# AN OBSERVATION ON THE POINCARÉ POLYNOMIALS OF MODULI SPACES OF ONE-DIMENSIONAL SHEAVES

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Abstract. We notice that for $0 < d \leqslant 6$ the Poincaré polynomial of Simpson moduli space $M_{dm+1}(\mathbb{P}_2)$ is divisible by the Poincaré polynomial of the projective space $\mathbb{P}_{3d-1}$. A somehow regular behaviour of the difference of the Poincaré polynomials of the Hilbert scheme of $\frac{(d-2)(d-1)}{2}$ points on $\mathbb{P}_2$ and the moduli space of Kronecker modules $N(3; d-2, d-1)$ is noticed for $d = 4, 5, 6$.



**Notations.** Fix an algebraically closed field $\Bbbk$, $\operatorname{char}\Bbbk = 0$. Let $V$ be a 3-dimensional vector space over $\Bbbk$ and let $\mathbb{P}_2 = \mathbb{P}V$ be the corresponding projective plane. Consider a linear polynomial $P(m) = dm + 1$ in $m$ with integer coefficients, $d > 0$. Let $M_{dm+1} = M_{dm+1}(\mathbb{P}_2)$ be the Simpson moduli space (cf. [16]) of semi-stable sheaves on $\mathbb{P}_2$ with Hilbert polynomial $dm + 1$.

**Moduli spaces description.** It has been shown in [12] that $M_{dm+1} \cong \mathbb{P}(S^dV^*)$ for $d = 1$, $d = 2$, and $M_{3m+1}$ is isomorphic to the universal cubic plane curve $\{(C, p) \in \mathbb{P}(S^3V^*) \times \mathbb{P}_2 \mid p \in C\}$, which is a $\mathbb{P}_8$-bundle over $\mathbb{P}_2$.

In [8], [14], and [15] the moduli spaces $M_{dm+1}(\mathbb{P}_2)$ are described in terms of stratifications for $d = 4$, $d = 5$, and $d = 6$ respectively. Similar stratifications are also obtained in [17]. A description of $M_{4m+1}(\mathbb{P}_2)$ as a blow-down of a blow-up of a certain projective bundle over a smooth 6-dimensional base is given in [3] and [11]. The moduli spaces $M_{dm+1}(\mathbb{P}_2)$ were also studied using wall-crossing techniques in [2] for $d = 4, 5$ and [1] for $d = 6$.

**Birational models.** As shown in [13], $M_{dm+1}$ is birational to a $\mathbb{P}_{3d-1}$-bundle over the moduli space of Kronecker modules $N(3; d-2, d-1)$(cf. [5, 7]). At the same time $M_{dm+1}$ is also birational to the flag Hilbert scheme $H(l, d)$ of pairs $Z \subseteq C$, where $Z$ is a zero-dimensional scheme of length $l = \frac{(d-2)(d-1)}{2}$ on a planar curve $C \subseteq \mathbb{P}_2$ of degree $d$. There is a natural morphism from $H(l, d)$ to the Hilbert scheme $\mathbb{P}_2^{[l]}$ of zero-dimensional subschemes in $\mathbb{P}_2$ of length $l$. As mentioned in [2], for $d < 6$, $H(l, d)$ is a $\mathbb{P}_{3d-1}$-bundle over $\mathbb{P}_2^{[l]}$.

**Poincaré polynomials.** Clearly,

$$P_{M_{m+1}}(t) = P_{\mathbb{P}_2}(t), \quad P_{M_{2m+1}}(t) = P_{\mathbb{P}_5}(t), \quad P_{M_{3m+1}}(t) = P_{\mathbb{P}_8}(t) \cdot P_{\mathbb{P}_2}(t).$$

The Poincaré polynomials $P_{M_{dm+1}}(t)$ of the moduli spaces $M_{dm+1}(\mathbb{P}_2)$, $d = 4, 5, 6$, have been computed by different authors using different methods. For example, for $d = 4, 5$ the corresponding values can be found in [2]. $P_{M_{6m+1}}(t)$ is computed in [1]. For completeness we provide here the corresponding expressions.

