) and [3.2.3,](#page-9-2) we have $a > 1$. Consequently, one may derive $J_{\Delta} \subseteq [x_g^-, x_g^+]$ and $x_g^- < 0$ as in Case [3.2.2.](#page-8-10) We shall handle the two inequalities involving h according to the value range of a. If $a = 2$, then the function $h(z) = -b$ reduces to a positive constant and we are done. Now we can suppose that $a \neq 2$.

(1) If $a > 2$, then

$$
h(x_{\Delta}^-) + h(x_{\Delta}^+) = \frac{4}{a^2}((a-2)c - ab) > 0.
$$

In view of Case [3.2.2,](#page-8-10) we have $F \ge 0$. By Eq. [\(3.9\)](#page-8-6), we have $h(x_{\Delta})h(x_{\Delta}^+) \ge 0$. Therefore, we infer that $h(x_{\Delta}^{\dagger}) \geq 0$. Since the polynomial $h(\overline{z})$ is strictly decreasing and $x_g^- < x_\Delta^+$, we have $h(x_g^-) > h(x_\Delta^+) > 0$.

(2) If $1 < a < 2$, by Ineq. [\(3.7\)](#page-8-1), it suffices to show that $h(x_g^{-}) > 0$. In view of Case [3.2.1,](#page-8-9) we have $F > 0$. By Ineq. [\(3.7\)](#page-8-1) and Eq. [\(3.9\)](#page-8-6), we have $h(x_{\Delta}) > 0$ and $x_h < x_\Delta^-.$ By Eq. [\(3.14\)](#page-9-1), we have $h(x_g^-)h(x_g^+) > 0$. Since $J_\Delta \subseteq [x_g^-., x_g^+]$, we deduce that $x_h < x_g^-$, i.e., $h(x_g^-) > 0$.

This completes the proof.

Let $X, Y \subset \mathbb{R}$ such that $|X| - |Y| \in \{0, 1\}$. We say that X interlaces Y, if the elements x_i of X and the elements y_j of Y can be arranged so that $x_1 \leq y_1 \leq$ $x_2 \leq y_2 \leq \cdots$, and that X strictly interlaces Y if no equality holds in the ordering. Lemma [3.3](#page-10-1) is Lemma 3.3 of $[12]$, wherein used in a proof of the real-rootedness of polynomials $W_n(z)$ defined by Recurrence [\(1.2\)](#page-1-1) with $a > 0$, $b \in \mathbb{R}$, $c = 0$ and $d < 0$ by induction.

Lemma 3.3 (Gross et al. [\[12\]](#page-12-8)). Let $\{W_n(z)\}_n$ be a sequence of polynomials sat-isfying Recurrence [\(1.1\)](#page-1-0). Let $m \geq 0$ and $\alpha, \beta \in \mathbb{R}$. Suppose that the polynomial $W_{m+2}(x)$ has degree $m+2$, and that $B(x) < 0$ for all $x \in R_{m+1}$, $W_m(\alpha)W_{m+2}(\alpha) > 0$, $W_m(\beta)W_{m+2}(\beta) > 0, |R_{m+1}| = m+1, R_{m+1} \subset (\alpha, \beta)$, and R_{m+1} strictly interlaces R_m . Then we have $|R_{m+2}| = m + 2$, $R_{m+2} \subset (\alpha, \beta)$, and R_{m+2} strictly interlaces R_{m+1} .

Now we are in a position to show the real-rootedness with the interlacing property and the best bound of all zeros.

Theorem 3.4. Let $a, c > 0$ and $b, d < 0$ such that $x_A < x_B$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. Then every polynomial $W_n(z)$ is real-rooted. Denote by R_n the zero set of $W_n(z)$. Then $R_n \subset (u, v)$, and the set R_{n+1} strictly interlaces R_n . Moreover, the bound (u, v) is sharp, in the sense that both the numbers u and v are limits of zeros.

Proof. We prove by induction with aid of Lemma [3.3](#page-10-1) for $(\alpha, \beta) = (u, v)$. Note that $R_1 = \{0\}$. By Lemma [3.2,](#page-7-2) we have $u < 0 < v$. From definition, any singleton set strictly interlaces the empty set R_0 . Now, we can suppose, for some $m \geq 0$, that $|R_{m+1}| = m+1$, $R_{m+1} \subset (u, v)$, and R_{m+1} strictly interlaces R_m . Let $n \geq 0$. From Recurrence [\(1.2\)](#page-1-1), every polynomial $W_n(z)$ is of degree n. By Lemma [3.2,](#page-7-2) we have $B(x) < 0$ for $x \in R_n$, $W_n(u)W_{n+2}(u) > 0$ and $W_n(v)W_{n+2}(v) > 0$. By Lemma [3.3,](#page-10-1) we obtain the real-rootedness, the bound (u, v) and the strict interlacing property. By Theorem [2.3,](#page-3-0) we have $\{x_{\Delta}^{\pm}\}\subseteq\clubsuit$. By Lemma [3.2,](#page-7-2) we have $\{u, v\}\backslash\{x_{\Delta}^{\pm}\}\subseteq\spadesuit$. Hence both the numbers u and v are limits of zeros. This completes the proof. \square

$$
\qquad \qquad \Box
$$

We remark that the sharpness of the bound (u, v) can be shown by using the totally different method demonstrated in the proof of Theorem 4.5 in [\[12\]](#page-12-8).

