STOCHASTIC SOLUTIONS FOR SPACE-TIME FRACTIONAL EVOLUTION EQUATIONS ON BOUNDED DOMAIN

Lorenzo Toniazzi

ABSTRACT. Space-time fractional evolution equations are a powerful tool to model diffusion displaying space-time heterogeneity. We prove existence, uniqueness and stochastic representation of classical solutions for an extension of Caputo evolution equations featuring time-nonlocal initial conditions. We discuss the interpretation of the new stochastic representation. As part of the proof a new result about inhomogeneous Caputo evolution equations is proven.

1. INTRODUCTION

It is a classical result that the solution to the standard heat equation $\partial_t u = \Delta u, u(0) = \phi_0$ allows the stochastic representation $u(t,x) = \mathbf{E}[\phi_0(X^{x,2}(t))]$, where $X^{x,2}$ is a Brownian motion started at $x \in \mathbb{R}^d$. Space-time fractional evolution equations (EEs) extend the heat equation by introducing space-time heterogeneity. This often is done by considering the Caputo EE $D_0^{\beta} u = -(-\Delta)^{\frac{\alpha}{2}} u$, where one substitutes the local operators ∂_t and Δ with fractional analogues. Respectively, the Caputo derivative $D_0^{\beta} u(t) = c_{\beta} \int_0^t u'(r)(t-r)^{-\beta} dr$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}u(x) = \mathcal{F}^{-1}(|\xi|^{\alpha}\mathcal{F}u(\xi))(x)$, where $\beta \in (0,1), \alpha \in (0,2)$, $c_{\beta} = \Gamma(1-\beta)^{-1}$ and \mathcal{F} is the Fourier transform (for standard references see [21, 14]). It is well known that the fundamental solution to the Caputo EE is the law of the non-Markovian anomalous diffusion $Y^{x}(t) = X^{x,\alpha}(\tau_0(t))$ (see, e.g., [39]). Here $X^{x,\alpha}$ is the rotationally symmetric α -stable Lévy process started at $x \in \mathbb{R}^d$ and $\tau_0(t)$ is the inverse process of the β -stable subordinator $X^{\beta}(t)$. The density of this beautiful formula was first observed in [43]. The time change interpretation first appeared in [35, 38], based on [5]. The process Y^x displays space-heterogeneity due to the jump nature of $X^{x,\alpha}$. Also time-heterogeneity features in Y^x , as the time change $t \mapsto \tau_0(t)$ is constant precisely when the subordinator $t \mapsto X^{\beta}(t)$ jumps, so that $t \mapsto Y^{x}(t)$ is trapped on such time intervals. This interesting trapping phenomenon leads to the process Y^x spreading at a slower rate than $X^{x,\alpha}$. Indeed, in the physics literature the anomalous diffusion Y^x is often referred

Date: July 7, 2021.

²⁰¹⁰ Mathematics Subject Classification. 26A33, 34A08, 35A09, 35C15, 60H30, 60G52.

Key words and phrases. Inhomogeneous Caputo evolution equation, restricted fractional Laplacian, Mittag-Leffler functions, stable Lévy processes, nonlocal boundary condition, time change, Feller semigroup.

The author is funded by the EPSRC, UK.

to as a sub-diffusion when $\alpha = 2$ (see, e.g., [48, 42, 33]). See [38] for a characterisation of Y^x as the scaling limit of continuous time random walks with heavy-tailed waiting times. See [7] for a characterisation of Y^x as the scaling limit of random conductance models or asymmetric Bouchaud's trap models ($\alpha = 2$). See [32, 34] for sample path properties of Y^x , and [20, 18] for heat kernel asymptotic formulas. Existence of classical solutions for Caputo EEs is generally a subtle problem. The works [24, 6, 2] tackle classical solutions on unbounded domains. Meanwhile the works [19, 36, 37, 31] consider bounded domains, and all their proofs rely on the spectral decomposition of the spatial operator. Stochastic representations for solutions to time-nonlocal equations is an active area of theoretical research (see, e.g., [4, 16, 28, 18]). Partly because they provide formulas in the general absence of closed forms along with suggesting probabilistic proof methods. Moreover, such representations can be useful for particle tracking codes (see, e.g., [49]). Let us remark that Caputo EEs are applied in a variety of fields, such as physics, finance, economics, biology and hydrogeology (see, e.g., [43, 45, 46, 8, 27]).

In this work we focus on the following extension of the Caputo EE: the inhomogeneous space-time fractional EE on bounded domain with Dirichlet boundary conditions and time-nonlocal initial condition

$$\begin{cases} D_{\infty}^{\beta}\tilde{u}(t,x) = \Delta_{\Omega}^{\frac{\alpha}{2}}\tilde{u}(t,x) + g(t,x), & \text{in } (0,T] \times \Omega, \\ \tilde{u}(t,x) = 0, & \text{in } [0,T] \times \partial\Omega, \\ \tilde{u}(t,x) = \phi(t,x), & \text{in } (-\infty,0] \times \Omega, \end{cases}$$
(1.1)

where $\Omega \subset \mathbb{R}^d$ is a regular domain, $\Delta_{\Omega}^{\frac{\alpha}{2}}$ is the restricted fractional Laplacian¹, and the time operator $-D_{\infty}^{\beta}$ is the generator of the inverted β -stable subordinator²

$$D_{\infty}^{\beta}f(t) = \int_0^{\infty} (f(t-r) - f(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}}, \quad t \in \mathbb{R}.$$
 (1.2)

As the main result of this work we prove existence and uniqueness of classical solutions to problem (1.1) along with the stochastic representation for the solution

$$\tilde{u}(t,x) = \mathbf{E} \left[\phi \left(-X^{t,\beta}(\tau_0(t)), X^{x,\alpha}(\tau_0(t)) \right) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}} \right] + \mathbf{E} \left[\int_0^{\tau_0(t) \wedge \tau_\Omega(x)} g \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right],$$
(1.3)

where the processes $-X^{t,\beta} = t - X^{\beta}$ and $X^{x,\alpha}$ are independent, and $\tau_{\Omega}(x)$ is the first exit time of $X^{x,\alpha}$ from Ω . To see why problem (1.1) extends the Caputo EE, let $\phi(t) = \phi(0)$

¹We define $\Delta_{\Omega}^{\frac{\alpha}{2}}$ on functions on Ω , so that the Euclidean boundary $\partial\Omega$ makes sense in (1.1). In the literature the operator $\Delta_{\Omega}^{\frac{\alpha}{2}}$ is often defined through the application of the singular integral definition of $-(-\Delta)^{\frac{\alpha}{2}}$ to functions vanishing outside Ω (see, e.g., [13]). ²The operator D_{∞}^{β} is often referred to as the Marchaud derivative in the fractional calculus literature

²The operator D_{∞}^{ρ} is often referred to as the Marchaud derivative in the fractional calculus literature (see, e.g., [44]).

for every $t \in (-\infty, 0)$ and g = 0 in both (1.1) and (1.3). Then

$$D_{\infty}^{\beta}\tilde{u}(t) = \int_{0}^{t} (\tilde{u}(t-r) - \tilde{u}(t)) \frac{\Gamma(-\beta)^{-1}dr}{r^{1+\beta}} - \frac{\phi(0) - \tilde{u}(t)}{\Gamma(1-\beta)} t^{-\beta} = D_{0}^{\beta}u(t),$$

where u is the restriction of \tilde{u} to $t \geq 0$, and one obtains the homogeneous Caputo EE and its solution, respectively. The recent works [17, 22] introduced a class of EEs that formally includes (1.1). They are motivated by the success of related nonlocal EEs arising in image processing, peridynamics and heat conduction (see, e.g., [25, 12, 47, 26]), and the general lack of alternatives to Caputo-type time-nonlocal models. Part of their intent is to introduce initial conditions on the 'past' (ϕ on $(-\infty, 0) \times \Omega$). Our stochastic solution (1.3) appears to be new, and it provides an interesting interpretation for the time-nonlocal initial condition ϕ . This is because the overshoot $W(t) = X^{t,\beta}(\tau_0(t))$ is the waiting/trapping time of the anomalous diffusion $X^{x,\alpha}(\tau_0(t))$. We discuss an interpretation where the values of ϕ on $(-\infty, 0) \times \Omega$ describe the initial condition at time 0 with respect to the 'depth' of Ω , rather than the 'past' of Ω . To the best of our knowledge, there are no classicalwellposedness results for the EE (1.1). Related weak-wellposedness results can be found in [17, 22] (for certain general Lévy kernels in (1.2)) and indirectly in [40] (for abstract Markovian generators), meanwhile [1] considers uniqueness of weak solutions. Worth mentioning that our simple Lemma 5.5 allows to obtain wellposedness and regularity results for EEs such as (1.1) as corollaries of theorems concerning inhomogeneous Caputo EEs (see, e.g., [24, 2]). To see why the stochastic representation (1.3) is natural, one can formally apply the classical probabilistic intuition for elliptic boundary value problems (see, e.g., $[23, Introduction, \S3]$) to problem (1.1) rewritten as

$$\begin{cases} \mathcal{L}\tilde{u} = -g, & \text{in } \Gamma, \\ \tilde{u} = \phi, & \text{in } \partial\Gamma, \end{cases}$$
(1.4)

where $\mathcal{L} = (-D_{\infty}^{\beta} + \Delta_{\Omega}^{\frac{\alpha}{2}})$ is the generator of the process $\{(-X^{t,\beta}(s), X^{x,\alpha}(s))\mathbf{1}_{\{s<\tau_{\Omega}(x)\}}\}_{s\geq 0}$ taking values in $(-\infty, T] \times \Omega$, $\Gamma = (0, T] \times \Omega$, and $\partial\Gamma := (-\infty, 0] \times \Omega \cup [0, T] \times \partial\Omega$, with $\phi = 0$ on $(0, T] \times \partial\Omega$.

To prove our main result, Theorem 5.6, we derive two results of independent interest. Namely:

• Theorem 4.6: the stochastic representation

$$u(t,x) = \mathbf{E} \left[\phi_0 \left(X^{x,\alpha}(\tau_0(t)) \right) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}} \right] + \mathbf{E} \left[\int_0^{\tau_0(t) \land \tau_\Omega(x)} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right],$$
(1.5)

is the unique classical solution to the inhomogeneous Caputo EE on bounded domain

$$\begin{cases} D_0^\beta u(t,x) = \Delta_\Omega^{\frac{\alpha}{2}} u(t,x) + f(t,x), & \text{in } (0,T] \times \Omega, \\ u(t,x) = 0, & \text{in } [0,T] \times \partial \Omega, \\ u(t,x) = \phi_0(x), & \text{in } \{0\} \times \Omega; \end{cases}$$
(1.6)

• Theorem 3.9: the stochastic representation (1.5) is a weak solution to problem (1.6).

Let us outline our proof strategy for Theorem 5.6. By plugging the values of ϕ in \tilde{u} , it is not hard to show the equivalence of classical solutions to problem (1.1) and to problem (1.6) with forcing term $f = g - D_{\infty}^{\beta} \phi$ and initial condition $\phi_0 = \phi(0)$ (see Lemma 5.5). Moreover, a Dynkin formula argument proves that the respective stochastic representations (1.3) and (1.5) agree (see Lemma 5.1). Hence, it is enough to prove Theorem 4.6. We do so by proving Theorem 3.9 and then showing the required regularity of the candidate solution (1.5). The main feature of our regularity assumption on the data ϕ and g is the differentiability in time. This is a consequence of the regularity assumption on f in Theorem 4.6, which we discuss now. Theorem 4.6 extends the proof of [19, Theorem 5.1], where problem (1.6) is treated for f = 0. This proof uses separation of variables combing eigenfunction expansions of $\Delta_{\Omega}^{\frac{\alpha}{2}}$ with Mittag-Leffler solutions to the Caputo initial value problem. Our separation of variables formula for the second term in (1.5) reads

$$\sum_{n=1}^{\infty} \psi_n(x) u_n(t) = \sum_{n=1}^{\infty} \psi_n(x) \int_0^t \langle f(s), \psi_n \rangle (t-s)^{\beta-1} \beta E_\beta'(-\lambda_n (t-s)^\beta) \, ds,$$

where $E_{\beta}(t) = \sum_{k=0}^{\infty} t^k \Gamma(k\beta + 1)^{-1}$ is a Mittag-Leffler function, $\{\lambda_n, \psi_n\}_{n \in \mathbb{N}}$ is the system of eigenvalues-eigenfunctions of $\Delta_{\Omega}^{\frac{\alpha}{2}}$ and $\langle \cdot, \cdot \rangle$ is the inner product on Ω . Unsurprisingly, each u_n is the solution to the inhomogeneous Caputo initial value problem $D_0^\beta u_n(t) =$ $-\lambda_n u_n(t) + \langle f(t), \psi_n \rangle, \ u_n(0) = 0$ (see [21, Theorem 7.2]). As we require differentiability of $t \mapsto u(t)$, we want to differentiate each $t \mapsto u_n(t)$. To compensate for the singularity of the Mittag-Leffler kernel $t^{\beta-1}E'_{\beta}(-\lambda_n t^{\beta})$ we require differentiability of $t \mapsto f(t)$. Note that for the space fractional heat equation $(\beta = 1)$ the Mittag-Leffler kernel is an exponential, and so continuity of f is enough to differentiate the u_n 's. Related results in the literature also require differentiability on f (see, e.g. [2, Theorem 7.3]). Briefly, the arguments for Theorem 3.9 reduce the Caputo EE (1.6) to a Poisson equation with zero boundary conditions on $\{0\} \times \Omega \cup [0,T] \times \partial \Omega$ by constructing space-time sub-Feller semigroups. We rely on the fact that the generator $-D_0^\beta$ only requires boundary conditions on the trivial set $\{0\}$. These arguments are an extension of the ideas in [28], and they appear versatile. For example, they can be used to prove stochastic weak solutions for problem (1.1) with general nonlocal operators in both space and time (ongoing work with the authors in [22]). As far as we know, stochastic representations for solutions such as (1.5) for time-nonlocal EEs appear in [28], meanwhile in [3] the solution is given a representation via the superposition

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principle. Possibly worth mentioning that we do not invoke [5, Theorem 3.1] and all our methods work for the standard Laplacian case $\alpha = 2$.

This work is structured as follows: in Section 2 we provide general notation and basic results about several stochastic processes obtained from $-X^{t,\beta}$ and $X^{x,\alpha}$, with a focus on semigroup results. In Section 3 we prove Theorem 3.9. In Section 4 we prove Theorem 4.6. In Section 5 we prove that the stochastic representation (1.3) is the unique classical solution to the EE (1.1). In Section 6 we discuss an interpretation of the stochastic representation (1.3).

