# A NOTE ON SMALL SETS OF REALS

TOMEK BARTOSZYNSKI AND SAHARON SHELAH

ABSTRACT. We construct an example of a combinatorially large measure zero set.

# 1. INTRODUCTION.

We will work in the space  $2^{\omega}$  equipped with standard topology and measure. More specifically, the topology is generated by basic open sets of form  $[s] = \{x \in$  $2^{\omega}: s \subset x$  for  $s \in 2^a$ ,  $a \in \omega^{<\omega}$ . The measure is the standard product measure such that  $\mu([s]) = 2^{-|\mathsf{dom}(s)|}$  and let  $\mathcal{N}$  be the collection of all measure zero sets.

Measure zero sets in  $2^{\omega}$  admit the following representation (see lemma 4):

- $X \in \mathcal{N}$  iff and only if there exists a sequence  $\{F_n : n \in \omega\}$  such that

- (1)  $F_n \subseteq 2^n$  for  $n \in \omega$ , (2)  $\sum_{n \in \omega} \frac{|F_n|}{2^n} < \infty$ , (3)  $X \subseteq \{x \in 2^\omega : \exists^\infty n \ x \upharpoonright n \in F_n\}.$

The main drawback of this representation is that sets  $F_n$  have overlapping domains. The following definitions from [1] and [3] offer a refinement.

(1) A set  $X \subseteq 2^{\omega}$  is small  $(X \in S)$  if there exists a sequence Definition 1.

- $\{I_n, J_n : n \in \omega\}$  such that
- (a)  $I_n \in [\omega]^{\langle \aleph_0}$  for  $n \in \omega$ ,
- (b)  $I_n \cap I_m = \emptyset$  for  $n \neq m$ , (c)  $J_n \subseteq 2^{I_n}$  for  $n \in \omega$ ,

- (d)  $\sum_{n \in \omega} \frac{|J_n|}{2^{|I_n|}} < \infty$ (e)  $X \subseteq \{x \in 2^{\omega} : \exists^{\infty} n \ x \upharpoonright I_n \in J_n\}$
- Without loss of generality we can assume that  $\{I_n : n \in \omega\}$  is a partition of  $\omega$  into finite sets. (2) We say that X is small<sup>\*</sup> (X  $\in S^*$ ) if in addition sets  $I_n$  are disjoint
- intervals, that is, if there exists a strictly increasing sequence of integers  $\{k_n : n \in \omega\}$  such that  $I_n = [k_n, k_{n+1})$  for each n.
- Let  $(I_n, J_n)_{n \in \omega}$  denote the set  $\{x \in 2^{\omega} : \exists^{\infty} n \ x \upharpoonright n \in J_n\}$ .

It is clear that  $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{N}$ .

Small sets are useful because of their combinatorial simplicity. To test that  $x \in X \in \mathcal{S}$  the real x must pass infinitely many independent tests as in Borel-Cantelli lemma. Furthermore, various structurally simple measure zero sets are small. In particular,

- (1) if  $X \in \mathcal{N}$  and  $|X| < 2^{\aleph_0}$  then  $X \in \mathcal{S}^{\star}$ , [1]
- (2) if X is contained in a countable union of closed measure zero sets then  $X \in \mathcal{S}^{\star}, [3]$

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(3) if F is a filter on  $\omega$  (interpreted as a subset of  $2^{\omega}$ ) and  $F \in \mathcal{N}$  then  $F \in \mathcal{S}^{\star}$ , [2], [4]

**Definition 2.** For families of sets  $\mathcal{A}, \mathcal{B}$  let  $\mathcal{A} \oplus \mathcal{B}$  be

$$\{X : \exists a \in \mathcal{A} \ \exists b \in \mathcal{B} \ (X \subset a \cup b)\}$$

Clearly, if  $\mathcal{J}$  is an ideal then  $\mathcal{J} \oplus \mathcal{J} = \mathcal{J}$ . Likewise,  $\mathcal{A} \cup (\mathcal{A} \oplus \mathcal{A}) \cup (\mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}) \cup \dots$ is an ideal for any  $\mathcal{A}$ .

## **Theorem 3.** [1]

 $\mathcal{S}^{\star} \oplus \mathcal{S}^{\star} = \mathcal{S} \oplus \mathcal{S} = \mathcal{N} = \mathcal{N} \oplus \mathcal{N}.$ 

The main result of this paper is to show that the above result is best possible, that is  $\mathcal{S}^* \subsetneq \mathcal{S} \subsetneq \mathcal{N}$ . It was known ([1]) that  $\mathcal{S}^* \subsetneq \mathcal{N}$ .

## 2. Preliminaries

To make the paper complete and self contained we present a review of known results.

