

GRADIENT ESTIMATES FOR STOKES SYSTEMS IN DOMAINS

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ABSTRACT. We study the stationary Stokes system with Dini mean oscillation coefficients in a domain having $C^{1,\text{Dini}}$ boundary. We prove that if (u, p) is a weak solution of the system with zero Dirichlet boundary condition, then (Du, p) is continuous up to the boundary. We also prove a weak type-(1, 1) estimate for (Du, p) .

1. INTRODUCTION AND MAIN RESULTS

We consider the stationary Stokes system with variable coefficients

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^d , $d \geq 2$. The differential operator \mathcal{L} is in divergence form acting on column vector-valued functions $u = (u^1, \dots, u^d)^\top$ as follows:

$$\mathcal{L}u = D_\alpha(A^{\alpha\beta}D_\beta u),$$

where the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ are $d \times d$ matrix-valued functions on Ω , which satisfy the strong ellipticity condition, i.e., there is a constant $\lambda \in (0, 1]$ such that for any $x \in \mathbb{R}^d$ and $\xi_\alpha \in \mathbb{R}^d$, $\alpha \in \{1, \dots, d\}$, we have

$$|A^{\alpha\beta}(x)| \leq \lambda^{-1}, \quad \sum_{\alpha, \beta=1}^d A^{\alpha\beta}(x)\xi_\beta \cdot \xi_\alpha \geq \lambda \sum_{\alpha=1}^d |\xi_\alpha|^2.$$

In a recent paper [3], we investigated minimal regularity assumptions on coefficients and data for $W^{1,\infty}$ and C^1 regularity of weak solutions to the Stokes system in a ball and a half ball. One of the results in [3] is that every weak solutions of (1.1) satisfy

$$(u, p) \in C^1(\Omega')^d \times C(\Omega'), \quad \Omega' \Subset \Omega$$

provided that the coefficients and data are of *Dini mean oscillation*. We say that a function is of Dini mean oscillation if its L^1 -mean oscillation satisfies the Dini condition; see Definition 1.1 for more precise definition. This class of functions was first introduced by Dong-Kim in [8] for C^1 and C^2 regularity of solutions to elliptic equations in divergence and nondivergence form. A local weak type-(1, 1) estimate for (Du, p) was also proved in [3].

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In this paper, we extend the aforementioned results in [3] to domains up to the boundary. More precisely, we prove that weak solutions of the Stokes system (1.1) with zero Dirichlet boundary condition satisfy

$$(u, p) \in C^1(\overline{\Omega})^d \times C(\overline{\Omega}) \quad (1.2)$$

provided that the coefficients and data are of Dini mean oscillation, and that Ω has $C^{1,\text{Dini}}$ boundary. As an application, we obtain Schauder estimate and regularity for weak solutions, which were studied in [9, Theorem 1.3, p. 198]. We also prove the global weak type-(1, 1) estimate for (Du, p) under a stronger assumption on the coefficients and the boundary.

Our argument in establishing (1.2) is based on the approach used in [6], where the authors proved boundary C^1 -estimates for divergence type elliptic equations

$$D_i(a^{ij}D_ju) = \text{div } f$$

with Dini mean oscillation coefficients on a domain having $C^{1,\text{Dini}}$ boundary. The key ingredient is L^q -mean oscillation estimates with $q \in (0, 1)$ for derivatives of solutions on the boundary. In [6], such mean oscillation estimates were obtained near a flat boundary and then the boundary C^1 -estimate follows from that on the half ball since the mapping of flattening boundary preserves the regularity assumptions on the coefficients and data. However, this argument does not work for the Stokes system because after the mapping the pressure term and the divergence equation give rise to extra terms which are *not* of Dini mean oscillation. In this paper, we establish the L^q -mean oscillation estimate near curved boundary. To this end, we fix a point $x_0 = (x_{01}, x'_0) \in \partial\Omega$ and a coordinate system so that the $C^{1,\text{Dini}}$ function χ defining $\partial\Omega$ near x_0 satisfies $|\nabla_{x'}\chi(x'_0)| = 0$. Then, in this coordinate system, we employ the mapping of flattening boundary to control the L^q -mean oscillation at x_0 . Therefore, our mean oscillation estimate at the boundary point x_0 depends on the coordinate system and the $C^{1,\text{Dini}}$ function χ associated with x_0 ; see Lemma 2.2. This makes the arguments much more involved.

The remainder of the paper is organized as follows. In the rest of this section, we state our main results along with some definitions and assumptions. In Section 2, we provide the proofs of the main theorems. In Appendix, we provide the proofs of some lemmas used in the paper.

For any $x \in \overline{\Omega}$ and $r > 0$, we denote $\Omega_r(x) = \Omega \cap B_r(x)$, where $B_r(x)$ is a usual Euclidean ball of radius r centered at x . We denote $B_r^+(x) = B_r(x) \cap \mathbb{R}_+^d$, where

$$\mathbb{R}_+^d = \{x = (x_1, x') \in \mathbb{R}^d : x_1 > 0, x' \in \mathbb{R}^{d-1}\}.$$

For $0 < q \leq \infty$, let $L^q(\Omega)$ be the space consisting of measurable functions on Ω that are q -th integrable. We define

$$\tilde{L}^q(\Omega) = \{f \in L^q(\Omega) : (f)_\Omega = 0\},$$

where $(f)_\Omega$ is the average of f over Ω , i.e.,

$$(f)_\Omega = \int_\Omega f \, dx = \frac{1}{|\Omega|} \int_\Omega f \, dx.$$

For $1 \leq q \leq \infty$, we denote by $W^{1,q}(\Omega)$ the usual Sobolev space and by $W_0^{1,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$. We define the Hölder semi-norm by

$$[f]_{C^\gamma(\Omega)} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}, \quad 0 < \gamma < 1.$$

We say that a measurable function $\omega : (0, a] \rightarrow [0, \infty)$ is a Dini function provided that there are constants $c_1, c_2 > 0$ such that

$$c_1\omega(t) \leq \omega(s) \leq c_2\omega(t) \quad \text{whenever} \quad \frac{t}{2} \leq s \leq t \leq a \quad (1.3)$$

and that ω satisfies the Dini condition

$$\int_0^a \frac{\omega(t)}{t} dt < \infty. \quad (1.4)$$

Definition 1.1. Let $f \in L^1(\Omega)$.

(i) We say that f is *uniformly Dini continuous* in Ω if the function $\varrho_f : (0, 1] \rightarrow [0, \infty)$ defined by

$$\varrho_f(r) := \sup_{x_0 \in \Omega} \sup_{x, y \in \Omega_r(x_0)} |f(x) - f(y)|$$

is a Dini function.

(ii) We say that f is of *Dini mean oscillation* in Ω if the function $\omega_f : (0, 1] \rightarrow [0, \infty)$ defined by

$$\omega_f(r) := \sup_{x \in \overline{\Omega}} \int_{\Omega_r(x)} |f - (f)_{\Omega_r(x)}| dy$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_f(t)}{t} dt < \infty.$$

Remark 1.2. Assume that $|\Omega_r(x)| \geq A_0 r^d$ for all $x \in \overline{\Omega}$ and $0 < r \leq 1$. If f is of Dini mean oscillation in Ω , then f is uniformly continuous in Ω with its modulus of continuity controlled by ω_f . Moreover, since ω_f satisfies the condition (1.3) with $(c_1, c_2) = (c_1, c_2)(d, A_0)$ (see, for instance, [10, p. 495]), we have that $\omega_f : (0, 1] \rightarrow [0, \infty)$ is a Dini function.

Definition 1.3. Let Ω be a domain in \mathbb{R}^d . We say that Ω has $C^{1, \text{Dini}}$ boundary if there exist a constant $R_0 \in (0, 1]$ and a Dini function $\varrho_0 : (0, 1] \rightarrow [0, \infty)$ such that the following holds: For any $x_0 = (x_{01}, x'_0) \in \partial\Omega$, there exists a $C^{1, \text{Dini}}$ function (i.e., C^1 function whose derivatives are uniformly Dini continuous) $\chi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ and a coordinate system depending on x_0 such that

$$\varrho_{\nabla_{x'} \chi}(r) \leq \varrho_0(r) \quad \text{for all } r \in (0, R_0), \quad (1.5)$$

and that in the new coordinate system, we have

$$|\nabla_{x'} \chi(x'_0)| = 0, \quad \Omega_{R_0}(x_0) = \{x \in B_{R_0}(x_0) : x_1 > \chi(x')\}. \quad (1.6)$$

Now, we state our main theorems.

Theorem 1.4. Let Ω be a bounded domain in \mathbb{R}^d having $C^{1, \text{Dini}}$ boundary. Assume that $(u, p) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ is the weak solution of

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g - (g)_\Omega & \text{in } \Omega, \end{cases} \quad (1.7)$$

where $f_\alpha \in L^2(\Omega)^d$ and $g \in L^2(\Omega)$.

(a) If $A^{\alpha\beta}$, f_α , and g are of Dini mean oscillation in Ω , then we have

$$(u, p) \in C^1(\overline{\Omega})^d \times C(\overline{\Omega}).$$

- (b) Let $0 < \gamma_0 < 1$ and $\partial\Omega$ be C^{1,γ_0} , i.e., $\varrho_0(r) = Nr^{\gamma_0}$ for some constant $N > 0$. If it holds that $[A^{\alpha\beta}]_{C^{\gamma_0}(\Omega)} + [f_\alpha]_{C^{\gamma_0}(\Omega)} + [g]_{C^{\gamma_0}(\Omega)} < \infty$, then we have

$$(u, p) \in C^{1,\gamma_0}(\overline{\Omega})^d \times C^{\gamma_0}(\overline{\Omega}).$$

Remark 1.5. By the same reasoning as [3, Remark 2.4], one can extend the results in Theorem 1.4 to the solution of the system

$$\begin{cases} \mathcal{L}u + \nabla p = f + D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g - (g)_\Omega & \text{in } \Omega, \end{cases}$$

where $f \in L^q(\Omega)^d$ with $q > d$.

In the next theorem, we prove the global weak type-(1, 1) estimate for Du and p .

Theorem 1.6. *Let Ω be a bounded domain in \mathbb{R}^d having $C^{1,\text{Dini}}$ boundary. Assume that $(u, p) \in W_0^{1,q}(\Omega)^d \times \tilde{L}^q(\Omega)$ is the weak solution of (1.7), where $f_\alpha \in L^q(\Omega)^d$, $g \in L^q(\Omega)$, and $q \in (1, \infty)$. If $A^{\alpha\beta}$ are of Dini mean oscillation in Ω and*

$$\varrho_0(r) + \omega_{A^{\alpha\beta}}(r) \leq C_0(\ln r)^{-2}, \quad \forall r \in (0, 1/2), \quad (1.8)$$

then for any $t > 0$, we have

$$|\{x \in \Omega : |Du(x)| + |p(x)| > t\}| \leq \frac{C}{t} \int_\Omega (|f_\alpha| + |g|) dx, \quad (1.9)$$

where the constant C depends only on $d, \lambda, \Omega, R_0, \varrho_0, \omega_{A^{\alpha\beta}}$, and C_0 .