2. Preliminaries

2.1. General notation. We denote by \mathbb{N} , \mathbb{R}^d , $\Gamma(\cdot)$, $\mathbf{1}_A(\cdot)$, $a \wedge b$, a.e., lhs and rhs, the set of natural numbers, the *d*-dimensional Euclidean space, the gamma function, the indicator function of the set A, the minimum between $a, b \in \mathbb{R}$, the statements almost everywhere with respect to Lebesgue measure, left hand side and right hand side, respectively. We define the one parameter Mittag-Leffler function for $\beta \in (0,1)$ as $E_{\beta}(t) = \sum_{k=0}^{\infty} t^k \Gamma(k\beta + 1)^{-1}$, $t \geq 0$. We define the Banach spaces

$$\begin{split} B(A) &= \{f: A \to \mathbb{R} \text{ is bounded and measurable}\},\\ C(K) &= \{f \in B(K): f \text{ is continuous}\},\\ C_{\partial\Omega}(\Omega) &= \{f \in C(\overline{\Omega}): f = 0 \text{ on } \partial\Omega\},\\ C_0([0,T]) &= \{f \in C([0,T]): f(0) = 0\},\\ C_\infty((-\infty,T]) &= \{f \in B((-\infty,T]): f \text{ is continuous and vanishes at infinity}\},\\ C_{\partial\Omega}([0,T] \times \Omega) &= \{f \in C([0,T] \times \overline{\Omega}): f = 0 \text{ on } \partial\Omega\},\\ C_{0,\partial\Omega}([0,T] \times \Omega) &= \{f \in C_{\partial\Omega}([0,T] \times \Omega): f(0) = 0\},\\ C_{\infty,\partial\Omega}((-\infty,T] \times \Omega) &= \{f \in B((-\infty,T] \times \Omega): f \text{ is continuous and vanishes at infinity}\},\\ C_{b,\partial\Omega}((-\infty,T] \times \Omega) &= \{f \in B((-\infty,T] \times \overline{\Omega}): f \text{ is continuous and } f = 0 \text{ on } \partial\Omega\}, \end{split}$$

all equipped with the supremum norm, where A is any subset of \mathbb{R}^d , the set $K \subset \mathbb{R}^d$ is compact, the set $\Omega \subset \mathbb{R}^d$ is bounded and open, $T \ge 0$. For a function $f : A \to \mathbb{R}$ we denote

its supremum norm by either $||f||_{\infty}$ or $||f||_{C(A)}$. We define the spaces

$$\begin{split} C(O) &= \{f: O \to \mathbb{R} \text{ is continuous}\},\\ C^k(\Omega) &= \{f \in C(\Omega): f \text{ is } k\text{-times continuously differentiable}\},\\ C^k_c(\Omega) &= \{f \in C(\Omega): f \in C^k(\Omega) \text{ and compactly supported}\},\\ C^\infty_c(\Omega) &= \{f \in C(\Omega): f \text{ is smooth and compactly supported}\},\\ C^1([0,T]) &= \{f, f' \in C([0,T])\},\\ C^1_0([0,T]) &= \{f, f' \in C_0([0,T])\},\\ C^1_\infty((-\infty,T]) &= \{f, f' \in C_\infty((-\infty,T])\},\\ C^{1,k}((0,T) \times \Omega) &= \{f \in C((0,T) \times \Omega): f \text{ is 1-time and } k\text{-times continuously}\\ &\quad \text{ differentiable in time and space, respectively}\},\\ C^{1,k}_c((0,T) \times \Omega) &= \{f \in C^{1,k}((0,T) \times \Omega): f \text{ is compactly supported}\},\\ C^{1,k}_{\partial\Omega}([0,T] \times \Omega) &= \{f \in C_{\partial\Omega}([0,T] \times \Omega): f \in C^{1,0}((0,T) \times \Omega), f' \in C_{\partial\Omega}([0,T] \times \Omega)\},\\ C^{n,k}_{\infty,\partial\Omega}((-\infty,T] \times \Omega) &= \{f \in C_{\infty,\partial\Omega}((-\infty,T] \times \Omega): \text{ all derivatives up to order } n \text{ in time}\\ &\quad \text{ and } k \text{ in space exist and belong to } C_{\infty,\partial\Omega}((-\infty,T] \times \Omega)\}, \end{split}$$

where the set $O \subset \mathbb{R}^d$ is open. We write $C^{1,0}_{\infty,\partial\Omega}((-\infty,T] \times \Omega) = C^1_{\infty,\partial\Omega}((-\infty,T] \times \Omega)$ Ω and $C^1_{b,\partial\Omega}((-\infty,T] \times \Omega) = \{f, \partial_t f \in C_{b,\partial\Omega}((-\infty,T] \times \Omega)\}$. By $(L^1(O), \|\cdot\|_{L^1(O)}), \|\cdot\|_{L^1(O)})$ $(L^2(O), \|\cdot\|_{L^2(O)})$ and $(L^{\infty}(O), \|\cdot\|_{L^{\infty}(O)})$ we mean the standard Banach spaces of realvalued Lebesgue integrable, square-integrable and essentially bounded functions on O, respectively. Without risk of confusion we write $\|\cdot\|_{L^{\infty}(O)} = \|\cdot\|_{\infty}$. We denote by $\|L\|$ the operator norm of a bounded linear operator L between Banach spaces. Given two sets of real-valued functions F and \tilde{F} , we define $F \cdot \tilde{F} := \{f\tilde{f} : f \in F, \tilde{f} \in \tilde{F}\}$, and by $\text{Span}\{F\}$ we mean the set of all linear combinations of functions in F. The notation we use for an E-valued stochastic process started at $x \in E$ is $X^x = \{X^x(s)\}_{s>0}$. Note that the symbol t will often be used to denote the starting point of a stochastic process with state space $E \subset \mathbb{R}$. By a strongly continuous contraction semigroup P we mean a collection of linear operators $P_s: B \to B, s \ge 0$, where B is a Banach space, such that $P_{s+r} = P_s P_r$, for every $s, r \ge 0$, P_0 is the identity operator, $\lim_{s\downarrow 0} P_s f = f$ in B, for every $f \in B$, and $\sup_s ||P_s|| \leq 1$. The generator of the semigroup P is defined as the pair $(\mathcal{L}, \text{Dom}(\mathcal{L}))$, where $\text{Dom}(\mathcal{L}) := \{f \in B : \mathcal{L}f := \lim_{s \downarrow 0} s^{-1}(P_s f - f) \text{ exists in } B\}$. We say that a set $C \subset \text{Dom}(\mathcal{L})$ is a core for $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ if the generator equals the closure of the restriction of \mathcal{L} to C. We say that a set $C \subset B$ is invariant under P if $P_s C \subset C$ for every s > 0. If a set C is invariant under P and a core for $(\mathcal{L}, \text{Dom}(\mathcal{L}))$, then we say that C is an invariant core for $(\mathcal{L}, \text{Dom}(\mathcal{L}))$. For a given $\lambda \geq 0$ we define the resolvent of P by $(\lambda - \mathcal{L})^{-1} := \int_0^\infty e^{-\lambda s} P_s ds$, and recall that for $\lambda > 0$, $(\lambda - \mathcal{L})^{-1} : B \to \text{Dom}(\mathcal{L})$ is a bijection and it solves the abstract resolvent equation

$$\mathcal{L}(\lambda - \mathcal{L})^{-1}f = \lambda(\lambda - \mathcal{L})^{-1}f - f, \quad f \in B_{2}$$

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see for example [23, Theorem 1.1]. By a *sub-Feller semigroup* we mean a strongly continuous contraction semigroup on any of the Banach spaces of continuous functions defined above such that P preserves non-negative functions. A *Feller semigroup* is a sub-Feller semigroup such that its extension to bounded measurable functions preserves constants.

2.2. Fractional derivatives, stable processes and related space-time semigroups.

Definition 2.1. For parameters $\beta \in (0,1)$ and $\alpha \in (0,2)$, we define: the Marchaud derivative D_{∞}^{β} by formula (1.2); the Caputo derivative D_{0}^{β} by

$$D_0^{\beta}f(t) = \int_0^t (f(t-r) - f(t)) \frac{\Gamma(-\beta)^{-1}dr}{r^{1+\beta}} + (f(0) - f(t)) \int_t^{\infty} \frac{\Gamma(-\beta)^{-1}dr}{r^{1+\beta}}, \quad t > 0,$$

and $D_0^{\beta}f(0) = \lim_{t\downarrow 0} D_0^{\beta}f(t)$; the restricted fractional Laplacian $\Delta_{\Omega}^{\frac{\alpha}{2}}$ by

$$\Delta_{\Omega}^{\frac{\alpha}{2}}f(x) = \lim_{\varepsilon \downarrow 0} \int_{\Omega \setminus B_{\varepsilon}(x)} (f(y) - f(x)) \frac{c_{\alpha,d} \, dy}{|x - y|^{d + \alpha}} - f(x) \int_{\mathbb{R}^d \setminus \Omega} \frac{c_{\alpha,d} \, dy}{|x - y|^{d + \alpha}}, \quad x \in \Omega,$$

and $\Delta_{\Omega}^{\frac{\alpha}{2}}f(z) = \lim_{x \to z} \Delta_{\Omega}^{\frac{\alpha}{2}}f(x)$ for $z \in \partial\Omega$, where $c_{\alpha,d}^{-1} = \int_{\mathbb{R}^d} \frac{1-\cos y_1}{|y|^{d+\alpha}} dy$, $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d and $B_{\varepsilon}(x)$ denotes the Euclidean ball of radius $\varepsilon > 0$ around $x \in \Omega$.

We now define several sub-Feller semigroups that relate to the fractional derivatives in Definition 2.1 and collect some results relevant for us. For $\beta \in (0,1)$, we denote by $X^{\beta} = \{X^{\beta}(s)\}_{s\geq 0}$ the standard β -stable subordinator, and by p_s^{β} the smooth density of $X^{\beta}(s), s > 0$.

Definition 2.2. For $\beta \in (0,1)$, we denote by $-X^{t,\beta} = \{-X^{t,\beta}(s) := t - X^{\beta}(s)\}_{s\geq 0}$ the inverted β -stable subordinator started at $t \in \mathbb{R}$, characterised by the Laplace transforms $\mathbf{E}[e^{-X^{0,\beta}(s)k}] = e^{-k^{\beta}s}, k, s > 0$. We define the first exit/passage times $\tau_0(t) = \inf\{s > 0 : t - X^{\beta}(s) \leq 0\}, t \in \mathbb{R}$.

Definition 2.3. For $\alpha \in (0,2)$, $d \in \mathbb{N}$, we denote by $X^{x,\alpha} = \{X^{x,\alpha}(s)\}_{s\geq 0}$ the rotationally symmetric α -stable Lévy process with values in \mathbb{R}^d , started at $x \in \mathbb{R}^d$, with characteristic functions $\mathbf{E}[e^{ik \cdot X^{0,\alpha}(s)}] = e^{-s|k|^{\alpha}}$, $k \in \mathbb{R}^d$, s > 0. We define the first exit times $\tau_{\Omega}(x) = \inf\{s > 0 : X^{x,\alpha}(s) \notin \Omega\}$, $x \in \mathbb{R}^d$.

Recall that the smooth density of $-X^{t,\beta}(s)$, s > 0, is supported $(-\infty, t)$ and it equals $p_s^{\beta}(t - \cdot)$, and that the law of $X^{x,\alpha}(s)$ is smooth for each s > 0 (see for example [14, page 10]).

Proposition 2.4. Fix T > 0. For the the inverted β -stable subordinator $-X^{t,\beta}$, denote the Feller semigroup $P^{\beta,\infty} = \{P_s^{\beta,\infty}\}_{s\geq 0}$ on $C_{\infty}((-\infty,T])$, by $P_s^{\beta,\infty}f(t) := \mathbf{E}[f(-X^{t,\beta}(s))]$, $s \geq 0$, denote by $(\mathcal{L}^{\infty}_{\beta}, \operatorname{Dom}(\mathcal{L}^{\infty}_{\beta}))$ the generator of $P^{\beta,\infty}$, and recall that $C^1_{\infty}((-\infty,T])$ is an invariant core for $(\mathcal{L}^{\infty}_{\beta}, \operatorname{Dom}(\mathcal{L}^{\infty}_{\beta}))$ with $\mathcal{L}^{\infty}_{\beta} = -D^{\beta}_{\infty}$ on $C^1_{\infty}((-\infty,T])$. (i) Define the absorbed process $-X_0^{t,\beta}$ by

$$-X_0^{t,\beta}(s) := \begin{cases} -X^{t,\beta}(s), & \text{if } s < \tau_0(t), \\ 0, & \text{if } s \ge \tau_0(t). \end{cases}$$
(2.1)

Then the process $-X_0^{t,\beta}$ induces a Feller semigroup on C([0,T]), denoted by $P^{\beta} = \{P_s^{\beta}\}_{s\geq 0}$, with generator $(\mathcal{L}_{\beta}, \text{Dom}(\mathcal{L}_{\beta}))$. Moreover, $C^1([0,T])$ is an invariant core for $(\mathcal{L}_{\beta}, \text{Dom}(\mathcal{L}_{\beta}))$ and

$$\mathcal{L}_{\beta} = -D_0^{\beta} \quad \text{on} \quad C^1([0,T]).$$

(ii) The sub-Feller semigroup $P^{\beta,\text{kill}} := P^{\beta}$ on $C_0([0,T])$ is the the sub-Feller semigroup induced by the killed version of the process (2.1), and its generator is $(\mathcal{L}_{\beta}^{\text{kill}}, \text{Dom}(\mathcal{L}_{\beta}^{\text{kill}})) = (\mathcal{L}_{\beta}, \text{Dom}(\mathcal{L}_{\beta}) \cap \{f(0) = 0\})$. Moreover, $C_0^1([0,T])$ is an invariant core for $(\mathcal{L}_{\beta}^{\text{kill}}, \text{Dom}(\mathcal{L}_{\beta}^{\text{kill}}))$ and

$$\mathcal{L}_{\beta}^{\text{kill}} = -D_0^{\beta} \quad \text{on} \quad C_0^1([0,T]).$$

(iii) The following three identities hold

$$\mathbf{E}\left[\tau_{0}(t)\right] = \frac{t^{\beta}}{\Gamma(\beta+1)}, \quad \mathbf{E}\left[e^{-\lambda\tau_{0}(t)}\right] = E_{\beta}(-\lambda t^{\beta}), \quad t, \lambda \ge 0, \text{ and}$$
(2.2)

$$\int_{0}^{\infty} p_{s}^{\beta}(t-r) \, ds = \frac{(t-r)^{\beta-1}}{\Gamma(\beta)}, \quad t > r.$$
(2.3)

(iv) The alternative representation of the Caputo derivative

$$D_0^{\beta} u(t) = \int_0^t u'(r) \, \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)}, \quad \text{for } 0 < t < T,$$

holds if $u \in C([0,T]) \cap C^1((0,T))$ and $u' \in L^1((0,T))$.

Proof.

(i) It is easy to prove that $P_s^{\beta}f(t) := \int_0^t f(r)p_s^{\beta}(t-r)\,dr + f(0)\int_{-\infty}^0 p_s^{\beta}(t-r)\,dr$ is a Feller semigroup on C([0,T]), and the corresponding process is indeed $-X_0^{t,\beta}$. By using the proof of [9, Proposition 14]³, it holds that $C^1([0,T]) \subset \text{Dom}(\mathcal{L}_{\beta})$, and that $\mathcal{L}_{\beta} = -D_0^{\beta}$ on $C^1([0,T])$. To prove that $C^1([0,T])$ is invariant under P^{β} , we directly compute for $g \in C^1([0,T])$, $t \in (0,T)$ and s > 0,

$$\partial_t P_s^{\beta} g(t) = \partial_t \left(\int_0^t g(t-r) p_s^{\beta}(r) \, dr + g(0) \int_{-\infty}^{-t} p_s^{\beta}(-r) \, dr \right)$$
$$= \int_0^t g'(t-r) p_s^{\beta}(r) \, dr \pm g(0) p_s^{\beta}(t).$$

³We select $c_+ = \Gamma(-\alpha)^{-1}$ and $c_- = 0$ in [9, Proposition 14]. In the statement of [9, Proposition 14] it is required that $F \in C^2([0,\infty))$, but $F \in C^1([0,\infty))$ is enough.

Then $C^1([0,T])$ is a dense subspace of $\text{Dom}(\mathcal{L}_\beta)$ which is invariant under P^β , and so it is a core for $(\mathcal{L}_\beta, \text{Dom}(\mathcal{L}_\beta))$ by [15, Lemma 1.34].

(ii) Similarly to part (i), it can be shown that $P_s^{\beta,\text{kill}}f(t) = \int_0^t f(r)p_s^{\beta}(t-r) dr$. To show $\text{Dom}(\mathcal{L}_{\beta}) \cap \{f(0) = 0\} \subset \text{Dom}(\mathcal{L}_{\beta}^{\text{kill}})$, let $f \in \text{Dom}(\mathcal{L}_{\beta}) \cap \{f(0) = 0\}$, then for some $\lambda > 0$, let $g \in C([0,T])$ such that

$$f(t) = \int_0^\infty e^{-\lambda s} P_s^\beta g(t) \, ds, \quad \text{and} \quad g(0) \frac{1}{\lambda} = \int_0^\infty e^{-\lambda s} P_s^\beta g(0) \, ds = f(0) = 0,$$

and so $g \in C_0([0,T])$. As $P_s^{\beta} = P_s^{\beta,\text{kill}}$ on $C_0([0,T])$, it follows that $f \in \text{Dom}(\mathcal{L}_{\beta}^{\text{kill}})$. The inclusion $\text{Dom}(\mathcal{L}_{\beta}) \cap \{f(0) = 0\} \supset \text{Dom}(\mathcal{L}_{\beta}^{\text{kill}})$ is immediate using $P_s^{\beta} = P_s^{\beta,\text{kill}}$ on $C_0([0,T])$. By equating a resolvent equation, it follows that $\mathcal{L}_{\beta}^{\text{kill}} = \mathcal{L}_{\beta}$ on $\text{Dom}(\mathcal{L}_{\beta}^{\text{kill}})$. Invariance of $C_0^1([0,T])$ can be proven as in part (i). The last statement now follows from part (i).

(iii) The first identity follows from the third identity (2.3). The second identity follows by [50, Theorem 2.10.2]. To prove the third identity (2.3), recall that

$$p_s^{\beta}(t-r) = s^{-1/\beta} p_1^{\beta}(s^{-1/\beta}(t-r)), \quad t > r,$$

and then compute

$$\int_0^\infty p_s^\beta(t,r)\,ds = (t-r)^{\beta-1} \int_0^\infty u^{-1/\beta} p_1^\beta(u^{-1/\beta})\,du = (t-r)^{\beta-1} \frac{1}{\Gamma(\beta)},$$

using the Mellin transform of the β -stable density p_1^{β} for the last equality (see for example [50, Theorem 2.6.3]).