**Lemma 4.** Suppose that  $X \subset 2^{\omega}$ . X has measure zero iff and only if there exists a sequence  $\{F_n : n \in \omega\}$  such that

- (1)  $F_n \subseteq 2^n$  for  $n \in \omega$ , (1)  $\sum_{n \in \omega} \frac{|F_n|}{2^n} < \infty,$ (3)  $X \subseteq \{x \in 2^\omega : \exists^\infty n \ x \upharpoonright n \in F_n\}.$

*Proof.*  $\leftarrow$  Note that  $\{x \in 2^{\omega} : \exists^{\infty} n \ x \upharpoonright n \in F_n\} = \bigcap_{m \in \omega} \bigcup_{n \ge m} \{x \in 2^{\omega} : x \upharpoonright n \in F_n\}$  $F_n$  }. Now,

$$\mu\left(\bigcup_{n\geq m} \{x\in 2^{\omega}: x\restriction n\in F_n\}\right) \leq \sum_{n\geq m} \mu\left(\{x\in 2^{\omega}: x\restriction n\in F_n\}\right) \leq \sum_{n\geq m} \frac{|F_n|}{2^n} \longrightarrow 0.$$

 $\longrightarrow$  If X has measure zero then there exists a sequence of open sets  $\{U_n : n \in \omega\}$ such that

(1)  $\mu(U_n) \leq 2^{-n}$ , for each n,

(2) 
$$X \subseteq \bigcap_{n \in \omega} U_n$$
.

Find a sequence of  $\{s_m^n : n, m \in \omega\}$  such that

- (1)  $s_m^n \in 2^{<\omega}$ ,
- (2)  $[s_m^n] \cap [s_k^n] = \emptyset$  when  $k \neq m$ , (3)  $U_n = \bigcup_{m \in \omega} [s_m^n].$

For  $k \in \omega$  let  $F_k = \{s_m^n : n, m \in \omega, |s_m^n| = k\}$ . Note that  $X \subseteq \{x \in 2^\omega :$  $\exists^{\infty} k \ x \upharpoonright k \in F_k$  and that  $\sum_{k \in \omega} \frac{|F_k|}{2^k} \leq \sum_{n \in \omega} \mu(U_n) \leq 1$ . 

Theorem 5. [1]  $\mathcal{S}^{\star} \oplus \mathcal{S}^{\star} = \mathcal{S} \oplus \mathcal{S} = \mathcal{N}$ .

*Proof.* Since  $\mathcal{N}$  is an ideal,  $\mathcal{N} \oplus \mathcal{N} = \mathcal{N}$ . Consequently, it suffices to show that  $\mathcal{S}^{\star} \oplus \mathcal{S}^{\star} = \mathcal{N}$ . The following theorem gives the required decomposition.

**Theorem 6** ([1]). Suppose that  $X \subseteq 2^{\omega}$  is a measure zero set. Then there exist sequences  $\langle n_k, m_k : k \in \omega \rangle$  and  $\langle J_k, J'_k : k \in \omega \rangle$  such that

(1)  $n_k < m_k < n_{k+1}$  for all  $k \in \omega$ ,

- $\begin{array}{ll} (2) \ J_{k} \subseteq 2^{[n_{k}, n_{k+1})}, \ J'_{k} \subseteq 2^{[m_{k}, m_{k+1})} \ for \ k \in \omega, \\ (3) \ the \ sets \ \left([n_{k}, n_{k+1}), J_{k}\right)_{k \in \omega} \ and \ \left([m_{k}, m_{k+1}), J'_{k}\right)_{k \in \omega} \ are \ small^{\star}, \ and \\ (4) \ X \subseteq \left([n_{k}, n_{k+1}), J_{k}\right)_{k \in \omega} \cup \left([m_{k}, m_{k+1}), J'_{k}\right)_{k \in \omega}. \end{array}$

In particular, every null set is a union of two small<sup> $\star$ </sup> sets.

*Proof.* Let  $X \subseteq 2^{\omega}$  be a null set.

We can assume that  $X \subseteq \{x \in 2^{\omega} : \exists^{\infty} n \ x \mid n \in F_n\}$  for some sequence  $\langle F_n : n \in F_n \rangle$  $\omega$  satisfying conditions of Lemma 4.

Fix a sequence of positive reals  $\langle \varepsilon_n : n \in \omega \rangle$  such that  $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ . Define two sequences  $\langle n_k, m_k : k \in \omega \rangle$  as follows:  $n_0 = 0$ ,

$$m_k = \min\left\{j > n_k : 2^{n_k} \cdot \sum_{i=j}^{\infty} \frac{|F_i|}{2^i} < \varepsilon_k\right\},\,$$

and

$$n_{k+1} = \min\left\{j > m_k : 2^{m_k} \cdot \sum_{i=j}^{\infty} \frac{|F_i|}{2^i} < \varepsilon_k\right\} \text{ for } k \in \omega.$$