Remark 1.7. Under the hypothesis of Theorem 1.4 (a), the unique solvability of the problem (1.7) is available in the solution space $W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ as well as $W_0^{1,q}(\Omega)^d \times \tilde{L}^q(\Omega)$ with $q \in (1, \infty)$, when $f_\alpha \in L^q(\Omega)^d$ and $g \in L^q(\Omega)$; see the proof of Theorem 1.6. Therefore, in Theorems 1.4 and 1.6, the weak solutions indeed exist.

We present the $W^{1,q}$ -estimate for a $W^{1,1}$ -weak solution, which follows from Theorem 1.4, the solvability result mentioned in Remark 1.7, and the argument in Brezis [2] (see also [1, Appendix]). For a proof, one may refer to the proofs of [3, Theorems 2.5 and 5.4], where we proved the $W^{1,q}$ -estimates for $W^{1,1}$ -weak solutions to the Stokes system with partially Dini mean oscillation coefficients in a ball and a half ball.

Corollary 1.8. *Let $q \in (1, \infty)$ and Ω be a bounded domain in \mathbb{R}^d having $C^{1,\text{Dini}}$ boundary. Assume that $(u, p) \in W_0^{1,1}(\Omega)^d \times \tilde{L}^1(\Omega)$ is a weak solution of (1.7), where $f_\alpha \in L^q(\Omega)^d$ and $g \in L^q(\Omega)$. If $A^{\alpha\beta}$ are of Dini mean oscillation in Ω , then we have $(u, p) \in W_0^{1,q}(\Omega)^d \times \tilde{L}^q(\Omega)$ with the estimate*

$$\|u\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq C(\|u\|_{W^{1,1}(\Omega)} + \|p\|_{L^1(\Omega)} + \|f_\alpha\|_{L^q(\Omega)} + \|g\|_{L^q(\Omega)}),$$

where the constant C depends only on $d, \lambda, \Omega, R_0, \varrho_0$, and $\omega_{A^{\alpha\beta}}$.

We finish this section with a remark that, by Corollary 1.8 the results in Theorems 1.4 and 1.6 still hold under the assumption that $(u, p) \in W_0^{1,1}(\Omega)^d \times \tilde{L}^1(\Omega)$.

2. PROOF OF MAIN THEOREMS

Hereafter in the paper, we use the following notation.

Notation 2.1. For nonnegative (variable) quantities A and B , we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \leq CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b .

2.1. Proof of Theorem 1.4. We shall derive a priori estimates for Du and p by assuming that $(u, p) \in C^1(\overline{\Omega})^d \times C(\overline{\Omega})$. The general case follows from a standard approximation argument.

Throughout this proof, we use the following notation and properties. Recall that ϱ_0 is the Dini function from Definition 1.3.

i. We set $q = 1/2$ and

$$\Phi(x_0, r) := \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^{d \times d}}} \left(\int_{\Omega_r(x_0)} |Du - \Theta|^q + |p - \theta|^q dx \right)^{1/q}.$$

ii. For any $x \in \overline{\Omega}$ and $r \in (0, 1]$, we have

$$r^d \lesssim_{d, R_0, \varrho_0} |\Omega_r(x)|. \quad (2.1)$$

iii. For $\gamma \in (0, 1)$ and $\kappa \in (0, 1/2]$, we define

$$\tilde{\varrho}_0(r) := \varrho_0(r) + \sum_{i=1}^{\infty} \kappa^{\gamma i} (\varrho_0(\kappa^{-i}r)[\kappa^{-i}r < 1] + \varrho_0(1)[\kappa^{-i}r \geq 1]),$$

where we use Inverse bracket notation; i.e., $[P] = 1$ if P is true and $[P] = 0$ otherwise. By Lemma 3.1, $\tilde{\varrho}_0 : (0, 1] \rightarrow [0, \infty)$ is a Dini function satisfying

$$\tilde{\varrho}_0(t) \lesssim_{\varrho_0} \tilde{\varrho}_0(s) \lesssim_{\varrho_0} \tilde{\varrho}_0(t) \quad \text{whenever } \frac{t}{2} \leq s \leq t \leq 1. \quad (2.2)$$

Moreover, by the comparison principle for Riemann integrals, we have

$$\sum_{j=0}^{\infty} \tilde{\varrho}_0(\kappa^j r) \lesssim_{\varrho_0, \kappa} \int_0^r \frac{\tilde{\varrho}_0(t)}{t} dt < \infty$$

for all $r \in (0, 1]$.

iv. For $\gamma \in (0, 1)$, $\kappa \in (0, 1/2]$, and $f \in L^1(\Omega)$, we denote

$$\tilde{\omega}_f(r) := \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega_f(\kappa^{-i}r)[\kappa^{-i}r < 1] + \omega_f(1)[\kappa^{-i}r \geq 1]).$$

By Remark 1.2, (2.1), and Lemma 3.1, if f is of Dini mean oscillation in Ω , then $\tilde{\omega}_f : (0, 1] \rightarrow [0, \infty)$ is a Dini function satisfying

$$\tilde{\omega}_f(t) \lesssim_{d, R_0, \varrho_0} \tilde{\omega}_f(s) \lesssim_{d, R_0, \varrho_0} \tilde{\omega}_f(t) \quad \text{whenever } \frac{t}{2} \leq s \leq t \leq 1.$$

Moreover, we have

$$\sum_{j=0}^{\infty} \tilde{\omega}_f(\kappa^j r) \lesssim_{d, R_0, \varrho_0, \kappa} \int_0^r \frac{\tilde{\omega}_f(t)}{t} dt < \infty \quad (2.3)$$

for all $r \in (0, 1]$.

To prove Theorem 1.4, we will use the following three lemmas related to L^q -mean oscillation estimates for Du and p . The first lemma is about the interior estimates, which is an adaptation of [3, Lemma 4.3].

Lemma 2.1. *Let $x_0 \in \Omega$ and $\gamma \in (0, 1)$. Under the same hypothesis of Theorem 1.4 (a), there exists a constant $\kappa_1 \in (0, 1/2]$ depending only on d, λ , and γ , such that the following hold.*

(i) *For any $0 < \kappa \leq \kappa_1$ and $0 < r \leq \min\{1, \text{dist}(x_0, \partial\Omega)/4\}$, we have*

$$\begin{aligned} \sum_{j=0}^{\infty} \Phi(x_0, \kappa^j r) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0, \kappa} \Phi(x_0, r) \\ &\quad + \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\tilde{\omega}_{A^{\alpha\beta}}(t)}{t} dt + \int_0^r \frac{\tilde{\omega}_{f_\alpha}(t) + \tilde{\omega}_g(t)}{t} dt. \end{aligned}$$

(ii) *For any $0 < \kappa \leq \kappa_1$ and $0 < \rho \leq r \leq \min\{1, \text{dist}(x_0, \partial\Omega)/4\}$, we have*

$$\Phi(x_0, \rho) \lesssim_{d, \lambda, \gamma, \kappa} \left(\frac{\rho}{r}\right)^\gamma \Phi(x_0, r) + \|Du\|_{L^\infty(B_r(x_0))} \tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho).$$

Proof. By following the proof of [3, Lemma 4.3], we see that

$$\begin{aligned} \Phi(x_0, \kappa r) &\leq C_0 \kappa \Phi(x_0, r) \\ &\quad + C_0 \kappa^{-d/q} (\|Du\|_{L^\infty(B_r(x_0))} \omega_{A^{\alpha\beta}}(r) + \omega_{f_\alpha}(r) + \omega_g(r)) \end{aligned}$$

for all $0 < \kappa \leq 1/2$ and $0 < r \leq \min\{1, \text{dist}(x_0, \partial\Omega)/4\}$, where $C_0 = C_0(d, \lambda) > 0$. We take $\kappa_1 = \kappa_1(d, \lambda, \gamma) \in (0, 1/2]$ such that $C_0 \kappa_1^{1-\gamma} \leq 1$. Then for any $0 < \kappa \leq \kappa_1$, we have

$$\Phi(x_0, \kappa r) \leq \kappa^\gamma \Phi(x_0, r) + C (\|Du\|_{L^\infty(B_r(x_0))} \omega_{A^{\alpha\beta}}(r) + \omega_{f_\alpha}(r) + \omega_g(r)),$$

where $C = C(d, \lambda, \gamma, \kappa)$. By iterating, we obtain for $j \in \{1, 2, \dots\}$ that

$$\begin{aligned} \Phi(x_0, \kappa^j r) &\leq \kappa^{\gamma j} \Phi(x_0, r) \\ &\quad + C (\|Du\|_{L^\infty(B_r(x_0))} \tilde{\omega}_{A^{\alpha\beta}}(\kappa^j r) + \tilde{\omega}_{f_\alpha}(\kappa^j r) + \tilde{\omega}_g(\kappa^j r)), \end{aligned} \tag{2.4}$$

where we used the fact that

$$\sum_{i=1}^j \kappa^{\gamma(i-1)} \omega_\bullet(\kappa^{j-i} r) \leq \kappa^{-\gamma} \tilde{\omega}_\bullet(\kappa^j r). \tag{2.5}$$

Taking the summations of both sides of (2.4) with respect to $j = 0, 1, 2, \dots$, and using (2.3), we see that the assertion (i) holds.

For given $\rho \in (0, r]$, let j be an integer such that

$$\kappa^{j+1} < \frac{\rho}{r} \leq \kappa^j.$$

If $j = 0$, then obviously we have

$$\Phi(x_0, \rho) \lesssim_{d, \kappa} \Phi(x_0, r) \lesssim_{d, \kappa, \gamma} \left(\frac{\rho}{r}\right)^\gamma \Phi(x_0, r).$$

On the other hand, if $j \geq 1$, then by (2.4) with ρ in place of $\kappa^j r$, we get

$$\begin{aligned} \Phi(x_0, \rho) &\lesssim \kappa^{\gamma j} \Phi(x_0, \kappa^{-j} \rho) + \|Du\|_{L^\infty(B_{\kappa^{-j} \rho}(x_0))} \tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho) \\ &\lesssim \left(\frac{\rho}{r}\right)^\gamma \Phi(x_0, r) + \|Du\|_{L^\infty(B_r(x_0))} \tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho). \end{aligned}$$

Therefore, the assertion (ii) holds. The lemma is proved. \square

In the next lemma, we prove L^q -mean oscillation estimates of linear combinations of Du and p at $x_0 \in \partial\Omega$. We note that the L^q -mean oscillation and its estimates depend on the coordinate system associated with x_0 .

Lemma 2.2. *Let $x_0 \in \partial\Omega$ and $\gamma \in (0, 1)$. Let us fix a $C^{1, \text{Dini}}$ function $\chi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ and a coordinate system associated with x_0 satisfying (1.5) and (1.6) in Definition 1.3. In this coordinate system, we define*

$$\Psi(x_0, r) := \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^d}} \left(\int_{\Omega_r(x_0)} |D_1 u - \Theta|^q + \sum_{i=2}^d |D_i \chi D_1 u + D_i u|^q + |p - \theta|^q dx \right)^{1/q}.$$

Then under the same hypothesis of Theorem 1.4 (a), there exist constants

$$R_1 = R_1(\varrho_0, R_0) \in (0, R_0/4) \quad \text{and} \quad \kappa_2 = \kappa_2(d, \lambda, \gamma, R_0, \varrho_0) \in (0, 1/8]$$

such that the following hold.