(iv) This is a standard computation and we omit it.

We say that a bounded open set $\Omega \subset \mathbb{R}^d$ is a *regular set* if Ω satisfies the exterior cone condition at every point $\partial\Omega$, i.e. for each $x \in \partial\Omega$ there exists a finite right circular open cone V_x with vertex x, such that $V_x \subset \Omega^c$ (see [19, end of Section 4]). From now on Ω is always a regular set.

Proposition 2.5. Define the sub-process $X_{\Omega}^{x,\alpha}$ started at $x \in \Omega$ by

$$X_{\Omega}^{x,\alpha}(s) := \begin{cases} X^{x,\alpha}(s), & s < \tau_{\Omega}(x), \\ \text{cemetery}, & s \ge \tau_{\Omega}(x), \end{cases}$$

(i) Then $X_{\Omega}^{x,\alpha}$ induces a sub-Feller semigroup on $C_{\partial\Omega}(\Omega)$, which we denote by $P^{\Omega} = \{P_s^{\Omega}\}_{s\geq 0}$, and we denote its generator by $(\mathcal{L}_{\Omega}, \text{Dom}(\mathcal{L}_{\Omega}))$. Moreover if $u \in \text{Dom}(\mathcal{L}_{\Omega})$ then there exists a sequence $u_n \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega)$ such that $u_n \to u$ uniformly and $\Delta_{\Omega}^{\frac{\alpha}{2}} u_n \to \mathcal{L}_{\Omega} u$ uniformly on compact subsets of Ω . The transition density of $X_{\Omega}^{x,\alpha}(s)$, denoted by $p_s^{\Omega}(x, y)$, is jointly continuous in x and y, for every s > 0.

(ii) For every $u \in \text{Dom}(\mathcal{L}_{\Omega})$ and $\varphi \in C_c^2(\Omega)$ it holds

$$\int_{\Omega} \mathcal{L}_{\Omega} u\varphi \, dx = \int_{\Omega} u \Delta_{\Omega}^{\frac{\alpha}{2}} \varphi \, dx.$$
(2.4)

(iii) The semigroup P^{Ω} induces a strongly continuous contraction semigroup on $L^{2}(\Omega)$, and we denote its generator by $(\mathcal{L}_{\Omega,2}, \operatorname{Dom}(\mathcal{L}_{\Omega,2}))$. Moreover there exists a sequence of positive numbers $0 < \lambda_{1} < \lambda_{2} \leq \lambda_{3} \leq \ldots$, and an orthonormal basis $\{\psi_{n}\}_{n \in \mathbb{N}}$ of $L^{2}(\Omega)$, so that $P_{s}^{\Omega}\psi_{n} = e^{-\lambda_{n}s}\psi_{n}$ in $L^{2}(\Omega)$, for every $n \in \mathbb{N}$, s > 0. For $k \geq 1$, we denote by $\operatorname{Dom}(\mathcal{L}_{\Omega,2}^{k})$ the subset of $L^{2}(\Omega)$ such that $\|f\|_{\mathcal{L}_{\Omega,2}^{k}} := \left(\sum_{n=1}^{\infty} \lambda_{n}^{2k} \langle f, \psi_{n} \rangle^{2}\right)^{1/2} < \infty$. Moreover, P^{Ω} on $C_{\partial\Omega}(\Omega)$ has the same set of eigenvalues and eigenfunctions as P^{Ω} on $L^{2}(\Omega)$.

Proof. (i) The first two statements are a consequence of [4, Lemma 2.2 and Theorem 2.7]. The last statement follows by the strong Markov property along with joint continuity of the transition densities of $X^{x,\alpha}$ (see for example [19, Section 4]).

(ii) The operator $\Delta_{\Omega}^{\frac{\alpha}{2}}$ is self-adjoint in the sense that

$$\int_{\Omega} \Delta_{\Omega}^{\frac{\alpha}{2}} u\varphi \, dx = \int_{\Omega} u \Delta_{\Omega}^{\frac{\alpha}{2}} \varphi \, dx, \qquad (2.5)$$

if $\varphi \in C_c^2(\Omega)$ and $u \in C_{\partial\Omega}(\Omega) \cap C^2(\Omega)$. Now use the approximating sequence from part (i) of the current proposition to conclude.

(iii) These results can be found in [19, Section 4] and references therein.

In the next lemma we construct three sub-Feller semigroups by combining in space-time the sub-Feller semigroups defined so far. We combine them in a way that allows us to describe the newly constructed space-time generator as the closure of the sum of the time and space generators. This is how we give meaning to the boundary value problem viewpoint formally presented in (1.4).

Lemma 2.6. Consider the four tuples

$$(P^{\beta,\infty}, C_{\infty}((-\infty,T]), \mathcal{L}^{\infty}_{\beta}, \operatorname{Dom}(\mathcal{L}^{\infty}_{\beta})), \qquad (P^{\beta}, C([0,T]), \mathcal{L}_{\beta}, \operatorname{Dom}(\mathcal{L}_{\beta})), (P^{\beta, \operatorname{kill}}, C_{0}([0,T]), \mathcal{L}^{\operatorname{kill}}_{\beta}, \operatorname{Dom}(\mathcal{L}^{\operatorname{kill}}_{\beta})), \qquad (P^{\Omega}, C_{\partial\Omega}(\Omega), \mathcal{L}_{\Omega}, \operatorname{Dom}(\mathcal{L}_{\Omega})),$$

defined in Proposition 2.4, Proposition 2.4-(i), Proposition 2.4-(ii) and Proposition 2.5-(i), respectively. Let $\mathcal{C}^{\infty}_{\beta}$, \mathcal{C}_{β} , $\mathcal{C}^{kill}_{\beta}$ and \mathcal{C}_{Ω} be invariant cores for $(\mathcal{L}^{\infty}_{\beta}, \text{Dom}(\mathcal{L}^{\infty}_{\beta}))$, $(\mathcal{L}_{\beta}, \text{Dom}(\mathcal{L}_{\beta}))$, $(\mathcal{L}^{kill}_{\beta}, \text{Dom}(\mathcal{L}^{kill}_{\beta}))$ and $(\mathcal{L}_{\Omega}, \text{Dom}(\mathcal{L}_{\Omega}))$, respectively.

(i) Then $P^{\beta,\Omega} = \{P_s^{\beta}P_s^{\Omega}\}_{s\geq 0}$ is a sub-Feller semigroup on $C_{\partial\Omega}([0,T]\times\Omega)$. The generator $(\mathcal{L}_{\beta,\Omega}, \operatorname{Dom}(\mathcal{L}_{\beta,\Omega}))$ of $P^{\beta,\Omega}$ is the closure of

$$(\mathcal{L}_{\beta} + \mathcal{L}_{\Omega}, \operatorname{Span} \{ \mathcal{C}_{\beta} \cdot \mathcal{C}_{\Omega} \})$$
 in $C_{\partial \Omega}([0, T] \times \Omega),$

where P^{β} and \mathcal{L}_{β} act on the [0, T]-variable, and P^{Ω} and \mathcal{L}_{Ω} act on the Ω -variable.

(ii) Then $P^{\beta,\Omega,\text{kill}} = \{P_s^{\beta,\text{kill}}P_s^{\Omega}\}_{s\geq 0}$ is a sub-Feller semigroup on $C_{0,\partial\Omega}([0,T]\times\Omega)$. The generator $(\mathcal{L}_{\beta,\Omega}^{\text{kill}}, \text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}}))$ of $P^{\beta,\Omega,\text{kill}}$ is the closure of

$$(\mathcal{L}_{\beta}^{\text{kill}} + \mathcal{L}_{\Omega}, \text{Span}\{\mathcal{C}_{\beta}^{\text{kill}} \cdot \mathcal{C}_{\Omega}\})$$
 in $C_{0,\partial\Omega}([0,T] \times \Omega),$

where $P^{\beta,\text{kill}}$ and $\mathcal{L}^{\text{kill}}_{\beta}$ act on the [0, T]-variable, and P^{Ω} and \mathcal{L}_{Ω} act on the Ω -variable.

(iii) Then $P^{\beta,\Omega,\infty} = \{P_s^{\beta,\infty}P_s^{\Omega}\}_{s\geq 0}$ is a sub-Feller semigroup on $C_{\infty,\partial\Omega}((-\infty,T]\times\Omega)$. The generator $(\mathcal{L}^{\infty}_{\beta,\Omega}, \operatorname{Dom}(\mathcal{L}^{\infty}_{\beta,\Omega}))$ of $P^{\beta,\Omega,\infty}$ is the closure of

$$(\mathcal{L}^{\infty}_{\beta} + \mathcal{L}_{\Omega}, \operatorname{Span} \{ \mathcal{C}^{\infty}_{\beta} \cdot \mathcal{C}_{\Omega} \})$$
 in $C_{\infty,\partial\Omega}((-\infty, T] \times \Omega),$

where $P^{\beta,\infty}$ and $\mathcal{L}^{\infty}_{\beta}$ act on the $(-\infty, T]$ -variable, and P^{Ω} and \mathcal{L}_{Ω} act on the Ω -variable.

(iv) It holds that $P_s^{\beta,\Omega} = P_s^{\beta,\Omega,\text{kill}}$ on $C_{0,\partial\Omega}([0,T] \times \Omega)$, $\mathcal{L}_{\beta,\Omega} = \mathcal{L}_{\beta,\Omega}^{\text{kill}}$ on $\text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}})$, and $\text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}}) = \text{Dom}(\mathcal{L}_{\beta,\Omega}) \cap \{f(0) = 0\}.$

Proof. The proofs of (i), (ii) and (iii) can be found in Appendix A.II.

(iv) The first claim is an immediate consequence of $P^{\beta,\text{kill}} = P^{\beta}$ on $C_0([0,T])$. The second claim follows from the third by considering a resolvent equation. To prove the third claim, we show the equivalent statement

$$\operatorname{Dom}(\mathcal{L}_{\beta,\Omega}^{\operatorname{kill}}) \subset \operatorname{Dom}(\mathcal{L}_{\beta,\Omega}), \text{ and } \text{ if } u \in \operatorname{Dom}(\mathcal{L}_{\beta,\Omega}), \text{ then } u - u(0) \in \operatorname{Dom}(\mathcal{L}_{\beta,\Omega}^{\operatorname{kill}}).$$

The first inclusion is immediate using $P_s^{\beta,\Omega} = P_s^{\beta,\Omega,\text{kill}}$, on $C_{0,\partial\Omega}([0,T] \times \Omega)$. For the second part, let $u \in \text{Dom}(\mathcal{L}_{\beta,\Omega})$ and consider its resolvent representation for some $\lambda > 0$ and $g \in C_{\partial\Omega}([0,T] \times \Omega)$. Then

$$u(0,x) = \int_0^\infty e^{-\lambda s} P_s^\beta P_s^\Omega g(0,x) \, ds = \int_0^\infty e^{-\lambda s} P_s^\beta P_s^\Omega(g(0))(t,x) \, ds,$$

as $P_s^{\beta}g(0,x) = P_s^{\beta}(g(0))(t,x)$. Now consider

$$\begin{split} u(t,x) - u(0,x) &= \int_0^\infty e^{-\lambda s} P_s^\Omega P_s^\beta (g - g(0))(t,x) \, ds \\ &= \int_0^\infty e^{-\lambda s} P_s^\Omega P_s^{\beta,\text{kill}} (g - g(0))(t,x) \, ds \in \text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}}), \end{split}$$

where we use the fact that $P^{\beta,\text{kill}} = P^{\beta}$ on $C_{0,\partial\Omega}([0,T] \times \Omega)$ and that $g-g(0) \in C_{0,\partial\Omega}([0,T] \times \Omega)$.

Remark 2.7. Note that

$$(-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}g(t,x) = \int_0^\infty P_s^{\beta,\Omega}g(t,x)\,ds = \mathbf{E}\left[\int_0^{\tau_0(t)\wedge\tau_\Omega(x)} g\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right)\,ds\right],$$

for $g \in C_{0,\partial\Omega}([0,T] \times \Omega)$. Also, from now on we might write $\tau_{t,x}$ for $\tau_0(t) \wedge \tau_\Omega(x)$.

3. Stochastic weak solution for problem (1.6)

3.1. Definition of weak solution. Define the operator

$$-D_0^{\beta,*}\varphi(s) := \partial_s I_T^{1-\beta}\varphi(s) + \delta_0(ds)I_T^{1-\beta}\varphi(0),$$

where δ_0 is the delta-measure at 0, and the Riemann-Liouville integral $I_T^{1-\beta}$ is defined as

$$I_T^{1-\beta} f(s) := \int_s^T f(t) \, \frac{(t-s)^{-\beta} dt}{\Gamma(1-\beta)}, \quad s < T.$$

In the current section only the pairing $\langle \cdot, \cdot \rangle$ is defined as

$$\langle f,g \rangle := \int_0^T \int_\Omega f(t,x)g(t,x)\,dx\,dt.$$

Definition 3.1. Let $f \in L^{\infty}((0,T) \times \Omega)$ and $\phi_0 \in C_{\partial\Omega}(\Omega)$. A function $u \in L^2((0,T) \times \Omega)$ is said to be a *weak solution to problem* (1.6) if

$$\langle u, (-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\omega}{2}})\varphi \rangle = \langle -f, \varphi \rangle, \quad \text{for every } \varphi \in C_c^{1,2}((0,T) \times \Omega), \tag{3.1}$$

and $u(t) \to \phi_0$ a.e. as $t \downarrow 0$.

The next proposition motivates Definition 3.1.

Proposition 3.2. Let $\varphi \in C_c^1((0,T))$ and $u \in C([0,T]) \cap C^1((0,T))$ such that $u' \in L^1((0,T))$. Then

$$\int_0^T D_0^\beta u(t)\varphi(t)\,dt = -\int_0^T u(t)\left(\partial_t I_T^{1-\beta}\varphi(t)\right)\,dt - u(0)I_T^{1-\beta}\varphi(0)\,dt$$

Proof. Using Proposition 2.4-(iv), Fubini's Theorem and integration by parts, compute

$$\int_0^T D_0^\beta u(t)\varphi(t) dt = \int_{\mathbb{R}} \int_{\mathbb{R}} u'(s) \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \varphi(t) \mathbf{1}_{\{0 < t < T\}} \mathbf{1}_{\{0 < s < t\}} ds dt$$
$$= \int_{\mathbb{R}} u'(s) \mathbf{1}_{\{0 < s < T\}} \left(\int_s^T \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \varphi(t) dt \right) ds$$
$$= \int_0^T u'(s) I_T^{1-\beta} \varphi(s) ds$$
$$= -\int_0^T u(s) \partial_s I_T^{1-\beta} \varphi(s) ds - u(0) I_T^{1-\beta} \varphi(0).$$

From Proposition 3.2 and the identity in (2.5), it is straightforward to prove the following lemma.

Lemma 3.3. Let $\varphi \in C_c^{1,2}((0,T) \times \Omega)$ and $u \in C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega)$ such that $\partial_t u \in L^1((0,T) \times \Omega)$. Then

$$\langle u, (-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\alpha}{2}})\varphi \rangle = \langle (-D_0^{\beta} + \Delta_{\Omega}^{\frac{\alpha}{2}})u, \varphi \rangle.$$

3.2. Existence of a weak solution. Following [28], we define two auxiliary notions of solution for problem (1.6), starting from the abstract evolution equation

$$\mathcal{L}_{\beta,\Omega}u = -f \text{ on } (0,T] \times \overline{\Omega}, \quad u = \phi_0 \text{ on } \{0\} \times \overline{\Omega}, \quad u \in \text{Dom}(\mathcal{L}_{\beta,\Omega}).$$
(3.2)

Definition 3.4. Let $f \in C_{\partial\Omega}([0,T] \times \Omega)$ and $\phi_0 \in \text{Dom}(\mathcal{L}_{\Omega})$ such that $f(0) = -\mathcal{L}_{\Omega}\phi_0$. We say that a function $u \in C_{\partial\Omega}([0,T] \times \Omega)$ is a solution in the domain of the generator to problem (1.6) if u satisfies (3.2).

The next solution concept for problem (1.6) is defined as a pointwise approximation of solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$ such that the approximating forcing term $\{f_n\}_{n\in\mathbb{N}}$ satisfies a dominated convergence type of condition.

Definition 3.5. Let $f \in B([0,T] \times \overline{\Omega})$ and $\phi_0 \in \text{Dom}(\mathcal{L}_{\Omega})$. We say that a function $u \in B([0,T] \times \overline{\Omega})$ is a generalised solution to problem (1.6) if

$$u = \lim_{n \to \infty} u_n$$
 pointwise,

where each u_n is the solution in the domain of the generator for a corresponding forcing term $f_n \in C_{\partial\Omega}([0,T] \times \Omega)$ such that

$$f_n \to f$$
 a.e. on $(0,T] \times \Omega$, $\sup_n ||f_n||_{\infty} < \infty$, and $f_n(0) = -\mathcal{L}_{\Omega}\phi_0$ for each $n \in \mathbb{N}$.