Let 
$$I_k = [n_k, n_{k+1})$$
 and  $I'_k = [m_k, m_{k+1})$  for  $k \in \omega$ . Define  
 $s \in J_k \iff s \in 2^{I_k} \& \exists i \in [m_k, n_{k+1}) \exists t \in F_i \ s \restriction \mathsf{dom}(t) \cap \mathsf{dom}(s) =$ 

$$t \restriction \mathsf{dom}(t) \cap \mathsf{dom}(s)$$

Similarly

$$s \in J'_k \iff s \in 2^{I'_k} \& \exists i \in [n_{k+1}, m_{k+1}) \exists t \in F_i \ s \restriction \mathsf{dom}(t) \cap \mathsf{dom}(s) = t \restriction \mathsf{dom}(t) \cap \mathsf{dom}(s)$$

It remains to show that  $(I_k, J_k)_{k \in \omega}$  and  $(I'_k, J'_k)_{k \in \omega}$  are small sets and that their union covers X.

Consider the set  $(I_k, J_k)_{k \in \omega}$ . Notice that for  $k \in \omega$ 

$$\frac{|J_k|}{2^{I_k}} \le 2^{n_k} \cdot \sum_{i=m_k}^{n_{k+1}} \frac{|F_i|}{2^i} \le \varepsilon_k.$$

Since  $\sum_{n=1}^{\infty} \varepsilon_n < \infty$  this shows that the set  $(I_n, J_n)_{n \in \omega}$  is null. An analogous argument shows that  $(I'_k, J'_k)_{k \in \omega}$  is null. Finally, we show that

$$X \subseteq (I_n, J_n)_{n \in \omega} \cup (I'_n, J'_n)_{n \in \omega}.$$

Suppose that  $x \in X$  and let  $Z = \{n \in \omega : x \mid n \in F_n\}$ . By the choice of  $F_n$ 's set Z is infinite. Therefore, one of the sets,

$$Z \cap \bigcup_{k \in \omega} [m_k, n_{k+1})$$
 or  $Z \cap \bigcup_{k \in \omega} [n_{k+1}, m_{k+1})$ ,

is infinite. Without loss of generality we can assume that it is the first set. It follows that  $x \in (I_n, J_n)_{n \in \omega}$  because if  $x \upharpoonright n \in F_n$  and  $n \in [m_k, n_{k+1})$ , then by the definition there is  $t \in J_k$  such that  $x | [n_k, n_{k+1}) = t$ . 

Now lets turn attention to the family of small sets  $\mathcal{S}$ . Observe that the representation used in the definition of small sets is not unique. In particular, it is easy to see that

**Lemma 7.** Suppose that  $(I_n, J_n)_{n \in \omega}$  is a small set and  $\{a_k : k \in \omega\}$  is a partition of  $\omega$  into finite sets. For  $n \in \omega$  define  $I'_n = \bigcup_{l \in a_n} I_l$  and  $J'_n = \{s \in 2^{I'_n} : \exists l \in \mathcal{I}_n : \exists l \in \mathcal{I}_n$  $a_n \exists t \in J_l \ s \upharpoonright I_l = t \upharpoonright I_l \}$ . Then  $(I_n, J_n)_{n \in \omega} = (I'_n, J'_n)_{n \in \omega}$ .

**Lemma 8.** Suppose that  $(I_n, J_n)_{n \in \omega}$  and  $(I'_n, J'_n)_{n \in \omega}$  are two small sets. If  $\{I_n :$  $n \in \omega$  is a finer partition than  $\{I'_n : n \in \omega\}$ , then  $(I_n, J_n)_{n \in \omega} \cup (I'_n, J'_n)_{n \in \omega}$  is a small set.

*Proof.* Define  $I''_n = I'_n$  for  $n \in \omega$  and let

$$J_n'' = J_n' \cup \left\{ s \in 2^{I_n'} : \exists k \; \exists s \in J_k (I_k \subseteq I_n' \& s \upharpoonright I_k \in J_k) \right\}.$$
  
see that  $(I_n, J_n)_{n \in \omega} \cup (I_n, J_n)_{n \in \omega}) = (I_n'', J_n'')_{n \in \omega}.$ 

It is easy to see that  $(I_n, J_n)_{n \in \omega} \cup (I_n, J_n)_{n \in \omega}) = (I''_n, J''_n)_{n \in \omega}$ .

Since members of  $\mathcal S$  do not seem to form an ideal we are interested in characterizing instances when a union of two sets in  $\mathcal{S}$  is in  $\mathcal{S}$ .

**Theorem 9.** Suppose that  $(I_n, J_n)_{n \in \omega}$  and  $(I'_n, J'_n)_{n \in \omega}$  are two small sets and  $(I_n, J_n)_{n \in \omega} \subseteq (I'_n, J'_n)_{n \in \omega}$ . Then there exists a set  $(I''_n, J''_n)_{n \in \omega}$  such that  $(I_n, J_n)_{n \in \omega} \subseteq (I'_n, J''_n)_{n \in \omega}$ .  $(I''_n, J''_n)_{n \in \omega} \subseteq (I'_n, J'_n)_{n \in \omega}$  and partition  $\{I''_n : n \in \omega\}$  is finer than both  $\{I_n : n \in \omega\}$ and  $\{I'_n : n \in \omega\}$ .