(i) For any $0 < \kappa \leq \kappa_2$ and $0 < r \leq 2R_1$, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \Psi(x_0, \kappa^j r) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0, \kappa} \Psi(x_0, r) \\ &\quad + (\|Du\|_{L^\infty(\Omega_r(x_0))} + \|p\|_{L^\infty(\Omega_r(x_0))}) \int_0^r \frac{\tilde{\varrho}_0(t) + \tilde{\omega}_{A^{\alpha\beta}}(t)}{t} dt \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_r(x_0))} \int_0^r \frac{\tilde{\varrho}_0(t)}{t} dt + \int_0^r \frac{\tilde{\omega}_{f_\alpha}(t) + \tilde{\omega}_g(t)}{t} dt. \end{aligned}$$

(ii) For any $0 < \kappa \leq \kappa_2$ and $0 < \rho \leq r \leq 2R_1$, we have

$$\begin{aligned} \Psi(x_0, \rho) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0, \kappa} \left(\frac{\rho}{r}\right)^\gamma \Psi(x_0, r) \\ &\quad + (\|Du\|_{L^\infty(\Omega_r(x_0))} + \|p\|_{L^\infty(\Omega_r(x_0))}) (\tilde{\varrho}_0(\rho) + \tilde{\omega}_{A^{\alpha\beta}}(\rho)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_r(x_0))} \tilde{\varrho}_0(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho). \end{aligned}$$

Proof. Recall that we use $0 = (0, 0')$, $x = (x_1, x')$, and $y = (y_1, y')$ to denote points in \mathbb{R}^d . Without loss of generality, we assume that $x_0 = 0 \in \partial\Omega$ and $\chi(0') = 0$. We denote $B_R = B_R(0)$, $B_R^+ = B_R^+(0)$, and $\Omega_R = \Omega_R(0)$. Since $|\nabla_{x'} \chi(0')| = 0$, it follows from (1.5) that there exists a constant $R_1 = R_1(\varrho_0, R_0) \in (0, R_0)$ satisfying

$$|\nabla_{x'} \chi(x')| \leq 1/2 \quad \text{if } |x'| \leq R_1. \quad (2.6)$$

Let $\Gamma(y) = (y_1 + \chi(y'), y')$ and $\Lambda(x) = \Gamma^{-1}(x) = (x_1 - \chi(x'), x')$. We divide the proof into several steps.

Step 1. In this step, we prove that

$$B_{R_1/2}^+ \subset \Lambda(\Omega_{R_1}), \quad (2.7)$$

$$\Omega_{r/2} \subset \Gamma(B_r^+) \subset \Omega_{2r} \quad \text{for } r \in (0, R_1/2]. \quad (2.8)$$

To prove (2.7), assume that $y \in B_{R_1/2}^+$. Then we have

$$\begin{aligned} |y_1 + \chi(y')|^2 + |y'|^2 &\leq 2|y_1|^2 + 2|\chi(y')|^2 + |y'|^2 \\ &\leq |y|^2 + |y_1|^2 + 2|\chi(y')|^2 \\ &< \frac{R_1^2}{2} + 2|\chi(y')|^2. \end{aligned}$$

Notice from (2.6) that

$$|\chi(y')|^2 = |\chi(y') - \chi(0')|^2 \leq \frac{|y'|^2}{4} < \frac{R_1^2}{4}.$$

Combining the above two inequalities, we have $|y_1 + \chi(y')|^2 + |y'|^2 < R_1^2$, which implies that $y \in \Lambda(\Omega_{R_1})$. Thus we get (2.7). Using a similar argument, we have (2.8).

Step 2. In this step, we use the standard technique of flattening the boundary. We denote

$$v(y) = u(\Gamma(y)), \quad \pi(y) = p(\Gamma(y)), \quad b(y) = (0, D_2\chi(y'), \dots, D_d\chi(y'))^\top.$$

Since (u, p) satisfies (1.7), we have that

$$\begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta v) + \nabla \pi = D_\alpha F_\alpha + D_1(\pi b) & \text{in } B_{R_1}^+, \\ \operatorname{div} v = G + D_1 v \cdot b & \text{in } B_{R_1}^+, \\ v = 0 & \text{on } B_{R_1} \cap \partial \mathbb{R}_+^d, \end{cases}$$

where we set

$$\mathcal{A}^{\alpha\beta} = D_\ell \Lambda^\beta D_k \Lambda^\alpha A^{k\ell}(\Gamma), \quad F_\alpha = D_k \Lambda^\alpha f_k(\Gamma), \quad G = g(\Gamma) - (g)_\Omega.$$

Let $0 < r \leq R_1/4$. For a given function f , we denote $\bar{f} = (f)_{B_r^+}$. Define an elliptic operator \mathcal{L}_0 by

$$\mathcal{L}_0 v = D_\alpha(\overline{\mathcal{A}^{\alpha\beta}} D_\beta v),$$

and observe that (v, π) satisfies

$$\begin{cases} \mathcal{L}_0 v + \nabla \pi = D_\alpha \mathcal{F}_\alpha & \text{in } B_{R_1}^+, \\ \operatorname{div} v = \mathcal{G} + \overline{\mathcal{G}} & \text{in } B_{R_1}^+, \\ v = 0 & \text{on } B_{R_1} \cap \partial \mathbb{R}_+^d, \end{cases}$$

where

$$\mathcal{F}_\alpha = (\overline{\mathcal{A}^{\alpha\beta}} - \mathcal{A}^{\alpha\beta}) D_\beta v + F_\alpha - \overline{F_\alpha} + \delta_{1\alpha} \pi b, \quad \mathcal{G} = G - \overline{\mathcal{G}} + D_1 v \cdot b.$$

Here, δ_{ij} is the usual Kronecker delta symbol. We decompose

$$(v, \pi) = (v_1, \pi_1) + (v_2, \pi_2), \tag{2.9}$$

where $(v_1, \pi_1) \in W_0^{1,2}(B_{4r}^+)^d \times \tilde{L}^2(B_{4r}^+)$ is the weak solution of the problem

$$\begin{cases} \mathcal{L}_0 v_1 + \nabla \pi_1 = D_\alpha (I_{B_r^+} \mathcal{F}_\alpha) & \text{in } B_{4r}^+, \\ \operatorname{div} v_1 = I_{B_r^+} \mathcal{G} - (I_{B_r^+} \mathcal{G})_{B_{4r}^+} & \text{in } B_{4r}^+. \end{cases}$$

Here, $I_{B_r^+}$ is the characteristic function. By [3, Lemma 6.5] with scaling, we have for $t > 0$ that

$$|\{y \in B_r^+ : |Dv_1(y)| + |\pi_1(y)| > t\}| \lesssim_{d,\lambda} \frac{1}{t} \int_{B_r^+} (|\mathcal{F}_\alpha| + |\mathcal{G}|) dy.$$

This inequality implies that for $\tau > 0$,

$$\begin{aligned} & \int_{B_r^+} (|Dv_1| + |\pi_1|)^q dy \\ &= \left(\int_0^\tau + \int_\tau^\infty \right) qt^{q-1} |\{y \in B_r^+ : |Dv_1(y)| + |\pi_1(y)| > t\}| dt \\ &\lesssim |B_r^+| \tau^q + \left(\int_{B_r^+} |\mathcal{F}_\alpha| + |\mathcal{G}| dy \right) \tau^{q-1}. \end{aligned}$$

By optimizing over τ and taking the q -th root, we have

$$\left(\int_{B_r^+} (|Dv_1| + |\pi_1|)^q dy \right)^{1/q} \lesssim \int_{B_r^+} (|\mathcal{F}_\alpha| + |\mathcal{G}|) dy. \quad (2.10)$$

Since $(v_2, \pi_2) = (v, \pi) - (v_1, \pi_1)$ satisfies

$$\begin{cases} \mathcal{L}_0 v_2 + \nabla \pi_2 = 0 & \text{in } B_r^+, \\ \operatorname{div} v_2 = (I_{B_r^+} \mathcal{G})_{B_{4r}^+} + \overline{G} & \text{in } B_r^+, \\ v_2 = 0 & \text{on } B_r \cap \partial \mathbb{R}_+^d, \end{cases}$$

by [3, Lemma 6.3], we have for any $\kappa \in (0, 1/2]$,

$$\begin{aligned} & \left(\int_{B_{\kappa r}^+} |D_1 v_2 - (D_1 v_2)_{B_{\kappa r}^+}|^q + |D_{y'} v_2|^q + |\pi_2 - (\pi_2)_{B_{\kappa r}^+}|^q dy \right)^{1/q} \\ &\lesssim_{d,\lambda} \kappa \inf_{\Theta \in \mathbb{R}^d} \left(\int_{B_r^+} |D_1 v_2 - \Theta|^q + |D_{y'} v_2|^q dy \right)^{1/q}. \end{aligned} \quad (2.11)$$

Observe from (2.9) that

$$\begin{aligned} & \left(\int_{B_{\kappa r}^+} |D_1 v - (D_1 v_2)_{B_{\kappa r}^+}|^q + |D_{y'} v|^q + |\pi - (\pi_2)_{B_{\kappa r}^+}|^q dy \right)^{1/q} \\ &\lesssim \left(\int_{B_{\kappa r}^+} |D_1 v_2 - (D_1 v_2)_{B_{\kappa r}^+}|^q + |D_{y'} v_2|^q + |\pi_2 - (\pi_2)_{B_{\kappa r}^+}|^q dy \right)^{1/q} \\ &\quad + \left(\int_{B_{\kappa r}^+} |Dv_1|^q + |\pi_1|^q dy \right)^{1/q}. \end{aligned}$$

Using this inequality together with (2.10) and (2.11), we obtain that

$$\begin{aligned} & \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^d}} \left(\int_{B_{\kappa r}^+} |D_1 v - \Theta|^q + |D_{y'} v|^q + |\pi - \theta|^q dy \right)^{1/q} \\ &\lesssim_{d,\lambda} \kappa \inf_{\Theta \in \mathbb{R}^d} \left(\int_{B_r^+} |D_1 v - \Theta|^q + |D_{y'} v|^q dy \right)^{1/q} + \kappa^{-d/q} \int_{B_r^+} (|\mathcal{F}_\alpha| + |\mathcal{G}|) dy. \end{aligned}$$

Thus, from the definitions of \mathcal{F}_α and \mathcal{G} , and the fact that

$$\int_{B_r^+} |b| dy = \int_{B_r^+} |b - b(0)| dy \leq \varrho_0(r),$$

we get

$$\begin{aligned}
& \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^d}} \left(\int_{B_{\kappa r}^+} |D_1 v - \Theta|^q + |D_{y'} v|^q + |\pi - \theta|^q dy \right)^{1/q} \\
& \lesssim \kappa \inf_{\Theta \in \mathbb{R}^d} \left(\int_{B_r^+} |D_1 v - \Theta|^q + |D_{y'} v|^q dy \right)^{1/q} \\
& \quad + \kappa^{-d/q} (\|Dv\|_{L^\infty(B_r^+)} + \|\pi\|_{L^\infty(B_r^+)}) \left(\varrho_0(r) + \int_{B_r^+} |\mathcal{A}^{\alpha\beta} - \overline{\mathcal{A}^{\alpha\beta}}| dy \right) \\
& \quad + \kappa^{-d/q} \int_{B_r^+} (|F_\alpha - \overline{F_\alpha}| + |G - \overline{G}|) dy.
\end{aligned} \tag{2.12}$$