$$\begin{aligned} P_{M_{4m+1}}(t) = &1 + 2t^2 + 6t^4 + 10t^6 + 14t^8 + 15t^{10} + 16t^{12} + 16t^{14} + 16t^{16} + \\ &16t^{18} + 16t^{20} + 16t^{22} + 15t^{24} + 14t^{26} + 10t^{28} + 6t^{30} + 2t^{32} + t^{34}, \end{aligned}$$

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$$P_{M_{5m+1}}(t) = 1 + 2t^2 + 6t^4 + 13t^6 + 26t^8 + 45t^{10} + 68t^{12} + 87t^{14} + 100t^{16} + 107t^{18} + 111t^{20} + 112t^{22} + 113t^{24} + 113t^{26} + 113t^{28} + 112t^{30} + 111t^{32} + 107t^{34} + 100t^{36} + 87t^{38} + 68t^{40} + 45t^{42} + 26t^{44} + 13t^{46} + 6t^{48} + 2t^{50} + t^{52},$$

$$P_{M_{6m+1}}(t) = (1 + t^2 + 4t^4 + 7t^6 + 16t^8 + 25t^{10} + 47t^{12} + 68t^{14} + 104t^{16} + 128t^{18} + 146t^{20} + 128t^{22} + 104t^{24} + 68t^{26} + 47t^{28} + 25t^{30} + 16t^{32} + 7t^{34} + 4t^{36} + t^{38} + t^{40}) \cdot \frac{t^{36}-1}{t^2-1}.$$

**Observing a regular behaviour.** Notice that the expression for $P_{M_{6m+1}}(t)$ in [1] is given as a multiple of polynomial

$$\frac{t^{36}-1}{t^2-1} = 1 + t^2 + \cdots + t^{32} + t^{34},$$

which is the Poincaré polynomial of the projective space $\mathbb{P}_{17}$.

Decomposing the polynomials $P_{M_{4m+1}}(t)$ and $P_{M_{5m+1}}(t)$ into irreducible factors using Singular [4], we notice that for every $0 < d \leqslant 6$ the Poincaré polynomial $P_{M_{dm+1}}(t)$ is divisible by the Poincaré polynomial of the projective space $\mathbb{P}_{3d-1}(t)$, i. e., $P_{M_{dm+1}}(t)$ looks for $0 < d \leqslant 6$ as the Poincaré polynomial of a projective $\mathbb{P}_{3d-1}$-bundle over some space.

Denote

$$P_{v,d}(t) = \frac{P_{M_{dm+1}}(t)}{P_{\mathbb{P}_{3d-1}}(t)}, \quad 0 < d \leqslant 6.$$

Then

$$P_{v,1}(t) = 1, \quad P_{v,2}(t) = 1, \quad P_{v,3}(t) = 1 + t^2 + t^4,$$
$$P_{v,4}(t) = t^{12} + t^{10} + 4t^8 + 4t^6 + 4t^4 + t^2 + 1,$$
$$P_{v,5}(t) = t^{24} + t^{22} + 4t^{20} + 7t^{18} + 13t^{16} + 19t^{14} + 23t^{12} + 19t^{10} + 13t^8 + 7t^6 + 4t^4 + t^2 + 1$$
$$P_{v,6}(t) = t^{40} + t^{38} + 4t^{36} + 7t^{34} + 16t^{32} + 25t^{30} + 47t^{28} + 68t^{26} + 104t^{24} + 128t^{22} + 146t^{20} + 128t^{18} + 104t^{16} + 68t^{14} + 47t^{12} + 25t^{10} + 16t^8 + 7t^6 + 4t^4 + t^2 + 1.$$