*Proof.* Let start with the following:

**Lemma 10.** Suppose that  $(I_n, J_n)_{n \in \omega}$  and  $(I'_n, J'_n)_{n \in \omega}$  are two small sets. The following conditions are equivalent:

- (1)  $(I_n, J_n)_{n \in \omega} \subseteq (I'_n, J'_n)_{n \in \omega},$
- (2) for all but finitely many  $n \in \omega$  and for every  $s \in J_n$  there exists  $m \in \omega$  and  $t \in J'_m$  such that
  - (a)  $I_n \cap I'_m \neq \emptyset$ ,

  - (b)  $s \upharpoonright (I_n \cap I'_m) = t \upharpoonright (I_n \cap I'_m),$ (c)  $\forall u \in 2^{I'_m \setminus I_n} t \upharpoonright (I_n \cap I'_m) \cap u \in J'_m.$

*Proof.* (2)  $\rightarrow$  (1) Suppose that  $x \in (I_n, J_n)_{n \in \omega}$ . Then for infinitely many  $n, x \upharpoonright I_n \in$  $J_n$ . For all but finitely many of those n's, conditions (b) and (c) of clause (2) guarantee that for some m such that  $I_n \cap I'_m \neq \emptyset$ ,  $x \upharpoonright (I_n \cap I'_m) \cap x \upharpoonright (I'_m \setminus I_n) \in J'_m$ . Consequently,  $x \in (I'_n, J'_n)_{n \in \omega}$ .

 $\neg(2) \rightarrow \neg(1)$  Suppose that condition (2) fails. Then there exists an infinite set  $Z \subseteq \omega$  such that for each  $n \in Z$  there is  $s_n \in J_n$  such that for every m such that  $I_n \cap I'_m \neq \emptyset$  exactly one of the following conditions holds:

- (1)  $s_n \upharpoonright (I_n \cap I'_m) \neq t \upharpoonright (I_n \cap I'_m)$  for every  $t \in J'_m$ , (2) there is  $t \in J'_m$  such that  $s_n \upharpoonright (I_n \cap I'_m) = t \upharpoonright (I_n \cap I'_m)$  but for some u = u $u_{n,m} \in 2^{I'_m \setminus I_n}, t \upharpoonright (I_n \cap I'_m)^\frown u_{n,m} \notin J'_m.$

By thinning out the set Z we can assume that no set  $I'_m$  intersects two distinct sets  $I_n$  for  $n \in \mathbb{Z}$ . Also for each  $m \in \omega$  fix  $t^m \in 2^{I'_m}$  such that  $t^m \notin J'_m$ .

Let  $x \in 2^{\omega}$  be defined as follows:

$$x(l) = \begin{cases} s_n(l) & n \in Z \text{ and } l \in I_n \text{ and } u_{n,m} \text{ is not defined} \\ 0 & \text{if } n \in Z \text{ and } l \in I'_m \setminus I_n \text{ and } I_n \cap I_m \neq \emptyset \text{ and } u_{n,m} \text{ is not defined} \\ s_n(l) & \text{if } n \in Z \text{ and } l \in I_n \cap I'_m \text{ and } u_{n,m} \text{ is defined} \\ u_{n,m}(l) & \text{if } n \in Z \text{ and } l \in I'_m \setminus I_n \text{ and and } I_n \cap I_m \neq \emptyset \text{ and } u_{n,m} \text{ is defined} \\ t^m(l) & \text{if } l \in I_m \text{ and } I_m \cap I_n = \emptyset \text{ for all } n \in Z \end{cases}$$

Observe that the first two clauses define  $x \upharpoonright I'_m$  when  $I'_m \cap I_n \neq \emptyset$  for some  $n \in Z$ and  $u_{n,m}$  is undefined, the next two clauses define  $x \upharpoonright I'_m$  when  $I'_m \cap I_n \neq \emptyset$  for some  $n \in Z$  and  $u_{n,m}$  is defined, and finally the last clause defines  $x \upharpoonright I'_m$  when  $I'_m \cap I_n = \emptyset$ for all  $n \in Z$ . It is easy to see that these cases are mutually exclusive and that  $x \in (I_n, J_n)_{n \in \omega}$  since  $x \upharpoonright I_n = s_n \in J_n$  for  $n \in Z$ . Finally note that  $x \notin (I'_n, J'_n)_{n \in \omega}$ since by the choice of  $u_{n,m}$  (or property of  $s_n$ )  $x \upharpoonright I'_m \notin J'_m$  for all m.  $\Box$ 