We note that

$$\sup_{y, z \in B_r^+} |D\Lambda(y) - D\Lambda(z)| \leq \varrho_0(r), \quad \sup_{y \in B_r^+} |D\Lambda(y)| \leq 1/2.$$

Using this and following the proof of [6, Lemma 2.1], we have

$$\int_{B_r^+} |\mathcal{A}^{\alpha\beta} - \overline{\mathcal{A}^{\alpha\beta}}| dy \lesssim_{d, \lambda} \varrho_0(r) + \int_{B_r^+} |A^{\alpha\beta}(\Gamma) - \overline{A^{\alpha\beta}(\Gamma)}| dy.$$

Hence, by the change of variables, (2.8), and $\varrho_0(r) \lesssim_{\varrho_0} \varrho_0(2r)$, we see that

$$\int_{B_r^+} |\mathcal{A}^{\alpha\beta} - \overline{\mathcal{A}^{\alpha\beta}}| dy \lesssim_{d, \lambda, \varrho_0} \varrho_0(2r) + \omega_{A^{\alpha\beta}}(2r).$$

Similarly, we have

$$\int_{B_r^+} (|F_\alpha - \overline{F_\alpha}| + |G - \overline{G}|) dy \lesssim_{d, \varrho_0} \|f_\alpha\|_{L^\infty(\Omega_{2r})} \varrho_0(2r) + \omega_{f_\alpha}(2r) + \omega_g(2r).$$

Therefore, using the change of variables, (2.1), and (2.8), we get from (2.12) that

$$\begin{aligned}
& \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^d}} \left(\int_{\Omega_{\kappa r/2}} |D_1 u - \Theta|^q + \sum_{i=2}^d |D_i \chi D_1 u + D_i u|^q + |p - \theta|^q dx \right)^{1/q} \\
& \lesssim_{d, \lambda, R_0, \varrho_0} \kappa \inf_{\Theta \in \mathbb{R}^d} \left(\int_{\Omega_{2r}} |D_1 u - \Theta|^q + \sum_{i=2}^d |D_i \chi D_1 u + D_i u|^q dx \right)^{1/q} \\
& \quad + \kappa^{-d/q} (\|Du\|_{L^\infty(\Omega_{2r})} + \|p\|_{L^\infty(\Omega_{2r})}) (\varrho_0(2r) + \omega_{A^{\alpha\beta}}(2r)) \\
& \quad + \kappa^{-d/q} (\|f_\alpha\|_{L^\infty(\Omega_{2r})} \varrho_0(2r) + \omega_{f_\alpha}(2r) + \omega_g(2r)).
\end{aligned} \tag{2.13}$$

for $0 < r \leq R_1/4$ and $\kappa \in (0, 1/2]$.

Step 3. We are ready to prove the lemma. By replacing $\kappa/4$, $2r$, and $R_1/2$ by κ , r , and $2R_1$ in (2.13), we obtain for $0 < r \leq 2R_1$ and $\kappa \in (0, 1/8]$ that

$$\begin{aligned}
\Psi(0, \kappa r) & \leq C_0 \kappa \Psi(0, r) \\
& \quad + C_0 \kappa^{-d/q} (\|Du\|_{L^\infty(\Omega_r)} + \|p\|_{L^\infty(\Omega_r)}) (\varrho_0(r) + \omega_{A^{\alpha\beta}}(r)) \\
& \quad + C_0 \kappa^{-d/q} (\|f_\alpha\|_{L^\infty(\Omega_r)} \varrho_0(r) + \omega_{f_\alpha}(r) + \omega_g(r)),
\end{aligned}$$

where $C_0 = C_0(d, \lambda, R_0, \varrho_0) > 0$. We take $\kappa_2 = \kappa_2(d, \lambda, \gamma, R_0, \varrho_0) \in (0, 1/8]$ so that $C_0 \kappa_2^{1-\gamma} \leq 1$. Then for any $0 < \kappa \leq \kappa_2$, we have

$$\begin{aligned} \Psi(0, \kappa r) &\leq \kappa^\gamma \Psi(0, r) + C(\|Du\|_{L^\infty(\Omega_r)} + \|p\|_{L^\infty(\Omega_r)})(\varrho_0(r) + \omega_{A^{\alpha\beta}}(r)) \\ &\quad + C(\|f_\alpha\|_{L^\infty(\Omega_r)} \varrho_0(r) + \omega_{f_\alpha}(r) + \omega_g(r)), \end{aligned}$$

where $C = C(d, \lambda, \gamma, R_0, \varrho_0, \kappa) > 0$. By iterating, we obtain for $j \in \{1, 2, \dots\}$ that

$$\begin{aligned} \Psi(0, \kappa^j r) &\leq \kappa^{\gamma j} \Psi(0, r) \\ &\quad + C(\|Du\|_{L^\infty(\Omega_r)} + \|p\|_{L^\infty(\Omega_r)})(\tilde{\varrho}_0(\kappa^j r) + \tilde{\omega}_{A^{\alpha\beta}}(\kappa^j r)) \\ &\quad + C(\|f_\alpha\|_{L^\infty(\Omega_r)} \tilde{\varrho}_0(\kappa^j r) + \tilde{\omega}_{f_\alpha}(\kappa^j r) + \tilde{\omega}_g(\kappa^j r)), \end{aligned} \quad (2.14)$$

where we used (2.5) and

$$\sum_{i=1}^j \kappa^{\gamma(i-1)} \varrho_0(\kappa^{j-i} r) \leq \kappa^{-\gamma} \tilde{\varrho}_0(\kappa^j r).$$

The estimate (2.14) corresponds to (2.4). The rest of the proof is identical to that of Lemma 2.1 and is omitted. \square

By combining Lemmas 2.1 and 2.2, we obtain the following L^q -mean oscillation estimates for Du and p .

Lemma 2.3. *Let $x_0 \in \Omega$ and $\gamma \in (0, 1)$. Under the same hypothesis of Theorem 1.4 (a), if $R_1 = R_1(\varrho_0, R_0)$ is the constant from Lemma 2.2 and*

$$\kappa = \kappa(d, \lambda, \gamma, R_0, \varrho_0) = \min\{\kappa_1, \kappa_2\},$$

where κ_1 and κ_2 are constants from Lemmas 2.1 and 2.2, then the following hold.

(i) For any $0 < r \leq R_1$, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \Phi(x_0, \kappa^j r) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0} r^{-d} (\|Du\|_{L^1(\Omega_{3r}(x_0))} + \|p\|_{L^1(\Omega_{3r}(x_0))}) \\ &\quad + (\|Du\|_{L^\infty(\Omega_{3r}(x_0))} + \|p\|_{L^\infty(\Omega_{3r}(x_0))}) \int_0^r \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} \int_0^r \frac{\varrho_0^\sharp(t)}{t} dt + \int_0^r \frac{\omega_{f_\alpha}^\sharp(t) + \omega_g^\sharp(t)}{t} dt, \end{aligned} \quad (2.15)$$

where each integration is finite; see Remark 2.4

(ii) For any $0 < \rho \leq r \leq R_1$, we have

$$\begin{aligned} \Phi(x_0, \rho) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0} \left(\frac{\rho}{r}\right)^\gamma r^{-d} (\|Du\|_{L^1(\Omega_{3r}(x_0))} + \|p\|_{L^1(\Omega_{3r}(x_0))}) \\ &\quad + (\|Du\|_{L^\infty(\Omega_{3r}(x_0))} + \|p\|_{L^\infty(\Omega_{3r}(x_0))}) (\varrho_0^\sharp(\rho) + \omega_{A^{\alpha\beta}}^\sharp(\rho)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} \varrho_0^\sharp(\rho) + \omega_{f_\alpha}^\sharp(\rho) + \omega_g^\sharp(\rho). \end{aligned} \quad (2.16)$$

Here, we set

$$\varrho_0^\sharp(\rho) := \sup_{\rho \leq R \leq R_1} \left(\frac{\rho}{R}\right)^\gamma \tilde{\varrho}_0(R), \quad \omega_\bullet^\sharp(\rho) := \sup_{\rho \leq R \leq R_1} \left(\frac{\rho}{R}\right)^\gamma \tilde{\omega}_\bullet(R).$$

Remark 2.4. Note that ϱ_0^\sharp is a Dini function; see [6, pp. 463–464]. By the definition of ϱ_0^\sharp and (2.2), we have

$$2^{-\gamma} \varrho_0^\sharp(t) \leq \varrho_0^\sharp(s) \lesssim_{\gamma, \varrho_0} \varrho_0^\sharp(t), \quad \frac{t}{2} \leq s \leq t \leq R_1.$$

Therefore, using the comparison principle for Riemann integrals, we get

$$\sum_{j=0}^{\infty} \varrho_0^\sharp(\kappa^j r) \lesssim_{\gamma, \varrho_0, \kappa} \int_0^r \frac{\varrho_0^\sharp(t)}{t} dt < \infty, \quad 0 < r \leq R_1. \quad (2.17)$$

Similarly, we have

$$\sum_{j=0}^{\infty} \omega_f^\sharp(\kappa^j r) \lesssim_{d, \gamma, R_0, \varrho_0, \kappa} \int_0^r \frac{\omega_f^\sharp(t)}{t} dt < \infty, \quad 0 < r \leq R_1, \quad (2.18)$$

for any f having Dini mean oscillation in Ω .

Proof of Lemma 2.3. The estimate (2.15) is an easy consequence of the estimate (2.16). Indeed, for $j \in \{0, 1, 2, \dots\}$, by taking $\rho = \kappa^j r$ in (2.16), we have

$$\begin{aligned} \Phi(x_0, \kappa^j r) &\lesssim \kappa^{\gamma j} r^{-d} (\|Du\|_{L^1(\Omega_{3r}(x_0))} + \|p\|_{L^1(\Omega_{3r}(x_0))}) \\ &\quad + (\|Du\|_{L^\infty(\Omega_{3r}(x_0))} + \|p\|_{L^\infty(\Omega_{3r}(x_0))}) (\varrho_0^\sharp(\kappa^j r) + \omega_{A^{\alpha\beta}}^\sharp(\kappa^j r)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} \varrho_0^\sharp(\kappa^j r) + \omega_{f_\alpha}^\sharp(\kappa^j r) + \omega_g^\sharp(\kappa^j r). \end{aligned} \quad (2.19)$$

Taking the summations of both sides of (2.19) with respect to $j = 0, 1, 2, \dots$, and using (2.17) and (2.18), we conclude (2.15).