Computing the Poincaré polynomials of $N(3; d-2, d-1)$ and $\mathbb{P}_2^{[l]}$ using the formulas from [6] and [9] and their computer algebra implementations in [10], one notices that all the coefficients of $P_{v,d}(t)$, $d = 3, 4, 5, 6$, are between the values of the corresponding Betti numbers of the moduli space of Kronecker modules $N(3; d-2, d-1)$ and the Hilbert scheme $\mathbb{P}_2^{[l]}$. More precisely,

$$P_{v,3}(t) = 1 + t^2 + t^4 = P_{N(3;1,2)}(t) = P_{\mathbb{P}_2^{[1]}}(t),$$

$$P_{v,4}(t) = P_{N(3;2,3)}(t) + t^4(1 + t^2 + t^4) = P_{\mathbb{P}_2^{[3]}}(t) - t^2(t^4+1)(t^4+t^2+1),$$

$$P_{v,5}(t) = P_{N(3;3,4)}(t) + t^4(1 + t^2 + t^4)(1 + t^2 + 3t^4 + 5t^6 + 3t^8 + t^{10} + t^{12}) = P_{\mathbb{P}_2^{[6]}}(t) - t^2(t^2+1)^2(t^{16} + 5t^{12} + 3t^{10} + 9t^8 + 3t^6 + 5t^4 + 1),$$

$$P_{v,6}(t) = P_{N(3;4,5)}(t) + t^4(1 + t^2 + t^4) \cdot f = P_{\mathbb{P}_2^{[10]}}(t) - t^2(t^4 + t^2 + 1) \cdot g,$$

where

$$\begin{aligned} f =&1 + t^2 + 4t^4 + 6t^6 + 14t^8 + 18t^{10} + 31t^{12} + 33t^{14}+\\ &31t^{16} + 18t^{18} + 14t^{20} + 6t^{22} + 4t^{24} + t^{26} + t^{28},\\ g =&1 + t^2 + 4t^4 + 8t^6 + 20t^8 + 35t^{10} + 66t^{12} + 93t^{14} + 108t^{16}+\\ &93t^{18} + 66t^{20} + 35t^{22} + 20t^{24} + 8t^{26} + 4t^{28} + t^{30} + t^{32}.\end{aligned}$$

**Questions to answer.** We formulate here some questions that seem reasonable to ask.

(1) Is it a coincidence that $P_{M_{dm+1}}(t)$ is divisible by $P_{\mathbb{P}_{3d-1}}(t)$ for $0 < d \leqslant 6$?
(2) Can one expect this also to be the case for $d > 6$?
(3) Are there meaningful geometric spaces with Poincaré polynomials $P_{v,d}(t)$?

**Remarks on the Poincaré polynomials of Hilberts schemes of points and moduli spaces of Kronecker modules.** As a somehow related side remark we share here some observations on the difference of the Poincaré polynomials of the Hilbert scheme of $l$ points on $\mathbb{P}_2$ and the moduli space of Kronecker modules $N(3; d-2, d-1)$.

Notice that the schemes $\mathbb{P}_2^{[l]}$ and $N(3; d-2, d-1)$ are birational. This can be explained as follows. Let $H'_d \subseteq H_d$ be the closed subscheme of schemes lying on a curve of degree $d-3$. It is an irreducible hypersurface in $H_d$. Let $N'_d \subseteq N_d$ be the closed subscheme consisting of the classes of Kronecker modules whose maximal minors have a common factor. Then $N_d \setminus N'_d$ is isomorphic to $H_d \setminus N'_d$, the isomorphism sends the a class of a Kronecker module to the vanishing scheme of its maximal minors.

For $d = 3$ there is clearly an isomorphism $\mathbb{P}_2^{[1]} \cong N(3; 1, 2)$. For $d = 4$ the Hilbert scheme $\mathbb{P}_2^{[3]}$ is a blow-up of $N(3; 2, 3)$ along a smooth subscheme that is isomorphic to a projective plane (cf. [6, Théorème 4]). Though the explicit description of this birational equivalence is unknown to the author for $d > 4$, we wish to provide here the following observations.