Suppose that  $(I_n, J_n)_{n \in \omega}$  and  $(I'_n, J'_n)_{n \in \omega}$  are two small sets and  $(I_n, J_n)_{n \in \omega} \subseteq (I'_n, J'_n)_{n \in \omega}$ . Consider the partition consisting of sets  $\{I_n \cap I'_m : n, m \in \omega\}$ . For each non-empty set  $I_n \cap I'_m$  we define  $J''_{n,m} \subseteq 2^{I_n \cap I'm}$  as follows:

 $s \in J''_{n,m}$  if there is  $t \in J'_m$  such that  $s \upharpoonright (I_n \cap I'_m) = t \upharpoonright (I_n \cap I'_m)$  and for all  $u \in 2^{I'_m \setminus I_n} t \upharpoonright (I_n \cap I'_m) \cap u \in J'_m$ .

Observe that the definition of  $J_{n,m}''$  does not depend on  $J_n$ . Note that

$$\sum_{m,n\in\omega,I_n\cap I'_m\neq\emptyset} \frac{|J''_{m,n}|}{2^{|I_n\cap I'_m|}} = \sum_{m\in\omega} \sum_{n\in\omega,I_n\cap I'_m\neq\emptyset} \frac{|J''_{m,n}|}{2^{|I_n\cap I'_m|}} =$$
$$\sum_{m\in\omega} \sum_{n\in\omega,I_n\cap I'_m\neq\emptyset} \frac{|J''_{n,m}|\cdot 2^{|I'_m\setminus I_n|}}{2^{|I''_k|}\cdot 2^{|I'_m\setminus I_n|}} \le \sum_{m\in\omega} \frac{|J''_m|}{2^{|I'_m|}} < \infty.$$

To finish the proof observe that for  $x \in 2^{\omega}$ , whenever  $x \upharpoonright (I_n \cap I'_m) \in J''_{n,m}$  then  $x \upharpoonright I'_m \in J'_m$ . Similarly, if  $x \upharpoonright I_n \in J_n$  then by Lemma 10 there is m such that  $x \upharpoonright (I_n \cap I'_m) \in J''_{m,n}$ . It follows that  $(I_n, J_n)_{n \in \omega} \subseteq (I_{n,m}, J''_{n,m})_{n,m \in \omega} \subseteq (I'_m, J'_m)_{m \in \omega}$ .  $\Box$ 

## 3. Small sets versus measure zero sets

In this section we will prove the main result.

**Theorem 11.** There exists a null set which is not small, that is  $S \subsetneq \mathcal{N}$ .

*Proof.* We will use the following:

**Lemma 12.** For every  $\varepsilon > 0$  and sufficiently large  $n \in \omega$  there exists a set  $A \subset 2^n$ such that  $\frac{|A|}{2^n} < \varepsilon$  and for every  $u \subset n$  such that  $\frac{n}{4} \leq |u| \leq \frac{3n}{4}$ , and  $B_0 \subset 2^u$  and  $B_1 \subset 2^{n\setminus u}$  such that  $\frac{|B_0|}{2^{|u|}} \geq \frac{1}{2}$  and  $\frac{|B_1|}{2^{|n\setminus u|}} \geq \frac{1}{2}$  we have  $(B_0 \times B_1) \cap A \neq \emptyset$ .

*Proof.* The key case is when  $\varepsilon$  is very small and sets  $B_0, B_1$  have relative measure approximately  $\frac{1}{2}$ . In such case  $B_0 \times B_2$  has relative measure  $\frac{1}{4}$  yet it intersects A. Fix large  $n \in \omega$  and choose  $A \subset 2^n$  randomly. That is, for each  $s \in 2^n$ , the probability  $\operatorname{Prob}(s \in A) = \varepsilon$  and for  $s, s' \in 2^n$ , events  $s \in A$  and  $s' \in A$  are independent. It is well known that for a large enough n the set constructed this way will have measure  $\varepsilon$  (with negligible error).

Fix  $n/4 \le |u| \le 3n/4$  and let

$$\mathcal{B}_{u} = \left\{ (B_{0}, B_{1}) : B_{0} \subset 2^{u}, B_{1} \subset 2^{n \setminus u} \text{ and } \frac{|B_{0}|}{2^{|u|}}, \frac{|B_{1}|}{2^{|n \setminus u|}} \geq \frac{1}{2} \right\}.$$

Note that  $|\mathcal{B}_u| \le 2^{2^{|u|} + 2^{|n \setminus u|}} \le 2^{2^{\frac{3n}{4} + 1}}.$ 

For  $(B_0, B_1) \in \mathcal{B}_u$ ,  $\mathsf{Prob}((B_0 \times B_1) \cap A = \emptyset) = (1 - \varepsilon)^{|B_0 \times B_1|} \leq (1 - \varepsilon)^{2^{n-2}}$ . Consequently.