To complete the proof, it suffices to prove that (2.16) holds. Without loss of generality, we assume that $x_0 = 0 \in \Omega$. We denote $B_R = B_R(0)$ and $\Omega_R = \Omega_R(0)$. Let $0 < \rho \leq r \leq R_1$. Note that if $r/6 < \rho \leq r$, then (2.16) follows from the definition of Φ . Hence we only need to consider the case of $0 < \rho \leq r/6$. We consider the following three cases:

$$r \leq \text{dist}(0, \partial\Omega), \quad \text{dist}(0, \partial\Omega) \leq 4\rho, \quad 4\rho < \text{dist}(0, \partial\Omega) < r.$$

i. $r \leq \text{dist}(0, \partial\Omega)$: Set $R = r/4$. Since $B_{4R} \subset \Omega$, by Lemma 2.1 (ii), we have

$$\Phi(0, \rho) \lesssim \left(\frac{\rho}{R}\right)^\gamma \Phi(0, R) + \|Du\|_{L^\infty(B_R)} \tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho).$$

Thus from the fact that

$$\tilde{\omega}_\bullet(\rho) \leq \omega_\bullet^\sharp(\rho), \quad \Phi(0, R) \lesssim R^{-d} (\|Du\|_{L^1(\Omega_R)} + \|p\|_{L^1(\Omega_R)}),$$

we get (2.16).

ii. $\text{dist}(0, \partial\Omega) \leq 4\rho$: We take $y_0 \in \partial\Omega$ such that $\text{dist}(0, \partial\Omega) = |y_0|$. We fix a $C^{1, \text{Dini}}$ function χ and a coordinate system associated with y_0 satisfying (1.5) and (1.6). In this coordinate system, using (2.1) and the fact that $\Omega_\rho \subset \Omega_{5\rho}(y_0)$, we have

$$\Phi(0, \rho) \lesssim_{d, R_0, \varrho_0} \Psi(y_0, 5\rho) + \left(\int_{\Omega_{5\rho}(y_0)} \sum_{i=2}^d |D_i \chi D_1 u|^q dx \right)^{1/q},$$

where Ψ is given in Lemma 2.2. Note that

$$|D_{x'} \chi(x')| = |D_{x'} \chi(x') - D_{x'} \chi(y'_0)| \leq \varrho_0(5\rho), \quad x' \in B'_{5\rho}(y'_0).$$

Using this together with Lemma 2.2 (ii), we obtain that

$$\Phi(0, \rho) \lesssim \Psi(y_0, 5\rho) + \varrho_0(5\rho) \|Du\|_{L^\infty(\Omega_{5\rho}(y_0))} \quad (2.20)$$

$$\begin{aligned} &\lesssim \left(\frac{\rho}{r}\right)^\gamma \Psi(y_0, r) + (\| |Du| + |p| \|_{L^\infty(\Omega_r(y_0))}) (\tilde{\varrho}_0(5\rho) + \tilde{\omega}_{A^{\alpha\beta}}(5\rho)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_r(y_0))} \tilde{\varrho}_0(5\rho) + \tilde{\omega}_{f_\alpha}(5\rho) + \tilde{\omega}_g(5\rho). \end{aligned} \quad (2.21)$$

Since it holds that

$$\Omega_r(y_0) \subset \Omega_{3r}, \quad \tilde{\varrho}(5\rho) \lesssim_\gamma \varrho_0^\sharp(\rho), \quad \tilde{\omega}_\bullet(5\rho) \lesssim_\gamma \omega_\bullet^\sharp(\rho),$$

$$\Psi(y_0, r) \lesssim r^{-d} (\|Du\|_{L^1(\Omega_{3r})} + \|p\|_{L^1(\Omega_{3r})}),$$

we get (2.16) from (2.21).

iii. $4\rho < \text{dist}(0, \partial\Omega) < r$: Set $R = \text{dist}(0, \partial\Omega)/4$, and observe that

$$\rho < R, \quad 5R < 2r \leq 2R_1.$$

Since $B_{4R} \subset \Omega$, by Lemma 2.1 (ii), we have

$$\Phi(0, \rho) \lesssim \left(\frac{\rho}{R}\right)^\gamma \Phi(0, R) + \|Du\|_{L^\infty(B_R)} \tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho). \quad (2.22)$$

We take $y_0 \in \partial\Omega$ such that $\text{dist}(0, \partial\Omega) = |y_0|$. We fix a $C^{1, \text{Dini}}$ function χ and a coordinate system associated with y_0 satisfying (1.5) and (1.6). In this coordinate system, similar to (2.21), we have

$$\begin{aligned} \Phi(0, R) &\lesssim \Psi(y_0, 5R) + \varrho_0(5R) \|Du\|_{L^\infty(\Omega_{5R}(y_0))} \\ &\lesssim \left(\frac{R}{r}\right)^\gamma \Psi(y_0, 2r) + (\| |Du| + |p| \|_{L^\infty(\Omega_{2r}(y_0))}) (\tilde{\varrho}_0(5R) + \tilde{\omega}_{A^{\alpha\beta}}(5R)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_{2r}(y_0))} \tilde{\varrho}_0(5R) + \tilde{\omega}_{f_\alpha}(5R) + \tilde{\omega}_g(5R). \end{aligned} \quad (2.23)$$

Combining (2.22) and (2.23), and using the fact that

$$\Omega_{2r}(y_0) \subset \Omega_{3r}, \quad \Psi(y_0, 2r) \lesssim r^{-d} (\|Du\|_{L^1(\Omega_{3r})} + \|p\|_{L^1(\Omega_{3r})}),$$

we get (2.16).

The lemma is proved. \square

Now we are ready to prove the assertion (a) in the theorem.

Proof of Theorem 1.4 (a). In this proof, we fix $\gamma \in (0, 1)$. Let $R_1 = R_1(\varrho_0, R_0) \in (0, R_0/4)$ be the constant from Lemma 2.2 and $\kappa = \kappa(d, \lambda, \gamma, R_0, \varrho_0) \in (0, 1/8]$ be the constant from Lemma 2.3. We denote

$$\mathcal{U} = |Du| + |p|, \quad \mathcal{G}(r) = \int_0^r \frac{\omega_{f_\alpha}^\sharp(t) + \omega_g^\sharp(t)}{t} dt.$$

We first derive L^∞ -estimates for Du and p . Let $x_0 \in \Omega$ and $0 < r \leq R_1$. We take $\theta_{x_0, r} \in \mathbb{R}$ and $\Theta_{x_0, r} \in \mathbb{R}^{d \times d}$ to be such that

$$\Phi(x_0, r) = \left(\int_{\Omega_r(x_0)} |Du - \Theta_{x_0, r}|^q + |p - \theta_{x_0, r}|^q dx \right)^{1/q}.$$

Similarly, we find $\theta_{x_0, \kappa^i r} \in \mathbb{R}$ and $\Theta_{x_0, \kappa^i r} \in \mathbb{R}^{d \times d}$ for $i \in \{1, 2, \dots\}$. Recall the assumption that $(u, p) \in C^1(\bar{\Omega})^d \times C(\bar{\Omega})$. Thus, since the right-hand side of (2.19) goes to zero as $j \rightarrow \infty$, we see that

$$\lim_{i \rightarrow \infty} \theta_{x_0, \kappa^i r} = p(x_0), \quad \lim_{i \rightarrow \infty} \Theta_{x_0, \kappa^i r} = Du(x_0). \quad (2.24)$$

By averaging the inequality

$$|\Theta_{x_0, \kappa r} - \Theta_{x_0, r}|^q \leq |Du - \Theta_{x_0, \kappa r}|^q + |Du - \Theta_{x_0, r}|^q$$

on $\Omega_{\kappa r}(x_0)$ and taking the q -th root, we have

$$|\Theta_{x_0, \kappa r} - \Theta_{x_0, r}| \lesssim \Phi(x_0, \kappa r) + \Phi(x_0, r).$$

Similarly, we have $|\theta_{x_0, \kappa r} - \theta_{x_0, r}| \lesssim \Phi(x_0, \kappa r) + \Phi(x_0, r)$. Thus by iterating and (2.24), we have

$$|Du(x_0) - \Theta_{x_0, r}| + |p(x_0) - \theta_{x_0, r}| \lesssim \sum_{j=0}^{\infty} \Phi(x_0, \kappa^j r). \quad (2.25)$$

This inequality together with Lemma 2.3 (i) implies

$$\begin{aligned} & |Du(x_0) - \Theta_{x_0, r}| + |p(x_0) - \theta_{x_0, r}| \\ & \lesssim r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x_0))} + \|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x_0))} \int_0^r \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \\ & \quad + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} \int_0^r \frac{\varrho_0^\sharp(t)}{t} dt + \mathcal{G}(r). \end{aligned}$$

Note that

$$|\Theta_{x_0, r}| + |\theta_{x_0, r}| \lesssim \Phi(x_0, r) + r^{-d} \|\mathcal{U}\|_{L^1(\Omega_r(x_0))} \lesssim r^{-d} \|\mathcal{U}\|_{L^1(\Omega_r(x_0))}.$$

Combining the above two inequalities, we have

$$\begin{aligned} \mathcal{U}(x_0) & \leq C_1 r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x_0))} + C_1 \|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x_0))} \int_0^r \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \\ & \quad + C_1 \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} \int_0^r \frac{\varrho_0^\sharp(t)}{t} dt + C_1 \mathcal{G}(r), \end{aligned}$$

where $C_1 = C_1(d, \lambda, \gamma, R_0, \varrho_0)$. We take $r_0 \in (0, R_1]$ so that

$$C_1 \int_0^{r_0} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \leq \frac{1}{3^d}.$$

Then for any $x_0 \in \Omega$ and $0 < r \leq r_0$, we have that

$$\begin{aligned} \mathcal{U}(x_0) & \leq C_1 r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x_0))} + 3^{-d} \|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x_0))} \\ & \quad + 3^{-d} \|f_\alpha\|_{L^\infty(\Omega_{3r}(x_0))} + C_1 \mathcal{G}(r). \end{aligned} \quad (2.26)$$

Here, the constant r_0 depends only on $d, \lambda, \gamma, R_0, \varrho_0$, and $\omega_{A^{\alpha\beta}}$.