3.2. Case $x_A = x_B$. Suppose that $x_A = x_B$. Then Eq. [\(3.1\)](#page-7-5) reduces to

$$
u = \begin{cases} x_{\Delta}^-, & \text{if } a < 2 \text{ and } F \le 0 \\ x_g^+, & \text{if } a < 1 \text{ and } F > 0 \\ x_g^-, & \text{otherwise} \end{cases} \quad \text{and} \quad v = x_{\Delta}^+ = x_A = x_B.
$$

In an analogue with Lemma [3.2,](#page-7-2) we have Lemma [3.5.](#page-11-1)

Lemma 3.5. Let $a, c > 0$ and $b, d < 0$. If $x_A = x_B$, then $u \le x_{\Delta}^-, u < 0$, $W_n(u)(-1)^n > 0$, and $u \in \spadesuit$ as if $u \neq x_\Delta^-$.

Proof. Same to the proof of Lemma [3.2.](#page-7-2)

Now we can demonstrate the root distribution of the polynomials $\{W_n(z)\}.$

Theorem 3.6. Let $a, c > 0$ and $b, d < 0$ such that $x_A = x_B$. Let $\{W_n(z)\}_n$ be a sequence of polynomials satisfying Recurrence [\(1.2\)](#page-1-1) with $W_0(z) = 1$ and $W_1(z) = z$. Then the function $U_n(z) = W_n(z)/A^{\lfloor n/2 \rfloor}(z)$ is a polynomial, with all its zeros lying in the interval (u, x_B) . Moreover, the interval (u, x_B) is sharp in the sense that both the numbers u and x_B are limits of zeros of the polynomials $U_n(z)$.

Proof. By Recurrence [\(1.2\)](#page-1-1), the functions $U_n(z)$ satisfy the recurrence

(3.15)
$$
U_n(z) = \begin{cases} U_{n-1}(x) + c' \cdot U_{n-2}(x), & \text{if } n \text{ is even,} \\ A(x)U_{n-1}(x) + c' \cdot U_{n-2}(x), & \text{if } n \text{ is odd,} \end{cases}
$$

where $c' = c/a$, with $U_0(z) = 1$ and $U_1(z) = z$. It follows immediately that the function $U_n(z)$ is a polynomial of degree $\lceil n/2 \rceil$. Let R'_n be the zero set of $U_n(z)$.

We shall show by induction that the zeros z_j of $U_n(z)$ strictly interlaces the zeros x_j of $U_{n-1}(z)$ from the left, in the interval (u, x_B) , i.e.,

$$
(3.16) \quad \begin{cases} u < z_1 < z_2 < \dots < z_{\lceil \frac{n}{2} \rceil} < x_{\lceil \frac{n-1}{2} \rceil} < x_B, \\ u < z_1 < z_2 < \dots < z_{\lceil \frac{n-1}{2} \rceil} < x_{\lceil \frac{n-1}{2} \rceil} < z_{\lceil \frac{n}{2} \rceil} < x_B, \quad \text{if } n \text{ is odd.} \end{cases}
$$

We make some preparations. First, by Recurrence (3.15) , it is direct to show by induction that $U_n(x_B) > 0$. Second, by Lemma [3.5,](#page-11-1) we have $u \le x_\Delta^ < x_\Delta^+ = x_A$ and $W_n(u)(-1)^n > 0$. Therefore, we have $A(u) < 0$ and thus

$$
U_n(u)(-1)^{|n/2|} > 0
$$

In particular, we have $U_2(u) < 0$. Since $U_2(u) = z + c'$, we have $u < -c' < 0 < x_B$. This checks the truth for $n = 2$. Let $n \geq 3$. By induction hypothesis, the set R'_{n-1} strictly interlaces R'_{n-2} from the left. Therefore, we have

$$
U_{n-2}(x_j)(-1)^{\lceil n/2+j \rceil} > 0
$$
 for $j \leq \lceil (n-1)/2 \rceil$.

By Recurrence [\(3.15\)](#page-11-2), the number $U_n(x_j)$ has the same sign as the number $U_{n-2}(x_j)$, that is, $U_n(x_j)(-1)^{\lceil n/2+j \rceil} > 0$. By using the intermediate value theorem, we derive the desired [\(3.16\)](#page-11-3).

Same to the proof of Theorem [3.4,](#page-10-0) one may show the minimality of the interval (u, x_B) as a bound of the zeros of polynomials $W_n(z)$. Note that $x_{\Delta}^- \neq x_{\Delta}^+$. By Theorem [2.3,](#page-3-0) each point in the interval J_{Δ} is a limit of zeros of the polynomials $W_n(z)$. Therefore, each point in J_Δ is a limit of zeros of the polynomials $U_n(z)$, and the interval $(u, x_B) = (u, x_{\Delta}^+)$ becomes the best bound of the union of zeros of all polynomials $U_n(z)$. This completes the proof.

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