 $\mathsf{Prob}(\exists (B_0, B_1) \in \mathcal{B}_u \ (B_0 \times B_1) \cap A = \emptyset) \le |\mathcal{B}_u| (1 - \varepsilon)^{2^{n-2}} \le 2^{2^{\frac{3n}{4}+1}} (1 - \varepsilon)^{2^{n-2}}.$ Finally, since we have at most  $2^n$  possible sets u,

$$\begin{aligned} \mathsf{Prob}(\exists u \; \exists (B_0, B_1) \in \mathcal{B}_u \; (B_0 \times B_1) \cap A = \emptyset) \leq \\ 2^n |\mathcal{B}_u| (1-\varepsilon)^{2^{n-2}} \leq 2^{2^{\frac{3n}{4}} + n + 1} (1-\varepsilon)^{2^{n-2}} \leq 2^{2^{\frac{7n}{8}}} (1-\varepsilon)^{2^{n-2}} \leq \\ 2^{2^{\frac{7n}{8}}} (1-\varepsilon)^{\frac{1}{\varepsilon} \epsilon 2^{n-2}} \leq \frac{2^{2^{\frac{7n}{8}}}}{2^{\varepsilon 2^{n-2}}} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore there is a non-zero probability that a randomly chosen set A has the required properties. In particular, such a set must exist.

Let  $\{k_n^0, k_n^1 : n \in \omega\}$  be two sequences defined as  $k_n^0 = n(n+1)$  and  $k_n^1 = n^2$  for n > 0.

Let  $I_n^0 = [k_n^0, k_{n+1}^0)$  and  $I_n^1 = [k_n^1, k_{n+1}^1)$  for  $n \in \omega$ . Observe that the sequences are selected such that

- $\begin{array}{ll} (1) & |I_n^0| = 2n+2 \text{ and } |I_n^1| = 2n+1 \text{ for } n \in \omega, \\ (2) & I_n^0 \subset I_n^1 \cup I_{n+1}^1 \text{ for } n > 0, \\ (3) & I_n^1 \subset I_{n-1}^0 \cup I_n^0 \text{ for } n > 1, \\ (4) & |I_n^0 \cap I_n^1| = |I_n^1 \cap I_{n-1}^0| = n \text{ for } n > 1, \\ (5) & |I_n^0 \cap I_{n+1}^1| = |I_n^1 \cap I_n^0| = n+1 \text{ for } n > 1. \end{array}$

Finally, for n > 0 let  $J_n^0 \subset 2^{I_n^0}$  and  $J_n^1 \subset 2^{I_n^1}$  be selected as in Lemma 12 for  $\varepsilon_n = \frac{1}{n^2}$ . Easy calculation shows that for  $n \ge 140$  the sets  $J_n^0$  and  $J_n^1$  are defined and have the required properties.

Suppose that  $(I_n^0, J_n^0)_{n \in \omega} \cup (I_n^1, J_n^1)_{n \in \omega} \subset (I_n^2, J_n^2)_{n \in \omega}$ . CASE 1 There exists  $i \in \{0, 1\}$  and infinitely many  $n, m \in \omega$  such that

$$\frac{|I_m^i|}{4} \le |I_m^i \cap I_n^2| \le \frac{3|I_m^i|}{4}.$$

Without loss of generality i = 0. Let  $\{a_k : k \in \omega\}$  be a partition of  $\omega$  into finite sets. For  $n \in \omega$  define  $I'_n = \bigcup_{l \in a_n} I^2_l$  and  $J'_n = \{s \in 2^{I'_n} : \exists l \in a_n \exists t \in J^2_l \ s \upharpoonright I^2_l = t \upharpoonright I^2_l\}$ . By Lemma 7, we know that  $(I'_n, J'_n)_{n \in \omega} = (I^2_n, J^2_n)_{n \in \omega}$  no matter what is the choice of the partition  $\{a_k : k \in \omega\}$ .

Consequently, let us choose  $\{a_k : k \in \omega\}$  and an infinite set  $Z \subseteq \omega$  such that

- (1) for every  $m \in Z$  there is  $n \in \omega$  such that  $\frac{|I_m^0|}{4} \le |I_m^0 \cap I_n'| \le \frac{3|I_m^0|}{4}$ .
- (2) for every  $m \in Z$  there exists  $n \in \omega$  such that  $I_m^0 \subset I'_n \cup I'_{n+1}$ , (3) for every  $n \in \omega$  there is at most one  $m \in Z$  such that  $I_m^0 \cap I'_n \neq \emptyset$ .

To construct the required partition  $\{a_k : k \in \omega\}$  we inductively glue together sets  $I_l^2$  as follows: suppose that m is such that there is n such that  $\frac{|I_m^0|}{4} \leq |I_m^0 \cap I_n^2| \leq I_m^0$  $\frac{3|I_m^0|}{4}.$  Then we define  $a_n = \{n\}$  and  $a_{n+1} = \{u : I_m^0 \cap I_u^2 \neq \emptyset \text{ and } u \neq n\}.$  Let Z be the subset of the collection of m's selected as above that is thin enough to satisfy condition (3).