Now let us fix $x_0 \in \Omega$ and $0 < R \leq R_1$. For $k \in \{2, 3, \dots\}$, we denote $r_k = R(1 - 2^{1-k})$. Since $r_{k+1} - r_k = 2^{-k}R$, we have $\Omega_{4r}(y) \subset \Omega_{r_{k+1}}(x_0)$ for any $y \in \Omega_{r_k}(x_0)$ and $r = 2^{-k-2}R$. We take k_0 sufficiently large such that $2^{-k_0-2}R_1 \leq r_0$. Then by (2.26) with $r = 2^{-k-2}R$, we have for $k \geq k_0$ that

$$\begin{aligned} \|\mathcal{U}\|_{L^\infty(\Omega_{r_k}(x_0))} & \leq C_1 \left(\frac{2^{k+2}}{R} \right)^d \|\mathcal{U}\|_{L^1(\Omega_{r_{k+1}}(x_0))} + 3^{-d} \|\mathcal{U}\|_{L^\infty(\Omega_{r_{k+1}}(x_0))} \\ & \quad + 3^{-d} \|f_\alpha\|_{L^\infty(\Omega_{r_{k+1}}(x_0))} + C_1 \mathcal{G}(R). \end{aligned}$$

By multiplying both sides of the above inequality by 3^{-dk} and summing the terms with respect to $k = k_0, k_0 + 1, \dots$, we see that

$$\begin{aligned} \sum_{k=k_0}^{\infty} 3^{-dk} \|\mathcal{U}\|_{L^\infty(\Omega_{r_k}(x_0))} &\leq CR^{-d} \|\mathcal{U}\|_{L^1(\Omega_R(x_0))} + \sum_{k=k_0+1}^{\infty} 3^{-dk} \|\mathcal{U}\|_{L^\infty(\Omega_{r_k}(x_0))} \\ &\quad + C \|f_\alpha\|_{L^\infty(\Omega_R(x_0))} + C\mathcal{G}(R), \end{aligned}$$

where each summation is finite and $C = C(d, \lambda, \gamma, R_0, \varrho_0) > 0$. By subtracting

$$\sum_{k=k_0+1}^{\infty} 3^{-dk} \|\mathcal{U}\|_{L^\infty(\Omega_{r_k}(x_0))}$$

from both sides of the above inequality, we get the following L^∞ -estimate for Du and p :

$$\|\mathcal{U}\|_{L^\infty(\Omega_{R/2}(x_0))} \leq C(R^{-d} \|\mathcal{U}\|_{L^1(\Omega_R(x_0))} + \|f_\alpha\|_{L^\infty(\Omega_R(x_0))} + \mathcal{G}(R)) \quad (2.27)$$

for any $x_0 \in \Omega$ and $R \in (0, R_1]$, where $C = C(d, \lambda, \gamma, R_0, \varrho_0, \omega_{A^{\alpha\beta}})$.

Next, we shall derive estimates of the modulus of continuity of Du and p . We first claim that for any $x \in \Omega$ and $0 < \rho \leq r \leq R_1/4$, we have

$$\begin{aligned} \sum_{j=0}^{\infty} \Phi(x, \kappa^j \rho) &\lesssim_{d, \lambda, \gamma, R_0, \varrho_0} \left(\frac{\rho}{r}\right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{10r}(x))} \\ &\quad + (\|\mathcal{U}\|_{L^\infty(\Omega_{10r}(x))} + \|f_\alpha\|_{L^\infty(\Omega_{10r}(x))}) \int_0^\rho \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\rho). \end{aligned} \quad (2.28)$$

We consider the following two cases:

$$4\rho \leq \text{dist}(x, \partial\Omega) \quad \text{and} \quad 4\rho > \text{dist}(x, \partial\Omega).$$

i. $4\rho \leq \text{dist}(x, \partial\Omega)$: Since $B_{4\rho}(x) \subset \Omega$, by Lemma 2.1 (i), we have

$$\begin{aligned} \sum_{j=0}^{\infty} \Phi(x, \kappa^j \rho) &\lesssim \Phi(x, \rho) + \|Du\|_{L^\infty(B_\rho(x))} \int_0^\rho \frac{\tilde{\omega}_{A^{\alpha\beta}}(t)}{t} dt \\ &\quad + \int_0^\rho \frac{\tilde{\omega}_{f_\alpha}(t) + \tilde{\omega}_g(t)}{t} dt. \end{aligned}$$

From Lemma 2.3 (ii), it follows that

$$\begin{aligned} \Phi(x, \rho) &\lesssim \left(\frac{\rho}{r}\right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x))} + \|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x))} (\varrho_0^\sharp(\rho) + \omega_{A^{\alpha\beta}}^\sharp(\rho)) \\ &\quad + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x))} \varrho_0^\sharp(\rho) + \omega_{f_\alpha}^\sharp(\rho) + \omega_g^\sharp(\rho). \end{aligned}$$

Combining the above two inequalities, and using the fact that

$$\tilde{\omega}_\bullet(\rho) \leq \omega_\bullet^\sharp(\rho) \lesssim \int_0^\rho \frac{\omega_\bullet^\sharp(t)}{t} dt, \quad \varrho_0^\sharp(\rho) \lesssim \int_0^\rho \frac{\varrho_0^\sharp(t)}{t} dt, \quad (2.29)$$

we get

$$\begin{aligned} \sum_{j=0}^{\infty} \Phi(x, \kappa^j \rho) &\lesssim \left(\frac{\rho}{r}\right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x))} \\ &\quad + (\|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x))} + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x))}) \int_0^\rho \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\rho). \end{aligned} \quad (2.30)$$

This inequality implies (2.28).

- ii. $4\rho > \text{dist}(x, \partial\Omega)$: Let i_0 be the integer such that $4\kappa^{i_0+1}\rho \leq \text{dist}(x, \partial\Omega) < 4\kappa^{i_0}\rho$. Since $B_{4\kappa^{i_0+1}\rho}(x) \subset \Omega$, by the same reasoning as in (2.30), we have

$$\begin{aligned} \sum_{j=i_0+1}^{\infty} \Phi(x, \kappa^j \rho) &= \sum_{j=0}^{\infty} \Phi(x, \kappa^{j+i_0+1} \rho) \lesssim \left(\frac{\kappa^{i_0+1} \rho}{r} \right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x))} \\ &+ (\|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x))} + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x))}) \int_0^{\kappa^{i_0+1}\rho} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\kappa^{i_0+1}\rho). \end{aligned}$$

Thus we get (using $\kappa^{i_0+1}\rho \leq \rho$)

$$\begin{aligned} \sum_{j=i_0+1}^{\infty} \Phi(x, \kappa^j \rho) &\lesssim \left(\frac{\rho}{r} \right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{3r}(x))} \\ &+ (\|\mathcal{U}\|_{L^\infty(\Omega_{3r}(x))} + \|f_\alpha\|_{L^\infty(\Omega_{3r}(x))}) \int_0^\rho \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\rho). \end{aligned} \quad (2.31)$$

We take $y_0 \in \partial\Omega$ such that $|y_0| = \text{dist}(x, \partial\Omega)$. We fix a coordinate system associated with y_0 satisfying (1.6). Observe that for $j \in \{0, 1, \dots, i_0\}$, we have

$$\Omega_{\kappa^j \rho}(x) \subset \Omega_{5\kappa^j \rho}(y_0).$$

Then similar to (2.20), we obtain

$$\Phi(x, \kappa^j \rho) \lesssim \Psi(y_0, 5\kappa^j \rho) + \varrho_0(5\kappa^j \rho) \|Du\|_{L^\infty(\Omega_{5\rho}(y_0))}.$$

Summing the terms with respect to $j = 0, 1, \dots, i_0$, and using the fact that

$$\sum_{j=0}^{i_0} \varrho_0(5\kappa^j \rho) \leq \sum_{j=0}^{\infty} \tilde{\varrho}_0(5\kappa^j \rho) \lesssim \int_0^{5\rho} \frac{\varrho_0^\sharp(t)}{t} dt,$$

we have

$$\sum_{j=0}^{i_0} \Phi(x, \kappa^j \rho) \lesssim \sum_{j=0}^{i_0} \Psi(y_0, 5\kappa^j \rho) + \|Du\|_{L^\infty(\Omega_{5\rho}(y_0))} \int_0^{5\rho} \frac{\varrho_0^\sharp(t)}{t} dt. \quad (2.32)$$

Recall that $0 < 5\rho \leq 5r \leq 2R_1$. Hence, by Lemma 2.2 and (2.29), we get the following two inequalities:

$$\begin{aligned} \sum_{j=0}^{i_0} \Psi(x, 5\kappa^j \rho) &\lesssim \Psi(y_0, 5\rho) \\ &+ (\|\mathcal{U}\|_{L^\infty(\Omega_{5\rho}(y_0))} + \|f_\alpha\|_{L^\infty(\Omega_{5\rho}(y_0))}) \int_0^{5\rho} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(5\rho), \\ \Psi(y_0, 5\rho) &\lesssim \left(\frac{\rho}{r} \right)^\gamma \Psi(y_0, 5r) \\ &+ (\|\mathcal{U}\|_{L^\infty(\Omega_{5r}(y_0))} + \|f_\alpha\|_{L^\infty(\Omega_{5r}(y_0))}) \int_0^{5\rho} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(5\rho). \end{aligned}$$

Combining these together, we get from (2.32) that

$$\begin{aligned} \sum_{j=0}^{i_0} \Phi(x, \kappa^j \rho) &\lesssim \left(\frac{\rho}{r} \right)^\gamma r^{-d} \|\mathcal{U}\|_{L^1(\Omega_{10r}(x))} \\ &+ (\|\mathcal{U}\|_{L^\infty(\Omega_{10r}(x))} + \|f_\alpha\|_{L^\infty(\Omega_{10r}(x))}) \int_0^\rho \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\rho), \end{aligned} \quad (2.33)$$

where we used the fact that $\Omega_{5r}(y_0) \subset \Omega_{10r}(x)$,

$$\int_0^{5\rho} \frac{\varrho_0^\sharp(t)}{t} dt \lesssim \int_0^\rho \frac{\varrho_0^\sharp(t)}{t} dt, \quad \int_0^{5\rho} \frac{\omega_\bullet^\sharp(t)}{t} dt \lesssim \int_0^\rho \frac{\omega_\bullet^\sharp(t)}{t} dt.$$

Therefore, we get (2.28) from (2.31) and (2.33).

Now we are ready to estimate the modulus of continuity of Du and p . Let $x_0 \in \Omega$ and $0 < R \leq R_1$. Let $x, y \in \Omega_{R/4}(x_0)$ with $\rho := |x - y| \leq R/40$. Then for any $z \in \Omega_\rho(x) \cap \Omega_\rho(y)$, we have

$$\begin{aligned} & |Du(x) - Du(y)|^q \\ & \leq |Du(x) - \Theta_{x,\rho}|^q + |\Theta_{x,\rho} - \Theta_{y,\rho}|^q + |Du(y) - \Theta_{y,\rho}|^q \\ & \leq 2 \sup_{y_0 \in \Omega_{R/4}(x_0)} |Du(y_0) - \Theta_{y_0,\rho}|^q + |Du(z) - \Theta_{x,\rho}|^q + |Du(z) - \Theta_{y,\rho}|^q. \end{aligned}$$

By taking average over $z \in \Omega_\rho(x) \cap \Omega_\rho(y)$ and taking the q -th root, we have

$$\begin{aligned} |Du(x) - Du(y)| & \lesssim \sup_{y_0 \in \Omega_{R/4}(x_0)} |Du(y_0) - \Theta_{y_0,\rho}| + \Phi(x, \rho) + \Phi(y, \rho) \\ & \lesssim \sup_{y_0 \in \Omega_{R/4}(x_0)} \left(\sum_{j=0}^{\infty} \Phi(y_0, \kappa^j \rho) + \Phi(y_0, \rho) \right) \\ & \lesssim \sup_{y_0 \in \Omega_{R/4}(x_0)} \sum_{j=0}^{\infty} \Phi(y_0, \kappa^j \rho) \end{aligned}$$

where we used (2.25) in the second inequality. Similarly, we get the same bound for p , and thus, by using (2.28) and the fact that

$$\Omega_{R/4}(y_0) \subset \Omega_{R/2}(x_0) \quad \text{for } y_0 \in \Omega_{R/4}(x_0),$$

we obtain

$$\begin{aligned} |Du(x) - Du(y)| + |p(x) - p(y)| & \lesssim \left(\frac{\rho}{R} \right)^\gamma R^{-d} \|\mathcal{U}\|_{L^1(\Omega_{R/2}(x_0))} \\ & + (\|\mathcal{U}\|_{L^\infty(\Omega_{R/2}(x_0))} + \|f_\alpha\|_{L^\infty(\Omega_{R/2}(x_0))}) \int_0^\rho \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + \mathcal{G}(\rho). \end{aligned}$$