Remark 3.6. Any generalised solution must satisfy the boundary conditions u = 0 on $[0, T] \times \partial \Omega$ and $u = \phi_0$ on $\{0\} \times \Omega$.

Lemma 3.7. Let $\phi_0 \in \text{Dom}(\mathcal{L}_{\Omega})$. Then

- (i) If $f + \mathcal{L}_{\Omega}\phi_0 \in C_{0,\partial\Omega}([0,T] \times \Omega)$, then there exists a unique solution in the domain of the generator to problem (1.6).
- (ii) If $f \in B([0,T] \times \overline{\Omega})$, then there exists a unique generalised solution to problem (1.6).
- (iii) Both solutions in part (i) and (ii) allow the stochastic representation (1.5).

Proof. (i) Observe that the potential $(-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}$ maps $C_{0,\partial\Omega}([0,T] \times \Omega)$ to itself. This follows from $P_s^{\beta,\Omega,\text{kill}}g \in C_{0,\partial\Omega}([0,T] \times \Omega)$ for $g \in C_{0,\partial\Omega}([0,T] \times \Omega)$, $s \ge 0$, and Dominated Convergence Theorem (DCT) with dominating function $G(s) := \|g\|_{\infty} \mathbf{P}[s < \tau_0(T)]$. Note that we use the first identity in (2.2) to prove that $G \in L^1((0,\infty))$. The potential $(-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}$ is also bounded by the inequality

$$\left| \left(-\mathcal{L}_{\beta,\Omega}^{\text{kill}} \right)^{-1} g(t,x) \right| \le \|g\|_{\infty} \mathbf{E} \left[\tau_0(T) \right], \quad g \in C_{0,\partial\Omega}([0,T] \times \Omega)$$

It then follows by [23, Theorem 1.1'] that $\bar{u} := (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(f + \mathcal{L}_{\Omega}\phi_0)$ is the unique solution to the abstract evolution equation

$$\mathcal{L}_{\beta,\Omega}^{\text{kill}}\bar{u} = -(f + \mathcal{L}_{\Omega}\phi_0) \text{ on } (0,T] \times \overline{\Omega}, \quad \bar{u} = 0 \text{ on } \{0\} \times \overline{\Omega}, \quad \text{and } \bar{u} \in \text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}}).$$
(3.3)

It is now enough to show that \bar{u} satisfies (3.3) if and only if $u = \bar{u} + \phi_0$ satisfies (3.2). For the 'if' direction, let $u \in \text{Dom}(\mathcal{L}_{\beta,\Omega})$ satisfy (3.2). Note that $u(0) = \phi_0$. Then $\bar{u} := u - \phi_0 \in \text{Dom}(\mathcal{L}_{\beta,\Omega}^{\text{kill}})$, and $\mathcal{L}_{\beta,\Omega}\bar{u} = \mathcal{L}_{\beta,\Omega}^{\text{kill}}\bar{u}$, by Lemma 2.6-(iv). So we can compute

$$\mathcal{L}_{\beta,\Omega}^{\text{kill}}\bar{u} = \mathcal{L}_{\beta,\Omega}(u - \phi_0) = \mathcal{L}_{\beta,\Omega}u - \mathcal{L}_{\Omega}\phi_0 = -f - \mathcal{L}_{\Omega}\phi_0$$

where we use

$$\mathcal{L}_{\beta,\Omega} 1 \phi_0 = (\mathcal{L}_\beta + \mathcal{L}_\Omega) 1 \phi_0 = \mathcal{L}_\Omega \phi_0,$$

from Lemma 2.6-(i) taking the invariant cores $C_{\beta} = \text{Dom}(\mathcal{L}_{\beta})$ and $C_{\Omega} = \text{Dom}(\mathcal{L}_{\Omega})$ (recalling that $\mathcal{L}_{\beta}1 = 0$). For the 'only if' direction, let \bar{u} satisfy (3.3), and define $u := \bar{u} + \phi_0$. Then with the same justifications as just above, compute

$$\mathcal{L}_{\beta,\Omega} u = \mathcal{L}_{\beta,\Omega}^{\text{kill}} \bar{u} + \mathcal{L}_{\beta,\Omega} \phi_0 = -(f + \mathcal{L}_\Omega \phi_0) + \mathcal{L}_\Omega \phi_0 = -f$$

It follows that

$$u = (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1} (f + \mathcal{L}_{\Omega} \phi_0) + \phi_0.$$

(ii) Let $f \in B([0,T] \times \overline{\Omega})$. Then $f + \mathcal{L}_{\Omega}\phi_0 \in B([0,T] \times \overline{\Omega})$. Now take a sequence $\{\tilde{f}_n\}_{n\mathbb{N}} \in C_{0,\partial\Omega}([0,T] \times \Omega)$ such that $\tilde{f}_n \to f + \mathcal{L}_{\Omega}\phi_0$ a.e., and $\sup_n \|\tilde{f}_n\|_{\infty} < \infty$. Define $f_n := \tilde{f}_n - \mathcal{L}_{\Omega}\phi_0$ for each $n \in \mathbb{N}$ and note that $f_n \to f$ a.e., $\sup_n \|\tilde{f}_n\|_{\infty} < \infty$ and $f_n(0) = -\mathcal{L}_{\Omega}\phi_0$, as required by Definition 3.5. Now, for each f_n consider the stochastic representation of the respective solution in the domain of the generator

$$u_n(t,x) = \mathbf{E}\left[\int_0^{\tau_{t,x}} f_n\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] + \mathbf{E}\left[\int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi_0\left(X^{x,\alpha}(s)\right) ds\right] + \phi_0(x).$$

Fix $(t, x) \in (0, T] \times \Omega$. Using absolute continuity with respect of Lebesgue measure of the laws of $-X^{t,\beta}(s)$ and $X^{x,\alpha}_{\Omega}(s)$ for each s > 0, and the bound $\mathbf{E}[\tau_{t,x}] \leq \mathbf{E}[\tau_0(t)] < \infty$, we can apply DCT twice to obtain as $n \to \infty$

$$\begin{split} \mathbf{E} \left[\int_0^{\tau_{t,x}} f_n \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right] &= \int_0^\infty P_s^{\beta, \text{kill}} P_s^\Omega f_n(t,x) \, ds \\ &\to \int_0^\infty P_s^{\beta, \text{kill}} P_s^\Omega f(t,x) \, ds \\ &= \mathbf{E} \left[\int_0^{\tau_{t,x}} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) \, ds \right], \end{split}$$

using as a dominating function $G := \sup_n \|f_n\|_\infty$ to show that for each s > 0

$$F_n(s) := P_s^{\beta,\text{kill}} P_s^{\Omega} f_n(t,x) \to P_s^{\beta,\text{kill}} P_s^{\Omega} f(t,x) =: F(s)$$

and the dominating function $G(s) := \sup_n ||f_n||_{\infty} \mathbf{P}[s < \tau_{t,x}]$ to show that

$$\int_0^\infty F_n(s)\,ds \to \int_0^\infty F(s)\,ds.$$

The convergence on $[0,T] \times \partial \Omega \cup \{0\} \times \overline{\Omega}$ is trivial. It follows that a generalised solution u exists and it is given by

$$u = (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(f + \mathcal{L}_{\Omega}\phi_0) + \phi_0.$$

Finally, independence of the approximating sequence proves uniqueness.

(iii) This is a standard application of Dynkin formula ([23, Theorem 5.1]) using the finite stopping times $\tau_{t,x}$, $(t,x) \in (0,T] \times \Omega$, namely

$$(-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(\mathcal{L}_{\Omega}\phi_{0})(t,x) = \mathbf{E}\left[\int_{0}^{\tau_{t,x}}\mathcal{L}_{\beta,\Omega}\phi_{0}\left(X^{x,\alpha}(s)\right)ds\right] = \mathbf{E}\left[\phi_{0}(X^{x,\alpha}(\tau_{t,x}))\right] - \phi_{0}(x).$$

We now show that the dual of $\mathcal{L}_{\beta,\Omega}$ is $(-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\alpha}{2}})$.

Lemma 3.8. Let $u \in \text{Dom}(\mathcal{L}_{\beta,\Omega})$. Then

$$\langle \mathcal{L}_{\beta,\Omega} u, \varphi \rangle = \langle u, (-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\alpha}{2}})\varphi \rangle, \text{ for every } \varphi \in C_c^{1,2}((0,T) \times \Omega).$$

Proof. By Lemma 2.6-(i) and Proposition 2.4-(i) we can pick a sequence

 $\{u_n\}_{n\in\mathbb{N}}\subset \operatorname{Span}\left\{C^1([0,T])\cdot\operatorname{Dom}(\mathcal{L}_\Omega)\right\},\$

such that $u_n \to u$ and $\mathcal{L}_{\beta,\Omega} u_n \to \mathcal{L}_{\beta,\Omega} u$ in $C_{\partial\Omega}([0,T] \times \Omega)$, with the additional property

$$\mathcal{L}_{\beta,\Omega}u_n = (-D_0^\beta + \mathcal{L}_\Omega)u_n, \quad \text{for every } n \in \mathbb{N}.$$
(3.4)

Hence, for every $\varphi \in C_c^{1,2}((0,T) \times \Omega)$, as $n \to \infty$

$$\langle \mathcal{L}_{\beta,\Omega} u, \varphi \rangle \leftarrow \langle \mathcal{L}_{\beta,\Omega} u_n, \varphi \rangle = \langle u_n, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi \rangle \rightarrow \langle u, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi \rangle,$$

where we use DCT for both limits, and for the equality we use the identity (3.4) along with Proposition 3.2 and the dual identity in Proposition 2.5-(ii).

We now combine Lemma 3.8 with the notion of generalised solution to obtain the main theorem of this section.

Theorem 3.9. Let $f \in L^{\infty}((0,T) \times \Omega)$ and $\phi_0 \in C_{\partial\Omega}(\Omega)$. Then the function $u \in B([0,T] \times \overline{\Omega})$ defined in (1.5) is a weak solution to problem (1.6).

Proof. Assume for the moment that $\phi_0 \in \text{Dom}(\mathcal{L}_\Omega)$. By the definition of a generalised solution we can take an approximating sequence of forcing terms $\{f_n\}_{n\in\mathbb{N}} \subset C_{\partial\Omega}([0,T]\times\Omega)$ such that $f_n \to f$ a.e., $\sup_n \|f_n\|_{\infty} < \infty$, and the respective solutions in the domain of the generator $\{u_n\}_{n\in\mathbb{N}}$ satisfy

 $u_n(0) = \phi_0 \text{ for all } n \in \mathbb{N}, \quad u_n \to u \text{ pointwise on } [0,T] \times \Omega, \quad \sup_n ||u_n||_{\infty} < \infty,$

where the last property is an immediate consequence of the stochastic representation (1.5). Hence, we obtain for every $\varphi \in C_c^{1,2}((0,T) \times \Omega)$, as $n \to \infty$

$$\langle -f,\varphi\rangle \leftarrow \langle -f_n,\varphi\rangle = \langle \mathcal{L}_{\beta,\Omega}u_n,\varphi\rangle = \langle u_n, (-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\alpha}{2}})\varphi\rangle \rightarrow \langle u, (-D_0^{\beta,*} + \Delta_{\Omega}^{\frac{\alpha}{2}})\varphi\rangle,$$

where we applied DCT for both limits, the first equality is due to the u_n 's being solutions in the domain of the generator, and the second equality holds as a consequence of Lemma 3.8.

Now, for $\phi_0 \in C_{\partial\Omega}(\Omega)$, let $\{\phi_{0,n}\}_{n\in\mathbb{N}} \subset \text{Dom}(\mathcal{L}_{\Omega})$ such that $\phi_{0,n} \to \phi_0$ in $C_{\partial\Omega}(\Omega)$. Let u_n be the generalised solution to problem (1.1) for $f \in B([0,T] \times \overline{\Omega})$ and $\phi_n \in \text{Dom}(\mathcal{L}_{\Omega})$, and u defined as in (1.5). Then $u_n \to u$ pointwise and $\sup_n ||u_n||_{\infty} < \infty$, which in turn implies by DCT

$$\langle -f,\varphi\rangle = \lim_{n\to\infty} \langle u_n, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle = \langle u, (-D_0^{\beta,*} + \Delta_\Omega^{\frac{\alpha}{2}})\varphi\rangle.$$

It is clear that the result holds for $f \in L^{\infty}((0,T) \times \Omega)$. Finally, the required convergence of u to the initial condition ϕ_0 follows by the argument in Remark 5.3, using the stochastic representation (1.5).

4. Stochastic classical solution for problem (1.6)

Definition 4.1. Let $f \in C((0,T] \times \Omega)$ and $\phi_0 \in C(\Omega)$. A function $u \in C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega)$, such that $|\partial_t u(t,x)| \leq Ct^{-\gamma}$, for every $(t,x) \in (0,T] \times \Omega$, for some $\gamma \in (0,1)$, C > 0, is said to be a *classical solution to problem* (1.6) if u satisfies the identities in (1.6), and for every $x \in \Omega$

$$\lim_{t \downarrow 0} |u(t, x) - \phi_0(x)| = 0.$$

In this section the pairing $\langle \cdot, \cdot \rangle$ is defined as

$$\langle f,g \rangle := \int_{\Omega} f(x)g(x) \, dx.$$

The proof of the main theorem of this section (Theorem 4.6), extends the eigenfunction expansion argument in [19, Theorem 5.1], using the next lemma as the key extra ingredient. Define for $\lambda \in \mathbb{R} \setminus \{0\}$ and $f \in C([0,T])$

$$F_{\lambda}[f](t) := (-\lambda)^{-1} \int_0^t f(r) \partial_t E_{\beta}(-\lambda(t-r)^{\beta}) dr, \quad t > 0.$$

Lemma 4.2. Let $\lambda > 0$ and $f \in C([0,T])$. Then

(i)

$$\mathbf{E}\left[\int_{0}^{\tau_{0}(t)}e^{-\lambda s}f(-X^{t,\beta}(s))\,ds\right]=F_{\lambda}\left[f\right](t),\quad t>0.$$

(ii) The bound

$$|F_{\lambda}[f](t)| \le \frac{c}{\lambda} ||f||_{\infty}, \quad t > 0,$$

$$(4.1)$$

holds, and if $f \in C^1([0,T])$ then

$$\left|\partial_t F_{\lambda}\left[f\right](t)\right| \le \frac{c}{\lambda} \left(\|f'\|_{\infty} + f(0)\frac{\lambda t^{\beta-1}}{1+\lambda t^{\beta}} \right), \quad t > 0,$$

$$(4.2)$$

for some positive constant c.

Proof. (i) Given the second identity in (2.2), it is enough to prove the equivalent identity

$$\mathbf{E}\left[\int_{0}^{\tau_{0}(t)} e^{-\lambda s} f(-X^{t,\beta}(s)) \, ds\right] + u_{0} \mathbf{E}\left[e^{-\lambda \tau_{0}(t)}\right] = F_{\lambda}\left[f\right](t) + u_{0} E_{\beta}(-\lambda t^{\beta}), \qquad (4.3)$$

where u_0 is some constant. We show that the lhs of (4.3) is the unique continuous solution to the Caputo initial value problem solved by the rhs of (4.3). Let $w \in C_0([0,T])$ such that $w' \in C([0,T])$. Then $u(t) := (\lambda - \mathcal{L}_\beta)^{-1}w(t) = \mathbf{E}[\int_0^{\tau_0(t)} e^{-\lambda s}w(-X^{t,\beta}(s)) ds]$ solves the resolvent equation

$$\mathcal{L}_{\beta}u = \lambda u - w, \quad u(0) = 0,$$

and $u \in \text{Dom}(\mathcal{L}_{\beta})$, by Proposition 2.4-(i). By the following computation

$$\begin{aligned} \partial_t u(t) &= \partial_t \int_0^t w(t-y) \left(\int_0^\infty e^{-\lambda s} p_s^\beta(y) \, ds \right) dy \\ &= w(0) \int_0^\infty e^{-\lambda s} p_s^\beta(t) \, ds + \int_0^t w'(t-y) \int_0^\infty e^{-\lambda s} p_s^\beta(y) \, ds \, dy, \qquad t > 0, \end{aligned}$$

it follows that $u \in C_0^1([0,T])$, and so $\mathcal{L}_{\beta}u = -D_0^{\beta}u$ by Proposition 2.4-(i). Let $u_0 \in \mathbb{R}$. Then $\bar{u} := u + u_0$ is a continuous solution to the Caputo initial value problem

$$-D_0^\beta \bar{u} = \mathcal{L}_\beta u - D_0^\beta u_0 = \lambda u - w = \lambda \bar{u} - (w + \lambda u_0),$$

with initial value $\bar{u}(0) = u_0$. By [21, Theorem 6.5 and Theorem 7.2] we obtain $\bar{u} =$ rhs of (4.3) for $f = w + \lambda u_0$. Now compute

$$\begin{split} \bar{u}(t) &= \mathbf{E} \left[\int_{0}^{\tau_{0}(t)} e^{-\lambda s} \left(w(-X^{t,\beta}(s)) \pm \lambda u_{0} \right) ds \right] + u_{0} \\ &= \mathbf{E} \left[\int_{0}^{\tau_{0}(t)} e^{-\lambda s} \left(w(-X^{t,\beta}(s)) + \lambda u_{0} \right) ds \right] - \lambda u_{0} \frac{\mathbf{E} \left[e^{-\lambda \tau_{0}(t)} \right] - 1}{-\lambda} + u_{0} \\ &= \mathbf{E} \left[\int_{0}^{\tau_{0}(t)} e^{-\lambda s} \left(w(-X^{t,\beta}(s)) + \lambda u_{0} \right) ds \right] + u_{0} \mathbf{E} \left[e^{-\lambda \tau_{0}(t)} \right]. \end{split}$$

Now, for an arbitrary $f \in C^1([0,T])$, by picking $w \equiv f - f(0)$ and $u_0 \equiv f(0)\lambda^{-1}$, we obtain the equality (4.3). A straightforward application of DCT proves the claim for $f \in C([0,T])$.