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Recall that  $(I_n^0, J_n^0)_{n \in \omega} \subseteq (I_n^2, J_n^2)_{n \in \omega} = (I'_n, J'_n)_{n \in \omega}$ . Working towards contradiction fix  $m \in \mathbb{Z}$ , and let  $I_m^0 \subseteq I'_n \cup I'_{n+1}$  (in this case  $I'_n = I^2_n$ ). By Lemma 10 it follows that if m is large enough then for every  $s \in J^0_m$ either

(1) for every  $u \in 2^{I'_n \setminus I^0_m}$  we have  $s \upharpoonright (I^0_m \cap I'_n) \cap u \in J'_n$ , or

(2) for every  $u \in 2^{I'_{n+1} \setminus I^0_m}$  we have  $s \upharpoonright (I^0_m \cap I'_{n+1}) \cap u \in J'_{n+1}$ .

Let  $J''_n = \{s \in 2^{I'_n \cap I'_n} : \forall u \in 2^{I'_n \setminus I'_n} \ s \cap u \in J'_n\}$  and  $J''_{n+1} = \{s \in 2^{I'_n \cap I'_{n+1}} :$ Let  $J_n = \{s \in \mathbb{Z} \ | \ u \in \mathbb{Z}^{I'_{n+1} \setminus I_m^0} \ s \cap u \in J'_{n+1} \}$ .  $\forall u \in 2^{I'_{n+1} \setminus I_m^0} \ s \cap u \in J'_{n+1} \}$ . Clearly  $\frac{|J''_n|}{2^{|I'_n \cap I_m^0|}} \le \frac{|J'_n|}{2^{|I'_n|}} \le \frac{1}{2}$  and  $\frac{|J''_{n+1}|}{2^{|I'_{n+1} \cap I_m^0|}} \le \frac{|J'_{n+1}|}{2^{|I'_{n+1}|}} \le \frac{1}{2}$ . Let  $B_n = 2^{I_m^0 \cap I'_n} \setminus J'_n$  and  $B_{n+1} = 2^{I_m^0 \cap I'_{n+1}} \setminus J'_{n+1}$ .

It follows that  $\frac{|B_n|}{2^{|I_m^0 \cap I_n'|}}, \frac{|B_{n+1}|}{2^{|I_m^0 \cap I_{n+1}'|}} \ge \frac{1}{2}$ . By Lemma 12 and the definition of set  $(I_m^0, J_m^0)_{m \in \omega}$  there is  $s_m \in (B_n \times B_{n+1}) \cap J_m^0$ . Consequently there is  $t_m \in 2^{I'_n \cup I'_{n+1}}$ such that  $t_m \upharpoonright I_m^0 = s_m \in J_0^m$  but  $t_m \upharpoonright I'_n \notin J'_n$  and  $t_m \upharpoonright I'_{n+1} \notin J'_{n+1}$ . For each  $n \in \omega$  choose  $r_n \in 2^{I'_n} \setminus J'_n$ . Define  $x \in 2^{\omega}$  as

$$x \restriction I'_n = \begin{cases} t_m \restriction I'_n & \text{if } I^0_m \cap I'_n \neq \emptyset \\ r_n & \text{if } I^0_m \cap I'_n = \emptyset \text{ for all } m \in Z \end{cases}$$

It follows that  $x \in (I_n^0, J_n^0)_{n \in \omega}$  but  $x \notin (I'_n, J'_n)_{n \in \omega} = (I_n^2, J_n^2)_{n \in \omega}$ , contradiction.

CASE 2 For every  $i \in \{0, 1\}$ , almost every  $n \in \omega$  and every  $m \in \omega$ ,

$$|I_n^2 \cap I_m^i| \le \frac{|I_m^i|}{4}.$$

This is quite similar to the previous case.

We inductively choose  $\{a_k : k \in \omega\}$  and define  $I'_n$ 's and  $J'_n$ 's as before. Next construct an infinite set  $Z\subseteq\omega$  such that

- (1) for every  $m \in Z$  there exists  $n \in \omega$  such that  $I_m^0 \subset I'_n \cup I'_{n+1}$  and  $\frac{|I_m^0|}{A} \leq$  $|I_m^0 \cap I_n'|, \ |I_m^0 \cap I_{n+1}'| \le \frac{3|I_m^0|}{4}.$ (2) for every  $n \in \omega$  there is at most one  $m \in Z$  such that  $I_m^0 \cap I_n' \neq \emptyset$ .

Since  $|I_k^2 \cap I_m^i| \leq \frac{|I_m^i|}{4}$  for each k, m we can get (1) by careful splitting  $\{k : I_m^0 \cap I_k^2 \neq 0\}$  $\emptyset$  into two sets.

The rest of the proof is exactly as before.