Therefore, by (2.27), we have

$$\begin{aligned} & |Du(x) - Du(y)| + |p(x) - p(y)| \\ & \leq CR^{-d} \|\mathcal{U}\|_{L^1(\Omega_R(x_0))} \left(\left(\frac{|x-y|}{R} \right)^\gamma + \int_0^{|x-y|} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \right) \\ & + C \|f_\alpha\|_{L^\infty(\Omega_R(x_0))} \int_0^{|x-y|} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \\ & + C\mathcal{G}(R) \int_0^{|x-y|} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt + C\mathcal{G}(|x-y|) \end{aligned} \tag{2.34}$$

for any $x, y \in \Omega_{R/4}(x_0)$ with $|x - y| \leq R/40$, where $x_0 \in \Omega$, $0 < R \leq R_1$, and $C > 0$ is a constant depending only on $d, \lambda, \gamma, R_0, \varrho_0$, and $\omega_{A^{\alpha\beta}}$. We note that if

$x, y \in \Omega_{R/4}(x_0)$ with $|x - y| > R/40$, then by (2.27), we have

$$\begin{aligned} & |Du(x) - Du(y)| + |p(x) - p(y)| \\ & \leq C \left(\frac{|x - y|}{R} \right)^\gamma (R^{-d} \|\mathcal{U}\|_{L^1(\Omega_R(x_0))} + \|f_\alpha\|_{L^\infty(\Omega_R(x_0))} + \mathcal{G}(R)). \end{aligned} \quad (2.35)$$

The assertion (a) in Theorem 1.4 is proved. \square

We now turn to the proof of the assertion (b) in the theorem.

Proof of Theorem 1.4 (b). In this proof, we set $\gamma = \frac{1+\gamma_0}{2}$ and $\varrho_0(r) = Nr^{\gamma_0}$, where $\gamma_0 \in (0, 1)$ and $N > 0$. Let $R_1 = R_1(\varrho_0, R_0) \in (0, R_0/4)$ be the constant from Lemma 2.2 and $\kappa = \kappa(d, \lambda, \gamma, R_0, \varrho_0) \in (0, 1/8]$ be the constant from Lemma 2.3. Here, we note that

$$R_1 = R_1(\gamma_0, N, R_0) \quad \text{and} \quad \kappa = \kappa(d, \lambda, \gamma_0, N, R_0).$$

By the same reasoning as in [3, Lemma 8.1 (b)], we have

$$\tilde{\varrho}_0(r) = \varrho_0(r) + \sum_{i=1}^{\infty} \kappa^{\gamma i} (\varrho_0(\kappa^{-i}r)[\kappa^{-i}r < 1] + \varrho_0(1)[\kappa^{-i}r \geq 1]) \lesssim_{\kappa, \gamma_0, N} r^{\gamma_0}$$

and

$$\tilde{\omega}_f(r) = \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega_f(\kappa^{-i}r)[\kappa^{-i}r < 1] + \omega_f(1)[\kappa^{-i}r \geq 1]) \lesssim_{\kappa, \gamma_0} [f]_{C^{\gamma_0}(\Omega)} r^{\gamma_0}$$

for any function f satisfying $[f]_{C^{\gamma_0}(\Omega)} < \infty$ and $0 < r \leq R_1$. Then it follows from the definitions of ϱ_0^\sharp and ω_f^\sharp that

$$\varrho_0^\sharp(r) \lesssim r^{\gamma_0}, \quad \omega_f^\sharp(r) \lesssim [f]_{C^{\gamma_0}(\Omega)} r^{\gamma_0}.$$

Therefore, by (2.27), (2.34), and (2.35), we conclude that

$$\begin{aligned} & \|Du\|_{L^\infty(\Omega_{R/2}(x_0))} + \|p\|_{L^\infty(\Omega_{R/2}(x_0))} + R^{\gamma_0} ([Du]_{C^{\gamma_0}(\Omega_{R/4}(x_0))} + [p]_{C^{\gamma_0}(\Omega_{R/4}(x_0))}) \\ & \leq CR^{-d} (\|Du\|_{L^1(\Omega_R(x_0))} + \|p\|_{L^1(\Omega_R(x_0))}) \\ & \quad + C\|f_\alpha\|_{L^\infty(\Omega_R(x_0))} + CR^{\gamma_0} ([f_\alpha]_{C^{\gamma_0}(\Omega)} + [g]_{C^{\gamma_0}(\Omega)}) \end{aligned}$$

for any $x_0 \in \Omega$ and $R \in (0, R_1]$, where $C > 0$ is a constant depending only on $d, \lambda, \gamma_0, N, R_0$, and $[A^{\alpha\beta}]_{C^{\gamma_0}(\Omega)}$. This completes the proof of the assertion (b) in Theorem 1.4, and that of Theorem 1.4. \square

2.2. Proof of Theorem 1.6. To prove the theorem, we consider the following two cases:

$$2 \leq q < \infty, \quad 1 < q < 2.$$

- i. $2 \leq q < \infty$: We only need to consider the case when $q = 2$. We adapt the arguments in the proof of [6, Theorem 1.9], where the authors proved the weak type-(1, 1) estimate for $W^{1,2}$ -weak solutions to elliptic equations. By the hypothesis of the theorem, Ω is a Lipschitz domain, which implies that the $W_0^{1,2}$ -solvability of the problem

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega \\ \operatorname{div} u = g - (g)_\Omega & \text{in } \Omega \end{cases} \quad (2.36)$$

is available (see, for instance, [4, Lemma 3.2]). Define a bounded linear operator T on $L^2(\Omega)^{d \times d} \times L^2(\Omega)$ by

$$T(f_1, \dots, f_d, g) = (D_1 u, \dots, D_d u, p),$$

where $(u, p) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ is the weak solution of (2.36). To get the desired estimate (1.9), it suffices to show that T satisfies the hypothesis of the following lemma.

Lemma 2.5. *Let Ω be a bounded domain in \mathbb{R}^d satisfying*

$$|\Omega_r(x)| \geq A_0 r^d \quad \text{for all } x \in \overline{\Omega} \text{ and } r \in (0, \text{diam } \Omega]. \quad (2.37)$$

Let T be a bounded linear operator from $L^2(\Omega)^k$ to $L^2(\Omega)^k$, where $k \in \{1, 2, \dots\}$. Suppose that for any $x_0 \in \Omega$, $0 < r < \mu \text{diam } \Omega$, and $g \in \tilde{L}^2(\Omega)^k$ with $\text{supp } g \subset \Omega_r(x_0)$, we have

$$\int_{\Omega \setminus B_{cr}(x_0)} |Tg| dx \leq C \int_{\Omega_r(x_0)} |g| dx,$$

where $\mu \in (0, 1)$, $c \in (1, \infty)$, and $C \in (0, \infty)$. Then for any $t > 0$ and $f \in L^2(\Omega)^k$, we have

$$|\{x \in \Omega : |Tf(x)| > t\}| \lesssim_{d, \Omega, k, \mu, c, C, A_0} \frac{1}{t} \int_{\Omega} |f| dx.$$

Proof. See [6, Lemma 4.1]. \square

We note that by (2.1), Ω satisfies (2.37) with $A_0 = A_0(d, R_0, \varrho_0, \text{diam } \Omega)$. We claim that T satisfies the hypothesis of Lemma 2.5 with

$$\mu = \frac{1}{4} \min \left\{ 1, \frac{R_1}{\text{diam } \Omega} \right\}, \quad c = 4, \quad C = C(d, \lambda, \Omega, R_0, \varrho_0, \omega_{A^{\alpha\beta}}, C_0) > 0.$$

Here and in this proof, R_1 , κ , $\tilde{\varrho}_0$, $\tilde{\omega}_\bullet$, ϱ_0^\sharp , and ω_\bullet^\sharp are those in the proof of Theorem 1.4. Fix $x_0 \in \Omega$ and $0 < r < \mu \text{diam } \Omega$. Assume that $(u, p) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ is the weak solution of (2.36), where $f_\alpha \in \tilde{L}^2(\Omega)^d$ and $g \in \tilde{L}^2(\Omega)$ are supported in $\Omega_r(x_0)$. Let $R \in [4r, \text{diam } \Omega)$ so that $\Omega \setminus B_R(x_0) \neq \emptyset$, and let \mathcal{L}^* be the adjoint operator of \mathcal{L} , i.e.,

$$\mathcal{L}^* v = D_\alpha (A_*^{\alpha\beta} D_\beta v), \quad A_*^{\alpha\beta} = (A^{\beta\alpha})^\top.$$

Then by [4, Lemma 3.2], for given

$$\phi_\alpha \in C_0^\infty(\Omega_{2R}(x_0) \setminus B_R(x_0))^d, \quad \psi \in C_0^\infty(\Omega_{2R}(x_0) \setminus B_R(x_0)),$$

there exists a unique $(v, \pi) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ satisfying

$$\begin{cases} \mathcal{L}^* v + \nabla \pi = D_\alpha \phi_\alpha & \text{in } \Omega, \\ \text{div } v = \psi - (\psi)_\Omega & \text{in } \Omega, \end{cases} \quad (2.38)$$

and

$$\| |Dv| + |\pi| \|_{L^2(\Omega)} \lesssim_{d, \lambda, \Omega} \| |\phi_\alpha| + |\psi| \|_{L^2(\Omega_{2R}(x_0) \setminus B_R(x_0))}. \quad (2.39)$$

By applying u and v as test functions to (2.38) and (2.36), respectively, we have

$$\begin{aligned} & \int_{\Omega} (D_\alpha u \cdot \phi_\alpha + p\psi) dx \\ &= \int_{\Omega_r(x_0)} (D_\alpha v - (D_\alpha v)_{\Omega_r(x_0)}) \cdot f_\alpha + (\pi - (\pi)_{\Omega_r(x_0)}) g dx. \end{aligned} \quad (2.40)$$