First of all consider the difference

$$\begin{aligned} P_{\mathbb{P}_2^{[3]}}(t) - P_{N(3;2,3)}(t) =&t^2(1 + t^2 + t^4)^2 = \\ &(1 + t^2 + t^4)(1 + t^2 + t^4 + t^6 - 1) = \\ &P_{\mathbb{P}_2}(t)(P_{\mathbb{P}_3}(t) - 1) = P_{\mathbb{P}_2}(t)P_{\mathbb{P}_3}(t) - P_{\mathbb{P}_2}(t),\end{aligned} \tag{1}$$

which indeed reflects the fact that $\mathbb{P}_2^{[3]}$ is obtained from $N(3; 2, 3)$ by a substitution of a subvariety isomorphic to a projective plane by a $\mathbb{P}_3$-bundle over it.

At the same time the differences

$$P_{\mathbb{P}_2^{[6]}}(t) - P_{N(3;3,4)}(t) = t^2(1 + t^2 + t^4)^2(1 + t^2 + 3t^4 + 7t^6 + 3t^8 + t^{10} + t^{12}) \tag{2}$$

and

$$P_{\mathbb{P}_2^{[10]}}(t) - P_{N(3;4,5)}(t) = t^2(1 + t^2 + t^4)^2 \cdot f, \tag{3}$$

with $f = 1+t^2+3t^4+8t^6+15t^8+26t^{10}+43t^{12}+55t^{14}+43t^{16}+26t^{18}+15t^{20}+8t^{22}+3t^{24}+t^{26}+t^{28}$, surprisingly turn out to be multiples of (1).

Concerning (2), one can easily notice that $N'_5$ contains a closed subvariety $N''$ that corresponds to the Kronecker modules with maximal minors having a common quadratic factor $q$. The corresponding points are the equivalence classes of the Kronecker modules

$$\begin{pmatrix} 0 & x_2 & -x_1 & l_0 \\ -x_2 & 0 & x_0 & l_1 \\ x_1 & -x_0 & 0 & l_2 \end{pmatrix}$$

such that $q = l_0x_0 + l_1x_1 + l_2x_2$. Here $x_0, x_1, x_2$ is a fixed basis of $V^*$. Then $N''$ is isomorphic to the space of conics, i. e., $N'' \cong \mathbb{P}_5$. Then

$$P_{\mathrm{Bl}_{N''} N_5}(t) - P_{N_5}(t) = P_{\mathbb{P}_5}(t)P_{\mathbb{P}_6}(t) - P_{\mathbb{P}_5}(t) = P_{\mathbb{P}_5}(P_{\mathbb{P}_6} - 1) = t^2 P^2_{\mathbb{P}_5} = t^2 P^2_{\mathbb{P}_2}(1+t^6)^2 = t^2 P^2_{\mathbb{P}_2}(1+2t^6+t^{12}) = t^2(1+t^2+t^4)^2(1+2t^6+t^{12})$$

because

$$P_{\mathbb{P}_5} = \frac{1-t^{12}}{1-t^2} = \frac{1-t^6}{1-t^2}(1+t^6) = P_{\mathbb{P}_2}\cdot(1+t^6).$$

So, indeed, $H_5$ seems to be not so far away from being the blow-up of $N_5$ along $N''$.

One could also expect (3) to bear some resemblances with the difference

$$P_{\mathbb{P}_9}(t)P_{\mathbb{P}_{10}}(t) - P_{\mathbb{P}_9}(t) = P_{\mathbb{P}_9}(t)(P_{\mathbb{P}_{10}}(t) - 1) = t^2 P_{\mathbb{P}_9}(t)^2$$

corresponding to a blow-up of $N_6$ at a subvariety isomorphic to the space of cubic planar curves. In this case, however, the factor $(1+t^2+t^4)^2$ does not appear immediately as a factor of $t^2P_{\mathbb{P}_9}(t)^2$.

One easily checks using [10] that for $d > 6$ the differences $P_{\mathbb{P}_2^{[l]}}(t) - P_{N(3;d-2,d-1)}(t)$ are not divisible by (1).

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