To conclude the proof it suffices to show that these two cases exhaust all possibilities. To this end we check that if for some  $i \in \{0, 1\}, m, n \in \omega, |I_n^2 \cap I_m^i| > \frac{3|I_m^1|}{4}$ then for some  $j \in \{0, 1\}$  and  $k \in \omega$ ,

$$\frac{3|I_k^j|}{4} \le |I_n^2 \cap I_k^j| \le \frac{3|I_k^j|}{4}.$$

This will show that potential remaining cases are already included in the CASE 1.

Fix i = 0 and  $n \in \omega$  (the case i = 1 is analogous.)

By the choice of intervals  $I_m^0$  and  $I_m^1$ , it follows that if  $|I_n^2 \cap I_m^0| > \frac{3|I_m^0|}{4}$  then  $|I_n^2 \cap I_m^1| > \frac{|I_m^1|}{4}$ . If  $|I_n^2 \cap I_m^1| \le \frac{3|I_m^1|}{4}$  then we are in CASE 1. Otherwise  $|I_n^2 \cap I_m^1| > \frac{3|I_m^1|}{4}$  and so  $|I_n^2 \cap I_{m+1}^0| > \frac{|I_{m+1}^1|}{4}$ . Continue inductively until the construction terminates after finitely many steps settling on j and k.

**Theorem 13.** Not every small set is small<sup>\*</sup>, that is  $S^* \subseteq S$ .

*Proof.* The proof is a modification of the previous argument.

Let  $I_n^0, I_n^1, J_n^0$  and  $J_n^1$  for  $n \in \omega$  be like in the proof of 11. Let  $\overline{I}_n^0 = \{2k : k \in I_n^0\}$ and  $\overline{I}_n^1 = \{2k + 1 : k \in I_n^1\}$  for  $n \in \omega$  and let  $\overline{J}_n^0 \subset 2^{\overline{I}_n^0}, \overline{J}_n^1 \subset 2^{\overline{I}_n^1}$  for  $n \in \omega$  be the induced sets. Note that  $(\{\overline{I}_n^0, \overline{I}_n^1\}, \{\overline{J}_n^0, \overline{J}_n^1\})_{n \in \omega}$  is a small set. We will show that this set is not small<sup>\*</sup>. Suppose that  $(\{\overline{I}_n^0, \overline{I}_n^1\}, \{\overline{J}_n^0, \overline{J}_n^1\})_{n \in \omega} \subseteq (I_n, J_n)_{n \in \omega}$ , where  $I_n = [k_n, k_{n+1})$  for an increasing sequence  $\{k_n : n \in \omega\}$ .

Without loss of generality we can assume that for every  $n\in\omega$  there exists  $i\in\{0,1\}$  and  $m\in\omega$  such that

(1)  $I_{m}^{i} \subseteq I_{n} \cup I_{n+1},$ (2)  $\frac{|I_{m}^{i}|}{4} \leq |I_{n} \cap I_{m}^{i}| \leq \frac{3|I_{m}^{i}|}{4},$ (3)  $\frac{|I_{m}^{i}|}{4} \leq |I_{n+1} \cap I_{m}^{i}| \leq \frac{3|I_{m}^{i}|}{4}.$ 

To get (1) we combine consecutive intervals  $I_n$  to make sure that each  $I_m^i$  belongs to at most two of them. Points (2) and (3) are a consequence of the properties of the original sequences  $\{I_n^0, I_n^1 : n \in \omega\}$ , namely that each integer belongs to exactly two of these intervals and that intersecting intervals cut each other approximately in half. The following example illustrates the procedure for finding *i* and *m*: If  $k_n$ is even then  $k_n/2$  belongs to  $I_j^0 \cap I_k^1$  with k - j equal to 0 or 1. The value of *i* and *m* depend on whether  $k_n/2$  belongs to the lower or upper half of the said interval. The case when  $k_n$  is odd is similar.

The rest of the proof is exactly like Case 1 of Theorem 11.

#### References

- Tomek Bartoszynski. On covering of real line by null sets. Pacific Journal of Mathematics, 131(1):1–12, 1988.
- [2] Tomek Bartoszynski. On the structure of measurable filters on a countable set. Real Analysis Exchange, 17(2):681–701, 1992.
- [3] Tomek Bartoszynski and Haim Judah. Set Theory: on the structure of the real line. A.K. Peters, 1995.
- [4] Michel Talagrand. On T. Bartoszynski structure theorem for measurable filters. Comptes Rendus Mathematique, 351(7-8):281–284, 2013.

National Science Foundation, Division of Mathematical Sciences, 2415 Eisenhower Avenue, Alexandria, VA 22314, USA

 $E\text{-}mail\ address: \texttt{tbartosz@nsf.gov}$ 

DEPARTMENT OF MATHEMATICS, HEBREW UNIVERSITY, JERUSALEM, ISRAEL *E-mail address*: shelah@math.huji.ac.il, http://math.rutgers.edu/~shelah/