Observe that

$$4r \leq \min\{R_1, R\} < \text{diam } \Omega.$$

Since $\phi_\alpha = \psi = 0$ in $\Omega_R(x_0)$, by (2.34), (2.35), and Hölder's inequality, we obtain that for any $x, y \in \Omega_r(x_0)$,

$$\begin{aligned} & |Dv(x) - Dv(y)| + |\pi(x) - \pi(y)| \\ & \leq CR^{-d/2} \| |Dv| + |\pi| \|_{L^2(\Omega_R(x_0))} \left(\left(\frac{r}{R} \right)^\gamma + \int_0^{2r} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \right), \end{aligned} \quad (2.41)$$

where $\gamma = 1/2$ and $C = C(d, \lambda, \Omega, R_0, \varrho_0, \omega_{A^{\alpha\beta}})$. Combining (2.39) – (2.41), and then using the duality, we see that

$$\int_{\Omega_{2R}(x_0) \setminus B_R(x_0)} (|Du| + |p|) dx \lesssim M \left(\left(\frac{r}{R} \right)^\gamma + \int_0^{2r} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \right), \quad (2.42)$$

where we set

$$M = \int_{\Omega_r(x_0)} (|f_\alpha| + |g|) dx.$$

Notice from (1.8) and [8, Eq. (3.5)] that

$$\tilde{\varrho}_0(\rho) + \tilde{\omega}_{A^{\alpha\beta}}(\rho) \leq C(\ln \rho)^{-2}, \quad \forall \rho \in (0, 1/2),$$

where $C = C(\gamma, \kappa, C_0) = C(d, \lambda, R_0, \varrho_0, C_0)$. Then it is routine to verify that

$$\varrho_0^\sharp(\rho) + \omega_{A^{\alpha\beta}}^\sharp(\rho) \leq C(\ln \rho)^{-2}, \quad \forall \rho \in (0, R_1],$$

and thus, we have

$$\int_0^{2r} \frac{\varrho_0^\sharp(t) + \omega_{A^{\alpha\beta}}^\sharp(t)}{t} dt \lesssim \left(\ln \frac{1}{r} \right)^{-1}.$$

This inequality together with (2.42) yields

$$\int_{\Omega_{2R}(x_0) \setminus B_R(x_0)} (|Du| + |p|) dx \lesssim \left(\left(\frac{r}{R} \right)^\gamma + \left(\ln \frac{1}{r} \right)^{-1} \right) M.$$

Let N be the smallest positive integer such that $\Omega \subset B_{2^{N+1}r}(x_0)$. By taking $R = 2^{i+1}r$, $i \in \{1, 2, \dots, N-1\}$, and using $N-1 \lesssim \ln(1/r)$, we have

$$\int_{\Omega \setminus B_{4r}(x_0)} (|Du| + |p|) dx \leq C \sum_{k=1}^{N-1} (2^{-k\gamma} + (\ln(1/r))^{-1}) M \leq CM,$$

where $C = C(d, \lambda, \Omega, R_0, \varrho_0, \omega_{A^{\alpha\beta}}, C_0)$. Therefore, the map T satisfies the hypothesis of Lemma 2.5.

- ii. $1 < q < 2$: In this case, we use an approximation argument together with the result in the first case, and the $W^{1,q}$ -estimate for the Stokes system in [4] (see also [7]). By [4, Theorem 5.1 and Corollary 5.3], the $W^{1,q}$ -estimate and solvability are available when the domain Ω has Lipschitz boundary with a small Lipschitz constant and the coefficients $A^{\alpha\beta}$ have vanishing mean oscillations (VMO):

$$\limsup_{\delta \rightarrow 0} \sup_{x \in \overline{\Omega}} \sup_{r \in (0, \delta]} \int_{B_r(x)} |A^{\alpha\beta} - (A^{\alpha\beta})_{B_r(x)}| dy = 0. \quad (2.43)$$

The coefficients $A^{\alpha\beta}$ considered in this paper are VMO in the sense that (see Remark 3.3)

$$\limsup_{\delta \rightarrow 0} \sup_{x \in \overline{\Omega}} \sup_{r \in (0, \delta]} \int_{\Omega_r(x)} |A^{\alpha\beta} - (A^{\alpha\beta})_{\Omega_r(x)}| dy = 0, \quad (2.44)$$

which is slightly weaker than (2.43). However, it is easy to check that the proofs of [4, Theorem 5.1 and Corollary 5.3] still work under the condition (2.44).

Now, we are ready to prove (1.9) when $q \in (1, 2)$. Assume that $(u, p) \in W_0^{1,q}(\Omega)^d \times \tilde{L}^q(\Omega)$ is the weak solution of (1.7), where $f_\alpha \in L^q(\Omega)^d$ and $g \in L^q(\Omega)$. Let $\{f_{\alpha,k}\} \subset L^2(\Omega)^d$ and $\{g_k\} \subset L^2(\Omega)$ be sequences such that

$$f_{\alpha,k} \rightarrow f_\alpha, \quad g_k \rightarrow g \quad \text{in } L^q(\Omega) \text{ as } k \rightarrow \infty. \quad (2.45)$$

By the $W_0^{1,2}$ -solvability of the problem (1.7), for $k \in \{1, 2, \dots\}$, there exists a unique weak solution $(u_k, p_k) \in W_0^{1,2}(\Omega)^d \times \tilde{L}^2(\Omega)$ of (1.7) with $f_{\alpha,k}$ and g_k in place of f_α and g . Then by the result in the first case, we see that

$$|\{x \in \Omega : |Du_k(x)| + |p_k(x)| > t\}| \leq \frac{C'}{t} \int_{\Omega} (|f_{\alpha,k}| + |g_k|) dx, \quad \forall t > 0,$$

where $C' = C'(d, \lambda, \Omega, R_0, \varrho_0, \omega_{A^{\alpha\beta}}, C_0)$. Moreover, since $(u - u_k, p - p_k) \in W_0^{1,q}(\Omega)^d \times \tilde{L}^q(\Omega)$ satisfies

$$\begin{cases} \mathcal{L}(u - u_k) + \nabla(p - p_k) = D_\alpha(f_\alpha - f_{\alpha,k}) & \text{in } \Omega, \\ \operatorname{div}(u - u_k) = g - g_k - (g)_\Omega + (g_k)_\Omega & \text{in } \Omega, \end{cases}$$

by the $W^{1,q}$ -estimate and (2.45), we have

$$\begin{aligned} & \|Du - Du_k\|_{L^q(\Omega)} + \|p - p_k\|_{L^q(\Omega)} \\ & \lesssim \|f_\alpha - f_{\alpha,k}\|_{L^q(\Omega)} + \|g - g_k\|_{L^q(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Observe that

$$\begin{aligned} & |\{x \in \Omega : |Du(x)| + |p(x)| > t\}| \\ & \leq |\{x \in \Omega : |Du_k(x)| + |p_k(x)| > t/2\}| \\ & \quad + |\{x \in \Omega : |Du(x) - Du_k(x)| + |p(x) - p_k(x)| > t/2\}| \\ & \lesssim_{C'} \frac{1}{t} \int_{\Omega} (|f_{\alpha,k}| + |g_k|) dx + \frac{1}{t^q} \int_{\Omega} (|Du - Du_k| + |p - p_k|)^q dx. \end{aligned}$$

Since the right-hand side of the above inequality converges to

$$\frac{1}{t} \int_{\Omega} (|f_\alpha| + |g|) dx,$$

we get the desired estimate (1.9).

The theorem is proved. \square

3. APPENDIX

In Appendix, we provide the proofs of some lemmas used in the previous section.

Lemma 3.1. *Let $\omega : (0, a] \rightarrow [0, \infty)$ be a Dini function satisfying (1.3) and (1.4). Set*

$$\tilde{\omega}(r) := \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega(\kappa^{-i} r) [\kappa^{-i} r < a] + \omega(a) [\kappa^{-i} r \geq a]),$$

where $\gamma \in (0, 1)$ and $\kappa \in (0, 1/2]$. Then $\tilde{\omega} : (0, a] \rightarrow [0, \infty)$ is also a Dini function satisfying

$$\tilde{\omega}(t) \lesssim_{c_1} \tilde{\omega}(s) \lesssim_{c_2} \tilde{\omega}(t) \quad \text{whenever } \frac{t}{2} \leq s \leq t \leq a \quad (3.1)$$

and that

$$\int_0^a \frac{\tilde{\omega}(t)}{t} dt < \infty. \quad (3.2)$$

Proof. Set

$$\hat{\omega}(r) = \begin{cases} \omega(r) & \text{if } r < a, \\ \omega(a) & \text{if } r \geq a, \end{cases}$$

and observe that

$$\tilde{\omega}(r) = \sum_{i=1}^{\infty} \kappa^{\gamma i} \hat{\omega}(\kappa^{-i} r).$$

Let $\frac{t}{2} \leq s \leq t \leq a$. To prove (3.1), it suffices to show that for any $i \in \{1, 2, \dots\}$, we have

$$\hat{\omega}(\kappa^{-i} t) \lesssim_{c_1} \hat{\omega}(\kappa^{-i} s) \lesssim_{c_2} \hat{\omega}(\kappa^{-i} t). \quad (3.3)$$

For i satisfying $\kappa^{-i} t < a$, by (1.3) and the fact that

$$\frac{\kappa^{-i} t}{2} \leq \kappa^{-i} s \leq \kappa^{-i} t,$$

we have

$$\hat{\omega}(\kappa^{-i} t) = \omega(\kappa^{-i} t) \lesssim_{c_1} \omega(\kappa^{-i} s) = \hat{\omega}(\kappa^{-i} s) \lesssim_{c_2} \omega(\kappa^{-i} t) = \hat{\omega}(\kappa^{-i} t),$$

which gives (3.3). On the other hand, for i satisfying $\kappa^{-i} t \geq a$, we consider the two cases:

$$\kappa^{-i} s < a, \quad \kappa^{-i} s \geq a.$$

If $\kappa^{-i} s < a$, then by (1.3) and the fact that

$$\frac{a}{2} \leq \kappa^{-i} s < a,$$

we have

$$\hat{\omega}(\kappa^{-i} t) = \omega(a) \lesssim_{c_1} \omega(\kappa^{-i} s) = \hat{\omega}(\kappa^{-i} s) \lesssim_{c_2} \omega(a) = \hat{\omega}(\kappa^{-i} t),$$

which implies (3.3). If $\kappa^{-i} s \geq a$, then by the definition of $\hat{\omega}$, we obtain that

$$\hat{\omega}(\kappa^{-i} t) = \hat{\omega}(\kappa^{-i} s).$$

Thus we prove that (3.3) holds. For the proof of (3.2), we refer to [5, Lemma 1]. The lemma is proved. \square

Lemma 3.2. *Let $\omega : (0, a] \rightarrow [0, \infty)$ be a Dini function satisfying (1.3) and (1.4). Then for any $\varepsilon > 0$, there exists $\delta \in (0, 1)$, depending only on c_1 and ε , such that*

$$\sup_{r \in (0, \delta]} \omega(r) < \varepsilon.$$

Proof. Observe that

$$\omega(r) \leq C_0 \inf_{s \in [r/2, r]} \omega(s) \leq C_0 \int_{r/2}^r \frac{\omega(s)}{s} ds$$

for any $r \in (0, a]$, where $C_0 = C_0(c_1)$. Therefore, for given $\varepsilon > 0$, if we take $\delta = \delta(c_1, \varepsilon) > 0$ such that

$$\int_0^\delta \frac{\omega(s)}{s} ds < \frac{\varepsilon}{C_0},$$

then $\omega(r) < \varepsilon$ for all $r \in (0, \delta]$. \square

Remark 3.3. From Remark 1.2 and Lemma 3.2, it follows that if f is of Dini mean oscillation in Ω satisfying Definition 1.1 (ii), then f has vanishing mean oscillation in the sense that

$$\lim_{\delta \rightarrow 0} \sup_{x \in \overline{\Omega}} \sup_{r \in (0, \delta]} \int_{\Omega_r(x)} |f - (f)_{\Omega_r(x)}| dy = 0.$$

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