(ii) Recall that there exists a constant c > 0 such that $0 \le -\partial_t E_\beta(-\lambda t^\beta) \le c \frac{\lambda t^{\beta-1}}{1+\lambda t^{\beta}}$ by [21, Theorem 7.3] and [30, Equation (17)], and $E_\beta(-\lambda t^\beta) \le \frac{c}{1+\lambda t^{\beta}}$. Then

$$\left| (-\lambda)^{-1} \int_0^t f(r) \partial_t E_\beta(-\lambda(t-r)^\beta) \, dr \right| \le \|f\|_\infty \frac{1 - E_\beta(-\lambda t^\beta)}{\lambda} \le \|f\|_\infty \frac{1 + c}{\lambda}.$$

For the second inequality we exploit the smoothness of f, computing for t > 0

$$\partial_t F_{\lambda} [f](t) = (-\lambda)^{-1} \partial_t \left(-\int_0^t f(r) \partial_r E_{\beta} (-\lambda(t-r)^{\beta}) dr \right)$$

= $(-\lambda)^{-1} \partial_t \left(\int_0^t f'(r) E_{\beta} (-\lambda(t-r)^{\beta}) dr - f(t) + f(0) E_{\beta} (-\lambda t^{\beta}) \right)$
= $(-\lambda)^{-1} \left(\int_0^t f'(r) \partial_t E_{\beta} (-\lambda(t-r)^{\beta}) dr \pm f'(t) + f(0) \partial_t E_{\beta} (-\lambda t^{\beta}) \right)$
= $F_{\lambda} [f'](t) - \lambda^{-1} f(0) \partial_t E_{\beta} (-\lambda t^{\beta}).$

Then

$$|\partial_t F_{\lambda}[f](t)| \le ||f'||_{\infty} \frac{1+c}{\lambda} + f(0)c \frac{t^{\beta-1}}{1+\lambda t^{\beta}}.$$

From the proof of [19, Theorem 5.1], we infer the following lemma.

Lemma 4.3. Working with the notation of Proposition 2.5-(iii):

(i) the system of eigenvectors $\{\psi_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis of $\text{Dom}(\mathcal{L}^k_{\Omega,2}) \subset L^2(\Omega)$. The corresponding eigenvalues can be ordered so that $\lambda_n \leq \lambda_{n+1}$, and also $\lambda_n \leq \tilde{c}_1 n^{\alpha/d}$ for some constant $\tilde{c}_1 > 0$. Also, for any compact subset K of $\Omega, j = 0, 1, 2$, there are constants $c_1 = c_1(K, j, d, \alpha)$ such that

$$|\nabla^j \psi_n(x)| \le c_1 \lambda_n^{(d+2j)/(2\alpha)},\tag{4.4}$$

where $c_1(K, 0, d, \alpha)$ is independent of K.

(ii) Suppose $\phi_0 \in \text{Dom}(\mathcal{L}^k_{\Omega,2})$ for $k > -1 + (3d+4)/(2\alpha)$. Then $N := \sum_{n=1}^{\infty} \lambda_n^{2k} \langle \phi_0, \psi_n \rangle^2 < \infty$, and the series

$$\sum_{n=1}^{\infty} E_{\beta}(-\lambda_n t^{\beta}) \langle \phi_0, \psi_n \rangle \psi_n(x) = \mathbf{E} \left[\phi_0(X^{x,\alpha}(\tau_0(t))) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}} \right],$$

defines a function in $C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega)$, with bounds for j = 1, 2, $\sum_{n=1}^{\infty} \left| E_{\beta}(-\lambda_n t^{\beta}) \langle \phi_0, \psi_n \rangle \nabla^j \psi_n(x) \right| \leq (c_2 \sqrt{N}) t^{-\beta} \sum_{n=1}^{\infty} \lambda_n^{(d+4)/(2\alpha)-1-k} < \infty, \quad t > 0,$ $\sum_{n=1}^{\infty} \left| \partial_t E_{\beta}(-\lambda_n t^{\beta}) \langle \phi_0, \psi_n \rangle \psi_n(x) \right| \leq c_3 t^{\gamma\beta-1}, \qquad x \in \Omega,$ where $c_2 = c_2(K, j, d, \alpha), c_3 = c_3(\Omega, \alpha)$, and $0 < \gamma < 1 \wedge (4/(2\alpha) - 1).$

We will assume that the forcing term f in (1.6) belongs to the space of functions

$$C^{1}([0,T]; \operatorname{Dom}(\mathcal{L}_{\Omega,2}^{k})) := \left\{ f \in C^{1}_{\partial\Omega}([0,T] \times \overline{\Omega}) : \sup_{t} \|f(t)\|_{\mathcal{L}_{\Omega,2}^{k}} + \sup_{t} \|\partial_{t}f(t)\|_{\mathcal{L}_{\Omega,2}^{k}} < \infty \right\}.$$

$$(4.5)$$

Note that if $f \in C^1([0,T]; \text{Dom}(\mathcal{L}^k_{\Omega,2}))$, then there exists M > 0 such that for every $n \in \mathbb{N}$

$$\sup_{t \in [0,T]} |\langle f(t), \psi_n \rangle| \le M \lambda_n^{-k}, \quad \text{and} \quad \sup_{t \in [0,T]} |\langle \partial_t f(t), \psi_n \rangle| \le M \lambda_n^{-k}.$$
(4.6)

Remark 4.4. The inclusion $\operatorname{Span}\{C^1([0,T])\cdot\operatorname{Dom}(\mathcal{L}^k_{\Omega,2})\} \subset C^1([0,T];\operatorname{Dom}(\mathcal{L}^k_{\Omega,2}))$ is clear. Moreover, if $k \in \mathbb{N}$, then the inclusion $C_c^{1,2k}([0,T] \times \Omega) \subset C^1([0,T];\operatorname{Dom}(\mathcal{L}^k_{\Omega,2}))$ holds⁴. To see this, let $f \in C_c^{1,2k}([0,T] \times \Omega)$ and compute for each $t \in [0,T]$

$$\sum_{n=1}^{\infty} \lambda_n^{2k} \langle f(t), \psi_n \rangle^2 = \sum_{n=1}^{\infty} \langle f(t), \mathcal{L}_{\Omega,2}^k \psi_n \rangle^2 = \sum_{n=1}^{\infty} \langle (\Delta_{\Omega}^{\frac{\alpha}{2}})^k f(t), \psi_n \rangle^2 = \| (\Delta_{\Omega}^{\frac{\alpha}{2}})^k f(t) \|_{L^2(\Omega)}^2 < \infty,$$

where the second equality holds by the same argument at the end the proof of Theorem 4.6, using $(\Delta_{\Omega}^{\frac{\alpha}{2}})^m f(t) \in L^2(\Omega)$ for each $t \in [0,T]$ and $m \leq k$. Now observe that by DCT the function $t \mapsto \|(\Delta_{\Omega}^{\frac{\alpha}{2}})^k f(t)\|_{L^2(\Omega)}$ is continuous on [0,T], because $(\Delta_{\Omega}^{\frac{\alpha}{2}})^k f \in C([0,T] \times \overline{\Omega})$. Repeat the argument for $\partial_t f$ to conclude.

Lemma 4.5. If $f(t) \in \text{Dom}(\mathcal{L}^k_{\Omega,2})$ for $k > -1 + (3d+4)/(2\alpha)$, for every $t \in [0,T]$, and $f \in C_{\partial\Omega}([0,T] \times \Omega)$, then

$$\mathbf{E}\left[\int_0^{\tau_{t,x}} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] = \sum_{n=1}^\infty \psi_n(x) F_{\lambda_n}\left[\langle f(\cdot), \psi_n \rangle\right](t).$$

If in addition $f \in C^1([0,T]; \text{Dom}(\mathcal{L}^k_{\Omega,2}))$, then there exists a constant C such that for $t \in (0,T]$

$$\sum_{n=1}^{\infty} |\psi_n(x)\partial_t F_{\lambda_n} \left[\langle f(\cdot), \psi_n \rangle \right](t) | \le C t^{\beta - 1}.$$
(4.7)

⁴We define $C_c^{1,2k}([0,T]\times\Omega) = C^{1,2k}((0,T)\times\Omega) \cap \{f,\partial_t f \in C([0,T]\times\Omega), \sup\{f\} \subset [0,T]\times\Omega \text{ is compact}\}.$

Proof. We justify the following equalities

$$\begin{split} \mathbf{E}\left[\int_{0}^{\tau_{t,x}} f\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] &= \int_{0}^{\infty} P_{s}^{\beta,\text{kill}} P_{s}^{\Omega} f(t,x) \, ds \\ &= \int_{0}^{\infty} P_{s}^{\beta,\text{kill}} \left(\sum_{n=1}^{\infty} \langle f(t), \psi_{n} \rangle \psi_{n}(x) e^{-s\lambda_{n}}\right) \, ds \\ &= \sum_{n=1}^{\infty} \psi_{n}(x) \int_{0}^{\infty} P_{s}^{\beta,\text{kill}} \langle f(t), \psi_{n} \rangle e^{-s\lambda_{n}} \, ds \\ &= \sum_{n=1}^{\infty} \psi_{n}(x) \mathbf{E}\left[\int_{0}^{\tau_{0}(t)} \langle f(-X^{t,\beta}(s)), \psi_{n} \rangle e^{-s\lambda_{n}} \, ds\right] \\ &= \sum_{n=1}^{\infty} \psi_{n}(x) F_{\lambda_{n}} \left[\langle f(\cdot), \psi_{n} \rangle\right](t). \end{split}$$

We can apply Fubini's Theorem in the third equality as

$$\sum_{n=1}^{\infty} |\langle f(t), \psi_n \rangle| \|\psi_n\|_{\infty} \le C \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k)} < \infty,$$

for some constant C > 0, each $t \ge 0$ and any $k > 3d/(2\alpha)$, using the bounds in Lemma 4.3-(i) and in (4.6). We apply Lemma 4.2-(i) in the fifth equality as $r \mapsto \langle f(r), \psi_n \rangle \in C([0,T])$ for each $n \in \mathbb{N}$. The other equalities are clear.

For the last claim we use the bounds in (4.2), (4.6) and Lemma 4.3-(i) to obtain

$$\begin{split} \sum_{n=1}^{\infty} |\psi_n(x)\partial_t F_{\lambda_n} \left[\langle f(t), \psi_n \rangle \right](t) | &\leq \sum_{n=1}^{\infty} |\psi_n(x)| \frac{c}{\lambda_n} \left(\sup_{r \in [0,T]} |\langle \partial_r f(r), \psi_n \rangle| + \frac{\lambda_n t^{\beta-1}}{1 + \lambda_n t^{\beta}} |\langle f(0), \psi_n \rangle| \right) \\ &\leq \sum_{n=1}^{\infty} |\psi_n(x)| \frac{cM\lambda_n^{-k}}{\lambda_n} \left(1 + \frac{\lambda_n t^{\beta-1}}{1 + \lambda_n t^{\beta}} \right) \\ &\leq (c_1 cM) \sum_{n=1}^{\infty} \frac{\lambda_n^{d/(2\alpha)} \lambda_n^{-k}}{\lambda_n} \left(1 + \frac{\lambda_n t^{\beta-1}}{1 + \lambda_n t^{\beta}} \right) \\ &\leq (c_1 cM) t^{\beta-1} \sum_{n=1}^{\infty} \lambda_n^{d/(2\alpha)-k} \\ &\leq (\tilde{c}_1 c_1 cM) t^{\beta-1} \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k)} < \infty, \end{split}$$

for any $k > 3d/(2\alpha)$, where the constants \tilde{c}_1, c_1, c and M follow the notation of the referenced inequalities, and a constant is omitted in the fourth inequality.

Theorem 4.6. Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi_0 \in Dom(\mathcal{L}^k_{\Omega,2})$, and $f \in C^1([0,T]; Dom(\mathcal{L}^k_{\Omega,2}))$ for some $k > -1 + (3d+4)/(2\alpha)$, where $C^1([0,T]; Dom(\mathcal{L}^k_{\Omega,2}))$ is

defined in (4.5). Then

$$u \in C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega), \quad and \\ |\partial_t u(t,x)| \le Ct^{-\gamma}, \text{ for every } (t,x) \in (0,T] \times \Omega, \text{ for some } \gamma \in (0,1), \ C > 0,$$

$$(4.8)$$

where u is defined in (1.5). Moreover, u is the unique classical solution to problem (1.6).

Proof. (The notation for constants is consistent with the referenced inequalities.) By Lemma 4.3-(ii) and Lemma 4.5 we can write our candidate solution (1.5) as

$$u(t,x) = \sum_{n=1}^{\infty} E_{\beta}(-\lambda_n t^{\beta}) \langle \phi_0, \psi_n \rangle \psi_n(x) + \sum_{n=1}^{\infty} F_{\lambda_n} \left[\langle f(\cdot), \psi_n \rangle \right](t) \psi_n(x)$$

and the first series enjoys the regularity properties stated in (4.8). We now prove the same regularity for the second series. Observe that $\sum_{n=1}^{\infty} F_{\lambda_n} \left[\langle f(\cdot), \psi_n \rangle \right](t) \psi_n(x)$ converges uniformly to a function in $C_{\partial\Omega}([0,T] \times \Omega)$, since we have the uniform bound

$$\sum_{n=1}^{\infty} |F_{\lambda_n} \left[\langle f(\cdot), \psi_n \rangle \right](t) \psi_n(x) | \leq \sum_{n=1}^{\infty} c \lambda_n^{-1} || \langle f(\cdot), \psi_n \rangle ||_{C([0,T])} c_1 \lambda_n^{d/(2\alpha)}$$
$$\leq (cc_1 M) \sum_{n=1}^{\infty} \lambda_n^{-1-k+d/(2\alpha)}$$
$$\leq (\tilde{c}_1 c_1 c M) \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k-1)} < \infty,$$

for any $k > -1 + 3d/(2\alpha)$, using the bounds in (4.6), (4.1) and Lemma 4.3-(i). Further, for j = 1, 2, and for any x in a compact subset K of Ω , the term-wise space derivative of u can be bounded as follows,

$$\sum_{n=1}^{\infty} |F_{\lambda_n} \left[\langle f(\cdot), \psi_n \rangle \right](t) |\| \nabla^j \psi_n \|_{\infty} \le \sum_{n=1}^{\infty} c \lambda_n^{-1} \| \langle f(\cdot), \psi_n \rangle \|_{C([0,T])} c_1 \lambda_n^{(d+4)/2\alpha} \le (\tilde{c}_1 c_1 c M) \sum_{n=1}^{\infty} n^{(\alpha/d)((d+4)/(2\alpha)-k-1)} < \infty,$$
(4.9)

as

$$\frac{\alpha}{d}\left(\frac{d+4}{2\alpha}-k-1\right)<-1\iff k>\frac{3d+4-2\alpha}{2\alpha},$$

where we use the bounds in (4.6), (4.1) and Lemma 4.3-(i). Thus, Weierstrass M-test implies that for any t > 0, u(t) is a C^2 function on every $K \subset \Omega$ compact. For the time regularity we use the inequality (4.7) from Lemma 4.5⁵.

By Theorem 3.9, u is also a weak solution to problem (1.6), and by Lemma 3.3 and standard approximation arguments, u satisfies the equalities in (1.6). Continuity at t = 0 can be proved as in Remark 5.3.

⁵From the proof of Lemma 4.5 it follows that if $\phi_0 = f(0) = 0$, then $\partial_t u$ is bounded.

To prove uniqueness, consider two classical solutions to problem (1.6), denoted by u, v. Then w := u - v is a classical solution to problem (1.6) with f = 0, $\phi_0 = 0$. Consider the continuous functions on [0, T], $t \mapsto \langle w(t), \psi_n \rangle$, $n \in \mathbb{N}$. If we can justify

$$D_0^\beta \langle w(t), \psi_n \rangle = \langle D_0^\beta w(t), \psi_n \rangle = \langle \Delta_\Omega^{\frac{\alpha}{2}} w(t), \psi_n \rangle = \langle w(t), \mathcal{L}_{\Omega,2} \psi_n \rangle = -\lambda_n \langle w(t), \psi_n \rangle, \quad (4.10)$$

for t > 0, it follows by [21, Theorem 6.5 and Theorem 7.2] that $\langle w(t), \psi_n \rangle = 0$ for every $t \in [0,T], n \in \mathbb{N}$, and we are done. The first equality is a consequence of $|\partial_r w(r,y)| \leq Cr^{-\gamma}$, for some $\gamma \in (0,1)$. The second and fourth equalities in (4.10) are clear. Now, as $\psi_n \in \text{Dom}(\mathcal{L}_{\Omega,2})$, there exists a sequence $\{\psi_{n,j}\}_{j\in\mathbb{N}} \subset C_c^{\infty}(\Omega)$, such that as $j \to \infty$

$$\psi_{n,j} \to \psi_n$$
, and $\Delta_{\Omega}^{\frac{\alpha}{2}} \psi_{n,j} = \mathcal{L}_{\Omega,2} \psi_{n,j} \to \mathcal{L}_{\Omega,2} \psi_n$, in $L^2(\Omega)$, (4.11)

where the equality in (4.11) holds by [19, Lemma 4.1]. Combining (4.11) with the equality (2.5) and $\Delta_{\Omega}^{\frac{\alpha}{2}}w(t) \in L^2(\Omega)$ for each t > 0, the third equality in (4.10) is proven. \Box

5. Stochastic classical solution for problem (1.1)

5.1. Stochastic representation and continuity at t = 0.

Lemma 5.1. Define the function $f_{\phi}: (0,T] \times \Omega \to \mathbb{R}$ as

$$f_{\phi}(t,x) := \int_{t}^{\infty} (\phi(t-r,x) - \phi(t,x)) \frac{-\Gamma(-\beta)^{-1} dr}{r^{1+\beta}},$$
(5.1)

assuming that $\phi \in C_{\infty,\partial\Omega}((-\infty,0] \times \Omega), \phi(0) \in \text{Dom}(\mathcal{L}_{\Omega})$, and the extension of ϕ to $\phi(0)$ on $(0,T] \times \overline{\Omega}$ is such that

$$\phi \in \text{Dom}(\mathcal{L}^{\infty}_{\beta,\Omega}), \text{ and } \mathcal{L}^{\infty}_{\beta,\Omega}\phi = (-D^{\beta}_{\infty} + \mathcal{L}_{\Omega})\phi.$$
 (5.2)

Then $f_{\phi} \in C([0,T] \times \overline{\Omega})$ and the function u defined in (1.5) for $f = f_{\phi}$ and $\phi_0 = \phi(0)$, equals the function \tilde{u} defined in (1.3) for g = 0, on $(0,T] \times \Omega$.

Proof. The first claim follows from $f_{\phi} = -D_{\infty}^{\beta}\phi \in C([0,T] \times \overline{\Omega})$, using (5.2) and $\mathcal{L}_{\Omega}\phi(t,x) = \mathcal{L}_{\Omega}\phi(0,x)$ for all $(t,x) \in [0,T] \times \overline{\Omega}$. Recall that we write $\tau_{t,x} = \tau_0(t) \wedge \tau_{\Omega}(x)$.

Fix $(t, x) \in (0, T] \times \Omega$. It is enough to justify the following equalities

$$\begin{split} u(t,x) &= \mathbf{E} \left[\phi(0, X^{x,\alpha}(\tau_0(t)) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}} + \int_0^{\tau_{t,x}} f_\phi \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right] \\ &= \mathbf{E} \left[\phi(0,x) + \int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi \left(0, X^{x,\alpha}(s) \right) ds + \int_0^{\tau_{t,x}} f_\phi \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right] \\ &= \mathbf{E} \left[\int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) - D_\infty^\beta \phi \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right] + \phi(0,x) \\ &= \mathbf{E} \left[\int_0^{\tau_{t,x}} \mathcal{L}_{\beta,\Omega}^\infty \phi \left(-X^{t,\beta}(s), X^{x,\alpha}(s) \right) ds \right] + \phi(0,x) \\ &= \mathbf{E} \left[\phi \left(-X^{t,\beta}(\tau_{t,x}), X^{x,\alpha}(\tau_{t,x}) \right) \right] \pm \phi(0,x). \end{split}$$

For the second equality we use Dynkin formula with Lemma 2.6-(i) and $\phi(0) \in \text{Dom}(\mathcal{L}_{\Omega})$; for the third equality, as we extended $\phi(t, x) = \phi(0, x)$ on $[0, T] \times \Omega$, we use the identities $f_{\phi}(t, x) = -D_{\infty}^{\beta}\phi(t, x)$ and $\mathcal{L}_{\Omega}\phi(0, x) = \mathcal{L}_{\Omega}\phi(t, x)$ on $(0, T] \times \Omega$; in the fourth equality we use assumption (5.2); the fifth equality is again an application of Dynkin formula with Lemma 2.6-(ii) and $\phi(t, x) = \phi(0, x)$ on $(0, T] \times \Omega$.

Corollary 5.2. If $\phi \in C^1_{b,\partial\Omega}((-\infty,0] \times \Omega)$, then for $(t,x) \in (0,T] \times \Omega$

$$\mathbf{E}\left[\phi\left(0, X^{x,\alpha}(\tau_{t,x})\right) + \int_{0}^{\tau_{t,x}} f_{\phi}\left(-X^{t,\beta}(s), X^{x,\alpha}(s)\right) ds\right] = \mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_{t,x}), X^{x,\alpha}(\tau_{t,x})\right)\right].$$
(5.3)

Proof. Step 1. We prove (5.3) for $\phi \in C^1_{\infty,\partial\Omega}((-\infty,0] \times \Omega) \cap \{\partial_t f(0) = 0\}$ with compact support in $(-\infty,0] \times \overline{\Omega}$. For such ϕ , let K > 0 such that ϕ is supported in $(-K,0] \times \overline{\Omega}$. By the same arguments as in the proof of Lemma 2.6-(ii), it follows that $\text{Span}\{C([-K,0]) \cap \{f(-K) = f(0) = 0\} \cdot C_{\partial\Omega}(\Omega)\}$ is dense in $C_{\partial\Omega}([-K,0] \times \Omega) \cap \{f(-K) = f(0) = 0\}$ with respect to the supremum norm. We can use this fact to construct a sequence $\{\phi_n\}_{n\in\mathbb{N}} \in$ $\text{Span}\{C^1_{\infty}(-\infty,0]) \cap \{f'(0) = 0\} \cdot C_{\partial\Omega}(\Omega)\}$ such that

$$\|\phi_n - \phi\|_{C((-\infty,0]\times\overline{\Omega})} + \|\partial_t(\phi_n - \phi)\|_{C((-\infty,0]\times\overline{\Omega})} \to 0, \quad \text{as } n \to \infty.$$

Moreover, it follows that $f_{\phi_n} \to f_{\phi}$ as $n \to \infty$ pointwise on $[0, T] \times \Omega$ and $\sup_n \|f_{\phi_n}\|_{C([0,T] \times \overline{\Omega})}$ is finite. It remains to show that (5.3) holds for functions in $\operatorname{Span}\{C_{\infty}^1(-\infty,0]) \cap \{f'(0) = 0\} \cdot C_{\partial\Omega}(\Omega)\}$, as DCT applied to the sequences above yields the claim. By Lemma 2.6-(iii) with $C_{\beta}^{\infty} = C_{\infty}^1((-\infty,T])$, Proposition 2.4 and Lemma 5.1, equality (5.3) holds for $\phi \in \operatorname{Span}\{C_{\infty}^1((-\infty,0]) \cap \{f'(0) = 0\} \cdot \operatorname{Dom}(\mathcal{L}_{\Omega}))\}$. As $\operatorname{Dom}(\mathcal{L}_{\Omega})$ is dense in $C_{\partial\Omega}(\Omega)$, equality (5.3) holds for $\phi \in \operatorname{Span}\{C_{\infty}^1((-\infty,0]) \cap \{f'(0) = 0\} \cdot C_{\partial\Omega}(\Omega)\}$ by DCT. Step 2. For $\phi \in C_{b,\partial\Omega}^1((-\infty,0] \times \Omega)$, take a sequence $\{\phi_n\}_{n \in \mathbb{N}} \subset C_{\infty,\partial\Omega}^1((-\infty,0] \times \Omega) \cap \{\partial_t f(0) = 0\}$ compactly supported in $(-\infty,0] \times \overline{\Omega}$, such that $\phi_n \to \phi$ pointwise on $(-\infty,0] \times \Omega$, and $\sup_n \|\phi_n\|_{C((-\infty,0] \times \overline{\Omega})} + \sup_n \|\partial_t \phi_n\|_{C((-\infty,0] \times \overline{\Omega})} < \infty$. Then $f_{\phi_n} \to f_{\phi}$ pointwise on $[0,T] \times \Omega$ and $\sup_n \|f_{\phi_n}\|_{C([0,T] \times \overline{\Omega})} < \infty$. Finally, apply DCT to both sides of (5.3).

Remark 5.3. If we can apply Corollary 5.2, then we can prove continuity at t = 0 for the solution (1.3) via the following argument

|Formula (1.5) -
$$\phi_0(x)$$
| \leq |**E** [$\phi_0(X^{x,\alpha}(\tau_0(t) \land \tau_\Omega(x))) - \phi_0(x)$] | + || f || $_{\infty}$ **E** [$\tau_0(t)$]
= $o_{t\downarrow 0}(1) + ||f||_{\infty} \frac{t^{\beta}}{\Gamma(\beta+1)}$,

for each $x \in \Omega$, using stochastic continuity of the process⁶ $t \mapsto X^{x,\alpha}(\tau_0(t))$ at t = 0. One could also use stochastic continuity at t = 0 of $-X^{t,\beta}(\tau_0(t)) = t - X^{\beta}(\tau_0(t))$, bypassing Corollary 5.2. In Proposition A.2 in the Appendix we prove continuity at t = 0 by proving a bound on big overshootings $-X^{t,\beta}(\tau_0(t))$ for small times.

5.2. Equivalence of the classical solutions to problems (1.1) and (1.6).

Definition 5.4. Let $\phi \in C_{b,\partial\Omega}((-\infty,0] \times \Omega)$ and $g \in C((0,T] \times \Omega)$. A function $\tilde{u} \in C_{b,\partial\Omega}((-\infty,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega)$ such that $|\partial_t \tilde{u}(t,x)| \leq Ct^{-\gamma}$, for every $(t,x) \in (0,T] \times \Omega$, for some $\gamma \in (0,1)$, C > 0, is said to be a *classical solution to problem* (1.1) if \tilde{u} satisfies the identities in (1.1), and for every $x \in \Omega$

$$\lim_{t \downarrow 0} |\tilde{u}(t,x) - \phi(0,x)| = 0.$$

Lemma 5.5. Let $\phi \in C_{b,\partial\Omega}((-\infty, 0] \times \Omega)$ such that $f_{\phi} \in C((0, T] \times \Omega)$, where f_{ϕ} is defined in (5.1), and let $g \in C((0, T] \times \Omega)$. Then, if u is a classical solution to problem (1.6) with $f = f_{\phi} + g$ and $\phi_0 = \phi(0)$, then the extension

$$\tilde{u} := \begin{cases} u, & \text{in } (0,T] \times \overline{\Omega}, \\ \phi, & \text{in } (-\infty,0] \times \Omega \end{cases}$$

is a classical solution to problem (1.1). Conversely, if \tilde{u} is a classical solution to problem (1.1), then the restriction of \tilde{u} to $[0,T] \times \overline{\Omega}$ is a classical solution to problem (1.6) with $f = f_{\phi} + g$ and $\phi_0 = \phi(0)$.

Proof. The equivalence of convergence to initial data and the required regularities are clear. It is also immediate that $\Delta_{\Omega}^{\frac{\alpha}{2}} u = \Delta_{\Omega}^{\frac{\alpha}{2}} \tilde{u}$ on $(0,T] \times \Omega$. Write $\nu(r) = -\Gamma(-\beta)^{-1}r^{-1-\beta}$.

⁶This follows as $X^{x,\alpha}(s)$ is right continuous and $\tau_0(t)$ is right continuous, non-decreasing with $\tau_0(0) = 0$.

On $(0,T] \times \Omega$ we have the equality

$$\begin{split} -D_{\infty}^{\beta}\tilde{u}(t,x) &= \int_{0}^{\infty} \left(\tilde{u}(t-r,x) - \tilde{u}(t,x)\right)\nu(r)dr\\ &= \int_{0}^{t} \left(\tilde{u}(t-r,x) - \tilde{u}(t,x)\right)\nu(r)dr + \int_{t}^{\infty}\phi(t-r,x)\nu(r)dr\\ &\quad -\tilde{u}(t,x)\right)\int_{t}^{\infty}\nu(r)dr \pm \phi(0,x)\int_{t}^{\infty}\nu(r)dr\\ &= -D_{0}^{\beta}\tilde{u}(t,x) + f_{\phi}(t,x). \end{split}$$

This is enough to prove both directions.

5.3. Main result.

Theorem 5.6. Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi \in C^1_{b,\partial\Omega}((-\infty, 0] \times \Omega)$ with $\phi(0) \in Dom(\mathcal{L}^k_{\Omega,2})$ and $f_{\phi}, g \in C^1([0,T]; Dom(\mathcal{L}^k_{\Omega,2}))$, for some $k > -1 + (3d+4)/(2\alpha)$, where f_{ϕ} is defined in (5.1) and $C^1([0,T]; Dom(\mathcal{L}^k_{\Omega,2}))$ is defined in (4.5). Then

$$\tilde{u} \in C_{b,\partial\Omega}((-\infty,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega), \quad and$$

 $|\partial_t \tilde{u}(t,x)| \leq Ct^{-\gamma}, \text{ for every } (t,x) \in (0,T] \times \Omega, \text{ for some } \gamma \in (0,1), \ C > 0,$

where \tilde{u} is defined as in (1.3). Moreover, \tilde{u} is the unique classical solution to problem (1.1).

Proof. By the assumptions on ϕ and g, and Lemma 5.5, existence and uniqueness of classical solutions follows by Theorem 4.6 with $\phi_0 = \phi(0)$ and $f = f_{\phi} + g$. Now apply Corollary 5.2 to obtain the stochastic representation (1.3) from the stochastic representation (1.5).

Remark 5.7. Using Corollary 5.2 (or [29, Theorem 1 for $\lambda = 0$]), $\mathbf{P}[-X^t(\tau_0(t)) \in \{0\}] = 0$ for every t > 0 (see [10, III, Theorem 4]) and the independence of $X^{x,\alpha}$ and $-X^{t,\beta}$, one can show that for $(t, x) \in (0, T] \times \Omega$

$$\mathbf{E}\left[\phi\left(-X^{t,\beta}(\tau_0(t)), X^{x,\alpha}(\tau_0(t))\right)\mathbf{1}_{\{\tau_0(t)<\tau_\Omega(x)\}}\right] = \int_{-\infty}^0 \int_\Omega \phi(r,y) H^{t,x}_{\beta,\alpha}(r,y) \, dr \, dy,$$

where

$$H^{t,x}_{\beta,\alpha}(r,y) = \int_0^t \frac{-\Gamma(-\beta)^{-1}}{(z-r)^{1+\beta}} \left(\int_0^\infty p_s^\Omega(x,y) p_s^\beta(t-z) \, ds \right) \, dz$$

It is straightforward to compute for $(t, x) \in (0, T] \times \Omega$

$$\mathbf{E}\left[\int_{0}^{\tau_{0}(t)\wedge\tau_{\Omega}(x)}g\left(-X^{t,\beta}(s),X^{x,\alpha}(s)\right)ds\right] = \int_{0}^{t}\int_{\Omega}g(z,y)\left(\int_{0}^{\infty}p_{s}^{\Omega}(x,y)p_{s}^{\beta}(t-z)\,ds\right)\,dz\,dy.$$

Remark 5.8. Notice that the value $\phi(0)$ does not contribute to the solution (1.3) because $\mathbf{P}[-X^t(\tau_0(t)) \in \{0\}] = 0$ for all t > 0. However, $u(t) \to \phi(0)$ as $t \downarrow 0$. We discuss the continuity of the solution at t = 0 in more detail in Appendix A.I.

Remark 5.9. We could drop the condition $\|\partial_t \phi\|_{\infty} < \infty$ in Theorem 5.6, by weakening Corollary 5.2, for example to ϕ being β^* -Hölder continuous at t = 0, for some $\beta^* > \beta$ and $\phi \in L^{\infty}((-\infty, 0) \times \Omega)$. This is essentially because $\lim_{t \downarrow 0} f_{\phi}(t)$ remains well-defined. However, in order to apply Theorem 4.6 in the proof of Theorem 5.6 we need to assume $f_{\phi} \in C^1([0, T]; \text{Dom}(\mathcal{L}^k_{\Omega, 2}))$. Hence, a minimal requirement is that ϕ is continuously differentiable in time and both ϕ and $\partial_t \phi$ are $\mathcal{O}(|r|^{\beta_*})$ at $-\infty$ and β^* -Hölder continuous at 0, for some $\beta_* < \beta < \beta^*$, as we need f_{ϕ} and $\partial_t f_{\phi}$ to be continuous on $[0, T] \times \overline{\Omega}$.

Remark 5.10. Suppose that $\phi \in C^{2,2k}_{\infty,\partial\Omega}((-\infty,0] \times \Omega)$ and $\phi(t)$ along with its partial derivatives in space are compactly supported in Ω , for each $t \in (-\infty,0]$, where $k \in \mathbb{N}$ and $k > -1 + (3d + 4)/(2\alpha)$. Then, an application of Remark 4.4 implies that $f_{\phi} \in C^1([0,T]; \text{Dom}(\mathcal{L}^k_{\Omega,2}))$.

6. Intuition for the stochastic solution (1.3)

We discuss the intuition for the stochastic representation (1.3) as the solution to the EE (1.1). Let us write $-W(t) = t - X^{\beta}(\tau_0(t)) = -X^{t,\beta}(\tau_0(t))$. Then W(t) is the overshoot of the subordinator X^{β} with respect to the barrier t, recalling that the first exit time/inverse subordinator is given by $\tau_0(t) = \inf\{s > 0 : t \leq X^{\beta}(s)\}$. To ease notation we write $Y^x := \{X^{x,\alpha}(\tau_0(t))\mathbf{1}_{\{\tau_0(t)<\tau_{\Omega}(x)\}}\}_{t\geq 0}$. Let us start from the intuition of Caputo EEs, as if $\phi(t,x) = \phi(0,x) =: \phi_0(x)$ for every $t \in (-\infty, 0] \times \Omega$, then the solution (1.3) reads

$$u(t,x) = \mathbf{E}\left[\phi_0(Y^x(t))\right],\tag{6.1}$$

and the EE (1.1) equals the Caputo EE (1.6) (for g = f = 0). The probabilistic object defining the solution (6.1) is the anomalous diffusion Y^x . Recall that the particle Y^x is either trapped or diffusing.

Key observation: reasoning path-wise, for some $\bar{x} \in \Omega$

the interval (t_1, t_2) is the maximal open interval so that $t \mapsto Y^x(t) = \bar{x}$ is constant

the interval
$$(t_1, t_2)$$
 is the maximal open interval so that $t \mapsto \tau_0(t)$ is constant

the interval (t_1, t_2) is the maximal open interval so that $t \mapsto X^{\beta}(\tau_0(t))$ is constant \iff

$$X^{\beta}(\tau_0(t)-) = t_1$$
 and $X^{\beta}(\tau_0(t)) = t_2$, (i.e. X^{β} jumped from t_1 to t_2).

The last statement implies that

$$W(t) = X^{\beta}(\tau_0(t)) - t = t_2 - t \in (0, t_2 - t_1) \text{ for every } t \in (t_1, t_2),$$

which is the trapping/waiting time of $Y^x(t)$. In words: the event of the diffusion Y^x being trapped at a point $\bar{x} \in \Omega$ at time t until time t + s happens precisely when W(t) = s. Hence the law of -W(t) provides a weighting of the initial condition $\phi(\bar{x})$ depending on the trapping/waiting time of $Y^x(t)$. Notice that the process $t \mapsto -W(t)$ is self-similar with

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index 1 and it is composed by right continuous 45 degrees increasing slopes with 0 leftmost limit (see Figure 1).



FIGURE 1. A typical path of the overshoot $t \mapsto -W(t) = -X^{t,\beta}(\tau_0(t)), \beta = 0.9.$

6.1. A non-memory interpretation. It is possibly appealing to think about the values $(-\infty, 0) \times \Omega$ for the initial condition ϕ as the 'depth' underneath the surface $\{0\} \times \Omega$ where the particle Y^x moves. Then one can think about the particle $Y^x(t)$ as falling instantaneously at the bottom of a hole/trap of depth $|t_2 - t_1|$, and then taking time $|t_2 - t_1|$ to climb back up to the surface. Then, at time t one can observe the particle being $|t_2 - t|$ -depth-units down in the hole. From this viewpoint, once the particle is in the hole it just drifts upward with unit speed. As a quick example, consider the variable separable initial condition $\phi(t, x) = p(t)q(x)$ where $p(t) = \mathbf{1}_{\{t < -1\}}$. Then the solution reads for t > 0

$$u(t,x) = \mathbf{E} \left[q(Y^{x}(t)) \mathbf{1}_{\{W(t)>1\}} \right]$$

= $\mathbf{E} \left[q(Y^{x}(t)) | Y^{x}(t) \text{ is more than 1 unit deep in a trap} \right]$
 $\left(= \mathbf{E} \left[q(Y^{x}(t)) | Y^{x}(t) \text{ is trapped for more than 1 time-unit} \right] \right).$

Hence, in this example the diffusive particle Y^x will have to be a least a unit deep in a hole (trapped for at least a unit time) for the values at its trapping point at its depth (in the past) to contribute to the solution.

A. Appendix

A.I. Continuity of solution (1.3) at t = 0.

Proposition A.1. For every $p, \varepsilon > 0$, the following bound on small overshootings holds,

$$\mathbf{P}[X^{t,\beta}(\tau_0(t)) \le \varepsilon] \ge (1-p), \quad \text{for every } t \le \varepsilon p^{\frac{1}{\beta}}.$$

Proof. With the first equality holding by [29, Theorem 1 for $\lambda = 0$] along with the identity (2.3), compute

$$\mathbf{P}[X^{t,\beta}(\tau_0(t)) \le \varepsilon] = \int_{-\varepsilon}^0 \left(\frac{1}{\Gamma(\beta)} \int_0^t (-\partial_y (y-r)^{-\beta}) \frac{(t-y)^{\beta-1}}{\Gamma(1-\beta)} \, dy\right) \, dr$$

$$= \int_{-\varepsilon}^0 \left(\frac{\beta}{\Gamma(\beta)\Gamma(1-\beta)} \int_0^t (y-r)^{-\beta-1} (t-y)^{\beta-1} \, dy\right) \, dr$$

$$= \frac{-\Gamma(\beta)^{-1}}{\Gamma(-\beta)} \int_0^t (t-y)^{\beta-1} \left(\int_{-\varepsilon}^0 (y-r)^{-\beta-1} dr\right) \, dy$$

$$= \frac{-\Gamma(\beta)^{-1}\beta^{-1}}{\Gamma(-\beta)} (a-a_\varepsilon(t)),$$

where $a_{\varepsilon}(t) := \int_0^t (t-y)^{\beta-1} (y+\varepsilon)^{-\beta} dy$ and $a := \int_0^t (t-y)^{\beta-1} y^{-\beta} dy = \Gamma(\beta)\Gamma(1-\beta)$ for every t > 0. Now pick $\tilde{t} = \varepsilon p^{1/\beta}$. Then for every $0 \le y \le \tilde{t}$

$$(y+\varepsilon)^{-\beta} = (y+p^{-1/\beta}\tilde{t})^{-\beta} \le p\tilde{t}^{-\beta} \le py^{-\beta},$$

hence for every $t \leq \tilde{t}$

$$\frac{a_{\varepsilon}(t)}{a} = \frac{\int_0^t (t-y)^{\beta-1} (y+\varepsilon)^{-\beta} dy}{\int_0^t (t-y)^{\beta-1} y^{-\beta} dy} \le p.$$

Then $a_{\varepsilon}(t) \leq pa$ for every $t \leq \tilde{t}$, which is equivalent to $a - a_{\varepsilon}(t) \geq (1 - p)a$ for every $t \leq \tilde{t}$. And so we obtain

$$\mathbf{P}[X^{t,\beta}(\tau_0(t)) \le \varepsilon] \ge (1-p) \frac{-\Gamma(\beta)^{-1}}{\Gamma(-\beta)} \beta^{-1} \Gamma(\beta) \Gamma(1-\beta) = (1-p).$$

We now use the bound in Proposition A.1 to prove the following continuity result

Proposition A.2. Consider the function \tilde{u} defined in (1.3), with an arbitrary Ω -valued stochastic (sub-)process X^x in place of $X^{x,\alpha}$, such that $t \mapsto X^x(\tau_0(t))$ is stochastically continuous at t = 0. Also assume $\phi \in B((-\infty, 0] \times \Omega))$ and ϕ is continuous at every point in $\{0\} \times \Omega$. Then for every $x \in \Omega$

$$\lim_{t \downarrow 0} |\tilde{u}(t, x) - \phi(0, x)| = 0.$$

Proof. Let $x \in \Omega$. Let $\delta > 0$ be arbitrary. Pick $\varepsilon, \varepsilon' > 0$ such that

$$\sup_{(s,y)\in(-\varepsilon,0]\times B_{\varepsilon'}(x)} |\phi(s,y) - \phi(0,x)| \le \delta.$$

Then

$$\begin{split} |\tilde{u}(t,x) - \phi(0,x)| &\leq \left| \mathbf{E} \left[(\phi(-X^{t,\beta}(\tau_{0}(t)), X^{x}(\tau_{0}(t)) - \phi(0,x)) \mathbf{1}_{\{X^{t,\beta}(\tau_{0}(t)) > \varepsilon\}} \right] \right| \\ &+ \left| \mathbf{E} \left[(\phi(-X^{t,\beta}(\tau_{0}(t)), X^{x}(\tau_{0}(t))) - \phi(0,x)) \mathbf{1}_{\{X^{t,\beta}(\tau_{0}(t)) \le \varepsilon\}} \right] \right| \\ &\leq 2 \|\phi\|_{\infty} \mathbf{P}[X^{t,\beta}(\tau_{0}(t)) > \varepsilon] \\ &+ \mathbf{E} \left[|\phi(-X^{t,\beta}(\tau_{0}(t)), X^{x}(\tau_{0}(t))) - \phi(0,x)| \mathbf{1}_{\{X^{t,\beta}(\tau_{0}(t)) \le \varepsilon, |X^{x}(\tau_{0}(t))) - x| \ge \varepsilon'\}} \right] \\ &+ \mathbf{E} \left[|\phi(-X^{t,\beta}(\tau_{0}(t)), X^{x}(\tau_{0}(t))) - \phi(0,x)| \mathbf{1}_{\{X^{t,\beta}(\tau_{0}(t)) \le \varepsilon, |X^{x}(\tau_{0}(t))) - x| \ge \varepsilon'\}} \right] \\ &\leq 2 \|\phi\|_{\infty} \mathbf{P}[X^{t,\beta}(\tau_{0}(t)) > \varepsilon] + \delta + 2 \|\phi\|_{\infty} \mathbf{P}[|X^{x}(\tau_{0}(t))) - x| > \varepsilon'] \end{split}$$

Now, by Proposition A.1, for all $t \leq \delta^{\frac{1}{\beta}} \varepsilon$ it holds that $\mathbf{P}[X^{t,\beta}(\tau_0(t)) > \varepsilon] \leq \delta$. Then the estimate above reads

$$|\tilde{u}(t,x) - \phi(0,x)| \le 2\|\phi\|_{\infty}\delta + \delta + 2\|\phi\|_{\infty}\mathbf{P}[|X^{x}(\tau_{0}(t))) - x| > \varepsilon'], \quad \text{for every } t \le \delta^{\frac{1}{\beta}}\varepsilon.$$

To conclude, by stochastic continuity, pick a possibly smaller threshold \bar{t} to obtain

$$\mathbf{P}[|X^x(\tau_0(t))) - x| > \varepsilon'] \le \delta \quad \text{for every } t \le \bar{t}.$$

Remark A.3. The continuity at t = 0 of Proposition A.2 is not obvious. For example it is clear that Proposition A.2 fails if we replace $-X^{t,\beta}$ with a decreasing Poisson process. In fact Proposition A.2 fails in general if we replace $-X^{t,\beta}$ with a decreasing compound Poisson process $-N^t(s)$ with generator

$$-D_{\infty}^{(\nu)}f(t) := \int_{0}^{\infty} (f(t-r) - f(t))\,\nu(dr), \quad \text{where} \quad 0 < \lambda := \int_{0}^{\infty} \nu(dr) < \infty.$$

To see this, observe that for every $\varepsilon, t > 0$

$$\mathbf{P}\left[N^{t}\left(\tau_{0}(t)\right) > \varepsilon\right] \geq \mathbf{P}\left[\text{first jump of } N^{t} \text{ is greater than } t + \varepsilon\right] = \int_{t+\varepsilon}^{\infty} \frac{\nu(dr)}{\lambda},$$

and note that the right hand side is non-decreasing as $t \downarrow 0$, where τ_0 is the left continuous inverse of N^0 . As $\int_0^\infty \nu(dr) > 0$ we can choose $\varepsilon_0 > 0$ and $\bar{t} > 0$ so that

$$\inf_{t \le \bar{t}} \mathbf{P} \left[N^t \left(\tau_0(t) \right) > \varepsilon_0 \right] \ge \int_{\bar{t} + \varepsilon_0}^{\infty} \frac{\nu(dr)}{\lambda} =: c > 0.$$

Now, consider a continuous non-negative ϕ with $\phi(0) = 0$, such that $\inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) > 0$. Then for every $t \leq \overline{t}$

$$\begin{split} |\tilde{u}(t) - \phi(0)| &= \mathbf{E} \left[\phi(-N^t \left(\tau_0(t)\right) \left(\mathbf{1}_{\{N^t(\tau_0(t)) > \varepsilon_0\}} + \mathbf{1}_{\{N^t(\tau_0(t)) \le \varepsilon_0\}} \right) \right] \\ &\geq \mathbf{E} \left[\phi(-N^t \left(\tau_0(t)\right)) \mathbf{1}_{\{N^t(\tau_0(t)) > \varepsilon_0\}} \right] \\ &\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) \mathbf{P} \left[N^t \left(\tau_0(t)\right) > \varepsilon_0 \right] \\ &\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) c > 0. \end{split}$$

A.II. **Proof of Lemma 2.6-(i)-(ii)-(iii).** The three proofs are essentially the same, hence we prove only (ii).

Note that $P_s^{\beta,\text{kill}}P_r^{\Omega} = P_r^{\Omega}P_s^{\beta,\text{kill}}$ for every $s, r \ge 0$, and that $\|P_s^{\Omega}f\|_{C([0,T]\times\overline{\Omega})}, \ \|P_s^{\beta,\text{kill}}f\|_{C([0,T]\times\overline{\Omega})} \le \|f\|_{C([0,T]\times\overline{\Omega})},$

for every $f \in C_{0,\partial\Omega}([0,T] \times \Omega)$, $s \ge 0$. It is then easy to prove that $P^{\beta,\Omega,\text{kill}}$ is sub-Feller semigruop on $C_{0,\partial\Omega}([0,T] \times \Omega)$. We denote the generator of $P^{\beta,\Omega,\text{kill}}$ by $(\mathcal{L}^{\text{kill}}_{\beta,\Omega}, \text{Dom}(\mathcal{L}^{\text{kill}}_{\beta,\Omega}))$. Let f = pq, where $p \in \mathcal{C}^{\text{kill}}_{\beta}$ and $q \in \mathcal{C}_{\Omega}$. Then, by a standard triangle inequality argument, we obtain

$$\begin{split} \left| \frac{P_h^{\beta,\mathrm{kill}} P_h^{\Omega} f(t,x) - f(t,x)}{h} - (\mathcal{L}_{\beta}^{\mathrm{kill}} + \mathcal{L}_{\Omega}) f(t,x) \right| \\ & \leq \|p\|_{C([0,T])} \left\| \frac{P_h^{\Omega} q - q}{h} - \mathcal{L}_{\Omega} q \right\|_{C(\overline{\Omega})} + \|\mathcal{L}_{\Omega} q\|_{C(\overline{\Omega})} \left\| P_h^{\beta,\mathrm{kill}} p - p \right\|_{C([0,T])} \\ & + \|q\|_{C(\overline{\Omega})} \left\| \frac{P_h^{\beta,\mathrm{kill}} p - p}{h} - \mathcal{L}_{\beta}^{\mathrm{kill}} p \right\|_{C([0,T])} \to 0, \end{split}$$

as $h \downarrow 0$. An induction argument proves that $\operatorname{Span}\{\mathcal{C}_{\beta}^{\operatorname{kill}} \cdot \mathcal{C}_{\Omega}\} \subset \operatorname{Dom}(\mathcal{L}_{\beta,\Omega}^{\operatorname{kill}})$ and $\mathcal{L}_{\beta,\Omega}^{\operatorname{kill}} = (\mathcal{L}_{\beta}^{\operatorname{kill}} + \mathcal{L}_{\Omega})$ on $\operatorname{Span}\{\mathcal{C}_{\beta}^{\operatorname{kill}} \cdot \mathcal{C}_{\Omega}\}$. Observing that $\operatorname{Span}\{\mathcal{C}_{\beta}^{\operatorname{kill}} \cdot \mathcal{C}_{\Omega}\}$ is invariant under $P^{\beta,\Omega\operatorname{kill}}$ and it is a subspace of $\operatorname{Dom}(\mathcal{L}_{\beta,\Omega}^{\operatorname{kill}})$, if we can prove that $\operatorname{Span}\{\mathcal{C}_{\beta}^{\operatorname{kill}} \cdot \mathcal{C}_{\Omega}\}$ is dense in $C_{0,\partial\Omega}([0,T] \times \Omega)$, we are done by [15, Lemma 1.34]. So proceed by noting that set $\operatorname{Span}\{\mathcal{C}^{\infty}([0,T]) \cdot \mathcal{C}^{\infty}(\overline{\Omega})\}$ is a sub-algebra of $C([0,T] \times \overline{\Omega})$ that contains constant functions and separates points. Hence $\operatorname{Span}\{\mathcal{C}^{\infty}([0,T]) \cdot \mathcal{C}^{\infty}(\overline{\Omega})\}$ is dense in $C([0,T] \times \overline{\Omega})$ by Stone-Weierstrass Theorem for compact⁷ Hausdorff spaces. We now prove density of the following set

$$\operatorname{Span}\{C_c^{\infty}((0,T]) \cdot C_c^{\infty}(\Omega)\} \subset C_{0,\partial\Omega}([0,T] \times \Omega).$$

For $f \in C_{0,\partial\Omega}([0,T] \times \Omega)$ we take a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \text{Span}\{C^{\infty}([0,T]) \cdot C^{\infty}(\overline{\Omega})\}$ such that $f_n \to f$, where $f_n(t,x) = \sum_{i=1}^{N_n} p_{i,n}(t)q_{i,n}(x)$, for some $N_n \in \mathbb{N}$ depending on $n \in \mathbb{N}$. Let $1_{T,n} \in C_c^{\infty}((0,T])$ and $1_{\Omega,n} \in C_c^{\infty}(\Omega)$ be smooth functions for each $n \in \mathbb{N}$, such that

⁷In the case of unbounded domains (part (iii) of the current lemma) use the Stone-Weierstrass Theorem for locally compact Hausdorff spaces.

 $0 \leq 1_{T,n}, 1_{\Omega,n} \leq 1, 1_{T,n}(t) = 1_{\Omega,n}(x) = 1$ for $t \in (\frac{1}{n}, T]$ and $x \in K_n$, and $1_{T,n}(t) = 1_{\Omega,n}(x) = 0$ for $t \in (0, \frac{1}{n+1}]$ and $x \in \Omega \setminus K_{n+1}$, where K_n is compact, $K_n \subset K_{n+1} \subset \Omega$ for each n, and $\bigcup_n K_n = \Omega$. Define for each $n \in \mathbb{N}$,

$$(t,x) \mapsto \tilde{f}_n(t,x) := \sum_{i=1}^{N_n} p_{i,n}(t) \mathbf{1}_{T,n}(t) q_{i,n}(x) \mathbf{1}_{\Omega,n}(x) \in \mathrm{Span}\{C_c^{\infty}((0,T]) \cdot C_c^{\infty}(\Omega)\}$$

Then, as $n \to \infty$

$$\|\tilde{f}_n - f\|_{C([0,T]\times\Omega)} \le \|f_n - f\|_{C([\frac{1}{n},T]\times K_n)} + \|\tilde{f}_n - f\|_{C((\frac{1}{n+1},\frac{1}{n}]\times\overline{\Omega}\cup[0,T]\times K_{n+1}\setminus K_n)} + \|f\|_{C([0,T]\times\overline{\Omega}\setminus K_{n+1}\cup[0,\frac{1}{n+1}]\times\overline{\Omega})} \to 0.$$

As $C_c^{\infty}(\Omega) \not\subset \operatorname{Dom}(\mathcal{L}_{\Omega})$ we need to work a bit more. For any $u \in C_{0,\partial\Omega}([0,T] \times \Omega)$ we can now take a uniformly approximating sequence $\{u_n\}_{n\in\mathbb{N}}\subset \operatorname{Span}\{C_c^{\infty}((0,T])\cdot C_c^{\infty}(\Omega)\}$. Denote $u_n(t,x) = \sum_{i=1}^{N_n} p_{i,n}(t)q_{i,n}(x)$, for some $N_n \in \mathbb{N}$ depending on $n \in \mathbb{N}$, where $p_{i,n} \in C_c^{\infty}((0,T]), q_{i,n} \in C_c^{\infty}(\Omega)$ are non-zero, for each $i \in \{1, ..., N_n\}$, $n \in \mathbb{N}$. As \mathcal{C}_{β} and \mathcal{C}_{Ω} are dense in $C_0([0,T]) \supset C_c^{\infty}((0,T])$ and $C_{\partial\Omega}(\Omega) \supset C_c^{\infty}(\Omega)$, respectively, we can pick $\{(\tilde{p}_{i,n}, \tilde{q}_{i,n}) : i \in \{1, ..., N_n\}, n \in \mathbb{N}\} \subset \mathcal{C}_{\beta} \times \mathcal{C}_{\Omega}$, in the following fashion: for each triplet $(N_n, p_{i,n}, q_{i,n})$, first pick $\tilde{p}_{i,n}$ so that

$$||p_{i,n} - \tilde{p}_{i,n}||_{C[0,T]} \le \frac{1}{nN_n||q_{i,n}||_{C[0,T]}}$$

secondly pick $\tilde{q}_{i,n}$ so that

$$\|q_{i,n} - \tilde{q}_{i,n}\|_{C[0,T]} \le \frac{1}{nN_n \|\tilde{p}_{i,n}\|_{C[0,T]}}$$

Then, after defining $\tilde{u}_n(t,x) := \sum_{i=1}^{N_n} \tilde{p}_{i,n}(t) \tilde{q}_{i,n}(x)$, we obtain

$$\begin{split} \|u - u_n\|_{\infty} &\leq \|u - u_n\|_{\infty} + \|u_n - u_n\|_{\infty} \\ &\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \|p_{i,n}q_{i,n} - \tilde{p}_{i,n}\tilde{q}_{i,n}\|_{\infty} \\ &\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \left(\|q_{i,n}\|_{\infty}\|p_{i,n} - \tilde{p}_{i,n}\|_{\infty} + \|\tilde{p}_{i,n}\|_{\infty}\|q_{i,n} - \tilde{q}_{i,n}\|_{\infty}\right) \\ &\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \left(\frac{\|q_{i,n}\|_{\infty}}{nN_n\|q_{i,n}\|_C[0,T]} + \frac{\|\tilde{p}_{i,n}\|_{\infty}}{nN_n\|\tilde{p}_{i,n}\|_C[0,T]}\right) \\ &= \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \frac{2}{nN_n} \\ &\leq \|u - u_n\|_{\infty} + \frac{2}{n} \to 0, \qquad \text{as } n \to \infty. \end{split}$$

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References

- [1] Allen, M.. Uniqueness for weak solutions of parabolic equations with a fractional time derivative. arXiv preprint arXiv:1705.03959 (2017).
- [2] Allen, M., Caffarelli L., Vasseur A. A parabolic problem with a fractional time derivative. Archive for Rational Mechanics and Analysis 221.2 (2016): 603-630.
- Baeumer, B., Kurita S., Meerschaert M. M.. Inhomogeneous fractional diffusion equations. Fractional Calculus and Applied Analysis 8.4 (2005): 371-386.
- Baeumer, Boris, Tomasz Luks, and Mark M. Meerschaert. Space-time fractional Dirichlet problems. arXiv preprint arXiv:1604.06421 (2016).
- [5] Baeumer, Boris, and Mark M. Meerschaert. Stochastic solutions for fractional Cauchy problems. Fractional Calculus and Applied Analysis 4.4 (2001): 481-500.
- [6] Baeumer B, Meerschaert M, Nane E. Brownian subordinators and fractional Cauchy problems. Transactions of the American Mathematical Society. 2009;361(7):3915-30.
- [7] Barlow, Martin T.; Černý, Jiří. Convergence to fractional kinetics for random walks associated with unbounded conductances. Probability theory and related fields, 2011, 149.3-4: 639-673.
- [8] Benson DA, Meerschaert MM, Revielle J. Fractional calculus in hydrologic modeling: A numerical perspective. Advances in water resources. 2013 Jan 1;51:479-97.
- [9] Bernyk, Violetta, Robert C. Dalang, and Goran Peskir. Predicting the ultimate supremum of a stable Lévy process with no negative jumps. The Annals of Probability 39.6 (2011): 2385-2423.
- [10] J. Bertoin (1996) Lévy processes. Cambridge University Press.
- [11] Bingham, N. H. Limit theorems for occupation times of Markov processes. Zeitschrift f
 ür Wahrscheinlichkeitstheorie und verwandte Gebiete 17.1 (1971): 1-22.
- [12] Bobaru F , Duangpanya M . The peridynamic formulation for transient heat conduction. Int J Heat Mass Transf 2010;53(19):404759 .
- [13] Bonforte, M., Vázquez, J. L. (2016). Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds. Nonlinear Analysis, 131, 363-398.
- [14] Bogdan, K., Byczkowski, T., Kulczycki, T., Ryznar, M., Song, R., Vondracek, Z. (2009). Potential analysis of stable processes and its extensions. Springer Science & Business Media.
- [15] Böttcher, Björn, R. L. Schilling, and Jian Wang. Lévy matters. III.: Lévy-type Processes: Construction, Approximation and Sample Path Properties. Cham: Springer (2013).
- [16] Chen Z-Q, (2017). Time fractional equations and probabilistic representation, Chaos, Solitons & Fractals, 102, 168-174.
- [17] Chen, A., Du, Q., Li, C., Zhou, Z. (2017). Asymptotically compatible schemes for space-time nonlocal diffusion equations. Chaos, Solitons & Fractals, 102, 361-371.
- [18] Chen Zhen-Qing, Panki Kim, Takashi Kumagai, Jian Wang (2017). Heat kernel estimates for time fractional equations.arXiv:1708.05863.
- [19] Chen, Zhen-Qing, Mark M. Meerschaert, and Erkan Nane. Space-time fractional diffusion on bounded domains. Journal of Mathematical Analysis and Applications 393.2 (2012): 479-488.
- [20] Deng CS, Schilling RL. Exact Asymptotic Formulas for the Heat Kernels of Space and Time-Fractional Equations. arXiv preprint:1803.11435. 2018 Mar 30.
- [21] Diethelm, K. (2010), The Analysis of Fractional Differential Equations, An application-oriented exposition using differential operators of Caputo Type, Lecture Notes in Mathematics, v. 2004, Springer.
- [22] Du, Q., Yang, V., Zhou, Z., Analysis of a nonlocal-in-time parabolic equation. Discrete and continuous dynamical systems series B, Vol 22, n. 2, 2017.
- [23] Dynkin, E. B. (1965), Markov processes, Vol. I, Springer-Verlag.
- [24] Eidelman SD, Kochubei AN. Cauchy problem for fractional diffusion equations. Journal of differential equations. 2004 May 20;199(2):211-55.
- [25] Gilboa G , Osher S . Nonlocal operators with applications to image processing. Multiscale Model Simul 20 08;7(3):10 0528 .

- [26] Du Q, Gunzburger M, Lehoucq RB, Zhou K. Analysis and approximation of nonlocal diffusion problems with volume constraints. SIAM Rev 2012;54(4):66796.
- [27] Fedotov S, Iomin A. Migration and proliferation dichotomy in tumor-cell invasion. Physical Review Letters. 2007 Mar 12;98(11):118101.
- [28] Hernández-Hernández, M.E., Kolokoltsov, V.N., Toniazzi, L., (2017). Generalised fractional evolution equations of Caputo type. Chaos, Solitons & Fractals, 102, 184-196.
- [29] Ikeda, Nobuyuki, and Shinzo Watanabe. On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes. Journal of Mathematics of Kyoto University 2.1 (1962): 79-95.
- [30] Krägeloh, Alexander M. Two families of functions related to the fractional powers of generators of strongly continuous contraction semigroups. Journal of mathematical analysis and applications 283.2 (2003): 459-467.
- [31] Leonenko NN, Meerschaert MM, Sikorskii A. Fractional pearson diffusions. Journal of mathematical analysis and applications. 2013 Jul 15;403(2):532-46.
- [32] M. Magdziarz: Path properties of subdiffusiona martingale approach. Stoch. Models 26 (2010) 256271.
- [33] Magdziarz M, Weron A, Weron K. Fractional Fokker-Planck dynamics: Stochastic representation and computer simulation. Physical Review E. 2007 Jan 26;75(1):016708.
- [34] M. Magdziarz, R.L. Schilling: Asymptotic properties of Brownian motion delayed by inverse subordinators. Proc. Amer. Math. Soc. 143 (2015) 44854501
- [35] Meerschaert, Mark M., David A. Benson, Hans-Peter Scheffler, Boris Baeumer. Stochastic solution of space-time fractional diffusion equations. Physical Review E 65, no. 4 (2002): 041103.
- [36] Meerschaert MM, Nane E, Vellaisamy P. Fractional Cauchy problems on bounded domains. The Annals of Probability. 2009;37(3):979-1007.
- [37] Meerschaert MM, Nane E, Vellaisamy P. Distributed-order fractional diffusions on bounded domains. Journal of Mathematical Analysis and Applications. 2011 Jul 1;379(1):216-28.
- [38] Meerschaert M.M., H.P. Scheffler: Limit theorems for continuous time random walks with infinite mean waiting times. J. Appl. Probab. 41 (2004) 623638.
- [39] Meerschaert, M.M., Sikorskii, A. (2012), Stochastic Models for Fractional Calculus, De Gruyter Studies in Mathematics, Book 43.
- [40] Ming, Liao. The Dirichlet problem of a discontinuous Markov process. Acta Mathematica Sinica 5.1 (1989): 9-15.
- [41] Podlubny, I. (1998). Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications (Vol. 198). Elsevier.
- [42] Piryatinska A, Saichev AI, Woyczynski WA. Models of anomalous diffusion: the subdiffusive case. Physica A: Statistical Mechanics and its Applications. 2005 Apr 15;349(3-4):375-420.
- [43] Saichev AI, Zaslavsky GM. Fractional kinetic equations: solutions and applications. Chaos: An Interdisciplinary Journal of Nonlinear Science. 1997 Dec;7(4):753-64.
- [44] Samko, Stefan G., Anatoly A. Kilbas, and Oleg I. Marichev. Fractional integrals and derivatives. Theory and Applications, Gordon and Breach, Yverdon 1993 (1993): 44.
- [45] Scalas E, Five years of continuous-time random walks in econophysics, Complex Netw. Econ. Interactions 567 (2006) 316
- [46] Scalas E, R Gorenflo, F Mainardi. Fractional calculus and continuous-time finance. Physica A: Statistical Mechanics and its Applications, 2000.
- [47] Silling SA, Lehoucq RB. Peridynamic theory of solid mechanics. Adv Appl Mech 2010;44:73168.
- [48] Zaslavsky GM. Fractional kinetic equation for Hamiltonian chaos. Physica D: Nonlinear Phenomena. 1994 Sep 1;76(1-3):110-22.
- [49] Zhang Y, Meerschaert MM, Baeumer B. Particle tracking for time-fractional diffusion. Physical Review E. 2008 Sep 19;78(3):036705.

SPACE-TIME FRACTIONAL EEs

[50] Zolotarev, V. M. (1986) One-dimensional stable distributions. Translations of Mathematical monographs, vol. 65, American Mathematical Society, 1986.

L. Toniazzi

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WARWICK, COVENTRY, UNITED KINGDOM.

E-mail address: l.toniazzi@warwick.ac.uk